On Parallels Between Words and Music

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Cover Page Footnote

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Synopsis

A generalised song is a means of drawing parallels between words and music. The parallels are encoded in a mathematical structure, which is interpreted in a verbal structure and a musical structure. Here we develop a number of new techniques for drawing such parallels, in giving two examples of generalised songs, 'Relation', and 'Merge/Split'.

The first five partials of a note played on a piano are roughly $0, 12, 19, 24, 28$ semitones above the fundamental.'Relation' is a generalised song, whose musical part is played on a piano, constructed from the mathematical relation $4.28 = 3.12 + 4.19$.

'Merge/Split' is a generalised song whose mathematical part consists of the mathematical operations of merging and splitting, the braid relation, and coassociativity.

1. Introduction.

A traditional song consists of some words and some music, united by a common mathematical structure: their meter. By a generalised song we mean a set of mathematical elements Ma, which are reflected in both a verbal structure V and a musical structure Mu.

$$Ma$$
$$\swarrow \qquad \searrow$$
$$Mu \qquad\qquad V$$
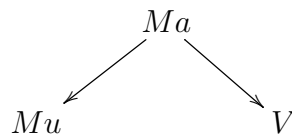
Thus, in a generalised song, we describe parallels between words and music. To give a simple example, we have a generalised song whose mathematical element is the number 22, whose verbal structure is the sixth chapter of the Book of Genesis, and whose musical structure is Bach's two part Invention No. 1. The number 22 is a unifying feature, since the sixth chapter of Genesis has 22 verses, whilst Bach's two part Invention No. 1 has 22 bars. This elementary example exhibits a feature of generalised songs: they can unite quite disparate musical and verbal parts.

To give another simple example, we have a generalised song whose mathematical element is the four letter word AABB, whose verbal structure is A.E. Housman's 'To an Athlete Dying Young', and whose musical structure is Mozart's Allegro K3. Housman's poem consists of stanzas with rhyming pattern AABB, whilst the Mozart piece consists of a 13 bar section (A) played twice consecutively, followed by a 19 bar section (B) played twice consecutively.

To demonstrate possibilities of the form, we have previously described a generalised song, 'Cube', whose main mathematical element was a cube [9]. Here we describe two further generalised songs, 'Relation' and 'Merge/Split'. The techniques used in 'Cube', 'Relation', and 'Merge/Split' are largely different from each other, and we thus introduce new generalised song techniques.

Our techniques are motivated in part by elementary features of category theory (perhaps unsurprising, given that category theory displays unifying elements). For example, in the verbal part of 'Relation', we lift the mathematical relation $4 \times 28 = 3 \times 12 + 4 \times 19$ to a translation of verbal parts. The number 4×28 is represented by four sentences of 28 German words, whilst the number $3 \times 12 + 4 \times 19$ is represented by three sentences of 12 English words and four sentences of 19 English words. The = symbol is represented by the fact that these two collections of sentences have the same meaning: each is a translation of the other. This lifting of an equality to a translation is reminiscent of categorification, in which we lift an equality of natural numbers to an isomorphism of finite sets [2].

To give another example, in 'Merge/Split', we interpret $x \Rightarrow y$ as (musical structure corresponding to x) followed by (musical structure corresponding to y), for certain higher dimensional arrows $x \Rightarrow y$.
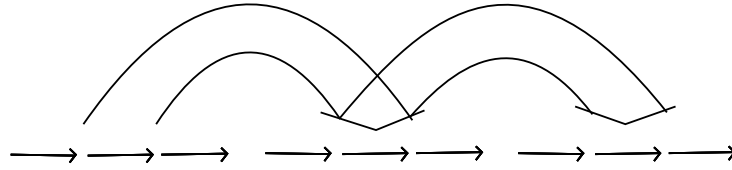
Figure 1: Higher dimensional arrows.

Our approach is to exhibit techniques with examples. In this respect, our generalised songs are similar to conventional songs, where techniques such as novel scales or instrument combinations can be introduced by example.

2. Musical preliminaries.

Here we gather some musical preliminaries. These are intended as explanatory material for the later sections of the paper, and can be skipped by a reader with some knowledge of music theory. For more details on harmonics etc., see Benson's book [4].

A computer and loudspeaker can be used to associate sounds to nice mathematical functions from $\mathbb{R}$ to $\mathbb{R}$. Indeed, such a function is converted by the computer to an electrical output, and this is in turn converted to pressure variations in the air by the loudspeaker. For example, for $f \in \mathbb{R}_{>0}$, the sound associated to the sine function $t \mapsto sin(2\pi ft)$ in this way is called a *pure tone* of frequency f. The pure tones given by frequencies outside the range $[20, 20000]$ are inaudible to the human ear.

The sound, or note, made by a string vibrating with frequency f can be modelled by the sound corresponding to a mathematical function

$$t \mapsto \sum_{n=1}^{10} a_n \sin(2\pi nft),$$

where $a_1, ..., a_{10} \in \mathbb{R}_{>0}$. The pure tones corresponding to the summands $t \mapsto \sin(2\pi nft)$, are called the *harmonics* of the note. The frequencies $nf, 1 \leq n \leq 10$ are also called the harmonics of the note. Harmonics are also known as *partials*. The first harmonic, corresponding to the summand $t \mapsto \sin(2\pi ft)$, with frequency f, is called the *fundamental*. The harmonics that are different from the fundamental are the *overtones* of the note.

We say a musical frequency f is *higher* than a musical frequency f' if $f > f'$ and f is *lower* than f' if $f < f'$. We say a musical frequency f is an *octave* higher than a musical frequency f' if $f = 2f'$. We say a musical frequency f is *n semitones* higher than a musical frequency f' if $f = 2^{\frac{n}{12}}f'$. The harmonics of a note with fundamental frequency f have frequencies $f, 2f, 3f, ..., 10f$, which lie approximately $0, 12, 19, 24, 28, 31, 34, 36, 38, 40$ semitones above the fundamental ($1 = 2^0, 2 = 2^{\frac{12}{12}}, 3 \approx 2^{\frac{19}{12}}, 4 = 2^{\frac{24}{12}}, 5 \approx 2^{\frac{28}{12}}$ etc.).

We say two notes are consonant if they resonate in the following way: the nth partial of one coincides with the mth partial of another, for some $n, m \le 6$. For example, two notes that are an octave apart have fundamental frequencies f and $2f$ for some f. The second harmonic of the lower note is equal to the first harmonic of the higher note; therefore these notes are consonant. Consider now two notes with fundamental frequencies f and $\frac{3}{2}f$ for some f. These are said to be a *perfect fifth* apart. The third harmonic of the lower note is equal to the second harmonic of the higher note; therefore these notes are consonant.

A *scale* is an increasing or decreasing sequence in $\mathbb{R}$, thought of as a set of frequencies. For example, we have the scale of notes on the *stave*, given by frequencies $440 \cdot 2^{\frac{n}{12}}$, for $n \in \mathbb{Z}$. The notes of an *equally tempered piano* have frequencies $440 \cdot 2^{\frac{n}{12}}, -48 \le n \le 39$ on the stave. Those notes which differ by a number of octaves are often identified. For example, those with frequency $440 \cdot 2^{\frac{3}{12}} \cdot 2^m, m \in \mathbb{Z}$ are all labelled C.

Our reason for limiting the number of our harmonics to 10 above is that an 11th harmonic would depart significantly from frequencies represented on the stave.

Perfect fifths can be used to construct some important scales. A perfect fifth can be approximated by seven semitones ($\frac{3}{2} \approx 2^{\frac{7}{12}}$). A *pentatonic scale* is given by frequencies $2^n \cdot 2^{\frac{7m}{12}} \cdot f$, $n \in \mathbb{Z}, 0 \le m \le 4$ for some f. A *major scale* is given by frequencies $2^n \cdot 2^{\frac{7m}{12}} \cdot f$, $n \in \mathbb{Z}, 0 \le m \le 6$ for some f. There are twelve major scales on the stave (take $f = 440 \cdot 2^{\frac{n}{12}}$ for some $n \in \mathbb{Z}$ and note you get the same scale if you do the same thing with an integer other than n, congruent to n modulo 12). Likewise there are twelve pentatonic scales on the stave. Sometimes we refer to a pentatonic scale as *the* pentatonic scale because pentatonic scales sound so similar to each other. Likewise we sometimes refer to a major scale as *the* major scale.

Notes are often given a duration. For example, the sound associated by our computer and loudspeaker to the function $t \mapsto \chi(t) \sin(2\pi f t)$, where χ is the characteristic function of an interval in $\mathbb{R}$ of the form $[a, a+d]$, is a pure tone of duration d. Let us fix a number $d > 0$. A *crotchet* is a note of duration d, a *quaver* is a note of duration $\frac{1}{2}d$, and a *minim* is a note of duration $2d$.

A silent note, associated to the constant function 0, is called a *rest*.

A musical *line* is a sequence of musical notes of finite duration, played successively. We say a pair of musical lines are in *counterpoint* if notes in the two lines that begin simultaneously are consonant (cf. [5]).

An *ostinato* is a continually repeated musical phrase.

A *glissando* is a continuous slide upwards or downwards between notes. For example, for $t \in [0, 1]$, the function $t \mapsto \sin(880\pi 2^t t)$, defines a glissando from a pure tone of frequency 440 to a pure tone of frequency 880.

The *fifties progression* is a certain musical progression found eg. in Earth Angel by The Penguins [8]. One form of this is the sequence of notes in the pentatonic scale: $f, 2^{\frac{4}{12}} f, 2^{\frac{-3}{12}} f, 2^{\frac{-10}{12}} f, 2^{\frac{-5}{12}} f, f$. Each pair of successive notes in the sequence is approximately consonant ($2^{\frac{4}{12}} \approx \frac{5}{4}, 2^{\frac{7}{12}} \approx \frac{3}{2}, 2^{\frac{5}{12}} \approx \frac{4}{3}$). Repeated, the sequence makes an ostinato.

3. 'Relation': the mathematical structure.

Here we describe the mathematical component of the generalised song 'Relation'. The structure of our generalised song is described by the following diagram:

	Verbal	Musical
1	$\{Tree\}$	$\{Scale\}$
2	$\{Human\}$	$\{Ostinato\}$
3	$\{Tree, Human\}$	$\{Scale, Ostinato\}$
4	$\{Scale, Human\}$	$\{Tree, Ostinato\}$

Both the verbal part and the musical part are thus split into four sections, indexed by the headings given in the diagram. The sections thus correspond to elements of the structure

$$\Omega = (\{a\}, \{b\}, \{a, b\}, \{(a, \sigma), b\}) \times \{\text{Verbal}, \text{Musical}\},$$

where σ is the nontrivial permutation of the set {Verbal, Musical}. The symmetry σ thus corresponds to a symmetry of our generalised song that swaps the verbal and musical parts. Ω, and the permutation σ form the mathematical component of 'Relation'.

The verbal part is built from sentences, whose underlying structures are given by syntactic *trees*. The musical part is built from piano notes, whose partials form *scales* of frequencies $1, 2, 3, 4, 5, ...$ times their fundamentals.

Both musical and verbal part are constructed by interpreting the relation $4 \times 28 = 3 \times 12 + 4 \times 19$ in some way.

4. 'Relation': the verbal part.

The four sections of the verbal part of 'Relation' have titles as specified in chapter 3. Each section consists of two paragraphs, one in German and one in English. The German paragraph consists of four sentences of 28 words, the English paragraph consists of three sentences of 12 words and four sentences of 19 words. The German paragraph and the English paragraph are translations of one another. The conceptual relation between the pair of paragraphs defines a strengthening of the weak relation $4 \times 28 = 3 \times 12 + 4 \times 19$ that equates their word counts.

For detail concerning the DNA analysis referred to in the text, see [7].

{Tree}

Von dem Stamm geht ein Stumpf aus, in einer Höhe von etwa 1,5 Metern; aus diesem Stumpf stechen einige grüne Zweige hervor, ungefähr in Richtung der untergehenden Sonne. Direkt unterhalb des Stumpfs erstreckt sich die Oberfläche einer rohgebauten Mauer aus Granit, wovon etliche Steine fehlen, und oberhalb erstrecken sich die Hauptäste des Baums und das Kronendach. Unter der Weide ist der Boden gestreut mit kleinen Zweigen, größeren Stöcken, grauen Kätzchen, die ganz am Frühlingsbeginn vom Baum abfallen und Blättern in verschiedenen Phasen des Zerfalls. Unter dem Baum sind dünne freiliegende Wurzeln auf dem Boden zu sehen und dazwischen, im schützenden Schatten des Blätterdachs, wächst ein hellgrünes Moos; weiter unten sieht man nichts.

There is a stump, around 1.5 metres up, emanating from the trunk. Some leafy stems protrude, approximately in the direction of the setting sun. Beneath the stump extends the top of a roughly built granite wall. There are some stones missing from the wall; above extend the main branches of the tree, and the canopy. Beneath the willow the ground is covered with small twigs, larger sticks, and catkins, which fall in the Spring. Some parts of the willow's slender roots are visible above the surface of the damp soil beneath the tree. Moss grows between them in the shade, and leaves gather in various stages of decay; lower down is invisible.

{*Human*}

Teile eines Skeletts wurden in einer Kalksteinhöhle endeckt, inklusiv der Schädeldach, der mit der langgestreckenden Stirn und prominenten Überaugenwülste von oben angesehen einem keramischen Gefäß der Glockenbecherkultur ähnelt. Dank der Radiokohlenstoffmethode wurde das Alter des Skeletts auf 39.900 Jahren geschätzt (bis auf einer Fehlergrenze von 620 Jahren) d.h. noch jung genung, dass eine DNA-Analyse möglich war. Mit großer Sorgfältigkeit, um Fehler zu vermeiden, wurde die ganze Nucleotidsequenz der HVR1 von der Kontrollregion mitochondrialer DNA ermittelt und mit den Linien von 994 zeitgenössischen Menschen verglichen. Während die Gensequenzen der zeitgenössischen Menschen die untereinander verglichen wurden, sich durchshnittlich um nur 8,0 Substitutionen unterschieden, unterschied sich davon die DNA des Skeletts durchschnittlich um 27,2 Positionen.

In a limestone cave, skeleton parts were found, including the skull top. From above it looks like an earthenware vessel of the Beaker people. The quantity of radiocarbon in the bones led to an estimated age. This was 39,900 years, up to an error of 620 years, youthful enough for an attempt at DNA analysis. After limiting error possibilities, the entire sequence of hypervariable region I of the mitochondrial DNA control region was determined. It was compared with 994 contemporary human lineages; these differed from our sequence by an average of 27.2 substitutions. The modern humans' sequences differed among themselves on average by 8.0 substitutions: further distinction was observed in the bones.

{Tree, Human}

Mit seinen ordentlich gekämmten Haaren sieht der aus Stein gemeisselte Kopf
ganz menschlich aus; dass er auch Baum ist erkennt man am Laub, das aus
dem Mund hervorspriesst. Die gebrochene Nase des Baummenscchen ist
früher irgendeinmal repariert worden, und andere Beschädigungsspuren, wie
Narben und Krater auf Stirn und Wangen, sind immer noch am Gesicht zu
erkennen. Mit der breiten erweiterten Stirn, dem Bart aus Blätter und den
starren Augen ist das Gesicht zum Teil raffinierte Eitelkeit, aber aufgeschlitzt
und mit einem Ausdruck der Überraschung. Solche Bildhauerkunst kommt
ab dem 12. Jahrhundert an in der Kirchenarchitektur häufig vor; Köpfe aus
Holz geschnitzt oder mit blattreichem Gesicht sind bekannt; man könnte
Erzahlungsparallele zu Dionysos ziehen.

*The stone head looks almost human, with his trim, carefully combed hair.
With vine leaves sprouting from the mouth, he is visibly a tree. The stone
tree-man has had a nose job sometime in the past. Further traces of damage,
such as scars and scratches of cheeks and forehead, are still visible on the
face. With a broad, extended forehead, the face is in part cultivated vanity,
but gashed and wearing a surprised expression. It is also part wild arboreal
man; man-tree sculptures are often found in church architecture from the
twelfth century. Examples in wood, or with leafy faces, are known; narrative
parallels can be drawn with the Greek God Dionysus.*

{Scale, Human}

Auch mit der Lautstärke auf Null gedreht, singen ihre Hände mir laut; ich erinnere mich vage an den Klang der Musik und beobachte die sorgfältigen Charakterisierungen ihrer Finger. Die Komposition dieser Passage, die aus einer chromatischen Tonleiter besteht, wäre einfach gewesen, es wäre nur darum gegangen, die Anfangs- und EndPunkte auszuwählen; sie ist eine standardmässige Übung. Unsere Tonleiter wird von zwei Passagen gleicher Tonart umrahmt, nur nach der schnell absteigenden Sequenz, die aus zwei und ein Drittel Oktaven besteht, verschnellt sich noch das Tempo. Das ganze Stück ist jetzt so berühmt, dass es selbst als eine Art Übung betrachtet werden mag, sowohl für Zuhörer als auch für Pianisten—schwieriger aber für Pianisten.

Even with the volume turned down to zero, her hands sing out. I can recall the music, and can observe her fingers' meticulous characterisations. The composition of this stretch, a chromatic scale, would have been elementary. It was fixed by a choice of beginning and endpoint; such a sequence is commonly used as a drill. The passage succeeding our rapidly descending scale is in the same key as that which precedes it, but faster. The entire piece is so famous by now, it can be seen as a sort of drill in itself. Both pianists and hearers attend to it frequently, with greater skill required from the players than from the listeners.

5. 'Relation': the musical part.

Music can be constructed via similarity (e.g., [1]). More specifically, notes which share a common partial possess a similarity that can be used to build musical compositions. This is the strategy we use to define the musical part of 'Relation', whose four sections are indexed as in section 3.

Consider the set of notes on a stave. The operation r_{12} (respectively r_{19}, r_{24}, r_{28}) of raising by 12 semitones (respectively 19, 24, 28 semitones) sends n to a note whose fundamental is approximately the second (respectively third, fourth, fifth) partial of n [4]. The operation l_{12} (respectively l_{19}, l_{24}, l_{28}) of lowering by 12 semitones (respectively 19, 24, 28 semitones) sends n to a note whose second (respectively third, fourth, fifth) partial is approximately the fundamental of n.

We can construct the pentatonic scale as follows: Take the lowest C on the piano, and all notes obtained from that C by raising it $12x$ semitones, $x \in \mathbb{Z}$, call this set of notes M_0; define $M_i = r_{19}^i(M_0)$ for $i = 1, 2, 3, 4$. We define M to be the union of the M_i, for $i = 0, ..., 4$. Then M is the pentatonic scale. The relation $4 \times 19 = 28 + 4 \times 12$ means that this scale contains notes whose fundamentals are approximately the fifth partials of the Cs in the scale. We can derive an ostinato, whose successive notes have common partials, from this relation as follows: Let t denote the note two octaves below middle C. The sequence

$$t, r_{28}(t), l_{19}r_{28}(t), r_{12}l_{19}r_{28}(t), l_{19}r_{12}l_{19}r_{28}(t), r_{24}l_{19}r_{12}l_{19}r_{28}(t),$$

$$l_{19}r_{24}l_{19}r_{12}l_{19}r_{28}(t), r_{12}l_{19}r_{24}l_{19}r_{12}l_{19}r_{28}(t), l_{19}r_{12}l_{19}r_{24}l_{19}r_{12}l_{19}r_{28}(t)$$

begins and ends with t, and when concatenated with itself successively gives an ostinato whose successive notes have common partials. The ostinato sounds resonant when played, if we allow successive notes to overlap. The resulting ostinato is a sort of pentatonic fifties progression (see eg. [8]). In our generalised song, we build a scale and an ostinato in an analogous way, using the relation $4 \times 28 = 3 \times 12 + 4 \times 19$ in place of the relation $4 \times 19 = 28 + 4 \times 12$.

The relation $4 \times 28 = 3 \times 12 + 4 \times 19$ implies that $l_{19}^4 l_{12}^3 r_{24}^4 = 1$. The musical part of 'Relation' is played in a scale, derived from this relation as follows: take the lowest C on the piano, and all notes obtained from that C by raising it $19x$ semitones, $x \in \mathbb{Z}$, call this set of notes N_0; define $N_1 = r_{28}(N_0)$, $N_2 = r_{28}(N_1)$, $N_3 = r_{28}(N_2)$; and $N_4 = r_{28}(N_3)$; define

$N_5 = l_{12}(N_4)$, $N_6 = l_{12}(N_5)$, and $N_7 = l_{12}(N_6)$. The relation $l_{19}^4 l_{12}^3 r_{24}^4 = 1$ implies that $N_7 = N_0$. We define N to be the union of the N_i, for $i = 0, ..., 6$. We define S to be the scale of notes in N whose frequencies are not too low or too high to be represented by notes on the piano.

Section $\{scale\}$ of the musical part of 'Relation' consists of the scale S played in quavers on a piano in descending order.

Suppose $w = w_{4+d}...w_2w_1$ is some word whose letters are four r_{28}s, together with a fixed number d of l_{19}s. Suppose $v = v_{3+e}...v_2v_1$ is a word whose letters are three l_{12}s, together with s fixed number e of l_{19}s. Suppose $d + e = 4$. If l_C is the lowest C on the piano, the sequence

$$l_C, w_1(l_C), w_2w_1(l_C), ..., w_{4+d}...w_2w_1(l_C),$$

$$v_1w_{4+d}...w_2w_1(l_C), ..., v_{3+e}...v_2v_1w_{4+d}...w_2w_1(l_C)$$

begins and ends with l_C. We can thus concatenate such sequences and maintain harmonious relations between successive notes.

Section $\{ostinato\}$ consists of a concatenation of ten such sequences, where $d = e = 2$. If we restrict our attention to sequences all of whose notes lie on the piano, there are five possible choices for w, which we order lexicographically from right to left with $l_{19} > r_{28}$, and ten possible choices for v, which we order lexicographically from right to left with $l_{19} > l_{12}$. Our ten sequences run through the the ten possible choices for v in descending lexicographic order, and twice through the five possible choices for w in descending lexicographic order. The result is an ostinato, which develops as the section progresses. The notes of our ostinato are minims, and successive notes overlap by a crotchet.

The ostinato continues in section $\{ostinato, scale\}$. This time we take a concatenation of seven sequences, where $d = 3$ and $e = 1$. If we restrict our attention to sequences all of whose notes lie on the piano, there are seven possible choices for w, which we order lexicographically from right to left with $l_{19} > r_{28}$, and four possible choices for v, which we order lexicographically from right to left with $l_{19} > l_{12}$. Our seven sequences run through the the seven possible choices for w in descending lexicographic order, and through the four possible choices for v in descending lexicographic order, followed by the last three of these choices for v in descending lexicographic order. In this section, the accompaniment consists of a concatenation of four note sequences

of successive quavers from the scale S. The first of the four quavers is either 19 semitones above the note it accompanies, in which case the sequence of quavers is ascending, or it is 19 semitones below the note it accompanies, in which case the sequence of quavers is descending. An ascending sequence is followed by an ascending sequence, unless such a sequence does not lie on the piano, in which case it is followed by a descending sequence. Likewise a descending sequence is followed by an descending sequence, unless such a sequence does not lie on the piano, in which case it is followed by an ascending sequence. The first note of our accompaniment is 19 semitones above the note it accompanies.

The ostinato continues in section $\{\text{ostinato}, \text{tree}\}$. We take a concatenation of five sequences, where $d = 4$ and $e = 0$. If we restrict our attention to sequences all of whose notes lie on the piano, there are five possible choices for w, which we order lexicographically from right to left with $l_{19} > r_{28}$, and only one possible choice for v. Our five sequences run through the five possible choices for w in descending lexicographic order. We have a graph whose vertices are the notes of the resulting concatenation, with edges connecting vertices corresponding to successive notes, which harmonise. This graph is thus a line, or a tree consisting of a single trunk. We wish to introduce branches to this tree by adding in further harmonies, emanating from certain notes among our concatenation that we will call *branch notes*. For our accompaniment we follow a branch note b in our ostinato by a minim $19x$ semitones above it, x crotchets later, for $x = 1, 2, 3, ...$ until after N crotchets the resulting notes do not lie on the piano. The next branch note is taken to be the note $N + 2$ crotchets after b. The first note of the section is a branch note. This determines our branch notes and our accompaniment completely.

6. 'Merge/Split': the mathematical structure.

The mathematical part of the generalised song 'Merge/Split' is the following sequence of mathematical notions: merging and splitting, coassociativity, and the braid relation. In the sequel, we exploit these mathematical notions to construct an associated verbal part 7, and an associated musical part 8.

7. 'Merge/Split': the verbal part.

The first section of the verbal structure is titled 'Coalgebra'. It looks like

$$s_1(\alpha, \beta, \alpha\beta$$

$$s_2(\alpha, \beta, \alpha\beta$$

$$s_1(\alpha, \beta, \alpha\beta$$

$$s_2(\alpha', \beta', \alpha'\beta'$$

$$s_1(\alpha', \beta', \alpha'\beta'$$

$$s_2(\alpha', \beta', \alpha'\beta'$$

$$\emptyset$$

$$s_1 * s_2(\alpha, \beta, \alpha\beta, \alpha', \beta', \alpha\beta'$$

Here, the lines $s_i(X$ of the diagram correspond to paragraphs of 'Coalgebra' featuring the words X. The symbols s_1 and s_2 represent the two distinct voices of a dialogue, whilst $s_1 * s_2$ represents those two voices speaking in unison, as a 'we'. We have $\alpha = coassociative$, $\beta = counit$, whilst $\alpha\beta = axiom$ is a conceptual level above these. We have $\alpha' = 1$, $\beta' = x$, whilst $\alpha'\beta' = C$ is a conceptual level above. We think of $\alpha\beta$ as obtained by merging α and β, and $\alpha'\beta'$ as obtained by merging α' and β'. For example, a set of axioms can be thought of as the *join* of the singleton set consisting of the coassociativity axiom and the singleton set consisting of the counit axiom, in the poset consisting of these three sets, ordered by inclusion. We think of the set of words to the right of $s_1 * s_2$ as obtained by merging those to the right of s_1 and s_2. The braid relation is represented as the sequence $s_1 s_2 s_1 s_2 s_1 s_2 \Rightarrow \emptyset$

In the second section, whose title is 'Bach's Two Part Invention No. 1', we use the diagram

$$s_1 * s_2(\alpha, \beta, \alpha\beta, \alpha', \beta', \alpha\beta'$$

$$\emptyset$$

$$s_1(\alpha\beta, \alpha, \beta$$

$$s_2(\alpha\beta, \alpha, \beta$$

$$s_1(\alpha\beta, \alpha, \beta$$

$$s_2(\alpha'\beta', \alpha', \beta'$$
$$s_1(\alpha'\beta', \alpha', \beta'$$
$$s_2(\alpha'\beta', \alpha', \beta'$$

In the second section, $\alpha = \textit{quaver}$, $\beta = \textit{semiquaver}$, whilst $\alpha\beta = \textit{bar}$ is a conceptual level above these. Likewise, $\alpha' = A$, $\beta' = C$, whilst $\alpha'\beta' = \textit{note}$ is a conceptual level above.

Coalgebra

'A K-coalgebra is a vector space C over K with a coproduct $\Delta : C \to C \otimes C$ and a counit $\epsilon : C \to K$. The coproduct obeys a coassociativity axiom, and the counit obeys a certain axiom as well.'

'Presumably the coassociativity axiom is dual to the associativity axiom for K-algebras, and the counit axiom is dual to the unit axiom for K-algebras.'

'Yes, and it is possible to visualise the maps using string diagrams. The co-product is represented as a splitting string, whilst the counit is represented as a vanishing string. The coassociativity axiom is represented as the topological equivalence of two string diagrams in which a string splits twice, whilst the counital axiom is represented as a pair of topological equivalences.'

'To pick an example, I suppose we can take C to be the vector space S with K-basis $\{x^i, i \in \mathbb{Z}_{\geq 0}\}$. For this to be a coalgebra dual to the symmetric algebra in a single variable, we should take $\Delta(x) = 1 \otimes x + x \otimes 1$.'

'The entire coproduct on S is given by

$$\Delta(x^i) = 1 \otimes x^i + x \otimes x^{i-1} + x^2 \otimes x^{i-2} + ... + x^i \otimes 1.'$$

'And we take ϵ to be the map from S to K that sends x^i to 1 for $i = 0$ and to 0 for $i > 0$.'

'We can check the coalgebra axioms directly on $C = S$. Indeed, the maps $(\Delta \otimes 1)\Delta$ and $(1 \otimes \Delta)\Delta$, evaluated at x^i, are both equal to $\sum_{j+k+l=i} x^j \otimes x^k \otimes x^l$, which verifies the coassociativity axiom. The composite of $(\epsilon \otimes id)\Delta$ and the canonical isomorphism $K \otimes S \cong S$ sends x^i to x^i, and is therefore the identity map, which verifies one side of the counit axiom. The other side is verified likewise.'

Bach's Two Part Invention No. 1

'During the first bar, we hear all of the notes of the C major triad: C, E, and G. In the right hand, we hear seven semiquavers and four quavers, whilst in the left hand we hear seven semiquavers.'

'Starting at the third bar, I see there follows a passage of two bars consisting of only semiquavers in the right hand, and only quavers in the left hand.'

'Yes, and after an intervening passage of six bars, there is a passage of two bars consisting of only quavers in the right hand, and only semiquavers in the left hand.'

'The two lines merge in the next bar, on the eighth semiquaver in the right hand, after three quavers, one of them dotted, in the left hand.'

'Looking back I can see the notes C, E, and G of the C major triad in bars 3,4,5,6,8,9,10 as well.'

'Yes, and in the other intervening bars there is only one of the three notes missing. For example, the second bar contains a C and some Gs, but no E.'

'The note on which the two lines merge is the E two whole tones above middle C.'

8. 'Merge/Split': the musical part.

Our musical part has three sections. The first section represents a sequence of six consecutive higher dimensional arrows between the following diagrams:
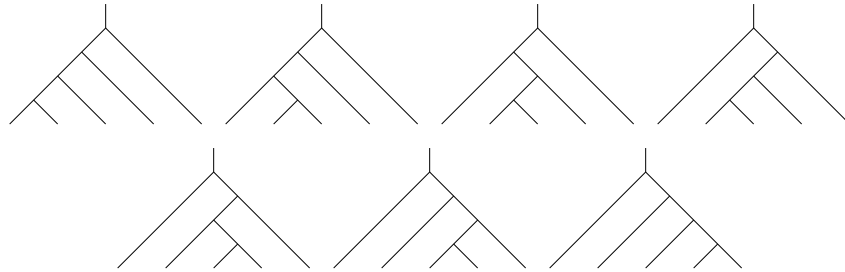

Figure 2: Coassociativity in action.

This diagram represents a proof that

$$(\Delta \otimes 1 \otimes 1 \otimes 1)(\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)(\Delta(x)) =$$

$$(1 \otimes 1 \otimes 1 \otimes \Delta)(1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Delta(x)),$$

for $\Delta : C \to C \otimes C$ coassociative and $x \in C$. Note that the coassociative law itself

$$(1 \otimes \Delta)(\Delta(x)) = (\Delta \otimes 1)(\Delta(x))$$

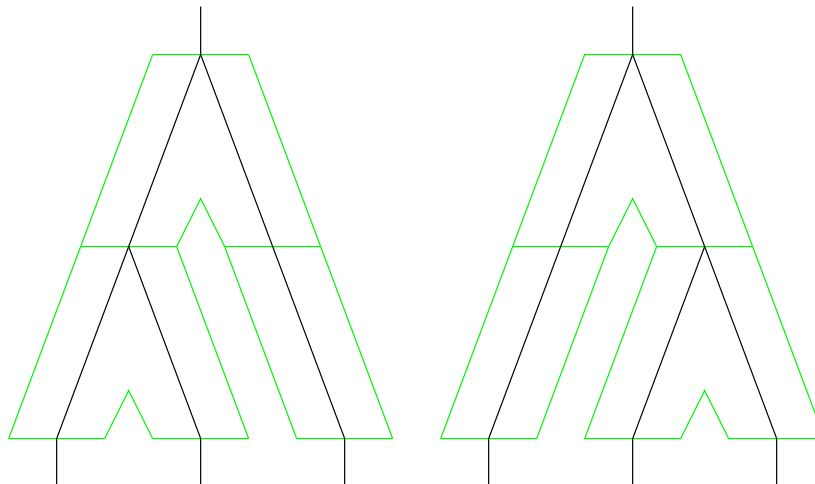for $x \in C$ can be depicted as follows:

Figure 3: The coassociative law.

where $\Delta : C \to C \otimes C$ is represented as a splitting string. Here the single point at the top of a diagram represents the coalgebra C on which Δ acts, and the three points at the bottom of the diagram represent the three components of $C \otimes C \otimes C$.

The second section represents a sequence of eight consecutive higher dimensional arrows between the following diagrams:
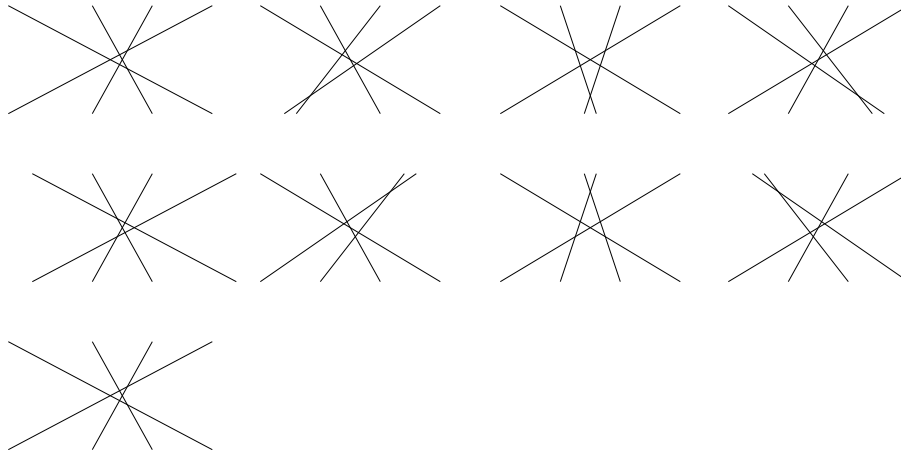
Figure 4: A sequence of compositions of braid relations.

Note that the braid relation itself can be depicted as follows (cf. [3], [6]):
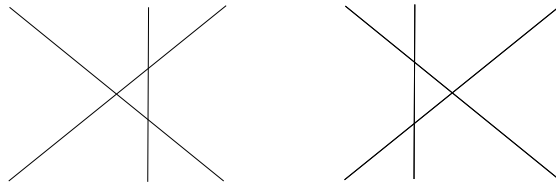
Figure 5: The braid relation $s_2 s_1 s_2 \Rightarrow s_1 s_2 s_1$.

How do we relate these diagrams to musical structures? We consider a diagram to represent a sum of functions $\sum_{i=1}^{n} \sin(2\pi f_i(t)t)$, where $f_i : \mathbb{R} \to \mathbb{R}$ is a function. Such a sum of functions is then played through a loudspeaker. We interpret a diagram as a representation of the graphs of the functions f_i, with the t axis running down the page. For example, the diagrams in Figure 2 each feature five overlapping graphs. In this way, a merging or splitting

of strings corresponds to a merging or splitting of sine wave components, and a crossing of strings corresponds to a merging of sine wave components, followed by a splitting of sine wave components.

The third section is slightly different from the first two, and we interpret counterpoint [5] as a form of splitting followed by merging. To see this, think of a musical note as the superposition of its harmonics as in section 2, and think of a pair of notes as being consonant when they share a common harmonic. In contrapuntal motion we have consecutive harmonies, which can thus be depicted as a merging and splitting of parts:
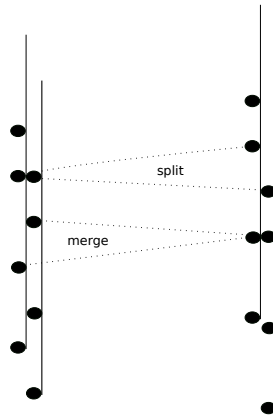

Figure 6: Contrapuntal motion.

Here a vertical line depicts a note, and the blobs attached to the line depict the harmonics of that note. The frequency of the harmonics increases up the page. The upper notes belong to one musical line, and the lower notes belong to another musical line. Adjacent blobs correspond to a consonance between the upper and lower left hand notes, and likewise there is a consonance between the upper and lower right hand notes.

To accentuate the merging and splitting, we introduce glissandi to connect consecutive notes, the second half of a note being interpreted as a glissando to the next. A rest is interpreted as a silent note whose pitch is that of the preceding note. To follow this procedure precisely, we must use a piece which uses the above interpretation of counterpoint precisely. We gave a number of examples in a previous work (minus glissandi) [10], and we select one (qn6) to give us our third section.

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