A New Ehrlich-Type Sixth-Order Simultaneous Method for Polynomial Complex Zeros

Um Novo Método Simultâneo de Sexta Ordem Tipo Ehrlich para Zeros Polinomiais Complexos

Luiz Guerreiro Lopes[1,†], Roselaine Neves Machado[2]

[1]*Faculty of Exact Sciences and Engineering, University of Madeira, Funchal, Madeira Is., Portugal*
[2]*Federal Institute of Rio Grande do Sul, Bento Gonçalves Campus, Bento Gonçalves, RS, Brazil*
[†]**Corresponding author:** lopes@uma.pt

Abstract

This paper presents a new iterative method for the simultaneous determination of simple polynomial zeros. The proposed method is obtained from the combination of the third-order Ehrlich iteration with an iterative correction derived from Li's fourth-order method for solving nonlinear equations. The combined method developed has order of convergence six. Some examples are presented to illustrate the convergence and efficiency of the proposed Ehrlich-type method with Li correction for the simultaneous approximation of polynomial zeros.

Keywords

Polynomial zeros • Simultaneous iterative methods • Ehrlich method • Li's fourth-order method

Resumo

Este artigo apresenta um novo método iterativo para a determinação simultânea de zeros polinomiais simples. O método proposto é obtido a partir da combinação da iteração de Ehrlich de terceira ordem com uma correção iterativa derivada do método de Li de quarta ordem para a resolução de equações não lineares. O método combinado desenvolvido tem ordem de convergência seis. Alguns exemplos são apresentados para ilustrar a convergência e eficiência do método tipo Ehrlich com correção de Li proposto para a aproximação simultânea de zeros polinomiais.

Palavras-chave

Zeros de polinômios • Métodos iterativos simultâneos • Método de Ehrlich • Método de Li de quarta ordem

1 Introduction

The problem of finding all zeros of a polynomial arises quite frequently in practice and is of great importance in many branches of science and engineering (see, e.g., [1, 2]). However, the apparent mathematical simplicity of the polynomials with real or complex coefficients is clearly belied by the difficulty of finding all their zeros.

Referring to the complexity involved in numerically approximating the zeros of complex polynomials, the renowned Swiss numerical analyst Peter Henrici (1923–1987) wrote in 1970 [3, p. 1]: "The problem of determining the zeros of a given polynomial with complex coefficients is a genuine nonlinear problem. At the same time, the problem is simple. It is so simple, in fact, that there is some hope that some day we may be able to solve it perfectly."

More than a half-century has passed since then, a period in which a large number of sequential and simultaneous iterative methods for finding polynomial zeros have been developed and published. However, the truth and hope that were hidden in those words are still as strong today as they were back then. This is especially true when the very ill-conditioned nature of the problem of finding polynomial zeros is considered.

A classic example of ill-conditioning is shown by the polynomial $P(x) = (x-1)(x-2)(x-3)\cdots(x-20) = x^{20} - 210x^{19} + 20615x^{18} - \ldots + (20!)$, given by Wilkinson [4, 5], whose zeros are, evidently, $x_j = j$, $j = 1, 2, 3, \ldots, 20$. Although the zeros of $P(x) = \prod_{j=1}^{20}(x-j)$ are real, simple, and well separated, a very small perturbation in one of the coefficients of the polynomial $P(x)$ can cause a drastic change in them, affecting not only their values but also their nature (see, e.g., [6, pp. 201–202], [7, pp. 266–268], and [8]). Referring to the strangeness caused by this unusual result when discussing the sensitivity of the zeros of the polynomial $P(x)$ and the errors involved in polynomial deflation, Wilkinson [8] described in the following way his experience at the beginning of the 1950s when implementing the Newton–Raphson method on an electronic computer aiming to find the largest zero of the mentioned polynomial, now known as Wilkinson's polynomial: "Speaking for myself I consider it as the most traumatic experience in my career as a numerical analyst."

However, even with polynomials of relatively low degree, difficulties similar to those found in the previous example may occur, especially in the case of polynomials with multiple zeros or clusters of zeros, which makes it clear that the problem of determining the zeros of polynomials is not trivial.

In view of that, there is still a high interest in developing new and more efficient numerical algorithms for approximating polynomial zeros, in particular iterative methods that allow finding all zeros of polynomials simultaneously. Although the simultaneous methods require appropriate starting approximations for all zeros in order to converge, they are innately parallel and have the advantage of avoiding the need for repeated deflations that are necessary for obtaining all zeros with sequential iterative methods, which can lead to very inaccurate results due to error accumulation in floating-point operations.

Considering this advantage, this paper presents and discusses a sixth-order simultaneous polynomial zero-finding method resulting from the combination of the well-known Ehrlich–Aberth iteration [9, 10] for the simultaneous approximation of polynomial zeros with the fourth-order Li's [11] method for nonlinear equations.

2 Preliminaries and notation

2.1 The third-order Ehrlich method

Let $P(z)$ be a monic complex polynomial of degree n with simple zeros $\zeta_1, \ldots, \zeta_n$, given by:

$$P(z) = \prod_{j=1}^{n}(z-\zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad a_i \in \mathbb{C}, \quad i \in \{0, \ldots, n-1\}. \tag{1}$$

Let us consider now the correction term $N(z) = P(z)/P'(z)$ from the well-known 2nd order Newton's method $\hat{z} = z - N(z)$, where, for simplicity of notation, $\hat{z}$ is a new approximation to a zero ζ. By taking its logarithmic derivative with respect to z, and given that, for $i \neq j$, the resulting sum can be separated into two parts, we obtain:

$$N(z) = \frac{P(z)}{P'(z)} = \left(\frac{d}{dz}\log P(z)\right)^{-1} = \left(\sum_{j=1}^{n}\frac{1}{z-\zeta_j}\right)^{-1} = \left(\frac{1}{z-\zeta_i} + \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z-\zeta_j}\right)^{-1}, \tag{2}$$

from where the following fixed-point relation can be easily derived:

$$\zeta_i = z - \left(\frac{1}{N(z)} - \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z-\zeta_j}\right)^{-1}, \quad i = 1, \ldots, n. \tag{3}$$

Let $z_1, \ldots, z_n$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of the polynomial $P(z)$.

The well-known Ehrlich–Aberth method [9, 10] for the simultaneous approximation of simple polynomial zeros can be obtained directly from (3) by putting $\zeta_i \simeq \hat{z}_i$ (where $\hat{z}_i$ is a new approximation to the zero ζ_i), setting $z = z_i$, and using the approximations z_j $(j \neq i)$ in place of the zeros ζ_j on the above identity. The resulting iterative method, also known simply as Ehrlich method, is as follows:

$$\hat{z}_i = z_i - \left(\frac{1}{N(z_i)} - \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-z_j}\right)^{-1}, \quad i = 1, \ldots, n. \tag{4}$$

This iterative formula was first derived by Maehly [12], and later independently by Börsch-Supan [13], Dočev and Byrnev [14], Ehrlich [10], Weißenhorn [15], Aberth [9], and Farmer and Loizou [16].

The convergence order of the Ehrlich method for simple zeros is three [17, 10].

2.2 The fourth-order Li method

A new two-step iterative method for nonlinear equations was introduced a few years ago by Li [11]. This method for finding a simple root of a nonlinear equation $f(x) = 0$ is defined by

$$\hat{x}_i = x_i - \frac{(f(x_i) - f(y_i))\, f(x_i)}{(f(x_i) - 2f(y_i))\, f'(x_i)}, \quad i = 0, 1, \dots, \tag{5}$$

where

$$y_i = x_i - \frac{f(x_i)}{f'(x_i)}, \tag{6}$$

with $f'(x_i) \neq 0$.

The first step, given by (6), is Newton's well-known second-order method. The derivation of this iterative method will be omitted here and can be found in Ref. [11]. The order of convergence of Li's method is four, as demonstrated in the same paper.

As can be seen above, this method involves only two function evaluations and one first derivative evaluation per iteration, not requiring the calculation of higher-order derivatives. Thus, according to the Kung–Traub conjecture [18], this iterative method is optimal because it requires a total of $n = 3$ functional evaluations per iteration, and its convergence order is equal to 2^{n-1}.

In this paper, an iterative correction derived from the optimal fourth-order Li's method will be applied to the Ehrlich–Aberth method (4) to increase its convergence rate, thus giving rise to a new high-order iterative method for the simultaneous approximation of polynomial complex zeros.

3 Ehrlich-type simultaneous method with Li correction

When examining the fixed-point relation (3), it becomes evident that more accurate approximations $\hat{z}_i$ for the zeros ζ_i of the polynomial $P(z)$ can be obtained by utilizing improved estimates z_j.

In order to improve the accuracy of estimates, the authors propose here the use of an iterative correction based on Li's optimal fourth-order two-step method (5), which is given by

$$x_j = z_j - \frac{P(z_j)}{P'(z_j)}, \tag{7}$$

$$K_L(z_j) = \frac{(P(z_j) - P(x_j))\, P(z_j)}{(P(z_j) - 2P(x_j))\, P'(z_j)}, \quad j = 1, \dots, n. \tag{8}$$

By substituting the Li approximation $z_j - K_L(z_j)$ in Eq. (4) instead of z_j, a new Ehrlich-type simultaneous method with Li correction is obtained, which is defined as follows:

$$\hat{z}_i = z_i - \left(\frac{1}{N(z_i)} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i - z_j + K_L(z_j)} \right)^{-1}, \quad i = 1, \dots, n. \tag{9}$$

The determination of the convergence order of the proposed iterative method will be addressed in the following section.

4 Convergence analysis

The following theorem establishes the order of convergence of the iterative scheme (9).

Theorem 4.1. *Consider a degree-n complex polynomial $P(z)$ with distinct simple zeros $\zeta_1, \dots, \zeta_n$ and let $z_1^{(0)}, \dots, z_n^{(0)}$ be initial guesses close enough to the zeros of $P(z)$. Then, the combined simultaneous method* (9) *has a convergence order of six.*

Proof. Substituting the right side of (2) into (9), we obtain

$$\hat{z}_i = z_i - \left(\frac{1}{z_i - \zeta_i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i - \zeta_j} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i - z_j + K_L(z_j)} \right)^{-1}. \tag{10}$$

Now, let us consider the numerical approximation errors

$$\epsilon_i = z_i - \zeta_i, \tag{11}$$

$$\hat{\epsilon}_i = \hat{z}_i - \zeta_i, \tag{12}$$

and, for the sake of convenience, the abbreviations

$$\gamma_{i,j} = z_i - z_j + K_L(z_j), \tag{13}$$

$$\lambda_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{z_j - \zeta_j - K_L(z_j)}{(z_i - \zeta_j)\gamma_{i,j}}. \tag{14}$$

Taking into account (13) and the approximation errors defined above, we have

$$\hat{\epsilon}_i = \epsilon_i - \left(\frac{1}{\epsilon_i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i - \zeta_j} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{\gamma_{i,j}} \right)^{-1}. \tag{15}$$

By combining the two sums and considering (13) again, we obtain

$$\hat{\epsilon}_i = \epsilon_i - \left(\frac{1}{\epsilon_i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{K_L(z_j) - z_j + \zeta_j}{(z_i - \zeta_j)\gamma_{i,j}} \right)^{-1}. \tag{16}$$

Considering (14), the previous sum is equal to $-\lambda_i$, which, after simple algebraic manipulations, leads to

$$\hat{\epsilon}_i = \frac{-{\epsilon_i}^2 \lambda_i}{1 - \epsilon_i \lambda_i}. \tag{17}$$

According to the theorem's assumption, the starting guesses are in close proximity to the zeros, resulting in small errors ϵ_i and $\hat{\epsilon}_i$ in terms of magnitude. Based on the given information, it can be inferred that ϵ_i is of the same order as ϵ_j and, similarly, $\hat{\epsilon}_i$ is also of the same order as $\hat{\epsilon}_j$, i.e., $\epsilon_i = \mathcal{O}_m(\epsilon_j)$ and $\hat{\epsilon}_i = \mathcal{O}_m(\hat{\epsilon}_j)$, for $i, j \in \{1, \dots, n\}$. In other words, the absolute values of ϵ_i and $\hat{\epsilon}_i$ are of the same order as the absolute values of ϵ_j and $\hat{\epsilon}_j$, respectively, i.e., the equalities $|\epsilon_i| = \mathcal{O}(|\epsilon_j|)$ and $|\hat{\epsilon}_i| = \mathcal{O}(|\hat{\epsilon}_j|)$ hold for $i, j \in \{1, \dots, n\}$.

Upon analyzing equations (13) and (14), it is evident that the denominator in Eq. (14) is limited and converges to $(\zeta_i - \zeta_j)^2$ for $i \neq j$ when it involves estimates that are sufficiently close to the zeros. In turn, Li's method exhibits a fourth-order convergence, i.e., $\hat{z}_i - \zeta = \mathcal{O}_m((z_i - \zeta)^4)$.

By considering the aforementioned two results, it can be deduced that the value of λ_i has an order of magnitude $\mathcal{O}_m(\epsilon^4)$. By applying the obtained result to (17), it can be concluded that the error term $\hat{\epsilon}$ is of the order $\mathcal{O}_m(\epsilon^6)$. This result serves as proof that the convergence order of the proposed simultaneous method, incorporating Li's correction as shown in (9), is six. □

5 Numerical examples

In order to illustrate the convergence and effectiveness of the proposed simultaneous iterative method, some real and complex polynomials with degrees between 5 and 20 are presented below. Table 1 presents the test polynomials and the references from which they were extracted. $P_5(z)$ is a Mignotte polynomial of the form $P(z) = z^n - (az - 1)^2$, with $n = 18$ and $a = 9$, whereas $P_6(z)$ is referred to as the scaled Wilkinson polynomial.

The proposed combined algorithm (9) was tested and compared with the well-known Ehrlich method using initial approximations to the zeros generated through the Aberth initialization scheme [9], with inclusion radii for the zeros given by the Guggenheimer bound [19].

Although such an initialization procedure is not the most suitable for the Ehrlich method and Ehrlich-type methods as it can lead to the non-convergence of these iterative methods, it was adopted here for illustrative purposes due to its simplicity. The initialization scheme proposed by Bini [20], which relies on a proper application of Rouché's theorem, is considered to be more adequate for this class of simultaneous iterative methods.

All the results presented in this section were obtained using double-precision floating-point arithmetic, with a numerical tolerance TOL $= 1\times10^{-12}$ and a maximum of MAXITER $= 50$ iterations.

Table 1: Test polynomials.

Polynomial	Ref.
$P_1(z) = z^5 - (4-i)z^4 + (6-4i)z^3 - (4-6i)z^2 - (15+4i)z - 15i$	[21]
$P_2(z) = z^5 - (4+5i)z^4 + (6+20i)z^3 - (4+30i)z^2 - (15-20i)z + 75i$	[22]
$P_2(z) = z^5 - (4+5i)z^4 + (6+20i)z^3 - (4+30i)z^2 - (15-20i)z + 75i$	[22]
$P_3(z) = z^{10} - 5iz^9 - 6z^8 - z^2 + 5iz + 6$	[23]
$P_4(z) = z^{15} + z^{14} + 1$	[24]
$P_5(z) = z^{18} - 81z^2 + 18z - 1$	[25]
$P_6(z) = \prod_{k=1}^{20}(z - \dfrac{k}{20})$	[26]

The stopping criterion adopted is defined as

$$E^{(k)} = \max_{1\le i\le n} |P(z_i^{(k)})| < \text{TOL} = 10^{-12}. \tag{18}$$

The number of iterations required for both methods to achieve convergence is shown in Table 2. The obtained results show that the proposed combined method consistently requires fewer iterations to meet the adopted stopping criterion, indicating an advantage in terms of convergence speed when compared to the Ehrlich method that served as the basis for it, which is in agreement with the theoretical result obtained in Section 4.

Table 2: Number of iterations for convergence.

Method	Eq.	P_1	P_2	P_3	P_4	P_5	P_6
Ehrlich	(4)	7	12	14	9	23	45
Ehrlich–Li	(9)	5	8	9	6	15	29

The observed differences were tested for statistical significance using the Wilcoxon signed-rank test [27]. This nonparametric test was designed for paired data and two related groups, making it the ideal option for comparing two algorithms on the same test problems. A significance level of $\alpha = 0.05$ was utilized. The results of the Wilcoxon signed-rank test (W = 21, p-value = 0.03125, which is less than α) indicated that, at the α-level of 0.05, there is a significant difference in performance between the two iterative methods on the test problems considered.

Figures 1 through 6 show the trajectories of approximations in the complex plane produced by the proposed iterative method (9) for the six test polynomials. The convergence trajectories for each of the polynomial zeros offer a visual representation of the behavior of the iterative method and are indicated with different colors.

The accuracy of the numerical approximations obtained is given by the maximal error between the numerical estimates and the exact values of the polynomial zeros,

$$\epsilon^{(k)} = \max_{1\le i\le n} |z_i^{(k)} - \zeta_i|, \tag{19}$$

where $k = 0, 1, \ldots$ is the iteration index.

Tables 3 and 4 show, by way of illustration, the maximal errors for the first five and first eight iterations of the Ehrlich method and the combined Ehrlich-Li method for the polynomials $P_1(z)$ (whose zeros are -1, $-i$, $1\pm 2i$, and 3) and $P_2(z)$ (with zeros at -1, $1\pm 2i$, 3, and $5i$), respectively. These results highlight the accuracy of the approximations generated by the proposed combined method.

6 Conclusion

The use of a correction term obtained from Li's optimal fourth-order method for nonlinear equations allows to increase the convergence order of the basic simultaneous method from three to six.

The provided examples illustrate the convergence and effectiveness of the proposed Ehrlich-like iterative method with Li's correction for approximating simple polynomial zeros simultaneously.

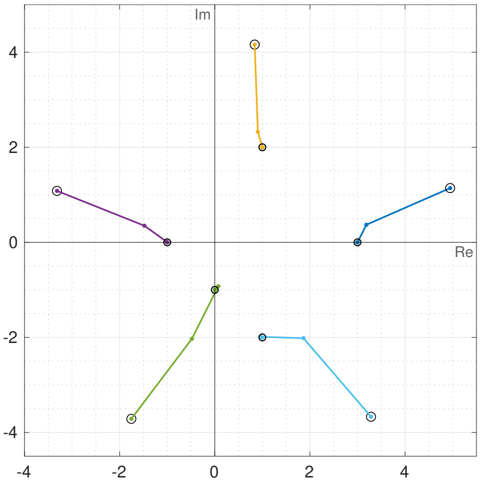

Figure 1: Approximation trajectories for polynom P_1.

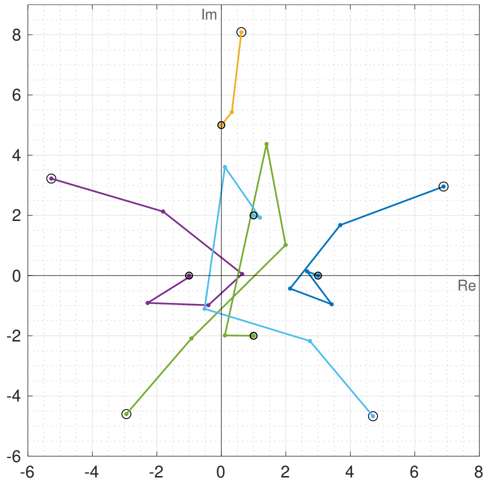

Figure 2: Approximation trajectories for polynom P_2.

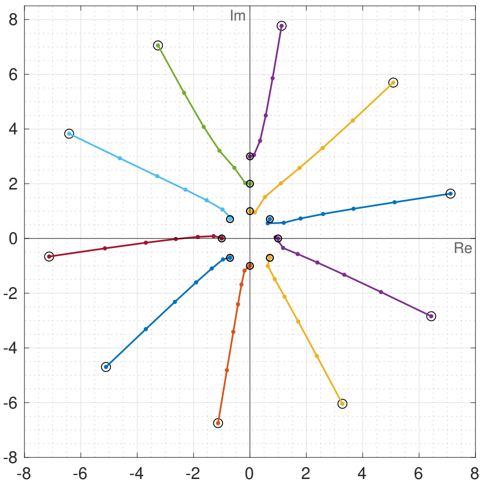

Figure 3: Approximation trajectories for polynom P_3.

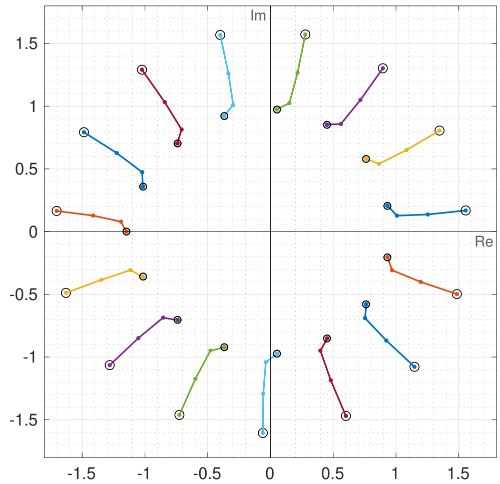

Figure 4: Approximation trajectories for polynom P_4.

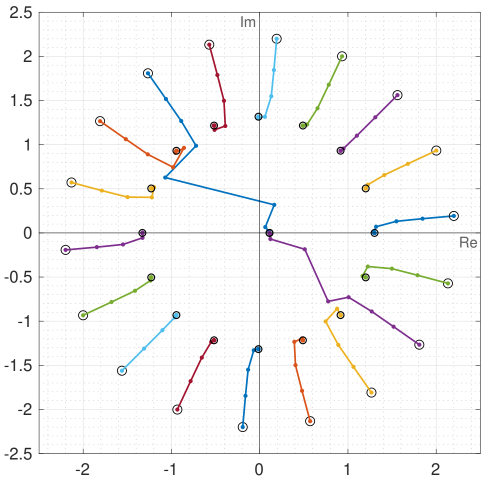

Figure 5: Approximation trajectories for polynom P_5.

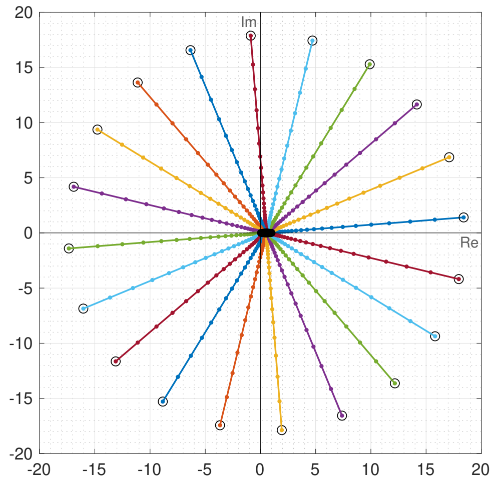

Figure 6: Approximation trajectories for polynom P_6.

Table 3: Maximal errors for the first five iterations of both methods for the polynomial $P_1(z)$.

k	$\epsilon^{(k)}_{\text{Ehrlich}}$	$\epsilon^{(k)}_{\text{Ehrlich–Li}}$
1	4.397 401 184 163 037	3.231 324 252 158 995
2	2.566 760 784 910 320	1.137 845 149 029 677
3	1.323 879 254 852 072	$1.063\,928\,273\,501\,572 \times 10^{-1}$
4	$4.598\,911\,197\,631\,101 \times 10^{-1}$	$2.003\,374\,465\,431\,683 \times 10^{-8}$
5	$4.349\,057\,266\,580\,498 \times 10^{-2}$	$2.220\,446\,049\,250\,313 \times 10^{-16}$

Table 4: Maximal errors for the first eight iterations of both methods for the polynomial $P_2(z)$.

k	$\epsilon^{(k)}_{\text{Ehrlich}}$	$\epsilon^{(k)}_{\text{Ehrlich–Li}}$
1	9.956 808 368 086 701	7.631 311 833 129 192
2	7.180 073 625 458 132	4.523 528 324 103 269
3	5.302 872 723 887 493	3.455 363 738 770 611
4	3.747 212 660 831 036	6.382 886 484 617 312
5	$1.167\,619\,561\,971\,287 \times 10^{1}$	1.572 922 295 722 127
6	4.749 679 952 242 196	$4.740\,868\,916\,357\,079 \times 10^{-2}$
7	2.187 307 532 257 996	$3.495\,706\,720\,081\,935 \times 10^{-10}$
8	2.568 938 041 932 459	$2.482\,534\,153\,247\,273 \times 10^{-16}$

References

[1] V. Y. Pan, "Old and new nearly optimal polynomial root-finders," in *Computer Algebra in Scientific Computing*, M. England, W. Koepf, T. M. Sadykov, W. M. Seiler, and E. V. Vorozhtsov, Eds. Cham, Switzerland: Springer, 2019, pp. 393–411.

[2] N. Rafiq, M. Shams, N. A. Mir, and Y. U. Gaba, "A highly efficient computer method for solving polynomial equations appearing in engineering problems," *Mathematical Problems in Engineering*, vol. 2021, 9826693, 2021. Available at: https://doi.org/10.1155/2021/9826693

[3] P. Henrici, "Uniformly convergent algorithms for the simultaneous determination of all zeros of a polynomial," in *Numerical Solution of Nonlinear Problems*, J. M. Ortega and W. C. Rheinboldt, Eds. Philadelphia, PA: SIAM, 1970, pp. 1–8.

[4] J. H. Wilkinson, "The evaluation of the zeros of ill-conditioned polynomials. Part I," *Numerische Mathematik*, vol. 1, no. 1, pp. 150–166, 1959. Available at: https://doi.org/10.1007/BF01386381

[5] ——, *Rounding Errors in Algebraic Processes*. Englewood Cliffs, NJ: Prentice-Hall, 1963.

[6] F. S. Acton, *Numerical Methods that Work*. Washington, DC: Mathematical Association of America, 1990.

[7] A. Neumaier, *Introduction to Numerical Analysis*. Cambridge, UK: Cambridge University Press, 2001.

[8] J. H. Wilkinson, "The perfidious polynomial," in *Studies in Numerical Analysis*, G. H. Golub, Ed. Washington, DC: Mathematical Association of America, 1984, pp. 1–28.

[9] O. Aberth, "Iteration methods for finding all zeros of a polynomial simultaneously," *Mathematics of Computation*, vol. 27, no. 122, pp. 339–344, 1973. Available at: https://doi.org/10.1090/S0025-5718-1973-0329236-7

[10] L. W. Ehrlich, "A modified Newton method for polynomials," *Communications of the ACM*, vol. 10, no. 2, pp. 107–108, 1967. Available at: https://doi.org/10.1145/363067.363115

[11] S. Li, "Fourth-order iterative method without calculating the higher derivatives for nonlinear equation," *Journal of Algorithms & Computational Technology*, vol. 13, pp. 1–8, 2019. Available at: https://doi.org/10.1177/1748302619887686

[12] H. J. Maehly, “Zur iterativen Auflösung algebraischer Gleichungen,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 5, no. 3, pp. 260–263, 1954. Available at: https://doi.org/10.1007/BF01600333

[13] W. Börsch-Supan, “A posteriori error bounds for the zeros of polynomials,” *Numerische Mathematik*, vol. 5, no. 1, pp. 380–398, 1963. Available at: https://doi.org/10.1007/BF01385904

[14] K. Dochev and P. Byrnev, “Certain modifications of Newton’s method for the approximate solution of algebraic equations,” *U.S.S.R. Computational Mathematics and Mathematical Physics*, vol. 4, no. 5, pp. 174–182, 1964. Available at: https://doi.org/10.1016/0041-5553(64)90148-X

[15] F. Weißenhorn, “Ein Beitrag zur Bestimmung der Nullstellen aus einer Polynom in Summenform und aus der Summe von Polynomen in Produktform,” *Archiv für Elektronik und Übertragungstechnik*, vol. 24, pp. 372–378, 1970.

[16] M. R. Farmer and G. Loizou, “A class of iteration functions for improving, simultaneously, approximations to the zeros of a polynomial,” *BIT*, vol. 15, no. 3, pp. 250–258, 1975. Available at: https://doi.org/10.1007/BF01933657

[17] G. Alefeld and J. Herzberger, “On the convergence speed of some algorithms for the simultaneous approximation of polynomial roots,” *SIAM Journal of Numerical Analysis*, vol. 11, no. 2, pp. 237–243, 1974. Available at: https://doi.org/10.1137/0711023

[18] H. T. Kung and J. F. Traub, “Optimal order of one-point and multipoint iteration,” *Journal of the ACM*, vol. 21, no. 4, pp. 643–651, 1974. Available at: https://doi.org/10.1145/321850.321860

[19] H. Guggenheimer, “Bounds for roots of algebraic equations,” *Archiv der Mathematik*, vol. 31, no. 6, pp. 568–569, 1978. Available at: https://doi.org/10.1007/BF01226493

[20] D. A. Bini, “Numerical computation of polynomial zeros by means of Aberth’s method,” *Numerical Algorithms*, vol. 13, no. 2, pp. 179–200, 1996. Available at: https://doi.org/10.1007/BF02207694

[21] M. Petković and L. Stefanović, “On the convergence order of accelerated root iterations,” *Numerische Mathematik*, vol. 44, no. 3, pp. 463–476, 1984. Available at: https://doi.org/10.1007/BF01405575

[22] M. S. Petković and L. V. Stefanović, “On the simultaneous method of the second order for finding polynomial complex zeros in circular arithmetic,” Freiburg im Breisgau, F.R.G., Freiburger Interval-Berichte 85/3, 63–95, 1985.

[23] I. Gargantini, “Parallel Laguerre iterations: the complex case,” *Numerische Mathematik*, vol. 26, no. 3, pp. 317–323, 1976. Available at: https://doi.org/10.1007/BF01395948

[24] P. D. Proinov and M. T. Vasileva, “On the convergence of high-order Ehrlich-type iterative methods for approximating all zeros of a polynomial simultaneously,” *Journal of Inequalities and Applications*, vol. 2015, 336, 2015. Available at: https://doi.org/10.1186/s13660-015-0855-5

[25] ——, “A new family of high-order Ehrlich-type iterative methods,” *Mathematics*, vol. 9, no. 16, 1855, 2021. Available at: https://doi.org/10.3390/math9161855

[26] H. Zhang, “Numerical condition of polynomials in different forms,” *Electronic Transactions on Numerical Analysis*, vol. 12, pp. 66–87, 2001. Available at: http://eudml.org/doc/121427

[27] J. D. Gibbons and S. Chakraborti, *Nonparametric Statistical Inference*. Boca Raton, FL: Chapman & Hall/CRC, 2010.