Static Analysis of Isotropic Beams Resting on Elastic Foundations of the Winkler-Pasternak Type☆

Análise Estática de Vigas Isotrópicas Apoiadas em Base Elástica Winkler-Pasternak

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Abstract

Beams resting on elastic foundations are widely used in engineering projects, so analyzing their displacement fields is very important. The present work presents solutions for the deflection of isotropic beams resting on elastic foundations of the Winkler-Pasternak type. The proposed formulation is based on the Euler-Bernoulli beam theory, and the governing equations and the boundary conditions are derived from the principle of virtual work. The direct integration method can decouple the deflections from axial displacement and twists. The system of deflection equations decouples into two principal directions and is transformed into a first-order system. The solution of this system of equations is obtained through the method of variation of parameters. When analyzing the results of the maximal deflections, it is observed that increasing values of the foundation stiffness provide decreasing deflections and that the influence of the Pasternak parameter is more significant on the results than that of the Winkler parameter.

Keywords

Euler-Bernoulli Theory • Isotropic Beams • Winkler-Pasternak Foundations

Resumo

Vigas apoiadas em fundações elásticas são amplamente utilizadas em projetos de engenharia, logo analisar os deslocamentos sofridos por elas é muito importante. Por isso, o presente trabalho apresenta soluções para a deflexão de vigas isotrópicas sobre fundações elásticas do tipo Winkler-Pasternak. A formulação proposta é baseada na teoria de vigas de Euler-Bernoulli e as equações que descrevem o problema e as condições de contorno são derivadas do princípio do trabalho virtual. O método de integração direta é utilizado para desacoplar as deflexões do deslocamento axial e da torção. O sistema de equações desacopladas de deflexão em duas direções principais é transformado em um sistema de primeira ordem. A solução deste sistema de equações é obtida através do método de variação de parâmetros. Ao analisar os resultados das deflexões máximas, observa-se que com o aumento dos

☆ This article is an extended version of the work presented at the Joint XXV ENMC National Meeting on Computational Modeling, XIII ECTM Meeting on Science and Technology of Materials, 9th MCSul South Conference on Computational Modeling and IX SEMENGO Seminar and Workshop on Ocean Engineering, held in webinar mode, from October 19th to 21th, 2022 valores de rigidez de fundação, os deslocamentos de flexão diminuem e que o parâmetro de rigidez de Pasternak tem maior influência aos resultados que a rigidez de Winkler.

Palavras-Chave

Teoria de Euler-Bernoulli • Vigas Isotrópicas • Fundações Winkler-Pasternak

1 Introduction

As there is transmission of efforts from the infrastructure to the superstructure, it is important to analyze the effects of the soil in the structural analysis. For this, there are physical models that try to describe the behavior of the base when subjected to a load, such as the models proposed by Winkler and Pasternak in 1867 and 1954, respectively [1].

According to Doeva, Masjedi and Weaver [2], the structural analysis of beams supported on elastic foundations attracts a lot of attention, and therefore, there are several analytical and numerical solutions to the problem in the literature. However, the vast majority of them are based on complex series techniques and are limited to specific boundary types or loading conditions.

In order to get around these problems, Doeva, Masjedi and Weaver [2] proposed a methodology for building analytical solutions for bending composite beams using the Euler-Bernoulli theory based on two-parameter elastic foundations. For this purpose, variational principles and fundamental matrices were used. To reduce the fourth-order terms that appear in the equations describing bending to first order, a grouping is performed between the transverse displacement fields in the thickness and width direction, and a second grouping composed of axial displacement and rotation. Thus, it is possible to build the fundamental matrices to obtain the analytical solution of the method of variation of parameters. The results were validated using the Chebyshev collocation method.

In [2], displacement results were presented for simply supported and clamped isotropic beams under uniformly distributed load on a Winkler-Pasternak elastic foundation, but the analytical solutions were not presented.

Thus, this work seeks to present the analytical solutions for the structural behavior when considering or not the elastic foundation through the methodology proposed by Doeva, Masjedi and Weaver [2]. Furthermore, the displacements for a cantilevered Euler-Bernoulli beam under point load at the free end and on a Winkler-Pasternak elastic foundation from this methodology are presented for the first time.

2 Mathematical Modeling

Consider a beam on an elastic foundation characterized by two parameters, namely, Winkler stiffness modulus k_w and Pasternak shear modulus k_p. The beam has length l, width b and height h, whose origin of the coordinate system is located on the axis of the beam, as shown in Fig. 1. Thus, $(x, y, z) \in [0, l] \times \left[-\frac{b}{2}, \frac{b}{2}\right] \times \left[-\frac{h}{2}, \frac{h}{2}\right]$.

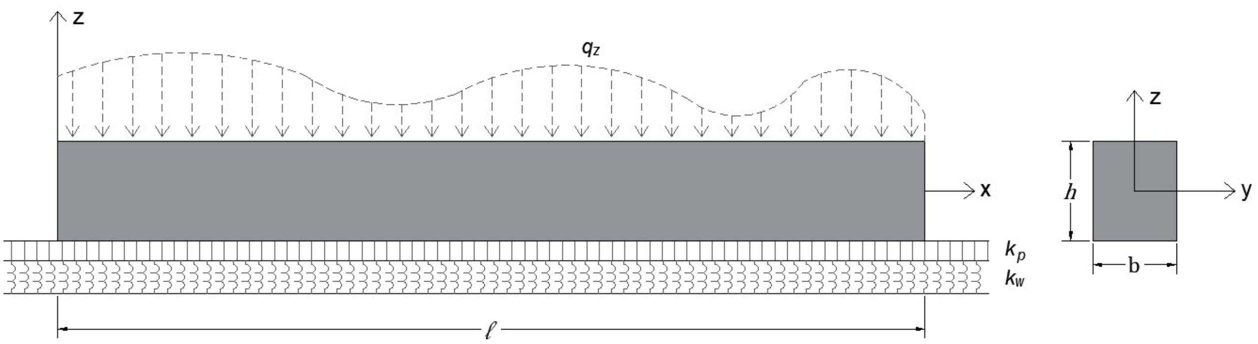

Figure 1: Beam on a two-parameter elastic base.

2.1 Winkler-type and Pasternak-type foundation models

Beams can be studied as structures that are in contact with a continuous medium. In order to simplify the analyses, this medium can be considered as an elastic base. Two of the models proposed for this are the Winkler-type foundation and the Pasternak-type foundation proposed in 1867 and 1954, respectively. Each model describes the behavior of the soil when subjected to loading and deduces the mathematical formulation derived from the physical model [3].

According to Selvadurai [1], the Winkler-type foundation model assumes that the stress applied at a point of the foundation is proportional to the transverse displacement suffered by this point and that the particles that make up the base behave like linear springs disconnected from each other. In order to combat the discontinuity at the interface between the loaded and unloaded regions of the Winkler model, the Pasternak foundation model proposes the interaction of the loaded region and its surroundings through the shear effect on the elastic elements.

2.2 Kinematic model for Euler-Bernoulli beam

One of the theories that describe displacements in beams is the Euler-Bernoulli model, which is the simplest and most used. The Euler-Bernoulli theory, when adopting a displacement field, hypothesizes that a straight line normal to the neutral surface remains straight and normal to it after the part is deformed [4].

According to Luo [5], the displacement of any point in the cross-section consists of the displacement of the rigid body on the reference line and the rotation of the cross-section of the beam. Therefore, it is possible to describe the components of the displacement field as

$$U_x(x,y,z) = u(x) + z\theta_y(x) - y\theta_z(x)\,, \tag{1}$$
$$U_y(x,y,z) = v(x) - z\varphi(x)\,, \tag{2}$$
$$U_z(x,y,z) = w(x) + y\varphi(x)\,, \tag{3}$$

where $U_x(x,y,z)$, $U_y(x,y,z)$ and $U_z(x,y,z)$ are the components of the displacement vector; $u(x)$ is the axial displacement of the beam in the x direction; $v(x)$ and $w(x)$ are the transverse displacements of the beam in the y and z directions, respectively; and $\varphi(x)$, $\theta_y(x)$ and $\theta_z(x)$ are the rotations of the beam cross-section around x, y and z, respectively. The rotations and displacements are related according to

$$\theta_y = -w'\,, \quad \theta_z = v'\,. \tag{4}$$

The number of lines superscribed to the displacements represents the order of the derivative with respect to x.

According to Luo [5], it is possible to adopt the relation between deformation and displacement as linear when assuming that the deformations are small. Thus, the components of the beam strain field can be expressed as

$$\varepsilon_{xx} = \frac{\partial U_x}{\partial x} = u' + z\theta_y' - y\theta_z', \tag{5}$$
$$\gamma_{xy} = \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} = (v' - \theta_z) - z\varphi' = -z\varphi', \tag{6}$$
$$\gamma_{xz} = \frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} = (w' + \theta_y) + y\varphi' = +y\varphi', \tag{7}$$

where ε_{xx} is the strain in x and γ_{xy} and γ_{xz} are strains in the xy and xz planes, respectively.

2.3 Internal work, external work and elastic work

The principle of virtual work for a beam on an elastic foundation is composed of the internal work, the external work, and the work caused by the elastic foundation [2]. Therefore, in this case, the principle of virtual work is

$$\int_0^l (\delta W_{int} + \delta W_f - \delta W_{ext})dx = 0, \tag{8}$$

where δW_{int}, δW_f and δW_{ext} are the variations of the internal, elastic foundation, and external works, respectively.

The stress on the beam under study is caused by bending in two main directions (y and z), by axial displacement, and by torsion. Therefore, according to Doeva, Masjedi and Weaver [2], the internal forces and moments in the beam can be calculated as

$$F_x = \int_A \sigma_{xx} dA, \tag{9}$$

$$M_x = \int_A (y\sigma_{xz} - z\sigma_{xy}) dA, \tag{10}$$

$$M_y = \int_A z\sigma_{xx} dA, \tag{11}$$

$$M_z = -\int_A y\sigma_{xx} dA. \tag{12}$$

Thus, the variation of the internal work can be obtained according to Eq. (13). Vectors and matrices are expressed in bold for a better visualization.

$$\int_0^l \delta W_{int} dx = \int_0^l \delta\epsilon^T \boldsymbol{N}\, dx = \int_0^l \delta\epsilon^T \boldsymbol{S}\epsilon\, dx, \tag{13}$$

where $\boldsymbol{\epsilon}$ is the strain vector, $\boldsymbol{N}$ is the vector of internal forces and moments, and $\boldsymbol{S}$ is the stiffness matrix, such that

$$\boldsymbol{\epsilon} = \begin{bmatrix} \varepsilon_x \\ k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} u' \\ \varphi' \\ -w'' \\ v'' \end{bmatrix}, \quad \boldsymbol{N} = \begin{bmatrix} F_x \\ M_x \\ M_y \\ M_z \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} S_{EA} & S_{ET} & S_{EF} & S_{EL} \\ S_{ET} & S_{GJ} & S_{FT} & S_{LT} \\ S_{EF} & S_{FT} & S_{EI_y} & S_{FL} \\ S_{EL} & S_{LT} & S_{FL} & S_{EI_z} \end{bmatrix}, \tag{14}$$

where S_{EA} is the extensional stiffness, S_{GJ} is the torsional stiffness, S_{EI_y} is the out-of-plane bending stiffness, S_{EI_z} is the in-plane bending stiffness, S_{ET} is the coupling between axial displacement and torsion, S_{EF} is the coupling between out-of-plane bending and axial displacement, S_{EL} is the coupling between in-plane bending and axial displacement, S_{FT} is the coupling between out-of-plane bending and torsion, S_{LT} is the coupling between bending and in-plane torsion and S_{FL} is the coupling between out-of-plane and in-plane bending.

The variation of the external work δW_{ext} is calculated as

$$\int_0^l \delta W_{ext} dx = \int_0^l \delta\bar{\boldsymbol{U}}^T \boldsymbol{Q}\, dx, \tag{15}$$

where $\bar{\boldsymbol{U}}$ is the vector of generalized displacements (translations and rotations), and $\boldsymbol{Q}$ is the vector of distributed loads, given by

$$\bar{\boldsymbol{U}} = \begin{bmatrix} u(x) \\ \varphi(x) \\ w(x) \\ v(x) \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} q_x(x) \\ q_\varphi(x) \\ q_z(x) \\ q_y(x) \end{bmatrix}, \tag{16}$$

where $q_x(x)$, $q_y(x)$ and $q_z(x)$ are the distributed loads and $q_\varphi(x)$ corresponds to the distributed torque.

Finally, according to Robinson and Adali [6], the variation of the work due to the Winkler-Pasternak elastic foundation is given by

$$\int_0^l \delta W_f dx = \int_0^l (\delta w\, k_w\, w + \delta w'\, k_p\, w')\, dx. \tag{17}$$

2.4 System of equations that describe the problem

The system of ordinary differential equations that represents the problem can be built using the variational principles to obtain Euler's Equation [7]. Then, from the fundamental lemma of the variational calculus to obtain the governing equations, that is, substituting Eqs. (13), (15), and (17) into Eq. (8), integrating by parts and collecting the coefficients of δu, $\delta\varphi$, δw, δv, $\delta w'$ and $\delta v'$, the system of governing equations can be obtained as

$$-S_{EA}u'' - S_{ET}\varphi'' + S_{EF}w''' - S_{EL}v''' - q_x = 0, \tag{18}$$
$$-S_{ET}u'' - S_{GJ}\varphi'' + S_{FT}w''' - S_{LT}v''' - q_\varphi = 0, \tag{19}$$
$$-S_{EF}u''' - S_{FT}\varphi''' + S_{EI_y}\, w^{(IV)} - S_{FL}v^{(IV)} + k_w w - k_p w'' - q_z = 0, \tag{20}$$
$$S_{EL}u''' + S_{LT}\varphi''' - S_{FL}w^{(IV)} + S_{EI_z}\, v^{(IV)} - q_y = 0, \tag{21}$$

with the boundary conditions

$$u = 0 \quad \text{or} \quad S_{EA}\, u' + S_{ET}\, \varphi' - S_{EF}\, w'' + S_{EL}\, v'' = f_x\, , \tag{22}$$
$$\varphi = 0 \quad \text{or} \quad S_{ET}\, u' + S_{GJ}\, \varphi' - S_{FT}\, w'' + S_{LT}\, v'' = m_x\, , \tag{23}$$
$$w' = 0 \quad \text{or} \quad -S_{EF}\, u' - S_{FT}\, \varphi' + S_{EI_y}\, w'' - S_{FL}\, v'' = -m_y \tag{24}$$
$$v' = 0 \quad \text{or} \quad S_{EL}\, u' + S_{LT}\, \varphi' - S_{FL}\, w'' + S_{EI_z}\, v'' = m_z\, , \tag{25}$$
$$\text{w} = 0 \quad \text{or} \quad S_{EF}\, u'' + S_{FT}\, \varphi'' - S_{EI_y}\, w''' + S_{FL}\, v''' + k_p w' = f_z\, , \tag{26}$$
$$\text{v} = 0 \quad \text{or} \quad -S_{EL}\, u'' - S_{LT}\, \varphi'' + S_{FL}\, w''' - S_{EI_z}\, v''' = f_y\, , \tag{27}$$

where f_x, f_y and f_z are concentrated boundary loads in the x, y and z directions, respectively. Also, m_x is the torque and m_y and m_z are the boundary moments about the y and z axes, respectively. An algebraic manipulation allows rewriting Eqs. (18)-(27) as

$$-\boldsymbol{A}\boldsymbol{U}'' + \boldsymbol{B}\boldsymbol{W}''' = \boldsymbol{Q_x}, \tag{28}$$
$$-\boldsymbol{B}^T\boldsymbol{U}''' + \boldsymbol{D}\boldsymbol{W}^{(IV)} + \boldsymbol{K_w}\boldsymbol{W} - \boldsymbol{K_p}\boldsymbol{W}'' = \boldsymbol{Q_z}, \tag{29}$$
$$\boldsymbol{A}\boldsymbol{U}' - \boldsymbol{B}\boldsymbol{W}'' = \boldsymbol{F_x}, \tag{30}$$
$$-\boldsymbol{B}^T\boldsymbol{U}' + \boldsymbol{D}\boldsymbol{W}'' = \boldsymbol{M_y}, \tag{31}$$
$$\boldsymbol{B}^T\boldsymbol{U}'' - \boldsymbol{D}\boldsymbol{W}''' + \boldsymbol{K_p}\boldsymbol{W}' = \boldsymbol{F_z}, \tag{32}$$

where

$$\boldsymbol{U} = \begin{bmatrix} u \\ \varphi \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} w \\ v \end{bmatrix}, \tag{33}$$
$$\boldsymbol{Q_x} = \begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}, \quad \boldsymbol{Q_z} = \begin{bmatrix} q_z \\ q_y \end{bmatrix}, \quad \boldsymbol{F_x} = \begin{bmatrix} f_x \\ m_x \end{bmatrix}, \quad \boldsymbol{F_z} = \begin{bmatrix} f_z \\ f_y \end{bmatrix}, \quad \boldsymbol{M_y} = \begin{bmatrix} -m_y \\ m_z \end{bmatrix}, \tag{34}$$
$$\boldsymbol{A} = \begin{bmatrix} EA & S_{ET} \\ S_{ET} & S_{GJ} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} S_{EF} & -S_{EL} \\ S_{FT} & -S_{LT} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} S_{EI_y} & -S_{FL} \\ -S_{FL} & S_{EI_z} \end{bmatrix}, \tag{35}$$
$$\boldsymbol{K_w} = \begin{bmatrix} k_w & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{K_p} = \begin{bmatrix} k_p & 0 \\ 0 & 0 \end{bmatrix}. \tag{36}$$

3 Solution procedure

From the methodology proposed by Doeva, Masjedi and Weaver [2], it is possible to derive Eq. (28) to obtain $\boldsymbol{U}'''$ and substitute it into Eq. (29). With this, it is obtained that

$$\boldsymbol{W}^{(IV)} = \boldsymbol{B_1}\boldsymbol{W}'' - \boldsymbol{B_2}\boldsymbol{W} + \boldsymbol{B_3}, \tag{37}$$

where

$$\boldsymbol{B_1} = (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{K_p}, \tag{38}$$
$$\boldsymbol{B_2} = (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{K_w}, \tag{39}$$
$$\boldsymbol{B_3} = (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{Q_z} - (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{Q}_x'. \tag{40}$$

A change of variable can be performed as

$$\boldsymbol{x_1} = \boldsymbol{W}, \quad \boldsymbol{x_2} = \boldsymbol{W}', \quad \boldsymbol{x_3} = \boldsymbol{W}'', \quad \boldsymbol{x_4} = \boldsymbol{W}''', \tag{41}$$

so that it can be stated that

$$\boldsymbol{x}_1' = \boldsymbol{W}' = \boldsymbol{x_2}, \quad \boldsymbol{x}_2' = \boldsymbol{W}'' = \boldsymbol{x_3}, \quad \boldsymbol{x}_3' = \boldsymbol{W}''' = \boldsymbol{x_4}, \quad \boldsymbol{x}_4' = \boldsymbol{W}^{(IV)} = \boldsymbol{B_1}\boldsymbol{x_3} - \boldsymbol{B_2}\boldsymbol{x_1} + \boldsymbol{B_3}. \tag{42}$$

Therefore, the system of linear differential equations defined by Eq. (37) is rewritten as [8]:

$$\boldsymbol{X}' = \boldsymbol{M}\boldsymbol{X} + \boldsymbol{f}, \tag{43}$$

where

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{0_{2x2}} & \boldsymbol{I} & \boldsymbol{0_{2x2}} & \boldsymbol{0_{2x2}} \\ \boldsymbol{0_{2x2}} & \boldsymbol{0_{2x2}} & \boldsymbol{I} & \boldsymbol{0_{2x2}} \\ \boldsymbol{0_{2x2}} & \boldsymbol{0_{2x2}} & \boldsymbol{0_{2x2}} & \boldsymbol{I} \\ -\boldsymbol{B_2} & \boldsymbol{0_{2x2}} & \boldsymbol{B_1} & \boldsymbol{0_{2x2}} \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} \boldsymbol{0_{2x1}} \\ \boldsymbol{0_{2x1}} \\ \boldsymbol{0_{2x1}} \\ \boldsymbol{B_3} \end{bmatrix}, \tag{44}$$

where $\boldsymbol{I}$ is the 2x2 identity matrix and $\boldsymbol{M}$ is called a complementary matrix.

3.1 Parameter variation method

According to Zill and Cullen [9], the solution of the non-homogeneous first-order linear system in Eq. (43) is

$$\boldsymbol{X} = \boldsymbol{\phi}(x)\boldsymbol{C} + \boldsymbol{\phi}(x)\int \boldsymbol{\phi}^{-1}(x)\boldsymbol{f}\,dx, \tag{45}$$

where $\boldsymbol{\phi}(x)$ is the fundamental matrix constituted in its columns by the solution vectors of the equivalent homogeneous system [9], and $\boldsymbol{C}$ is a vector formed by constants to be determined by the boundary conditions.

3.2 Fundamental matrices and displacement fields $w(x)$ and $v(x)$

Three combinations of the foundation parameters are studied here, namely, $k_w = 0$ and $k_p = 0$ (no foundation), $k_w = 0$ and $k_p \neq 0$, and $k_w \neq 0$ and $k_p \neq 0$, which generate different fundamental matrices and, consequently, different solutions because the corresponding complementary matrix $\boldsymbol{M}$ is different in each case.

In the more general case ($k_w \neq 0$ and $k_p \neq 0$) the complementary matrix $\boldsymbol{M}$ has four different nonzero eigenvalues $\lambda_i \neq 0$, $i = 1,2,3,4$, and one null eigenvalue $\lambda = 0$ of multiplicity 4, so there are four linearly independent eigenvectors $\boldsymbol{K_i}$, $i = 1, \dots, 4$, complemented by the following basis vectors:

$$\begin{aligned} \boldsymbol{K_5} &= [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \boldsymbol{K_6} &= [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \boldsymbol{K_7} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0]^T, \\ \boldsymbol{K_8} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T. \end{aligned} \tag{46}$$

Thus, the fundamental matrix $\boldsymbol{\phi}(x)$ for this case is given by

$$\boldsymbol{\phi}(x) = \left[\boldsymbol{K_1}e^{\lambda_1 x}\ \boldsymbol{K_2}e^{\lambda_2 x}\ \boldsymbol{K_3}e^{\lambda_3 x}\ \boldsymbol{K_4}e^{\lambda_4 x}\ \boldsymbol{K_5}\ (\boldsymbol{K_5}x + \boldsymbol{K_6})\left(\frac{\boldsymbol{K_5}}{2}x^2 + \boldsymbol{K_6}x + \boldsymbol{K_7}\right)\left(\frac{\boldsymbol{K_5}}{6}x^3 + \frac{\boldsymbol{K_6}}{2}x^2 + \boldsymbol{K_7}x + \boldsymbol{K_8}\right)\right]. \tag{47}$$

Then, it is possible to obtain $\boldsymbol{x_1} = \boldsymbol{W} = [w(x) \quad v(x)]^T$ from Eq. (45), which provides

$$\begin{aligned} w(x) &= \frac{4\sqrt{a - \sqrt{a^2 - 4b}}\,e^{-\lambda_1 x}\,(C_2 - C_1 e^{2\lambda_1 x})}{2\sqrt{2}\,a\left(a - \sqrt{a^2 - 4b}\right) - 4\sqrt{2}b} + \frac{4\sqrt{a + \sqrt{a^2 - 4b}}\,e^{-\lambda_3 x}\,(C_4 - C_3 e^{2\lambda_3 x})}{2\sqrt{2}a\left(a + \sqrt{a^2 - 4b}\right) - 4\sqrt{2}b} + \frac{FF1(x)}{2\sqrt{2}}, \\ v(x) &= \frac{1}{6}\left[6\,C_5 + 6\,C_6 x + 3\,C_7 x^2 + C_8 x^3 + FF2(x)\right], \end{aligned} \tag{48}$$

where

$$a = \frac{k_p}{S_{EIy}}, \quad b = \frac{k_w}{S_{EIy}}, \quad m(x) = \frac{q_z(x)}{S_{EIy}}, \quad n(x) = \frac{q_y(x)}{S_{EIz}}, \quad \lambda_1 = \frac{\sqrt{a - \sqrt{a^2 - 4b}}}{-\sqrt{2}}, \quad \lambda_3 = \frac{\sqrt{a + \sqrt{a^2 - 4b}}}{-\sqrt{2}},$$

$$\begin{aligned} FF1(x) = & \frac{\left(a - \sqrt{a^2 - 4b}\right)^{\frac{3}{2}}\left(\mathrm{e}^{-x\lambda_1} \int \mathrm{e}^{x\lambda_1} m(x)dx - \mathrm{e}^{x\lambda_1} \int \mathrm{e}^{-x\lambda_1} m(x)dx\right)}{\sqrt{a^2 - 4b}\left(-a^2 + a\sqrt{a^2 - 4b} + 2b\right)} \\ & + \frac{\left(a + \sqrt{a^2 - 4b}\right)^{\frac{3}{2}}\left(\mathrm{e}^{-x\lambda_3} \int \mathrm{e}^{x\lambda_3} m(x)dx - \mathrm{e}^{x\lambda_3} \int \mathrm{e}^{-x\lambda_3} m(x)dx\right)}{\sqrt{a^2 - 4b}\left(a^2 + a\sqrt{a^2 - 4b} - 2b\right)} \\ FF2(x) = & x^3 \int n(x)dx - 3x^2 \int xn(x)dx + 3x \int x^2 n(x)dx - \int x^3 n(x)dx \end{aligned} \quad . \quad (49)$$

3.3 Displacement $\boldsymbol{u}(x)$ and rotation $\boldsymbol{\varphi}(\boldsymbol{x})$ fields

Vector $\boldsymbol{U} = [u(x) \quad \varphi(x)]^T$ is obtained by integrating twice Eq. (28) as

$$\boldsymbol{U} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{W}' - \int \left(\int \boldsymbol{A}^{-1}\boldsymbol{Q_x} dx \right) dx + \boldsymbol{C_9}x + \boldsymbol{C_{10}}, \quad (50)$$

where $\boldsymbol{x_2} = \boldsymbol{W}'$ is obtained via Eq. (45) and vectors $\boldsymbol{C_9}$ and $\boldsymbol{C_{10}}$ are obtained from the boundary conditions.

4 Numerical results

4.1 Isotropic beam under uniformly distributed load $\boldsymbol{q_z}$

Consider a dimensionless form for transverse deflection $w(x)$ and foundation parameters k_w and k_p as

$$\bar{w}(x) = \frac{w(x)S_{EI_y}}{q_z l^4}, \quad \overline{k_w} = \frac{k_w l^4}{S_{EI_y}}, \quad \bar{k_p} = \frac{k_p l^2}{S_{EI_y}}. \quad (51)$$

In this case, displacements $v(x)$, $u(x)$ and $\varphi(x)$ are null. In turn, for nonzero $\overline{k_w}$ and $\overline{k_p}$, deflections $\bar{w}(x)$ for the simply supported beam and the doubly clamped beam are given, respectively, by

$$\bar{w}_{SS}(x) = \frac{4}{p^2 q^2}\left[1 + \frac{\mathrm{e}^{\frac{p(l-x)}{\sqrt{2}l}} q^2 + \mathrm{e}^{\frac{px}{\sqrt{2}l}} q^2 - \mathrm{e}^{\frac{q(l-x)}{\sqrt{2}l}} p^2 - \mathrm{e}^{\frac{qx}{\sqrt{2}l}} p^2}{\left(p^2 - q^2\right)\left(1 + \mathrm{e}^{\frac{p}{\sqrt{2}}}\right)}\right], \quad (52)$$

and

$$
\begin{aligned}
\bar{w}_{DC}(x) = \Bigg\{ & \frac{(p^2-q^2)pq}{2}\left(1-\cosh\frac{q}{\sqrt{2}}\cosh\frac{p}{\sqrt{2}}+2\cosh\frac{p}{2\sqrt{2}}\cosh\frac{p(l-2x)}{2l\sqrt{2}}\sinh\left(\frac{q}{2\sqrt{2}}\right)^2\right) \\
& -\frac{p^4-q^4}{4}\cosh\frac{p(l-2x)}{2l\sqrt{2}}\sinh\frac{q}{\sqrt{2}}\sinh\frac{p}{2\sqrt{2}}+\frac{\left(p^2-q^2\right)^2}{4}\cosh\frac{p(l-2x)}{2l\sqrt{2}}\sinh\frac{q}{\sqrt{2}}\sinh\frac{p}{\sqrt{2}} \\
& +\left(p^2-q^2\right)pq\cosh\frac{q}{2\sqrt{2}}\cosh\frac{q(l-2x)}{2l\sqrt{2}}\sinh\left(\frac{p}{2\sqrt{2}}\right)^2 \\
& -\frac{p^4-q^4}{4}\cosh\frac{q(l-2x)}{2l\sqrt{2}}\sinh\frac{q}{2\sqrt{2}}\sinh\frac{p}{\sqrt{2}}-\frac{\left(p^2-q^2\right)^2}{4}\cosh\frac{q(l-2x)}{2l\sqrt{2}}\sinh\frac{q}{2\sqrt{2}}\sinh\frac{p}{\sqrt{2}} \\
& +\frac{p^4-q^4}{4}\sinh\frac{q}{\sqrt{2}}\sinh\frac{p}{\sqrt{2}}\Bigg\} \\
& /\left\{\frac{(p^2-q^2)p^6q^6}{8}\left(1-\cosh\frac{q}{\sqrt{2}}\cosh\frac{p}{\sqrt{2}}\right)+\frac{(p^4-q^4)p^2q^2}{16}\sinh\frac{q}{\sqrt{2}}\sinh\frac{p}{\sqrt{2}}\right\},
\end{aligned}
\tag{53}
$$

where

$$
p=\sqrt{\bar{k}_p+\sqrt{\bar{k}_p^{\,2}-4\overline{k_w}}},\quad q=\sqrt{\bar{k}_p-\sqrt{\bar{k}_p^{\,2}-4\overline{k_w}}}. \tag{54}
$$

Maximal deflections for different combinations of $\overline{k_w}$ and $\bar{k}_p$ are compared with those from [2] in Table 1 for the simply supported beam $\left(\bar{w}_{SS}(l/2)\right)$ and the doubly clamped beam $\left(\bar{w}_{DC}(l/2)\right)$ under uniform load. The number of decimal places adopted for the analysis is the same used by the authors from [2].

Table 1: Maximal deflections of the simply supported beam and the doubly clamped beam under uniform load.

$\overline{k_w}$	$\overline{k_p}$	Simply supported beam $\left(\bar{w}_{SS}(l/2)\right)$		Doubly clamped beam $\left(\bar{w}_{DC}(l/2)\right)$	
		Present work	From [2]	Present work	From [2]
0	0	0.013021	0.013021	0.002604	0.002604
	10	0.006448	0.006448	0.002085	0.002085
	25	0.003661	0.003661	0.001607	0.001607
10	0	0.011804	0.011804	0.002553	0.002553
	10	0.006133	0.006133	0.002051	0.002051
	25	0.003556	0.003556	0.001587	0.001587
100	0	0.006400	0.006400	0.002165	0.002165
	10	0.004256	0.004256	0.001792	0.001792
	25	0.002828	0.002828	0.001426	0.001426

In both cases, the three sets of results obtained for the different values of foundation parameters are in agreement with the results presented by Doeva, Masjedi and Weaver [2]. Therefore, the equations presented in Eqs. (52) and (53) are, in fact, the solutions.

4.2 Cantilevered isotropic beam with a point load at the free end

One of the numerical examples presented by Eisenberger [10] is the maximum deflection suffered by a cantilevered Euler-Bernoulli beam with a point load at the free edge and without an elastic foundation. In this example, units are all compatible and will be omitted. The beam material properties are $E = 2.9 \cdot 10^7$ and $\upsilon = 0.3$. The deflections were analyzed for a beam with unit dimension for thickness, height fixed at 12, with lengths of 12, 40, 80, and 160 for a load of $f_z = 100$.

It is possible to find the maximum deflection results of this beam for each length l. The results obtained agree with those of Eisenberger [10], as shown in Table 2. The number of decimal places adopted for the analysis is the same used by the cited author.

Table 2: Maximal deflection for cantilever beam with a point load at free end.

Length (l)	Maximal deflection (w_{max})	
	Present Work	Eisenberger [10]
12	0.013793	0.013793
40	0.510856	0.510855
80	4.086800	4.086800
160	32.694800	32.694800

In order to analyze the effect of an elastic foundation, Table 3 presents the maximal deflections of the beam with $l = 160$ and $h = 12$ for different combinations of $\overline{k_w}$ and $\bar{k_p}$. To the best of our knowledge, there are no other published works on this case. It is observed that the Pasternak parameter $\bar{k_p}$ causes larger reductions in the maximal deflection than the Winkler parameter $\overline{k_w}$.

Table 3: Maximal deflection for cantilever beam with elastic foundation with a point load at the free end.

$\overline{k_w}$	$\bar{k_p}$	Maximal deflection (w_{max})
	0	32.694800
0	10	6.717827
	25	3.138769
	0	18.486274
10	10	5.720577
	25	2.886946
	0	4.309194
100	10	2.642665
	25	1.748078

5 Conclusions

This work proposed to develop analytical solutions for the structure of isotropic beams, when supported or not on a Winkler-Pasternak elastic foundation. The equations describing the problem and the boundary conditions were derived from the principle of virtual work under the assumptions of Euler-Bernoulli beam theory. The solution of the system of equations was obtained using the method of variation of parameters.

Analyzing numerically the results of the maximal deflections obtained, it was verified that with the increase of the foundation stiffness values, the flexural displacements decrease. Furthermore, it was found that the effect of the Pasternak stiffness parameter (k_p) on displacement is greater than the Winkler stiffness parameter (k_w).

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