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PAPER

On Some Advantages of the Predictor-Corrector Methods

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ABSTRACT

Usually, all numerical methods are divided into two sets known as explicit and implicit methods. Explicit methods (EM) are used to find a solution to a problem directly, without requiring initial preparation. But when using the implicit method (IM), other methods can sometimes be employed. Implicit methods are known to be more accurate than explicit ones. Therefore, the question arises about finding the golden mean. To accomplish this, we utilize certain properties of the predictor and corrector methods. We take into account that in forecasting methods, we use EM. However, I will show here that in some cases, IMs can be used as correction methods. It is clear that the results obtained here are fully consistent with the theoretical ones. To address the aforementioned issues, we employ the initial value problem (IVP) for a first-order ordinary differential equation (ODE). Conventional methods compare various nanomaterials (NMs) using multi-step, extended, and hybrid approaches.

KEYWORDS

initial value problem (IVP), ordinary differential equation (ODE), multi-step method (MSM), stability and degree, predictor-corrector methods (PCM), hybrid method (HM)

1 INTRODUCTION

One of the classical problems that many famous scientists, starting with Newton, have investigated is the initial value problem (IVP) for ordinary differential equations (ODEs), which is presented as follows:

$$\eta' = \psi(\xi,\eta),\ \eta(\xi_0) = \eta_0,\ \xi \in [\xi_0, \Omega]. \tag{1}$$

It is assumed that problem (1) has a unique continuous solution defined on the interval, $[\xi_0, \Omega]$, and the function, which is $\psi(\xi, \eta)$, continuous in the set of arguments, is defined in some closed set and has continuous partial derivatives up to ϑ and including. Let us divide interval $[\xi_0, \Omega]$ to Γ-parts by using the mesh-points $\xi_{p+1} = \xi_p + \gamma\ (p = \overline{0,\Gamma})$, here $0 < \gamma$ is the step size. To find numerical methods (NM) to

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study equation (1), we denote $\eta(\xi_p)$ the specific values for (1) at grid points ξ_p, and we will also denote the corresponding approximate values by η_p. Numerical solutions (NS) for the stated problem (1) have been carried out by many renowned scientists. Usually, the solution to problem (1) comes down to studying the following integral identity:

$$\eta(\xi)=\eta(\xi_0)+\int_{\xi_0}^{\xi}\psi(\tau,\eta(\tau))d\tau,\ \ x\in[\xi_0,\Omega]. \tag{2}$$

As indicated here, one can solve equation (2) by employing a quadrature formula (QF). To solve problem (1), we use a QF. To solve problem (1), many scientists have proposed using constant MSM coeffecients for convenience, as depicted below:

$$\sum_{p=0}^{i}\zeta_p\eta_{l+p}=\gamma\sum_{p=0}^{i}\varsigma_p\psi_{l+p},\ \ l=\overline{0,\Gamma-i}, \tag{3}$$

here $\psi_t=(\xi_t,\ \eta_t)$ $(t=0,1,2,\ldots)$.

This method was considered in previous works [1–10]. Some initial boundary value problems were solved in works [11] [13] [17] [31] [37–39]. The famous scientist Dahlquist, in order to assess the accuracy of method (3), highlighted that if (3) is stable and its order is equal to ϑ, then the following statement holds true:

$$\vartheta\le 2\left[\frac{i}{2}\right]+2, \tag{4}$$

and for all i there is a stable method of degree $\vartheta\le i+2$. Separately, we will note that when comparing numerical methods, the concepts of stability and degree are usually used, which are defined as:

Definition 1. (Dahlquist) We will call MSM (3) stable if the solution to the polynomial $\omega(\theta)=\zeta_i\theta^i+\zeta_{i-1}\theta^{i-1}+\ldots+\zeta_1\theta+\zeta_0$ lie in the unit circle, which does not have a multiple root on the boundary.

Definition 2. The integer value ϑ is called a power for method (3) if the asymptotic equality of the form is true:

$$\sum_{p=0}^{i}(\zeta_p\eta(\xi+p\gamma)-\gamma\varsigma_p\eta'(\xi+p\gamma))=O(\gamma^{\vartheta+1}),\ \gamma\to 0. \tag{5}$$

It can be seen that if method (3) converges, then the method can be stable, and vice versa. Therefore, stability is considered necessary and sufficient for method (3) to converge. Note that method (3) can be assumed to be true if the values of the coefficients are known. Thus, the main task in the study of the method (3) is to define the values of the coefficients. $\zeta_p,\varsigma_p(p=\overline{0,i})$.

The relationship between degree p and order i has been investigated by Bakhalov, Dahlquist, and others [1–3] [12–18].

Dahlquist justified that for all, i there exists a method1 such that $\vartheta\le 2\left[\frac{i}{2}\right]+2$, if this method is stable, it has a degree ϑ. And for all the i- there are stable methods with the degree $\vartheta_{\max}=2\left[\frac{i}{2}\right]+2$ if $\varsigma_i\ne 0$, $\zeta_i\ne 0$. If $\varsigma_i=0$ and $\zeta_i\ne 0$ only if there are stable methods with degree $\vartheta\le i$. If ϑ is even, then there are stable methods of type (3) with degree $\vartheta = i+2$, and in this case there are stable explicit methods with degree $\vartheta = i+1$. It is noted that the PCM was studied in [12] [13] [14] [15] [16] [17]. It has been proven that if the PCM is constructed in thefollowing:

$$\hat{\eta}_{l+i} = -\sum_{p=0}^{i-1} \frac{\hat{\zeta}_p}{\hat{\zeta}_i} \eta_{l+p} + h \sum_{p=0}^{i-1} \frac{\hat{\varsigma}_i}{\hat{\zeta}_i} \psi_{l+p}, \tag{6}$$

$$\eta_{l+i} = -\sum_{p=0}^{i-1} \frac{\zeta_p}{\zeta_i} \eta_{l+p} + \gamma \sum_{p=0}^{i-1} \frac{\varsigma_p}{\zeta_i} \psi_{l+p} + \gamma \frac{\varsigma_i}{\zeta_i} \psi(\xi_{l+i}, \hat{\eta}_{l+i}), \tag{7}$$

Then one can provethat for the convergence of this method, method (7) must be stable.

Noted that η_t the value of the solution of problem (1) is calculated by the predictor method. For the convergence of this method with the rate of $i + 2$, the method (6) must have the degree $i + 2$. Hence, method (6) must be unstable. Note that the method (6) can be unstable in the PCM. To illustrate the above-discribed, let's consider the following simple methods.

2 CONSTRUCTION PCMS BY USING THE KNOWN SIMPLE METHODS

To illustrate the place of the PCM in the development of NMs, consider simple methods of the form:

$$\hat{\eta}_{l+1} = \eta_l + \gamma \psi(\xi_l, \eta_l), \tag{8}$$

$$\eta_{l+1} = \mu_l + \gamma \psi(\xi_{l+1}, \eta_{l+1}). \tag{9}$$

Here, I have given two explicit and implicit Euler's methods for IMs. It is eviden that these methods are stable and possess equal capabilities $\vartheta = 1$. The method (8) is implicit; therefore, the ratio (9) can be considered nonlinear algebraic equation, the exact solution of which is difficult to find. As is known, one of the simplest methods for finding a solution to a nonlinear algebraic equation is the simple iteration method, which is represented in the following form:

$$\eta_{l+1}^{(j+1)} = \eta_l^{(j)} + \gamma \psi(\xi_{l+1}, \eta_{l+1}^{(j)}) \ (j = 0,1,2,...), \ \eta_{l+1}^{(0)} - is\ given. \tag{10}$$

By taking into account a sufficiently small step-size $\gamma > 0$, we can conclude that this process always converges. However, this process can be simplified. For this purpose, t is sufficient to replace value $\eta_{l+1}^{(0)}$ by the actual value $\hat{\eta}_{l+1}$. In this case, we obtain a result like:

$$\eta_{l+1} = \eta_l + \gamma \psi(\xi_{l+1}, \hat{\eta}_{l+1}). \tag{11}$$

From here one can be write:

$$\eta_{l+1} = \eta_l + \gamma \psi(\xi_{l+1}, \eta_l + \gamma \psi(\xi_l, \eta_l)). \tag{12}$$

Based on method (10), we will use method (8) to solve the emerging problem (1) . But by using the method of (11), there is no such need. Therefore, the processof compiling the algorithm is essential. Note that methods (8) and (11) make up the PCM.

The convergence of the PCM can be defined the approach described in [17]. Let's consider PCM based on methods 8 and 9. Likewise, these methods are robust, and they also have a degree $\vartheta = 1$. It is clear that the trapezoidal rule can be used to apply this algorithm. In this case, the following algorithm can be proposed:

$$\eta_{l+1} = \eta_l + \frac{\gamma(\psi(\xi_l, \eta_l) + \psi(\xi_{l+1}, \hat{\eta}_{l+1}))}{2}, \tag{13}$$

Here, $\hat{\eta}_{l+1}$ calculate using the formula (8). In this case, method (8) serves as the predictor, while method (9) functions as the corrector technique. Let's compare methods (11) and (13). These methods used the same forecasting method. It is clear that (11) has degree $\vartheta = 1$. But (13) has no degree $\vartheta = 2$. If we compare the expected methods, it turns out that method (13) is more accurate than method (11). The complexity of these methods is the same. Anyway, (13) is considered more accurate. Therefore, this method can be considered optimal. The most popular solution to problem (1) is the midpoint rule, which differs from other methods. For comparison, consider the following example:

$$\eta_{l+1} = \eta_l + \gamma\psi(\xi_{l+1/2}, \eta_{l+1/2}), \ l = \overline{0, \Gamma - 1}. \tag{14}$$

(14) is commonly referred to as the one-step EM. But in really, in order to find the value η_{l+1}using method (14), it is necessary to fame values η_l and $\eta_{l+1/2}$. Thus, formally, one can say that method (14) is a one-step method. However, we have showhere that this is not so. Let us in the method (14) γ replace in 2γ. Under such circumstances, (14) is described as follows:

$$\eta_{l+2} = \eta_l + \gamma\psi(\xi_{l+1}, \eta_{l+1}), \ l = \overline{0, \Gamma - 2}. \tag{15}$$

This is an explicit, stable method of degree $\vartheta = 2$. In the case under consideration, method (15) is two-step and explicit. To apply method (15) to solve the initial problem, two initial values must be known in order to apply the method for solving problem (1). We note that method (14) is used to solve similar problems. Typically, methods like (14) are referred to as fractional step size methods. Methods that possess such properties in research are considered independent. As a result, some authors proposed a new class of methods called HMs.

When comparing identities (11–14), the HMs type is considered the best among MSMs. Some of the weaknesses of these methods can be found in [18–31]. Note that point $\xi_l + \frac{\gamma}{2}$ for the method (14) is a hybrid point. To correct the aforementioned disadvantages of hypothetical models (HMs), an exception can be made. To illustrate this correction, it is proposed to use a modified version of known methods if the value of the hybrid points can be expresseas a rational number. For example, let's look at a HM of the form:

$$\eta_{l+1} = \eta_l + \frac{\gamma(\psi_l + 4\psi_{l+1/2} + \psi_{l+1})}{6}, \tag{16}$$

which is stable having degree $\vartheta = 4$. If here to replace the value $\psi_{l+1/2}$ trough ψ_{l+1} by the replace $\gamma$by 2γ, then the method will look like this:

$$\eta_{l+2} = \eta_l + \frac{\gamma(\psi_l + 4\psi_{l+1} + \psi_{l+2})}{3}, \tag{17}$$

which is the Simpson's method. Similarly, this method is also stable and its degree is $\vartheta = 4$. Now, we will consider a HM of the form

$$\eta_{l+2} = \eta_{l+1} + \frac{\gamma(5\psi_{l+3/2-\sqrt{15}/10} + 8\psi_{l+3/2} + 5\psi_{l+3/2+\sqrt{15}/10})}{18}, \tag{18}$$

This method is stable and its degree is equal to $\vartheta = 6$. In this method, a combination irrational and rational points are used. To use method (18), it is necessary to construct hybrid methods from a simple form. For simplicity, let's consider an example:

$$\eta_{l+1} = \eta_l + \frac{\gamma(\psi_{l+1} + 3\psi_{l+1/3})}{4}, \tag{19}$$

$$\eta_{l+1} = \eta_l + \frac{\gamma(\psi_l + 3\psi_{l+2/3})}{4}. \tag{20}$$

These methods reminds us of Runge-Kutta methods. For example, let us to conside the following Runge-Kutta method (RKM):

$$\eta_{l+1} = \eta_l + \frac{\gamma(K_1^{(l)} + 3K_2^{(l)})}{4}, \tag{21}$$

Here, $K_1^{(l)} = \psi(\xi_l, \eta_l)$, $K_2^{(l)} = \psi(\xi_{l+2/3}, \eta_l + 2\gamma K_1^{(l)}/3)$.

It is not difficult to prove that these quantities can be described as follows:

$$K_2^{(l)} = \psi(\xi_{l+2/3}, \eta_{l+2/3}) \text{ or } K_2^{(l)} = \psi_{l+2/3}.$$

By substituting these values into equation (21), we obtain the result from equation (20). It is easily understood that method (16) cannot be represented using explicit RKMs. Method (16) is implicit and it can be presented by using an explicit RKM with some errors. By using method (18), it is observed that there a direct relationship between the explicit RKM and the hybrid method. Note that within the class of RKMs, there are certain methods can be compared to the HMs. HMs are typically more specific than RKMs. HMs are also explicit. But these methods are more

Consider a method of the following form [32–36]

$$\eta_{l+1} = \eta_l + \frac{\gamma(\psi_{l+1/2+\zeta} + \psi_{l+1/2-\zeta})}{2}, \quad \zeta = \frac{\sqrt{3}}{6}. \tag{22}$$

This is a one-step method with $\vartheta = 4$, an explicit degree. One of the simple ways to constructstable methods with the degree is by using forward-jumping or advanced methods for study, which will be discussed inthe following section.

3 CONSTRUCTION OF SOME SIMPLE NMS WITH THE HELP OF ADVANCED METHODS

By utilizing Dahlquist's findings, we discover that the precision of stable MSMs is constrained by the value $2\left[\frac{i}{2}\right]+2$. For more accurate construction methods, V.

Ibrahimov recommends using stable and advanced techniques. Proved some theorems for defining the maximal value of the degree for the forward-jumping methods. We will look at a forward jumping method like this.

$$\sum_{p=0}^{\tau} \zeta_p \eta_{l+p} = \gamma \sum_{p=0}^{i} \varsigma_p \psi_{l+p}, \ l = \overline{0, \Gamma - i}, \ 0 < \tau < i, \tag{23}$$

V. Ibragimov studied in detail the stability of method (23) and found that if it is stable there exist stable methods with degree. Furthermore, $\vartheta \le i + t + 1$ ($t = i - \tau$ and $i \ge 3t$). For the convergence of method (23), its stability is necessary and sufficient. Obviously, if (23) it unstable, then $\vartheta \le i + \tau$ is satisfies. You will note that the maximum degree for a stable method of type (23) $k = 3$ will be equal to 5. It is known that if $\tau = i$, then the largest value of the stability degree of the method will be equal to 4. Based on this, the above method with degree will $i = 5$ be represented as:

$$\eta_{l+2} = \frac{11\eta_l + 8\eta_{l+1}}{19} + \frac{\gamma(10\psi_l + 57\psi_{l+1} + 24\psi_{l+2} - \psi_{l+3})}{57}, \tag{24}$$

local truncation error for which has the following form:

$$R_l = -\frac{11}{3420}\gamma^6 D_\delta^5 \psi \Big|_{\substack{\delta=\delta_l \\ \xi=\xi_l}} + O(\gamma^7).$$

It is clear that the complexity of applying method (24) lies in calculating the value of η_{l+3}. It is not difficult to show that if in the method of (24), the value η_{l+3} is changed by the formula:

$$\eta_{l+3} = \eta_l + \frac{\gamma(23\psi_{l+2} - 16\psi_{l+1} + 5\psi_l)}{12},$$

then (24) will be A-stability.

To illustrate some of the benefits of advanced methods, let's look at a method like:

$$\eta_{l+1} = \eta_l + \frac{\gamma(8\psi_{l+1} + 5\psi_l - \psi_{l+2})}{12}. \tag{25}$$

By using this method, one can be calculate the value of η_{l+1} "if" the values η_l and η_{l+2}, which are considered to be the solution values of the problem being studie at certain points. Therefore, some specialists refer to the method (25) as symmetrical. Noted that the method (25) is stable, but not A-stable. If in equation (25), the value is η_{l+2} to be replaced with the following form:

$$\eta_{l+2} = \eta_{l+1} + \frac{\gamma(3\psi_{l+1} - \psi_l)}{2},$$

Then the method (25) becomes A-stable. This result shows that using PCM is indeed beneficial. Such attempts were used early on by various authors [42] [43] [7] [18]. We often use such approaches to start correcting that. The following section is dedicated to the findings of our research.

4 NUMERICAL RESULTS

To substantiate the obtained methods, we study the solution of the IVP for a linear ODE of the 1st order of the form:

1. $\eta'(\xi) = \xi + \eta(\xi),\ \ \eta(0) = 1,\ \ \xi \in [0,1].$

   It is known that the ES for this task will be something like this:

$$\eta(\xi) = 2e^{\xi} - \xi - 1.$$

2. $\eta'(\xi) = \theta\eta(\xi),\ \ \eta(1) = \exp(1).$

   Similarly, the ES for this task will be something like this: $\eta(\xi) = e^{\theta\xi}$.

To solve these problems, have used the hybrid method (22), which has degree $p = 4$. Obviously, utilizing these techniques requires using some methods as a predictor. For this aim, here have used the midpoint method and one modification of Simpson's method. The results we obtained are summarized in Tables 1 and 2.

Table 1. Results for example 1

ξ_3	Step $\gamma = 0.1$	Step $\gamma = 0.02$
0.2	3.3E–8	5.4E–11
0.4	7.1E–8	1.1E–10
0.6	1.1E–7	1.8E–10
0.8	1.6E–7	2.5E–10
1.0	2.1E–7	3.4E–10

The receiving results corresponds to theoretical.

Table 2. Results for example 2

ξ_p	$\theta = 1$	$\theta = -1$	$\theta = 5$	$\theta = -5$
1.1	1.5E–7	2.0E–8	2.9E–2	1.1E–6
1.4	8.9E–7	5.8E–8	6.0E–1	7.0E–7
1.7	2.1E–6	7.5E–8	4.8E–0	2.6E–7
2.0	4.1E–6	7.9E–8	3.1E–1	8.2E–8

By these results one can receive a wide range of information about the methods constructed here.

As is known from the results presented in Tables 1 and 2, obtaining HMs can be considered appropriate

5 CONCLUSION

The range of applications for ODEs is very broad. Therefore, new papers in this field by specialists are always well received. Thus, it can be concluded that the topic of this paper is considered relevant. There are still various methods available for solving IVPs for first-order ODEs. Typically, problems arise that require the development of NMs to solve them. Here have constructed a new method for known methods to solve problems in areas of the natural sciences. Note that here, we are primarily focused on developing straightforward methods that will be more accurate than the ones currently known. Often, there is a need to determine the reliability of the results obtained. In such cases, one can employ bilateral methods. Similar methods can be found in the class of PCMs. Among the methods constructed here, there are bilateral methods. For example, Euler's explicit and implicit methods. Here we have demonstrated that PCMs offer a wide range of opportunities for constructing NMs (nanostructures) with vast possibilities. Due to the numerical experiments conducted, we can confidently state that the results align perfectly with the theoretical aspects. We hope that the results obtained here will have practical applications.

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