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The Distortion of Distributed Facility Location*

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Abstract

We study the *distributed facility location problem*, where a set of agents with positions on the line of real numbers are partitioned into disjoint districts, and the goal is to choose a point to satisfy certain criteria, such as optimize an objective function or avoid strategic behavior. A mechanism in our distributed setting works in two steps: For each district it chooses a point that is representative of the positions reported by the agents in the district, and then decides one of these representative points as the final output. We consider two classes of mechanisms: *Unrestricted* mechanisms which assume that the agents directly provide their true positions as input, and *strategyproof* mechanisms which deal with strategic agents and aim to incentivize them to truthfully report their positions. For both classes, we show tight bounds on the best possible approximation in terms of several minimization social objectives, including the well-known average social cost (average total distance of agents from the chosen point) and max cost (maximum distance among all agents from the chosen point), as well as other fairness-inspired objectives that are tailor-made for the distributed setting, in particular, the max-of-average and the average-of-max.

1 Introduction

The theory of social choice deals with the fundamental question of how to aggregate the opinions or *preferences* of diverse individuals into a collective decision. The quality of such a social decision can be measured in several ways, such as based on axiomatic properties, as is usually the case in economics, or qualitative metrics, an approach mainly stemming from the literature in computer science. The most prominent such metric is that of *distortion* [Procaccia and Rosenschein, 2006], which captures precisely the (in)efficiency of a social choice rule, or a class of such rules that often operate under some restrictions, such as the lack of expressive elicitation of the preferences of the agents.

The distortion of social choice rules (or *mechanisms*) has been a focal point of research over the past decade, for many different settings; see the recent survey of Anshelevich et al. [2021] for an overview. The vast majority of previous works assume a basic setting in which a set of agents have cardinal (i.e., numerical) preferences over a set of possible outcomes (alternatives), and the goal is to quantify the best possible distortion of mechanisms that are given as input limited information about the preferences of the agents (usually rankings that are consistent with the cardinal values) in terms of the *social welfare* objective, the total value of the agents for the chosen outcome [Anshelevich et al., 2015, Boutilier et al., 2015, Ebadian et al., 2022, Gkatzelis et al., 2020].

*This paper unifies and partially extends earlier versions that appeared in *Proceedings of the 14th International Symposium on Algorithmic Game Theory* [Filos-Ratsikas and Voudouris, 2021] and *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems* [Filos-Ratsikas et al., 2023].

In many cases, however, the situation is not that simple. For example, in elections, the agents (now voters) are naturally or artificially partitioned into districts, which elect their representatives, and based on these representatives only a final winner is chosen. More generally, the decision-making process is often *distributed*, in the sense that decisions are first made at a local level, among disjoint sets of agents, and then these decisions are aggregated into a collective outcome. These types of situations are not captured by the simple setting laid out above, and bring forward important challenges and complications when measuring the efficiency of social choice mechanisms.

To capture problems of a more complex nature like the ones mentioned above, Filos-Ratsikas et al. [2020] initiated the study of the distortion in *distributed social choice*, where decisions are made by mechanisms that operate as follows: The mechanism first chooses a representative alternative for each district according to a local election with the agents of the district as voters, and then chooses one of the representatives as the winner. In their work, Filos-Ratsikas et al. [2020] considered a setting with agents that have normalized cardinal valuations over the possible outcomes. In follow-up work, Anshelevich et al. [2022] studied the same question in the very popular *metric social choice* setting, which has dominated the literature of the distortion over the years. In this setting, agents and alternatives are points on a metric space, and distances capture either physical or ideological distances along different axes. Anshelevich et al. [2022] explored the possibilities and limitations in the design of distributed mechanisms by showing (almost) tight bounds on their distortion, not only for the social cost (total distance), but also for several other objectives which are appropriate for the distributed setting; in particular, they also consider the maximum cost of any agent, the maximum of the sum of costs of the agents in each district, and the sum of the maximum costs of the agents in each district.

Importantly, the work of Anshelevich et al. [2022] only considers a *discrete* social choice setting, in which there is a finite set of alternatives over which the agents are required to choose. Many real-world problems are better modeled as settings where there is a *continuum* of alternatives (e.g., captured by the line of real numbers). Traditionally, this setting has become known as *facility location* [Procaccia and Tennenholtz, 2013] and its centralized variant is one of the most well-studied topics in social choice theory; see the recent survey of Chan et al. [2021] for a detailed overview. In this paper, we study the distributed variant of the continuous setting. We consider two types of mechanisms: (a) mechanisms that are only constrained by the fact that they operate in a distributed environment, and (b) mechanisms that are also constrained to be *strategyproof*, i.e., they do not provide incentives to the agents to lie about their preferences. We show *tight* bounds on the distortion of distributed mechanisms within these two classes in terms of (variations of) the four social objectives considered by Anshelevich et al. [2022] as mentioned above. We highlight the distributed facility location setting that we focus on, as well as our results, in more detail below.

1.1 Setting and results

We consider a facility location setting with a set of *agents* that are positioned in the line of real numbers (capturing 1-Euclidean preferences) and are partitioned into disjoint *districts*. The composition of the districts is fixed and independent of the positions of the agents. In general, the districts are exogenous and could be assembled based on geographical locations that might be different from the positions of the agents on the line, as the latter may refer to opinions about an issue. As an example, consider the case where the position of the facility corresponds to the time at which an online meeting between groups of agents located in different cities must be scheduled. Here, the districts consist of agents in the same city, while the positions of the agents on the line correspond to their ideal times for holding the meeting.

A *distributed mechanism* takes as input the positions of the agents and outputs a single point of the line where a public facility is to be located. This decision is made as follows: For each district, the mechanism chooses a location that is representative of the positions of the agents therein. Afterwards,

	Unrestricted	Strategyproof
Average	2 (Section 3.1)	3 (Section 3.2)
Max	2 (Section 4)	2 (Section 4)
Average-of-Max	1 (Section 5.1)	$1+\sqrt{2}$ (Section 5.2)
Max-of-Average	2 (Section 6.1)	$1+\sqrt{2}$ (Section 6.2)

Table 1: Overview of our tight distortion bounds for deterministic distributed mechanisms.

it chooses the output to be one of the locations that represent the districts. The mechanism is distributed in the sense that the choice of the representative location of each district depends only on the positions reported by the agents that belong to the district. In the meeting scheduling example presented above, each district (consisting of agents located in the same city) jointly proposes a candidate meeting time as a function of the personal ideal times of the agents therein, and then one among these candidate times is chosen as the final one for the meeting.

We design deterministic distributed mechanisms that satisfy various criteria of interest and achieve the best possible distortion bounds. First, we aim to design distributed mechanisms to approximately optimize social objectives that are functions of the distances between the chosen locations and the positions of the agents. Following the work of Anshelevich et al. [2022], we focus on the following objectives:

- The average distance of the agents (*Average cost*).
- The maximum distance among all agents (*Max cost*).
- The average, over all districts, maximum agent distance in each district (*Average-of-Max cost*)
- The maximum, over all districts, average agent distance in each district (*Max-of-Average cost*).

We consider both the class of *unrestricted* mechanisms and the class of *strategyproof* mechanisms (mechanisms that do not incentivize the agents to misreport their locations). We show tight bounds on the best possible distortion of mechanisms in these classes for all aforementioned objectives of interest, which are all small constants; this showcases that the distributed nature of mechanisms in our setting leads to higher inefficiency compared to centralized mechanisms, but not by much. The precise bounds are shown in Table 1. Quite interestingly, and perhaps unexpectedly, our unrestricted mechanism for the Average-of-Max objective is *optimal*, that is, it achieves a distortion of 1. This demonstrates that, for this particular objective, the distributed nature of the decision making does not influence the quality of the decision at all, and stands in contrast to the results of Anshelevich et al. [2022] for the same objective in the discrete setting. Our strategyproof mechanisms are designed by carefully composing centralized statistics mechanisms for choosing the district representatives and the final location; in particular, depending on the objective at hand, we appropriately choose the values of two parameters p and q_d to define mechanisms that work by choosing the position of the q_d-th agent in a district d as its representative, and then select the p-th representative as the output location.

1.2 Related work

The distortion was originally defined by Procaccia and Rosenschein [2006] to quantify the loss in social welfare due to social choice mechanisms having access only to preference rankings over the possible outcomes, rather than to the complete cardinal structure of the preferences. The distortion of mechanisms has been studied for several social choice problems, including single-winner voting, multi-winner voting, participatory budgeting, and matching in both the normalized utilitarian setting [Boutilier et al., 2015, Caragiannis et al., 2017, Benadè et al., 2017, Ebadian et al., 2022, Filos-Ratsikas et al., 2014], as well as the metric setting [Anshelevich et al., 2015, Anshelevich and Postl, 2017, Caragiannis et al., 2022, Charikar and Ramakrishnan, 2022, Gkatzelis et al., 2020, Kempe, 2020, Kizilkaya and Kempe, 2022]. Recently, the notion of the distortion has been more broadly interpreted as capturing the deterioration of an aggregate objective due to limited information, giving rise to works on communication complexity [Mandal et al., 2019, 2020], query complexity [Amanatidis et al., 2021, 2022a,b, Ma et al., 2021], and other tradeoffs between information and distortion [Abramowitz et al., 2019], as well as the distortion of distributed mechanisms that we study in the present paper [Filos-Ratsikas et al., 2020, Anshelevich et al., 2022, Voudouris, 2023]. We refer the reader to the survey of Anshelevich et al. [2021] for a detailed exposition.

The literature on strategyproof facility location is also rather extensive. Procaccia and Tennenholtz [2013] were the first to study strategyproof facility location problems on the line as part of their agenda on approximate mechanism design without money. Since then, several variants of the problem have been proposed and studied, including settings in which there are several facilities to locate [Lu et al., 2009, 2010, Fotakis and Tzamos, 2016], the space of possible locations is restricted [Feldman et al., 2016, Serafino and Ventre, 2016, Kanellopoulos et al., 2023a,b], the agents have heterogeneous preferences over the facilities [Anastasiadis and Deligkas, 2018, Feigenbaum and Sethuraman, 2015, Xu et al., 2021], only some of the available facilities can be located [Deligkas et al., 2023, Elkind et al., 2022], or the aim is to optimize different objectives [Filos-Ratsikas et al., 2017, Cai et al., 2016, Zhou et al., 2022]. We refer the reader to the survey of Chan et al. [2021] for more details.

With very few exceptions, the aforementioned works on strategyproof facility location problems focus on the case where the agents are positioned on a line. Preferences induced by the line metric are often referred to as 1-*Euclidean* and have been instrumental in the development of some of the most fundamental models studied in economics [Hotelling, 1929, Downs, 1957], psychology [Coombs, 1964], political science [Stokes, 1963], and computer science [Elkind et al., 2016]. The pioneering ideas of Hotelling [1929] use the (continuous) line to explain the placement of firms on spatial markets, where the space is either physical, or might be determined by the characteristics of consumers or products. Downs [1957] then refined this model to explain the convergence of political party members towards the same points along an ideological axis, especially in bipartisan elections. In the model of Downs, the line can capture any political issue, such as the government's intervention in the economy, ranging from full control to very little or no intervention (see also [Stokes, 1963]). As another example captured by 1-Euclidean preferences, consider the question of how much environmental issues should affect the implementation of a policy, with extreme left corresponding to the opinion that "environmental issues should have no effect on the policy" and extreme right corresponding to the opinion that "if the policy has any effect on the environment whatsoever, it should not be implemented".

Generally speaking, while more general metric spaces allow for the definition of more fine-grained preferences along several dimensions, the (continuous) line metric is quite often expressive enough to accurately capture preferences on specific issues that might be put to a vote. From a purely social choice perspective, 1-Euclidean preferences have also attracted attention since they are simultaneously single-peaked and single-crossing, thus leading to several attractive properties from an axiomatic and a computational standpoint (e.g., see [Coombs, 1964, Grandmont, 1978, Elkind and Faliszewski, 2014]).

2 Preliminaries

An instance of our problem is a tuple $I = (N, \mathbf{x}, D)$, where

- N is a set of n *agents*.
- $\mathbf{x} = (x_i)_{i \in N}$ is a vector containing the *position* $x_i \in \mathbb{R}$ of agent i on the line of real numbers.
- $D = \{d_1, ..., d_k\}$ is a set of $k \geq 1$ given *districts*. Each district $d \in D$ contains a set $N_d \subseteq N$

of agents such that $N_d \cap N_{d'} = \varnothing$ and $\bigcup_{d \in D} N_d = N$. By $n_d = |N_d|$ we denote the number of agents in d; when $n_d := \lambda := n/k$ for every $d \in D$, we say that the districts are *symmetric*.

A *distributed mechanism* M is used to decide the location of a facility based on the positions reported by the agents and the composition of the districts. In particular, given an instance I, a distributed mechanism works by implementing the following two steps:

- Step 1: For each district $d \in D$, using only the positions of the agents in N_d, the mechanism chooses a *representative location* $y_d \in \mathbb{R}$ for the district.
- Step 2: Given the size and the representative locations of the districts, the mechanism outputs a single location $M(I) \in \{y_d\}_{d \in D}$ as the *winner*.

If a location z is chosen, then the distance $\delta(x_i, z) = |x_i - z|$ between the position x_i of agent i and z is the *individual cost* of agent i for z.

Remark 2.1. Embedded in the definition of distributed mechanisms are the following two properties. First, distributed mechanisms are *anonymous over districts*, meaning that for two districts in which the (reported) positions of the agents are identical, the mechanism will output the same location as the representative. Second, the mechanism is *independent over districts*, meaning that the representative chosen for a district d is independent not only of the reported positions of agents in other districts, but also of the number of other districts and their sizes. These assumptions are necessary for our lower bounds to work. At the same time, they are also (implicitly) present in previous works on distributed distortion [Filos-Ratsikas et al., 2020, Anshelevich et al., 2022], and capture the essence of the information loss in distributed decision making.

2.1 Social objectives and strategyproofness

We want to design mechanisms that output locations which are efficient according to a social objective. Let $z \in \mathbb{R}$ be any location. We consider the following four social minimization objectives:

- The *Average* cost (or *average social cost*) of location z is the average total individual cost of all agents for z:
$$\frac{1}{n} \sum_{i \in N} \delta(x_i, z) = \frac{1}{n} \sum_{d \in D} \sum_{i \in N_d} \delta(x_i, z).$$
- The *Max* cost of location z is the maximum individual cost over all agents for z:
$$\max_{i \in N} \delta(x_i, z) = \max_{d \in D} \max_{i \in N_d} \delta(x_i, z).$$
- The *Average-of-Max* cost of location z is the average sum over each district of the maximum individual cost therein:
$$\frac{1}{k} \sum_{d \in D} \left\{ \max_{i \in N_d} \delta(x_i, z) \right\}.$$
- The *Max-of-Average* cost of location z is the maximum over each district of the average total individual cost therein:
$$\max_{d \in D} \left\{ \frac{1}{n_d} \sum_{i \in N_d} \delta(x_i, z) \right\}.$$

To simplify our notation, whenever the social objective is clear from context, we will use $\text{cost}(z|I)$ to denote the cost of $z \in \mathbb{R}$ according to the objective function at hand in instance I. Whenever I is clear from context, we will drop it from notation and simply write $\text{cost}(z)$; this will mostly be done in the proofs of our upper bounds.

Another goal is to design mechanisms that are resilient to strategic manipulation, that is, they do not allow the agents to unilaterally affect the outcome in their favor (i.e., lead to a location with smaller individual cost) by reporting false positions. Formally, a mechanism is *strategyproof* if for any pair of instances $I = (N, (\mathbf{x}_{-i}, x_i), D)$ and $J = (N, (\mathbf{x}_{-i}, x_i'), D)$ that differ in the position of a single agent i, it holds that $\delta(x_i, M(I)) \leq \delta(x_i, M(J))$.

2.2 Distortion of mechanisms

The *distortion* of a distributed mechanism M with respect to some social objective (which defines the cost of each possible location) is the worst case (over all instances) of the ratio between the cost of the location chosen by the mechanism and the minimum cost of any location:

$$\sup_{I=(N,\mathbf{x},D)} \frac{\text{cost}(M(I)|I)}{\min_{z\in\mathbb{R}} \text{cost}(z|I)}$$

By definition, the distortion of is always at least 1. Our goal is to design distributed mechanisms that have an as low distortion as possible with respect to the social objectives defined above. We will consider both unrestricted mechanisms which assume that the agents act truthfully, as well as strategyproof mechanisms which aim to avoid strategic manipulations.

2.3 Useful observations

Before we proceed with the presentation of our main technical results in the upcoming sections, we first state some useful properties. The bounds on the distortion of some of our mechanisms will follow by characterizing worst-case instances, and for that we will need the inequality

$$\frac{\alpha+\gamma}{\beta+\gamma} < \frac{\alpha}{\beta}, \tag{1}$$

which holds for any $\alpha > \beta \geq 0$ and $\gamma > 0$.

Another useful observation is that any distributed mechanism with finite distortion with respect to any of the social objectives that we consider must be cardinally unanimous. Formally, a mechanism is *cardinally-unanimous* if it chooses the representative location of a district to be z whenever all agents in the district are positioned at z.

Lemma 2.2. *Any distributed mechanism that achieves finite distortion with respect to any social objective* $F \in \{$*Average*, *Max*, *Average-of-Max*, *Max-of-Average*$\}$ *must be cardinally-unanimous.*

Proof. Let M be a distributed mechanism that is *not* cardinally-unanimous. Consequently, there must exist a location z such that when all the agents of a district are positioned at z, the mechanism decides the representative location of the district to be some $y \neq z$. Now, consider an instance in which all agents (no matter which district they belong to) are positioned at z. Given the behavior of the mechanism, y is the representative location of all districts, and thus it must be the winner. However, $\text{cost}(z) = 0$ and $\text{cost}(y) > 0$ for any social objective F, and thus the distortion is infinite. So, to achieve finite distortion, any mechanism must be cardinally-unanimous. □

We next show that each member of a class of intuitive distributed mechanisms is strategyproof. Consider an ordering of the districts according to their representative locations (from left to right on the line), i.e., $y_d \le y_{d+1}$ for any $d \in [k-1]$, and assign weight $\alpha_d \ge 0$ to each district d. Fix $q_d \in [n_d]$ for any district d. Fix $t \in (0, \sum_d \alpha_d]$ and let $p(t,(\alpha_d)_d)$ be the maximum $\ell \in [k]$ such that $\sum_{d=1}^{\ell-1} \alpha_d < t$. The $p(t,(\alpha_d)_d)$-Statistic-of-q_d-Statistic mechanism works as follows: It first chooses the representative location of each district d to be the position of the q_d-th ordered agent therein, and then outputs the p-th ordered representative location as the winner. Whenever the district weights are $\alpha_d = 1$ for every district d, we have that $p = \lceil t \rceil$, and we will thus simplify our terminology and refer to these mechanisms as p-Statistic-of-q_d-Statistic; most of our strategyproof mechanisms belong to this subclass. For example, with unit weights, $q_d = \lfloor (n_d+1)/2 \rfloor$ and $p = \lfloor (k+1)/2 \rfloor$, we have a mechanism that selects the position of the (leftmost) median agent in each district to be its representative location and then selects the (leftmost) median representative location as the winner. The next lemma shows that any $p(t,(\alpha_d)_d)$-Statistic-of-q_d-Statistic mechanism is strategyproof, and will allow us to only focus on bounding the distortion in the next sections.

Lemma 2.3. *For fixed weights α_d for any district d, fixed $t \in [0, \sum_d \alpha_d]$, and fixed $q_d \in [n_d]$ for any district d, the $p(t,(\alpha_d)_d)$-Statistic-of-q_d-Statistic mechanism is strategyproof.*

Proof. Consider any instance $I = (N, \mathbf{x}, D)$ and let w be the location chosen by the mechanism. Let i be any agent that belongs to some district $d \in D$ that has representative y. If the position of i is the final winner, then i clearly has no incentive to deviate. So, without loss of generality, assume that the winner is some location $w > x_i$. Observe that to affect the outcome of the mechanism, agent i must first be able to affect the representative of d. We distinguish between the following cases.

- If $y < x_i$, then agent i would have to report a position $x_i' < y$ to change the representative of d, but such a position cannot affect the final winner since the total weight of the representatives to the left of w remains the same.

- If $y > w$, then agent i would have to report a position $x_i' > y$ to change the representative of d to x_i'. However, this again cannot affect the final winner as the total weight of the representatives to the left of w has not changed.

- If $y \in [x_i, w]$, then agent i could report $x_i' > w$ and change the representative of d to y'. If $y' \le w$, then w remains the winner since the total weight of the representatives to its left remains the same. If $y' > w$, then the total weight of the representatives to the left of w cannot increase, leading to some representative $z \ge w$ being chosen as the (new) winner. However, this would mean that the individual cost of i does not decrease (since the winner has potentially moved farther to the right).

Hence, agent i has no incentive to deviate, thus proving that the mechanism is strategyproof. $\square$

3 Average social cost

We begin with the Average social cost objective, for which we show a tight bound of 2 for the class of unrestricted mechanisms, and a tight bound of 3 for strategyproof mechanisms.

3.1 Unrestricted mechanisms

We start with the upper bound. We consider the Weighted-Median-of-TruncatedAvg mechanism, which works as follows: For each district d, the mechanism considers a set of $n_d/2$ agents ranging from the $(n_d/4+1)$-th leftmost to the $(3 \cdot n_d/4)$-th leftmost[1], and chooses their average as the representative location y_d of the district. Then, it considers a set of locations consisting of n_d copies of each

[1]For simplicity, we present the mechanism assuming that the number of agents in each district is a multiple of 4; extending the description of the mechanism and the proof is straightforward.

representative location y_d and chooses the median of them as the overall winner. See Mechanism 1 for a detailed description.

Mechanism 1: Weighted-Median-of-TruncatedAvg

for each district $d \in D$ **do**

  $S_d := \{i \in N_d : i$ is at least the $(n_d/4+1)$-th and at most the $(3 \cdot n_d/4)$-th leftmost agent$\}$;

  $y_d := \frac{\sum_{i \in S_d} x_i}{|S_d|}$;

return $w := \text{Median}_{d \in D}\{y_d^{n_d}\}$;

To bound the distortion of Weighted-Median-of-TruncatedAvg, we characterize the structure of worst-case instances, where the distortion of the mechanism is maximized and is strictly larger than 1. For such an instance I, let w be the location chosen by the mechanism when given as input a worst-case instance, and denote by o the optimal location; since the objective is the average social cost, o is the position of the median agent (or any point between the positions of the median agents in case of an even total number of agents). Without loss of generality, we assume that $w < o$; the case $w > o$ is symmetric.

We first show that there are cases where, starting from an instance with distortion strictly larger than 1, moving particular agents to appropriate intervals, leads to new instances that have strictly worse distortion. This transformation will be useful when characterizing the worst-case instances for the mechanism.

Lemma 3.1. *Let I and J be two instances that differ on the position of a single agent i, such that w is the location chosen by the mechanism when given any of the two instances as input, and o is the optimal location for I. The distortion of the mechanism when given J as input is strictly larger than its distortion when given I as input in the following cases:*

(a) i is positioned at $x_i < o$ in I, and at $x'_i \in (x_i, o]$ in J;

(b) i is positioned at $x_i > o$ in I, and at $x'_i \in [o, x_i)$ in J.

Proof. Since the optimal location o' for J satisfies the inequality $\text{cost}(o'|J) \leq \text{cost}(o|J)$, it suffices to show that

$$\frac{\text{cost}(w|I)}{\text{cost}(o|I)} < \frac{\text{cost}(w|J)}{\text{cost}(o|J)},$$

which would then imply that

$$\frac{\text{cost}(w|I)}{\text{cost}(o|I)} < \frac{\text{cost}(w|J)}{\text{cost}(o'|J)}.$$

For (a), we have that $\delta(x_i, w) \leq \delta(x_i, x'_i) + \delta(x'_i, w)$ by the triangle inequality, and also $\delta(x_i, o) = \delta(x_i, x'_i) + \delta(x'_i, o)$; recall our assumption that $w < o$. So,

$$\frac{\text{cost}(w|I)}{\text{cost}(o|I)} = \frac{\sum_{j \neq i} \delta(x_j, w) + \delta(x_i, w)}{\sum_{j \neq i} \delta(x_j, o) + \delta(x_i, o)} \leq \frac{\sum_{j \neq i} \delta(x_j, w) + \delta(x_i, x'_i) + \delta(x'_i, w)}{\sum_{j \neq i} \delta(x_j, o) + \delta(x_i, x'_i) + \delta(x'_i, o)}.$$

Since the distortion of the mechanism when given I as input is strictly larger than 1 and the distances are non-negative, we can apply Inequality (1) with $\alpha = \sum_{j \neq i} \delta(x_j, w) + \delta(x'_i, w)$, $\beta = \sum_{j \neq i} \delta(x_j, o) + \delta(x'_i, o)$ and $\gamma = \delta(x_i, x'_i)$, to obtain

$$\frac{\text{cost}(w|I)}{\text{cost}(o|I)} < \frac{\sum_{j \neq i} \delta(x_j, w) + \delta(x'_i, w)}{\sum_{j \neq i} \delta(x_j, o) + \delta(x'_i, o)} = \frac{\text{cost}(w|J)}{\text{cost}(o|J)}.$$

For (b), observe that $\delta(x_i, w) = \delta(x_i, x_i') + \delta(x_i', w)$ and $\delta(x_i, o) = \delta(x_i, x_i') + \delta(x_i', o)$. Therefore, the desired inequality again follows by appropriately applying Inequality (1). $\square$

We are now ready to show that the worst-case instance I has the following properties:

- At least $k/2$ districts have representative w (Lemma 3.2);
- o can be the only other district representative and all agents in such districts are positioned at o (Lemma 3.3).

Lemma 3.2. *In the worst-case instance I, there are no district representatives to the left of w.*

Proof. Suppose towards a contradiction that the worst-case instance I is such that there is a district d with representative $y < w$. Since y is an average of some agent positions in d, there is a set of agents $S \subseteq S_d$ with $x_i \leq w$ for every $i \in S$. We move each agent $i \in S$ to a new position x_i' such that $x_i < x_i' \leq w$ and the truncated average of the agents in d becomes w. Clearly, the outcome of the mechanism, as well as the optimal location, remain the same in the new instance; w is still the median representative, and the position of the overall median agent did not change. By Lemma 3.1(a) and since $w < o$, moving any agent $i \in S$ to $x_i' \leq w$ leads to a new instance with strictly larger distortion, which contradicts the fact that we start from a worst-case instance. $\square$

Lemma 3.3. *In the worst-case instance I, besides w, the only other district representative can be o, and all agents in such districts are positioned on o.*

Proof. Suppose towards a contradiction that the worst-case instance I is such that there exists a district d with representative $y \notin \{w, o\}$. We move every agent $i \in N_d$ from x_i to $x_i' = o$. Hence, the truncated average of the agents in d changes from y to o. By Lemma 3.2 and since w is the (weighted) median representative, we have that at least half of the multiset defined by district representatives coincide with w. Consequently, the outcome of the mechanism is not affected when we move the agents of d. The optimal location also remains the same as the median agent location does not change. By Lemma 3.1, the distortion of the new instance we obtain after moving each agent i (irrespective of whether $x_i < o$ or $x_i > o$) is strictly larger than the distortion of instance I, contradicting the fact that it is a worst-case instance. $\square$

We also argue that it suffices to focus on the case where the worst-case instance I consists of just two districts that are in fact symmetric; this will simplify the last part of our proof.

Lemma 3.4. *There exists a worst-case instance with two symmetric districts, one with representative w and one with representative o.*

Proof. Consider any worst-case instance, and let D_w and D_o denote the sets of districts that have representative w and o, respectively. We first argue that $\sum_{d \in D_w} n_d = \sum_{d \in D_o} n_d$. Note that since w is a median among all copies of representatives, we have $\sum_{d \in D_w} n_d \geq \sum_{d \in D_o} n_d$. Let us assume that $\sum_{d \in D_w} n_d > \sum_{d \in D_o} n_d$; we will reach a contradiction by creating a new instance, with strictly larger distortion, that has one additional district with $\sum_{d \in D_w} n_d - \sum_{d \in D_o} n_d$ agents positioned at o. Clearly, in this new instance the mechanism again outputs w, while the optimal location remains o. Since the agents in the newly added district contribute 0 to the optimal cost and strictly greater than 0 to the social cost of w, the distortion is strictly larger.

Now, since $\sum_{d \in D_w} n_d = \sum_{d \in D_o} n_d$ and all agents in districts with representative o are positioned at o (by Lemma 3.3), we can redistribute the agents in districts with representative o in a different set of districts, so that for any district $d \in D_w$ there is a dedicated district $d' \in D_o$ with $n_d = n_{d'}$. Note that w and o remain the same in this instance and so does the distortion. We can then, without loss of generality, limit our focus on worst-case instances with just two symmetric districts, one with representative w and one with representative o. $\square$

Having shown that it suffices to consider a worst-case instance with two symmetric districts, where district d_w has representative w while district d_o has all agents positioned at o, we now argue about the agent positions in d_w; recall that each district has size $\lambda = n/2$ in this case. Let ℓ and r be the locations of the $(\lambda/4+1)$- and $3\lambda/4$-leftmost agent, respectively, in d_w (i.e., the leftmost and rightmost location among agents in S_{d_w}). Clearly, it holds that $\ell \leq w \leq r$. We argue that $r \leq o$, and that all agents not in S_{d_w} are either at ℓ or at o.

Lemma 3.5. *In district d_w, $r \leq o$.*

Proof. Suppose towards a contradiction that the worst-case instance I is such that $r > o$ in d_w, and thus $\ell < w$. Let L be the set of agents in S_{d_w} that are positioned to the left of or at w, and R the set of agents in S_{d_w} that are positioned to the right of o. By the definition of w, for any set $Q \subseteq L$, we have

$$\begin{aligned}
w &= \frac{2}{\lambda}\left(\sum_{i\in L} x_i + \sum_{i \in R} x_i + \sum_{i\in S_{d_w}\setminus(L\cup R)} x_i\right)\\
&= \frac{2}{\lambda}\left(\sum_{i\in L} x_i + \sum_{i \in R} (x_i - o) + \sum_{i\in R} o + \sum_{i\in S_{d_w}\setminus(L\cup R)} x_i\right)\\
&= \frac{2}{\lambda}\left(\sum_{i\in L\setminus Q} x_i + \sum_{i \in Q}\left(x_i + \frac{1}{|Q|}\sum_{j\in R}(x_j - o)\right) + \sum_{i\in R} o + \sum_{i\in S_{d_w}\setminus(L\cup R)} x_i\right).
\end{aligned}$$

Consequently, there must exist a set $L_< \subseteq L$ such that $x_i + \frac{1}{|L_<|}\sum_{j\in R}(x_j - o) \leq w < o$ for every $i \in L_<$; if no such set exists, then the last expression above would be strictly larger than w. We obtain a new instance J by moving all agents in R from x_i to $x_i' = o$ and all agents in $L_<$ from x_i to $x_i' = x_i + \frac{1}{|L_<|}\sum_{j\in R}(x_j - o)$. Clearly, w is still the representative of d_w and o the optimal location. By Lemma 3.1, since all agents that moved are closer to o in J that in I, J must have distortion strictly larger than I, a contradiction. □

Lemma 3.6. *In district d_w, the $\lambda/4$ leftmost agents are positioned at ℓ and the $\lambda/4$ rightmost agents are positioned at o.*

Proof. Assume otherwise and note that all these agents are not in S_{d_w} and, hence, do not affect w. By repeatedly applying Lemma 3.1 and moving each agent i with $x_i < \ell$ to ℓ and each agent i with $x_i > r$ at o, we reach an instance with strictly larger distortion; a contradiction. □

We are finally ready to prove the main result of this section.

Theorem 3.7. *For Average, the distortion of* Weighted-Median-of-TruncatedAvg *is at most* 2.

Proof. By Lemmas 3.3, 3.4 and 3.6, we have that the 2λ agents in the worst-case instance I are distributed on the line as follows: $\lambda/4$ agents are positioned at ℓ, $5\lambda/4$ agents are positioned at o (λ agents from d_o and $\lambda/4$ agents from d_w), and $\lambda/2$ agents are positioned in $[\ell, r]$. We partition the $\lambda/2$ agents in S_{d_w} into two sets: $L = \{i \in S_{d_w} : x_i \leq w\}$ and $R = \{i \in S_{d_w} : x_i > w\}$. Since $r \leq o$ (due to Lemma 3.5) and $w = \sum_{i\in L\cup R} x_i$ (by definition), the optimal cost is

$$\begin{aligned}
\text{cost}(o|I) &= \frac{1}{2\lambda}\left(\frac{\lambda}{4}(o-\ell) + \sum_{i\in L}(o - x_i) + \sum_{i\in R}(o - x_i)\right)\\
&= \frac{1}{2\lambda}\left(\frac{\lambda}{4}(o-\ell) + \frac{\lambda}{2}(o - w)\right)
\end{aligned}$$

$$= \frac{1}{2\lambda}\left(\frac{\lambda}{4}(w-\ell) + \frac{3\lambda}{4}(o-w)\right). \tag{2}$$

Similarly, the cost of the mechanism is

$$\text{cost}(w|I) = \frac{1}{2\lambda}\left(\frac{\lambda}{4}(w-\ell) + \sum_{i\in L}(w-x_i) + \sum_{i\in R}(x_i-w) + \frac{5\lambda}{4}(o-w)\right). \tag{3}$$

By the definition of w, $\sum_{i\in L}(w-x_i) = \sum_{i\in R}(x_i-w)$. Also, again by definition, $|L| \geq 1$. If $R = \varnothing$, it must be the case that $\ell = w = r$, and the distortion is at most $5/3$ as Equations (2) and (3) are simplified to $\text{cost}(o|I) = 3(o-w)/8$ and $\text{cost}(w|I) = 5(o-w)/8$, respectively. Hence, in the rest of the proof we will assume that $|R| \geq 1$.

Since $x_i \leq o$ for each agent $i \in R$ and $|L| + |R| = \lambda/2$, we have

$$\sum_{i\in L}(w-x_i) = \sum_{i\in R}(x_i-w) \leq |R|(o-w) \Leftrightarrow o-w \geq \frac{\sum_{i\in L}(w-x_i)}{\lambda/2-|L|}.$$

Similarly, as $x_i \geq \ell$ for each agent $i \in L$, we obtain

$$\sum_{i\in L}(w-x_i) \leq |L|(w-\ell) \Leftrightarrow w-\ell \geq \frac{\sum_{i\in L}(w-x_i)}{|L|}.$$

Let $o - w = \frac{\sum_{i\in L}(w-x_i)}{\frac{\lambda}{2}-|L|} + \xi_1$ and $w - \ell = \frac{\sum_{i\in L}(w-x_i)}{|L|} + \xi_2$, where $\xi_1, \xi_2 \geq 0$. Therefore, Equations (2) and (3) can be rewritten as

$$\text{cost}(o|I) = \frac{1}{2\lambda}\left(\frac{\lambda}{4}\left(\frac{\sum_{i\in L}(w-x_i)}{|L|} + \xi_2\right) + \frac{3\lambda}{4}\left(\frac{\sum_{i\in L}(w-x_i)}{\frac{\lambda}{2}-|L|} + \xi_1\right)\right)$$

$$\text{cost}(w|I) = \frac{1}{2\lambda}\left(\frac{\lambda}{4}\left(\frac{\sum_{i\in L}(w-x_i)}{|L|} + \xi_2\right) + \frac{5\lambda}{4}\left(\frac{\sum_{i\in L}(w-x_i)}{\frac{\lambda}{2}-|L|} + \xi_1\right) + 2\sum_{i\in L}(w-x_i)\right).$$

It is not hard to see that, unless the distortion is at most $5/3$ and the claim holds trivially, the ratio is maximized when $\xi_1 = \xi_2 = 0$. We can then obtain the following upper bound on the distortion.

$$\frac{\text{cost}(w|I)}{\text{cost}(o|I)} \leq \frac{\frac{\lambda}{4|L|} + \frac{5\lambda}{2\lambda-4|L|} + 2}{\frac{\lambda}{4|L|} + \frac{3\lambda}{2\lambda-4|L|}} \leq 2,$$

where the last inequality follows since $\frac{\lambda}{4|L|} + \frac{5\lambda}{2\lambda-4|L|} + 2 \leq 2\left(\frac{\lambda}{4|L|} + \frac{3\lambda}{2\lambda-4|L|}\right) \Leftrightarrow (\lambda - 4|L|)^2 \geq 0$. This concludes the proof. $\square$

We now argue that Weighted-Median-of-TruncatedAvg is best possible by showing a matching lower bound of 2 on the distortion of all unrestricted mechanisms. To show the lower bound, we will argue about the representative locations that any mechanism with small distortion (less than 2) must choose when given as input particular symmetric districts. To simplify the presentation of our proofs, we will consider the total distance as the objective rather than the average; the ratio is exactly the same as the factor of $1/n$ simplifies. Without loss of generality, we assume that, for any instance with two symmetric districts that have different representative locations, the overall winner is the leftmost of them.

Lemma 3.8. *Let M be any mechanism with distortion strictly less than 2 in terms of Average. Let I be any instance with $k = 2\mu + 1$ symmetric districts of size λ such that (a) the representative of μ districts is some location y_1, (b) the representative of the remaining $\mu + 1$ districts is some location y_2, and (c) $y_1 < y_2$. Then,*

(i) $M(I) = y_2$, and

(ii) for the representative z of any district in which

- $\frac{\lambda}{2} + \frac{\lambda}{4(\mu+1)}$ *agents are positioned at some z_1,*
- $\frac{\lambda}{2} - \frac{\lambda}{4(\mu+1)}$ *agents are positioned at some $z_2 > z_1$,*

it holds that $z < \frac{z_1+z_2}{2}$.

Proof. We will prove the statement by induction on μ.

Base case: $\mu = 1$. For part (i), let y_1 and $y_2 > y_1$ be any real numbers. Consider an instance I with the following three districts:

- In the first district, all λ agents are positioned at y_1. Due to unanimity, the representative of this district is y_1.
- In the other two districts, all λ agents are positioned at y_2. Due to unanimity, the representative of these districts is y_2.

Clearly, the cost of the two representative locations are $\text{cost}(y_1|I) = 2\lambda(y_2 - y_1)$ and $\text{cost}(y_2|I) = \lambda(y_2 - y_1)$. Since the distortion of M is strictly less than 2, it must be the case that the overall winner in this instance is $M(I) = y_2$.

For part (ii), let z_1 and $z_2 > z_1$ be any real numbers. We will show that some location $z < \frac{z_1+z_2}{2}$ must be the representative of a district d such that (a) $5\lambda/8$ agents are positioned at z_1 and (b) $3\lambda/8$ agents are positioned at z_2. Consider the following instance J with three districts:

- In the first district, all λ agents are positioned at z_1. Due to unanimity, the representative of this district is z_1.
- The other two districts are identical to district d that is described above. Let z be the representative of these districts.

Assume towards a contradiction that $z \geq \frac{z_1+z_2}{2}$. Then, by part (i) of the statement that is proved above for the base case (which holds for any y_1 and $y_2 > y_1$), we know that $M(J) = z$. We have

$$\text{cost}(z_1|J) = 2 \cdot \frac{3\lambda}{8}(z_2 - z_1) = \frac{3\lambda}{4}(z_2 - z_1)$$

and

$$\text{cost}(z|J) = 2 \cdot \frac{5\lambda}{8}(z - z_1) + 2 \cdot \frac{3\lambda}{8}|z_2 - z| + \lambda(z - z_1).$$

Observe that $\text{cost}(z|J)$ is an increasing function of z, no matter whether $z < z_2$ or $z \geq z_2$. Since $z \geq \frac{z_1+z_2}{2}$,

$$\text{cost}(z|J) \geq \text{cost}\left(\frac{z_1 + z_2}{2}\middle| J\right) = \frac{6\lambda}{4}(z_2 - z_1).$$

Therefore, we have that the distortion of M is at least 2, a contradiction.

Induction step: We will prove the statement for $\mu = \ell$, assuming that it holds for $\mu = \ell - 1$. For part (i), let y_1 and $y_2 > y_1$ be any real numbers. Consider the following instance I with $2\ell + 1$ districts:

- In each of the first ℓ districts, $\frac{\lambda}{2}+\frac{\lambda}{4\ell}$ agents are positioned at y_1 and $\frac{\lambda}{2}-\frac{\lambda}{4\ell}$ agents are positioned at y_2. By part (ii) of the induction hypothesis, the representative of these districts is some location $z \leq \frac{y_1+y_2}{2}$.
- In each of the other $\ell+1$ districts, all λ agents are positioned at y_2. Due to unanimity, the representative of these districts is y_2.

By the range of possible values of z, we have

$$\begin{aligned}\text{cost}(z|I) &= \ell \cdot \left(\frac{\lambda}{2}+\frac{\lambda}{4\ell}\right) \cdot |z-y_1| + \ell \cdot \left(\frac{\lambda}{2}-\frac{\lambda}{4\ell}\right) \cdot (y_2 - z) + (\ell+1)\cdot \lambda \cdot (y_2 - z) \\ &\geq \lambda \cdot \frac{(2\ell+1)(y_2-y_1)}{2}\end{aligned}$$

and

$$\text{cost}(y_2|I) = \ell \cdot \left(\frac{\lambda}{2}+\frac{\lambda}{4\ell}\right) \cdot (y_2-y_1) = \lambda \cdot \frac{2\ell+1}{4} \cdot (y_2-y_1).$$

If the overall winner is z, then the distortion of M is at least 2, a contradiction. Consequently, it must be the case that the overall winner for instance I is $M(I) = y_2$.

For part (ii), let z_1 and $z_2 > z_1$ be any real numbers, and consider the following instance J with $2\ell+1$ districts:

- In the first ℓ districts, all λ agents are positioned at z_1. By unanimity, the representative of these districts is z_1.
- In each of the remaining $\ell+1$ districts, $\frac{\lambda}{2}+\frac{\lambda}{4(\ell+1)}$ agents are positioned at z_1 and $\frac{\lambda}{2}-\frac{\lambda}{4(\ell+1)}$ agents are located at z_2. Let z be the representative of these districts.

Assume towards a contradiction (to part (ii) of the lemma) that $z \geq \frac{z_1+z_2}{2}$. Then, by the proof of part (i) of the induction step above (which holds for any y_1 and $y_2 > y_1$), we know that the overall winner is $M(J) = z$. By the range of possible values of z, we have

$$\begin{aligned}\text{cost}(z_1|J) &= (\ell+1)\cdot\left(\frac{\lambda}{2}-\frac{\lambda}{4(\ell+1)}\right)\cdot(z_2-z_1) \\ &= \lambda \cdot \frac{2\ell+1}{4}\cdot(z_2-z_1)\end{aligned}$$

and

$$\begin{aligned}\text{cost}(z|J) &= \ell\cdot\lambda\cdot(z-z_1) + (\ell+1)\cdot\left(\frac{\lambda}{2}+\frac{\lambda}{4(\ell+1)}\right)\cdot(z-z_1) + (\ell+1)\cdot\left(\frac{\lambda}{2}-\frac{\lambda}{4(\ell+1)}\right)\cdot|z_2-z| \\ &\geq \lambda\cdot\frac{2\ell+1}{2}\cdot(z_2-z_1).\end{aligned}$$

Therefore, the distortion of M is at least 2, a contradiction. □

We are now ready to prove the lower bound.

Theorem 3.9. *For Average, the distortion of any mechanism is at least* $2-\varepsilon$, *for any* $\varepsilon > 0$.

Proof. Assume towards a contradiction that there exists some $\varepsilon > 0$ such that a mechanism M has distortion smaller than $2-\varepsilon$.

First, we will prove that the representative location y of a district d in which $\lambda/2$ agents are positioned at 0 and $\lambda/2$ agents are positioned at 1, must satisfy $y \geq 1/2$, as otherwise the distortion of M would be at least 2. Assume that $y < 1/2$ and consider the following instance I with two districts:

- The first district is identical to district d as defined above.
- In the second district, all λ agents are positioned at 1. By unanimity, the representative of this district is 1.

Recall that, for any instance with two districts that have different representative locations, it is without loss of generality to assume that the overall winner is the leftmost of the representatives, that is, $M(I) = y$ in our case. We have

$$\text{cost}(y|I) = \frac{\lambda}{2} \cdot |y| + \frac{\lambda}{2}(1-y) + \lambda(1-y) \geq \frac{\lambda(3-2y)}{2}$$

and

$$\text{cost}(1|I) = \frac{\lambda}{2}.$$

Therefore, the distortion of M is at least $3 - 2y > 2$, a contradiction. Hence, $y \geq 1/2$.

Now consider the following instance J with $2\mu + 1$ districts:

- In each of the first μ districts, there are λ agents positioned at 0. By unanimity, the representative of these districts is location 0.
- Each of the remaining $\mu + 1$ districts is identical to d: There are $\lambda/2$ agents positioned at 0 and $\lambda/2$ agents positioned at 1. By the above discussion, the representative of these districts is some location $y \geq 1/2$.

By Lemma 3.8, the overall winner of instance J is $M(J) = y$. We have

$$\text{cost}(0|J) = (\mu+1) \cdot \frac{\lambda}{2}$$

and

$$\begin{aligned} \text{cost}(y|J) &= (\mu+1) \cdot \frac{\lambda}{2} y + (\mu+1) \cdot \frac{\lambda}{2} |1-y| + \mu \cdot \lambda y \\ &\geq (\mu+1) \cdot \frac{\lambda}{2} + \mu \cdot \lambda y. \end{aligned}$$

Hence, the distortion of M is at least

$$1 + \frac{2\mu y}{\mu+1} \geq 1 + \frac{\mu}{\mu+1},$$

which, when μ becomes arbitrarily large, is at least $2 - \varepsilon$, a contradiction. □

3.2 Strategyproof mechanisms

For the class of strategyproof mechanisms, we will show a tight bound of 3. For the upper bound, we consider the Weighted-Median-of-Medians mechanism. For each district d, the mechanism chooses the position of the median agent in d as the representative location y_d of d. Then, it considers a multiset of locations consisting of n_d copies of each representative location y_d, and chooses the median of them as the overall winner. See Mechanism 2. This mechanism is strategyproof as it is equivalent to the $p(n/2, (n_d)_d)$-Statistic-of-$\lceil n_d/2 \rceil$-Statistic mechanism.

Theorem 3.10. *For Average, the distortion of* Weighted-Median-of-Medians *is at most* 3.

Mechanism 2: Weighted-Median-of-Medians

for each district $d \in D$ **do**

$\quad y_d :=$ Median$_{i \in N_d}\{x_i\}$;

return $w :=$ Median$_{d \in D}\{y_d^{n_d}\}$;

Proof. Let w be the overall winner chosen by the mechanism when given some instance as input, and let o be an optimal location. For a given set of points on the line, it is well-known that the median of them minimizes the total distance. Hence, for every district d, since y_d is the median agent position in district d, we have that

$$\sum_{i \in N_d} \delta(i, y_d) \leq \sum_{i \in N_d} \delta(i, o).$$

By adding these inequalities over all districts, we obtain

$$\sum_{d \in D} \sum_{i \in N_d} \delta(i, y_d) \leq \sum_{d \in D} \sum_{i \in N_d} \delta(i, o) = n \cdot \text{cost}(o). \tag{4}$$

Similarly, since w is the median of the multiset consisting of n_d copies of y_d for every $d \in D$, we have that

$$\sum_{d \in D} n_d \cdot \delta(y_d, w) \leq \sum_{d \in D} n_d \cdot \delta(y_d, o).$$

Equivalently (since $n_d = |N_d|$), and using the triangle inequality as well as Inequality (4), we have

$$\begin{aligned}
\sum_{d \in D} \sum_{i \in N_d} \delta(y_d, w) &\leq \sum_{d \in D} \sum_{i \in N_d} \delta(y_d, o) \\
&\leq \sum_{d \in D} \sum_{i \in N_d} \delta(i, y_d) + \sum_{d \in D} \sum_{i \in N_d} \delta(i, o) \\
&\leq 2n \cdot \text{cost}(o). \qquad (5)
\end{aligned}$$

Now, by applying the triangle inequality, Inequality (4) and Inequality (5), we can bound the average cost of w as follows:

$$\begin{aligned}
\text{cost}(w) &= \frac{1}{n} \sum_{i \in N} \delta(i, w) = \frac{1}{n} \sum_{d \in D} \sum_{i \in N_d} \delta(i, w) \\
&\leq \frac{1}{n} \sum_{d \in D} \sum_{i \in N_d} \delta(i, y_d) + \frac{1}{n} \sum_{d \in D} \sum_{i \in N_d} \delta(y_d, w) \\
&\leq 3 \cdot \text{cost}(o),
\end{aligned}$$

and the proof is complete. □

Next, we show a matching lower bound of 3 on the distortion of any strategyproof mechanism, thus establishing that Weighted-Median-of-Medians is best possible within this class. The construction of the lower bound is similar to the one for the class of unrestricted mechanisms, in the sense that we will show properties that strategyproof mechanisms with small distortion (less than 3) must satisfy when given as input instances with symmetric districts of size λ. As we did in the proof of the lower bound for unrestricted mechanisms, we consider the total distance as the objective rather than the average, and also assume without loss of generality that, for any instance with two symmetric districts that have different representative locations, the overall winner is the leftmost of them.

We first show the following lemma stating that, when moving *any subset* of the agents in a district to the representative location of the district, strategyproofness dictates that the representative remains the same for the newly acquired district. We remark that the proof of the lemma can also be obtained as a corollary of a more general result in [Border and Jordan, 1983] (see Proposition 3.2.); we present a short proof that is sufficient for our purposes here.

Lemma 3.11. *Let M be a strategyproof mechanism that chooses location y to be the representative of some district d. Then, y must also be chosen as the representative location of the district d_{-S} which is the same as d with the difference that all agents in some subset $S \subseteq N_d$ are positioned at y.*

Proof. Consider any agent $i \in N_d$ that is positioned at x_i in d, and the district $d_{-\{i\}}$ that is the same as d with the difference that agent i is positioned at y; recall that y is the location chosen by M to be representative of d. Now, suppose that M chooses some location $z \neq y$ to be representative of $d_{-\{i\}}$. Hence, when M is given as input a single-district instance consisting of $d_{-\{i\}}$, it chooses the overall winner to be z. However, agent i can misreport her position as x_i so that the district is changed to d, leading M to output y as the overall winner rather than x. This way, i has decreased her cost from $|z - y|$ to 0, contradicting the strategyproofness of M. Consequently, M must choose y as the representative of $d_{-\{i\}}$ as well. By induction, the same must be true when each of the agents in S moved, one by one, from their positions in d to y to form district d_{-S}. □

Our next lemma is very similar to Lemma 3.8, but applies only to *strategyproof* mechanisms with distortion strictly less than 3.

Lemma 3.12. *Let M be any strategyproof mechanism with distortion strictly less than 3 in terms of Average. Let I be any instance with $k = 2\mu+1$ symmetric districts of size λ such that (a) the representative of μ districts is some location y_1, (b) the representative of the remaining $\mu + 1$ districts is some location y_2, and (c) $y_1 < y_2$. Then, the overall winner must be $M(I) = y_2$.*

Proof. We will prove the statement by induction on μ.

Base case: $\mu = 1$**.** Consider an instance I_1 with a single district in which there are $3\lambda/4$ agents positioned at y_1 and $\lambda/4$ agents that are positioned at y_2. Observe that y_1 is the location that minimizes the total distance of the agents, in particular,

$$\text{cost}(y_1|I_1) = \frac{\lambda}{4} \cdot (y_2 - y_1).$$

We will argue that the representative z of this district must be such that $z < y_2$. Assume, otherwise, that $z \geq y_2$ and note that in that case

$$\text{cost}(z|I_1) = \frac{3\lambda}{4} \cdot (z - y_1) + \frac{\lambda}{4} \cdot (z - y_2) \geq \frac{3\lambda}{4} \cdot (y_2 - y_1),$$

which contradicts the assumption that the distortion of M is strictly less than 3.

Next, consider an instance I_2 with a single district such that there are $3\lambda/4$ agents positioned at z and $\lambda/4$ agents positioned at y_2. Observe that the districts of I_1 and I_2 are the same with the only difference that the $3\lambda/4$ agents who are positioned at y_1 in I_1 are positioned at z in I_2. Since z is the representative of the district in I_1, by Lemma 3.11, the representative of the district in I_2 must also be z.

Finally, consider an instance I_3 with the following three districts:

- The first district is identical to the district in I_2: $3\lambda/4$ agents are positioned at z and $\lambda/4$ agents are positioned at y_2. By the above discussion, the representative of this district is $z < y_2$.

- In the remaining two districts, all λ agents are positioned at y_2. By unanimity, the representative of these districts is y_2.

We have

$$\text{cost}(z|I_3) = \frac{\lambda}{4} \cdot (y_2 - z) + 2\lambda \cdot (y_2 - z) = \frac{9\lambda}{4} \cdot (y_2 - z)$$

and

$$\text{cost}(y_2|I_3) = \frac{3\lambda}{4} \cdot (y_2 - z).$$

If the overall winner is z, then the distortion is at least 3, and thus it must to be the case that the overall winner is $M(I_3) = y_2$.

Induction step: We will now prove the statement for $\mu = \ell$, assuming that it holds for $\mu = \ell - 1$. First, consider an instance I_ℓ with the following $2\ell - 1$ districts:

- In each of the first $\ell - 1$ districts, all λ agents are positioned at some y. By unanimity, the representative of these districts is y.

- In each of the remaining ℓ districts, $\lambda/2 + \lambda/(4\ell)$ agents are positioned at y, and $\lambda/2 - \lambda/(4\ell)$ agents are positioned at some $y_2 > y$. Let z be the representative of these districts.

Again, we will argue that $z < y_2$. Assume, otherwise, that $z \geq y_2$ and note that, in that case, the overall winner must be $M(I_\ell) = z$ by the induction hypothesis. We have

$$\begin{aligned}\text{cost}(z|I_\ell) &= (\ell - 1) \cdot \lambda \cdot (z - y) + \ell \cdot \left(\frac{\lambda}{2} + \frac{\lambda}{4\ell}\right) \cdot (z - y) + \ell \cdot \left(\frac{\lambda}{2} - \frac{\lambda}{4\ell}\right) \cdot (z - y_2) \\ &\geq \lambda \cdot \frac{6\ell - 3}{4} \cdot (y_2 - y).\end{aligned}$$

At the same time, we have that

$$cost(y|I_\ell) = \lambda \cdot \frac{2\ell - 1}{4} \cdot (y_2 - y),$$

and we reach a contradiction on the distortion of M being strictly less than 3; hence, it must be $z < y_2$.

Our next goal is to identify a district d such that $\lambda/2 + \lambda/(4\ell)$ agents are positioned at some location $y_1 < y_2$, $\lambda/2 - \lambda/(4\ell)$ agents are positioned at y_2, and the representative of the district is y_1.

- If $z = y$, then any of the last ℓ districts in I_ℓ is such a district.

- If $z \neq y$, consider a district d such that $\lambda/2 + \lambda/(4\ell)$ agents are positioned at z and $\lambda/2 - \lambda/(4\ell)$ agents are positioned at y_2. Observe that this district is similar to each of the last ℓ districts in I_ℓ, with the difference that the $\lambda/2 + \lambda/(4\ell)$ agents who are positioned at y in I_ℓ are now moved to z. Therefore, by Lemma 3.11, the representative of d must be z, and since $z < y_2$, d satisfies the property.

So, in any case we have identified the district d we have been looking for, with $y_1 = z$.

Finally, consider an instance J_ℓ with the following $2\mu + 1$ districts and recall that $\mu = \ell$:

- Each of the first ℓ districts is identical to d above: $\lambda/2 + \lambda/(4\ell)$ agents are positioned at y_1 and $\lambda/2 - \lambda/(4\ell)$ agents are positioned at y_2. So, the representative of these districts is y_1.

- In each of the other $\ell + 1$ districts, all λ agents are positioned at y_2. By unanimity, the representative of these districts is y_2.

We have

$$\text{cost}(y_1|J_\ell) = \ell \cdot \left(\frac{\lambda}{2} - \frac{\lambda}{4\ell}\right) \cdot (y_2 - y_1) + (\ell + 1) \cdot \lambda \cdot (y_2 - y_1) = \frac{3\lambda(2\ell + 1)}{4} \cdot (y_2 - y_1)$$

and

$$\text{cost}(y_2|J_\ell) = \ell \cdot \left(\frac{\lambda}{2} + \frac{\lambda}{4\ell}\right) \cdot (y_2 - y_1) = \frac{\lambda(2\ell + 1)}{4} \cdot (y_2 - y_1).$$

If the overall winner is y_1 then the distortion is at least 3, a contradiction. Hence, the overall winner must be $M(J_\ell) = y_2$. □

We are now ready to prove the lower bound.

Theorem 3.13. *For Average, the distortion of any strategyproof mechanism is at least $3 - \varepsilon$, for any $\varepsilon > 0$.*

Proof. Assume towards a contradiction that there exists some $\varepsilon > 0$ such that a strategyproof mechanism M has distortion smaller than $3 - \varepsilon$. We start with an instance I_1 with a single district in which $\lambda/2$ agents are positioned at 0 and $\lambda/2$ agents are positioned at 1. Let y be the representative of the district (and thus the overall winner). We will argue that it must be $y \geq 1$. Assume otherwise that $y < 1$, and let I_2 be an instance with a single district that is obtained from the district of I_1 by moving the first $\lambda/2$ agents from 0 to y (the remaining $\lambda/2$ agents are still positioned at 1). Note that if $y = 0$, then I_1 and I_2 are equivalent. Therefore, by Lemma 3.11, the representative of (the district of) I_2 is y as well. Next, consider an instance I_3 with the following two districts:

- The first district is identical to the district of I_2: $\lambda/2$ agents are positioned at y and $\lambda/2$ agents are positioned at 1. So, the representative of this district is y.

- In the second district, all λ agents are positioned at 1. By unanimity, the representative of this district is 1.

We have

$$\text{cost}(y|I_3) = \frac{\lambda}{2} \cdot (1 - y) + \lambda \cdot (1 - y) = \frac{3\lambda}{2} \cdot (1 - y)$$

and

$$\text{cost}(1|I_3) = \frac{\lambda}{2} \cdot (1 - y).$$

Recall that it is without loss of generality to assume that M selects the leftmost representative for any instance with two districts such that their representatives are different. So, in our case, the overall winner is $M(I_3) = y$. However, this leads to a distortion of at least 3, a contradiction. We have now established that the representative of the district of I_1 must be $y \geq 1$.

To complete the proof, consider an instance I with the following $2\mu + 1$ districts:

- In each of the first μ districts, all λ agents are positioned at 0. By unanimity, the representative of these districts is 0.

- Each of the remaining $\mu + 1$ districts is identical to the district of I_1: $\lambda/2$ agents are positioned at 0 and $\lambda/2$ agents are positioned at 1. By the above discussion, the representative of these districts is $y \geq 1$.

By Lemma 3.12, the overall winner must be $M(I) = y$. We have

$$\text{cost}(0|I) = (\mu + 1) \cdot \frac{\lambda}{2}$$

and

$$\text{cost}(y|I) = \mu \cdot \lambda \cdot y + (\mu + 1) \cdot \frac{\lambda}{2} \cdot y + (\mu + 1) \cdot \frac{\lambda}{2} \cdot (y - 1) \geq (3\mu + 1) \cdot \frac{\lambda}{2}.$$

Therefore, the distortion is at least $\frac{3\mu+1}{\mu+1}$, which becomes arbitrarily close to 3 as μ becomes arbitrarily large. □

4 Max cost

We now consider the Max cost objective, for which we show a tight bound of 2 for both unrestricted and strategyproof mechanisms. For the upper bound, we consider the Arbitrary mechanism, which chooses the representative of each district to be the position of any agent therein, and then chooses any representative as the final winner. See Mechanism 3 for a specific implementation of this mechanism using the position of the leftmost agent from each district as the district representative, and then the leftmost representative as the final winner. Clearly, Arbitrary is equivalent to some p-Statistic-of-q-Statistic mechanism depending on the choices within and over districts; for example, the particular implementation of Arbitrary as Mechanism 3 is equivalent to 1-Statistic-of-1-Statistic.

Mechanism 3: Arbitrary (Leftmost-of-Leftmost)

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for each district d **do**
  $y_d := \min_{i \in N_d}\{x_i\}$;
return $w := \min_d\{y_d\}$;

Theorem 4.1. *For Max, the distortion of* Arbitrary *is at most* 2.

Proof. Given any instance, let ℓ and r denote the positions of the leftmost and the rightmost agent, respectively. Clearly, the optimal location is $o = \frac{r-\ell}{2}$, and thus $\text{cost}(o) = \frac{r-\ell}{2}$. On the other hand, the Arbitrary mechanism will necessarily return the location of some agent as the winner w, and hence $\text{cost}(w) \leq r - \ell$; the claim follows. □

We also show a matching lower bound for all mechanisms, thus completing the picture.

Theorem 4.2. *For Max, the distortion of any mechanism (unrestricted or strategyproof) is at least* 2.

Proof. Consider any mechanism and the following instance I with two districts. The agents in the first district are all positioned at -1, while the agents in the second district are all positioned at 1. Due to unanimity (Lemma 2.2), the representatives of the two districts must be -1 and 1, respectively. Hence, the winner is either -1 or 1. However, $\text{cost}(-1|I) = \text{cost}(1|I) = 2$, whereas $\text{cost}(0|I) = 1$, leading to a distortion of 2. □

5 Average-of-Max

Here, we focus on the Average-of-Max objective; recall that this objective is the average sum over each district of the maximum agent cost therein. For unrestricted mechanisms, we show that it is possible to compute the optimal location (and thus achieve a distortion of 1), whereas, for strategyproof mechanisms, we show a tight distortion bound of $1 + \sqrt{2}$.

5.1 Unrestricted mechanisms

We will show that the Median-of-Midpoints mechanism optimizes the Average-of-Max objective. This mechanism chooses the representative of each district to be the midpoint of the interval defined by the positions of the agents therein, and then chooses the median representative (breaking ties in favor of the leftmost median in case there are two) as the final winner. See Mechanism 4 for a detailed description.

Mechanism 4: Median-of-Midpoints

for each district d **do**

  $y_d := \frac{1}{2} \cdot \left(\max_{i \in N_d} x_i + \min_{i \in N_d} x_i \right)$;

return $w :=$ Median$_{d \in D}\{y_d\}$;

Theorem 5.1. *For Average-of-Max, the distortion of* Median-of-Midpoints *is* 1.

Proof. For any district d, let ℓ_d and r_d be the (positions of the) leftmost and rightmost agents therein, respectively. The Average-of-Max cost of an arbitrary location z is

$$\frac{1}{k}\sum_{d \in D} \max_{i \in N_d} \delta(x_i, z) = \frac{1}{k}\sum_{d \in D} \max\{\delta(\ell_d, z), \delta(r_d, z)\}$$
$$= \frac{1}{k}\sum_{d \in D} \delta\left(\frac{\ell_d + r_d}{2}, z\right) + \frac{1}{k}\sum_{d \in D} \frac{r_d - \ell_d}{2}.$$

Since the second term is a constant in terms of z, the above expression is minimized when the first term is minimized, which is done when z is chosen to minimize the average distance from the midpoints of the intervals defined by the agents in each district. Consequently, it suffices to choose the median district midpoint as the winner. This is precisely the definition of Median-of-Midpoints. □

5.2 Strategyproof mechanisms

For strategyproof mechanisms, we will show a tight bound of $1 + \sqrt{2}$. For the upper bound, we consider the $\left(1 - 1/\sqrt{2}\right) k$-Leftmost-of-Rightmost mechanism, which chooses the representative of each district to be the position of the rightmost agent therein, and then chooses the $\lceil \left(1 - 1/\sqrt{2}\right) k \rceil$-th leftmost representative as the final winner. See Mechanism 5 for a detailed description. Clearly, the mechanism is an implementation of p-Statistic-of-q-Statistic with $p = \lceil \left(1 - 1/\sqrt{2}\right) k \rceil$ and $q_d = n_d$, and is thus strategyproof. So, it suffices to show that it achieves a distortion of at most $1 + \sqrt{2}$.

Mechanism 5: $\left(1 - 1/\sqrt{2}\right) k$-Leftmost-of-Rightmost

for each district $d \in D$ **do**

  $y_d :=$ rightmost agent;

return $w := \lceil \left(1 - 1/\sqrt{2}\right) k \rceil$-th leftmost representative;

Theorem 5.2. *For the Average-of-Max cost, the distortion of* $\left(1 - 1/\sqrt{2}\right) k$-Leftmost-of-Rightmost *is at most* $1 + \sqrt{2}$.

Proof. Let w be the location chosen by the mechanism when given some instance as input, and o an optimal location. For each district d, let i_d be the most distant agent from w, and i_d^* the most distant agent from o. So, $\text{cost}(w|I) = \frac{1}{k}\sum_{d\in D}\delta(i_d, w)$, and $\text{cost}(o|I) = \frac{1}{k}\sum_{d\in D}\delta(i_d^*, o) \geq \frac{1}{k}\sum_{d\in D}\delta(j, o)$ for any agent $j \in N_d$. We consider the following two cases depending on the relative positions of w and o.

Case 1: $o < w$.
Let $S = \{d \in D : y_d \geq w\}$ be the set of district representatives to the right of w. By the definition of w, we have that $|S| = k + 1 - \lceil (1 - 1/\sqrt{2})\, k\rceil = 1 + \lfloor k/\sqrt{2}\rfloor \geq \frac{k}{\sqrt{2}}$. Since $o < w \leq y_d$ for every $d \in S$ and $y_d \in N_d$, we have that

$$\text{cost}(o) \geq \frac{1}{k}\sum_{d\in S}\delta(y_d, o) \geq \frac{1}{k}\cdot |S| \cdot \delta(w, o) \geq \frac{1}{\sqrt{2}}\cdot \delta(w, o) \Leftrightarrow \delta(w, o) \leq \sqrt{2}\cdot \text{cost}(o).$$

By the triangle inequality and since $i_d \in N_d$, we have

$$\begin{aligned}\text{cost}(w) = \frac{1}{k}\sum_{d\in D}\delta(i_d, w) &\leq \frac{1}{k}\sum_{d\in D}\delta(i_d, o) + \frac{1}{k}\sum_{d\in D}\delta(w, o)\\ &\leq \text{cost}(o) + \delta(w, o)\\ &\leq (1+\sqrt{2})\cdot \text{cost}(o).\end{aligned}$$

Case 2: $w < o$.
We partition the districts into a set L that includes $\lceil (1 - \frac{1}{\sqrt{2}})k\rceil$ districts from the one with the leftmost representative until the one with the $\lceil (1 - \frac{1}{\sqrt{2}})k\rceil$-th leftmost representative (that is, w), and a set R that includes the remaining districts. By definition, we have that $|R|/|L| = (k - \lceil (1 - \frac{1}{\sqrt{2}})k\rceil)/(\lceil (1 - \frac{1}{\sqrt{2}})k\rceil) \leq 1 + \sqrt{2}$. For every district d, let ℓ_d and r_d be the leftmost and rightmost agents in d, respectively. We make the following observations:

- For every $d \in L$, since y_d is the rightmost agent of d and $y_d \leq w < o$, it must be the case that $i_d = i_d^* = \ell_d$. Due to the positions of ℓ_d, w and o, we have that $\delta(\ell_d, o) = \delta(\ell_d, w) + \delta(w, o)$.
- For every $d \in R$, by the triangle inequality, we have that $\delta(i_d, w) \leq \delta(i_d, o) + \delta(w, o)$. Since $\delta(i_d, o) \leq \delta(i_d^*, o)$ by the definition of i_d^*, we further have that $\delta(i_d, w) \leq \delta(i_d^*, o) + \delta(w, o)$.

Hence,

$$\begin{aligned}\text{cost}(w) &= \frac{1}{k}\sum_{d\in D}\delta(i_d, w) = \frac{1}{k}\sum_{d\in L}\delta(\ell_d, w) + \frac{1}{k}\sum_{d\in R}\delta(i_d, w)\\ &\leq \frac{1}{k}\sum_{d\in L}\Big(\delta(\ell_d, w) + \delta(w, o)\Big) - \frac{|L|}{k}\cdot\delta(w, o) + \frac{1}{k}\sum_{d\in R}\Big(\delta(i_d^*, o) + \delta(w, o)\Big)\\ &= \text{cost}(o) + \frac{|R| - |L|}{k}\cdot \delta(w, o).\end{aligned}$$

Since $y_d \leq w < o$ for every $d \in L$ and $y_d \in N_d$, we have that

$$\text{cost}(o) \geq \frac{1}{k}\sum_{d\in L}\delta(y_d, o) \geq \frac{|L|}{k}\cdot\delta(w, o) \Leftrightarrow \delta(w, o) \leq \frac{k}{|L|}\cdot \text{cost}(o).$$

Therefore, we obtain

$$\text{cost}(w) \leq \text{cost}(o|I) + \frac{|R| - |L|}{|L|}\cdot \text{cost}(o) = \frac{|R|}{|L|}\cdot\text{cost}(o) \leq (1+\sqrt{2})\cdot\text{cost}(o),$$

as desired. □

We also show a matching lower bound on the distortion of all strategyproof mechanisms.

Theorem 5.3. *For Average-of-Max, the distortion of any strategyproof mechanism is at least* $1+\sqrt{2}-\varepsilon$*, for any* $\varepsilon > 0$*.*

Proof. Assume towards a contradiction that there exists some $\varepsilon > 0$ and a strategyproof mechanism with distortion strictly smaller than $1+\sqrt{2}-\varepsilon$. Without loss of generality, we assume that when there are two symmetric districts with different representatives, we choose the leftmost as the final winner. We will prove the statement by showing some properties about the behavior of strategyproof mechanisms in particular instances.

Property (P1): We claim that there is a district with two agents such that the mechanism chooses some agent position as the district representative. Consider a district d with one agent positioned at x and one agent positioned at $y > x$. If the mechanism chooses the representative to be x or y, then we are done. Otherwise, suppose that the representative is chosen to be some $z \notin \{x, y\}$. Due to strategyproofness, z must also be the representative of the district d' where any of the two agents has been moved to z; otherwise, in the single-district instance consisting of d', the agent that is moved would have incentive to report that she is positioned as in d to change the outcome to z.

Property (P2): By Property (P1) there exists a district with two agents such that the mechanism chooses the district representative to be the position of one of the agents; without loss of generality we assume that the agents are positioned at 0 and 1. We claim that the representative of this district must be 1 as otherwise the distortion would be at least 3. Indeed, suppose otherwise that the representative is 0, and consider the following instance I_1 with two districts:

- In the first district, there is an agent at 0 and an agent at 1. By the above discussion, the representative is 0.
- In the second district, there are two agents at $1/2$. Due to unanimity, the representative is $1/2$ (otherwise the distortion would be infinite due to Lemma 2.2).

Since there are only two districts and two different representatives, the overall winner is 0. But,

$$\text{cost}(0|I_1) = \frac{1}{2}\bigg((1-0) + (1/2-0)\bigg) = 3/4$$

and

$$\text{cost}(1/2|I_1) = \frac{1}{2}\bigg((1-1/2) + (1/2-1/2)\bigg) = 1/4,$$

leading to a distortion of 3.

Property (P3): Let $\alpha < \beta$ be two (large) integers such that $\beta/\alpha = 1+\sqrt{2}-\delta$, for some arbitrarily small $\delta > 0$. We claim that in instances with $\alpha+\beta$ districts such that $1/2$ is the representative of α districts and 1 is the representative of β districts, the overall winner must be 1 as otherwise the distortion would be $\beta/\alpha = 1+\sqrt{2}-\delta$. Indeed, suppose that the winner is $1/2$ in such a case, and consider the following instance I_2 with $\alpha+\beta$ districts:

- In α districts, there are two agents at $1/2$.
- In β districts, there are two agents at 1.

Due to unanimity (Lemma 2.2), the representatives are $1/2$ and 1, respectively, and the overall winner is $1/2$ by assumption. Then, $\text{cost}(1/2|I_2) = \frac{1}{2}\cdot\beta/2$ and $\text{cost}(1|I_2) = \frac{1}{2}\cdot\alpha/2$. So, the distortion is at least $\beta/\alpha = 1+\sqrt{2}-\delta$.

Reaching a contradiction: Now, we consider the following instance I_3 with $\alpha + \beta$ districts:

- In α districts, there are two agents at $1/2$. Due to unanimity the representative of all these districts is $1/2$.
- In β districts, there is one agent at 0 and one agent at 1. By property (P2), the representative of all these districts is 1.

Since $1/2$ is the representative of α districts and 1 is the representative of β districts, by property (P3), the overall winner is 1. We have that

$$\text{cost}(1|I_3) = \frac{1}{2}\left(\frac{\alpha}{2} + \beta\right)$$

and

$$\text{cost}(1/2|I_3) = \frac{1}{2} \cdot \frac{\beta}{2}.$$

That is, the distortion is at least $2 + \frac{\alpha}{\beta} > 2 + \frac{1}{1+\sqrt{2}} = 1 + \sqrt{2}$; a contradiction. $\square$

6 Max-of-Average

We now turn our attention to the last objective, Max-of-Average, which is the maximum over each district of the average total individual cost therein. We show a tight bound of 2 for unrestricted mechanisms and a tight bound of $1 + \sqrt{2}$ for strategyproof mechanisms.

6.1 Unrestricted mechanisms

Since the lower bound of 2 for the Max cost objective holds even when there is a single agent in each district, it extends to the case of Max-of-Average as well. For the upper bound, we consider the Arbitrary-of-Avg mechanism, which chooses the representative of each district to be the average of the positions of the agents in the district, and then chooses an arbitrary representative (e.g., the leftmost) as the final winner. See Mechanism 6 for a detailed description.

Mechanism 6: Arbitrary-of-Avg

for each district $d \in D$ **do**

  $y_d := \frac{\sum_{i \in N_d} x_i}{n_d}$;

return $w := \min_{d \in D} y_d$;

Theorem 6.1. *For Max-of-Average, the distortion of* Arbitrary-of-Avg *is at most* 2.

Proof. Let w be the location chosen by the mechanism when given some instance as input, and o an optimal location; without loss of generality, we assume that $w < o$. Denote by d^* a district that defines the cost of w, that is, $d^* \in \arg\max_{d \in D} \frac{1}{n_d} \sum_{i \in N_d} \delta(x_i, w)$. Also, denote by d_w a district that has representative w, that is,

$$w = \frac{1}{n_{d_w}} \sum_{i \in N_{d_w}} x_i \Leftrightarrow \frac{1}{n_{d_w}} \sum_{i \in N_{d_w}} (w - x_i) = 0.$$

By the triangle inequality, we have that

$$\text{cost}(w) = \frac{1}{n_{d^*}} \sum_{i \in N_{d^*}} \delta(x_i, w)$$

$$\leq \frac{1}{n_{d^*}} \sum_{i \in N_{d^*}} \delta(x_i, o) + \frac{1}{n_{d^*}} \sum_{i \in N_{d^*}} \delta(w, o)$$
$$\leq \text{cost}(o) + \delta(w, o).$$

By the definition of d_w, we have that

$$\begin{aligned}
\delta(w, o) &= o - w \\
&= o - w + \frac{1}{n_{d_w}} \sum_{i \in N_{d_w}} (w - x_i) \\
&= \frac{1}{n_{d_w}} \sum_{i \in N_{d_w}} (o - x_i) \\
&\leq \frac{1}{n_{d_w}} \sum_{i \in N_{d_w}} \delta(x_i, o) \\
&\leq \text{cost}(o),
\end{aligned}$$

where the inequality follows since $\delta(x_i, o) = o - x_i$ when $x_i \leq o$ and $\delta(x_i, o) = x_i - o \geq o - x_i$ when $x_i \geq o$. Therefore, we obtain that $\text{cost}(w|I) \leq 2 \cdot \text{cost}(o)$, as desired. □

6.2 Strategyproof mechanisms

We now turn out attention to strategyproof mechanisms and show a tight bound of $1 + \sqrt{2}$. For the upper bound, we consider the Rightmost-of-$\left(1 - 1/\sqrt{2}\right) n_d$-Leftmost mechanism, which chooses the representative of each district d to be the position of the $\lceil \left(1 - 1/\sqrt{2}\right) n_d \rceil$-th leftmost agent therein, and then chooses the rightmost representative as the final winner. See Mechanism 7 for a detailed description. This mechanism is an implementation of p-Statistic-of-q_d-Statistic with $p = k$ and $q_d = \lceil \left(1 - 1/\sqrt{2}\right) n_d \rceil$, and is thus strategyproof. So, it suffices to show that it achieves a distortion of at most $1 + \sqrt{2}$.

Mechanism 7: Rightmost-of-$\left(1 - 1/\sqrt{2}\right) n_d$-Leftmost

for each district $d \in D$ **do**
  $y_d := \lceil \left(1 - 1/\sqrt{2}\right) n_d \rceil$-th leftmost agent;
return $w :=$ rightmost representative;

Theorem 6.2. *For the Max-of-Average cost, the distortion of* Rightmost-of-$\left(1 - 1/\sqrt{2}\right) n_d$-Leftmost *is at most* $1 + \sqrt{2}$.

Proof. Let w be the location chosen be the mechanism when given some instance as input, and o an optimal location. Denote by d^* a district that gives the max average sum for w, and by d_w a district with representative w. Also, for any district d, we denote by $\text{cost}_d(x) = \frac{1}{n_d} \sum_{i \in N_d} \delta(i, x)$ the average total distance of the agents in d from location x, and let o_d be the location that minimizes this distance (that is, o_d is the median agent of d). Clearly, by definition, we have that $\text{cost}(w) = \text{cost}_{d^*}(w)$, and $\text{cost}_d(o) \leq \text{cost}(o)$ for every district d. We consider the following two cases:

Case 1: $o < w$**.**
By the definition of d^* and the triangle inequality, we have

$$\text{cost}(w) = \frac{1}{n_{d^*}} \sum_{i \in N_{d^*}} \delta(i, w)$$

$$
\begin{aligned}
&\leq \frac{1}{n_{d^*}} \sum_{i \in N_{d^*}} \delta(i,o) + \frac{1}{n_{d^*}} \sum_{i \in N_{d^*}} \delta(o,w) \\
&\leq \text{cost}(o) + \delta(o,w).
\end{aligned}
$$

Let $S = \{i \in N_{d_w} : x_i \geq w\}$ be the set of agents that are positioned at the right of (or exactly at) w in d_w. Since $o < w$, $\delta(i,o) \geq \delta(w,o)$ for every $i \in S$. Also, by the definition of w, $|S| = n_{d_w} + 1 - \lceil (1 - 1/\sqrt{2})\, n_{d_w} \rceil = 1 + \lfloor \frac{1}{\sqrt{2}} \cdot n_{d_w} \rfloor \geq \frac{1}{\sqrt{2}} \cdot n_{d_w}$. Hence,

$$
\begin{aligned}
&\text{cost}_{d_w}(o) = \frac{1}{n_{d_w}} \sum_{i \in N_{d_w}} \delta(i,o) \geq \frac{1}{n_{d_w}} \cdot |S| \cdot \delta(w,o) \geq \frac{1}{\sqrt{2}} \cdot \delta(w,o) \\
&\Leftrightarrow \delta(w,o) \leq \sqrt{2} \cdot \text{cost}_{d_w}(o) \leq \sqrt{2} \cdot \text{cost}(o).
\end{aligned}
$$

By combining everything together, we obtain a bound of $1 + \sqrt{2}$.

Case 2: $w < o$**.**
We consider the following two subcases:

- $o_{d^*} \leq w < o$. By the monotonicity of the (average) social cost[2] for the agents in district d^*, we have that $\text{cost}_{d^*}(o_{d^*}) \leq \text{cost}_{d^*}(w) \leq \text{cost}_{d^*}(o)$, and thus $\text{cost}(w) \leq \text{cost}(o)$.
- $w < o_{d^*}$. Since w is the rightmost representative, it must be the case that $y_{d^*} \leq w < o_{d^*}$. So, again by the monotonicity of the (average) social cost within the district d^*, we have that $\text{cost}_{d^*}(o_{d^*}) \leq \text{cost}_{d^*}(w) \leq \text{cost}_{d^*}(y_{d^*})$. We will argue that $\text{cost}_{d^*}(y_{d^*}) \leq (1+\sqrt{2}) \cdot \text{cost}_{d^*}(o_{d^*})$. Let L be the set that includes $\lceil (1-\frac{1}{\sqrt{2}}) n_{d^*} \rceil$ agents of d^* from the leftmost to the $\lceil (1-\frac{1}{\sqrt{2}}) n_{d^*} \rceil$-th leftmost agent (that is, y_{d^*}), and the set R that includes the remaining agents. By definition, we have that $|R|/|L| = (n_{d^*} - \lceil (1-\frac{1}{\sqrt{2}}) n_{d^*} \rceil) / \lceil (1-\frac{1}{\sqrt{2}}) n_{d^*} \rceil \leq 1 + \sqrt{2}$. Now, observe that
  - For every agent $i \in L$, $i \leq y_{d^*}$, and thus $\delta(i, o_{d^*}) = \delta(i, y_{d^*}) + \delta(y_{d^*}, o_{d^*})$.
  - For every agent $i \in R$, $i \geq y_{d^*}$, and thus $\delta(i, y_{d^*}) \leq \delta(i, o_{d^*}) + \delta(y_{d^*}, o_{d^*})$.

  Hence,

$$
\begin{aligned}
\text{cost}_{d^*}(y_{d^*}) &= \frac{1}{n_{d^*}} \sum_{i \in N_{d^*}} \delta(i, y_{d^*}) \\
&= \frac{1}{n_{d^*}} \sum_{i \in L} \delta(i, y_{d^*}) + \frac{1}{n_{d^*}} \sum_{i \in R} \delta(i, y_{d^*}) \\
&\leq \frac{1}{n_{d^*}} \sum_{i \in L} \delta(i, y_{d^*}) + \frac{1}{n_{d^*}} \sum_{i \in R} \Big(\delta(i, o_{d^*}) + \delta(y_{d^*}, o_{d^*}) \Big) \\
&= \frac{1}{n_{d^*}} \sum_{i \in L} \Big(\delta(i, y_{d^*}) + \delta(y_{d^*}, o_{d^*}) \Big) + \frac{1}{n_{d^*}} \sum_{i \in R} \delta(i, o_{d^*}) + \frac{|R| - |L|}{n_{d^*}} \cdot \delta(y_{d^*}, o_{d^*}) \\
&= \text{cost}_{d^*}(o_{d^*}) + \frac{|R| - |L|}{n_{d^*}} \cdot \delta(y_{d^*}, o_{d^*}).
\end{aligned}
$$

  Since $y_{d^*} < o_{d^*}$, we also have that $\text{cost}_{d^*}(o) \geq \frac{1}{n_{d^*}} \cdot |L| \cdot \delta(y_{d^*}, o_{d^*})$, and thus

$$
\text{cost}_{d^*}(y_{d^*}) \leq \text{cost}_{d^*}(o_{d^*}) + \frac{|R| - |L|}{|L|} \cdot \text{cost}_{d^*}(o_{d^*})
$$

[2] It is a well-known fact that the social cost objective is monotone in the locations. In particular, for any set of agents S, if $y_1 \in \arg\min_x \sum_{i \in S} \delta(i,x)$, then $\sum_{i \in S} \delta(i, y_1) \leq \sum_{i \in S} \delta(i, y_2) \leq \sum_{i \in S} \delta(i, y_3)$ for any $y_1 \leq y_2 \leq y_3$ or $y_3 \leq y_2 \leq y_1$.

$$= \frac{|R|}{|L|} \cdot \text{cost}_{d^*}(o_{d^*})$$
$$\leq (1+\sqrt{2}) \cdot \text{cost}_{d^*}(o_{d^*}).$$

From this, we finally get that

$$\text{cost}_{d^*}(w) \leq (1+\sqrt{2}) \cdot \text{cost}_{d^*}(o_{d^*}) \leq (1+\sqrt{2}) \cdot \text{cost}_{d^*}(o),$$

and thus $\text{cost}(w) \leq (1+\sqrt{2})\text{cost}(o)$. □

Finally, we show a matching lower bound for any strategyproof mechanism.

Theorem 6.3. *For Max-of-Average, the distortion of any strategyproof mechanism is at least $1+\sqrt{2}-\varepsilon$, for any $\varepsilon > 0$.*

Proof. Suppose towards a contradiction that there is a strategyproof mechanism with distortion strictly smaller than $1+\sqrt{2}-\varepsilon$, for any $\varepsilon > 0$. We will reach a contradiction by showing several properties that any strategyproof mechanism must satisfy when given particular instances with symmetric districts consisting of $\lambda = (2+\sqrt{2})x$ agents, where x is an arbitrarily large integer.

Property (P1): Consider a district with $(1+\sqrt{2})x$ agents at 0 and x agents at 1.[3] We claim that the mechanism must choose 0 as the representative of this district as otherwise the distortion would be at least $1+\sqrt{2}$. Indeed, suppose that the representative is some $y \neq 0$. By moving one of the agents at 1 to y, we obtain a new district whose representative must still be y; otherwise, in the instance that consists only of this new district, the agent at y would have incentive to misreport her position as 1, thus leading to the representative (and the final winner) to change to y. By induction, we obtain that y must be the representative of the district with $(1+\sqrt{2})x$ agents at 0 and x agents at y. In the instance I that consists of only the latter district, the winner is y with $\text{cost}(y|I) = \frac{1}{\lambda} \cdot (1+\sqrt{2})x \cdot |y|$, whereas $\text{cost}(0|I) = \frac{1}{\lambda} \cdot x \cdot |y|$, leading to a distortion of at least $1+\sqrt{2}$.

Property (P2): Consider a district with x agents at 1 and $(1+\sqrt{2})x$ agents at 2. We claim that the mechanism must choose 2 as the representative of this district as otherwise the distortion would be at least $1+\sqrt{2}$. This follows by arguments similar to those for property (P1).

Reaching a contradiction: Consider the following instance J with two districts:

- In the first district, there are $(1+\sqrt{2})x$ agents at 0 and x agents at 1.
- In the second district, there are x agents at 1 and $(1+\sqrt{2})x$ agents at 2.

By properties (P1) and (P2), the representatives of the two districts must be 0 and 2, respectively, and thus one of these two locations is chosen as the final winner. However, $\text{cost}(0|J) = \text{cost}(2|J) = \frac{1}{\lambda} \cdot \big(2(1+\sqrt{2})x + x\big)$, while $\text{cost}(1|J) = \frac{1}{\lambda} \cdot (1+\sqrt{2})x$, leading to a distortion of $2 + \frac{1}{1+\sqrt{2}} = 1+\sqrt{2}$. □

7 Open problems

In this paper we settled the distortion of unrestricted and strategyproof mechanisms for the distributed single-facility location problem in terms of social objectives that are combinations of average and max (within and over the districts). There are several interesting directions for future work, such as to

[3] To be precise, since the number of agents must be an integer, we would need to have $\lceil (1+\sqrt{2})x \rceil$ agents at 0. We simplify our notation by dropping the ceilings, but it should be clear that this does not affect our arguments.

extend our work to more general metric spaces, or to define further meaningful objectives and study similar questions about efficiency and strategyproofness. It would also be quite interesting to try to obtain a characterization of strategyproof distributed mechanisms, assuming some natural properties (such as anonymity or cardinal unanimity), potentially making use of classic results in the centralized setting [Moulin, 1980, Massó and De Barreda, 2011]. Another direction is to consider a more general class of distributed mechanisms that are not restricted to choosing one of the district representatives as the winner, but can choose any location on the line according to some function of the representatives, similarly to how mechanisms operate in the distributed social choice setting of Anshelevich et al. [2022]. Beyond the single-facility location problem that we studied here, one could consider settings with more facilities and agents that have heterogeneous preferences over the facilities.

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