Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(1): 2454–2472.
DOI: 10.3934/math.2024121
Received: 14 November 2023
Revised: 14 December 2023
Accepted: 18 December 2023
Published: 25 December 2023

Research article

An analytic pricing formula for timer options under constant elasticity of variance with stochastic volatility

Sun-Yong Choi[1], **Donghyun Kim**[2] **and Ji-Hun Yoon**[2,3,*]

[1] Department of Financial Mathematics, Gachon University, Gyeoggi 13120, Republic of Korea
[2] Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea
[3] Institute of Mathematical Science, Pusan National University, Busan 46241, Republic of Korea

* **Correspondence:** Email: yssci99@pusan.ac.kr.

Abstract: Timer options, which were first introduced by Société Générale Corporate and Investment Banking in 2007, are financial securities whose payoffs and exercise are determined by a random time associated with the accumulated realized variance of the underlying asset, unlike vanilla options exercised at the prescribed maturity date. In this paper, taking account of the correlation between the underlying asset price and volatility, we investigate the pricing of timer options under the constant elasticity of variance (CEV) model, proposed by Cox and Ross [10], taking advantage of the approach of asymptotic analysis. Additionally, we validate the pricing precision of the approximate formula for timer options using the Monte Carlo method. We conduct numerical experiments based on our corrected prices and analyze price sensitivities concerning various model parameters, with a focus on the value of elasticity.

Keywords: timer options; stochastic volatility; constant elasticity of variance; asymptotic analysis; Monte-Carlo simulation
Mathematics Subject Classification: 91G20, 91G60

1. Introduction

In quantitative finance, volatility is a crucial factor in pricing derivatives, dynamic hedging, and portfolio management within the financial market. For FX (foreign exchange) options, prices are quoted in volatility. Given its significance in evaluating financial derivatives, volatility has been a focal point in both academia and practice.

For many years, academics have been studying how to model volatility. One of the most famous volatility models is the Black-Scholes model [1]. They described the dynamic of the asset price in terms of geometric Brownian model (GBM) in which volatility of the underlying asset is assumed to be a constant and the stock price has a log-normal distribution. Because of this, the Black-Scholes model has the advantage of being tractable, but has the disadvantage of not reflecting the time-varying volatility in the real market. In fact, in the real market, there is high probability that excess skewness and leptokurtosis occur as properties of the risk-neutral probability distribution unlike the log-normal distribution. Since then, to overcome these shortcomings of the Black-Scholes model, two major types of volatility models have been developed: Stochastic volatility (SV) models and local volatility models.

In SV models, taking account of many extraordinary volatility behaviors in the financial market after 1987 Financial Crash, the existence of a nonflat implied volatility has become remarkable. Therefore, participants interested in financial transactions have taken notice of the models that can predict the movement of financial assets, noting that the volatility of an underlying asset follows a stochastic process, and then the (pure) SV model has been proposed for describing and reflecting real situations in financial markets. In fact, the Heston model (cf. Heston [2]) and the fast mean-reverting SV model given by Fouque et al. [3] have become representative SV models, which are designed to capture the phenomenon of the mean-reversion of volatility in the real market. In addition, the Hull and White model [4] adopted the instantaneous variance process as a geometric Brownian motion. The mean-reverting Ornstein-Uhlenbeck process was also used in several models as a stochastic process (see Scott [5]; Chesney and Scott [6]; Schöbel and Zhu [7]). In addition, Heston model [2] assumed that the volatility is a Cox-Ingersoll-Ross (CIR) process and the Heston model is the one of most popular stochastic models because of the tractability.

The local volatility models were developed by Dupire [8] and Derman and Kani [9] for the continuous and discrete cases, respectively, which are also called non-parametric local volatility models. In this model, the volatility depends on the both asset price and time, addressing the importance of the correlation between the change in the underlying asset price and the randomness of volatility to price options. In addition, Cox and Ross [10] proposed the constant elasticity of variance (CEV) model as the parametric local volatility models. Especially, as seen in Tian et al. [11], the CEV model has been known to generate the U-shaped implied volatility contrary to the Black-Scholes model. However, in the CEV model mentioned by Cox and Ross [10], the movement in volatility and underlying asset price has perfect correlation, either positively or negatively, relying on the elasticity parameter. However, in the empirical study, there is definite correlation at all times, showing volatility is time-varying as shown in Ghysels et al. [12].

Recently, considering that the local volatility models or the SV models fail to capture the empirical evidence demonstrating that the implied volatility of equity options exhibits smile and skew curves simultaneously. A model that combines these two volatility models, called a hybrid model, has been proposed by many researchers, concentrating that the mixed models were designed to have the advantages of these two type of volatility models. First, one hybrid model was developed based on the non-parametric local volatility model and the stochastic model (see Van der Stoep et al. [13]; Tian et al. [11]; Cui et al. [14]), namely stochastic local volatility models. Second, another hybrid model that combines the CEV model and the SV model has also been developed (e.g., Andersen and Piterbarg [15]; Lord et al. [16]; Choi et al. [17]; Cui et al. [18]).

As volatility has a direct impact on pricing and hedging performance, several volatility derivatives have been introduced. In particular, these products enable investors to manage the volatility risk, that is, they are used to hedge volatility risk. Among the volatility derivatives, variance swap is the most famous [19, 20]. The payoff of variance swap is calculated from the difference between the realized variance and a strike price of the contract. Starting with the studies of Carr and Madan [19] and Demeterfi et al. [21], who evaluated variance swap by replicating the swap payoff with a portfolio of European options, various studies have been introduced. Zhu and Lian [22] priced the variance swap under the Heston model. Zheng and Kwok [23] developed a probabilistic approach to evaluate the variance swap under an SV model with jumps in the underlying asset price. A closed-form pricing formula for the discrete-time variance swap under general affine GARCH (generalized autoregressive conditional heteroskedasticity) type models was introduced by Badescu et al. [20]. Issaka [24] evaluate the variance swap by assuming that the underlying assets follow the multiple SV models. Xi and Wong [25] priced the variance swap in a rough Heston model [26] and investigated the sensitivity to roughness on the variance swap.

In addition, as another volatility derivative, there is a timer option. The payoff of the timer option is similar to that of the vanilla option, but it has a random maturity contrary to the vanilla option, whose payoff is determined by only the fixed maturity embedded in the option. The timer option expires when the accumulated realized variance of the underlying asset reaches a predetermined level. In particular, timer options were first studied by Neuberger [27] as the "mileage option", another name for timer options, in academia and launched by the Société Générale Corporate and Investment Banking (SG CIB) in 2007. Since April 2007, timer options have been traded on the market (see [28]). After the trade began, several studies have been conducted on the evaluation of time options. Bick [29] considered the continuous version of timer options, and Li [30] implemented the Monte Carlo simulation to price timer options. Saunders [31] investigated the pricing of timer options under the fast mean-reverting SV and provided the closed-form approximation formulas. Liang et al. [32] obtained closed-form pricing formulas for timer options under the 3/2 and Heston models by using the path-integral techniques. Bernard and Cui [33] developed a one-dimensional problem for pricing of a timer call option under a general SV model. In addition, they showed the empirical results in the Hull-White and Heston models. Li [34] studied perpetual timer options under general SV models. Thus, they obtained pricing formulas for the Heston model and the 3/2 model. Ma et al. [35] derived a fast approximate analytic method to evaluate timer options under the Vasicek interest rate model. Li [36] applied the Heston model to price the timer options. In addition, as mentioned in Zhang et al. [37], the perpetual timer option was examined based on the Hull-White SV model. [38] developed concise pricing formulas for timer options using a probabilistic method to explicitly derive probability densities in stochastic volatility models, including Heston and 3/2. Recently, Wang et al. [39] employed multiscale volatility models to price timer options, stressing that the factor of the volatility is assumed to follow both fast mean-reverting and slowly varying factors. In particular, [40] introduced the return timer option, a financial derivative with a random expiry triggered by the first occurrence of a return exceeding upper or lower barriers.

In this study, we investigate the timer option prices under the CEV model. In previous studies, the pricing of the time options has been only evaluated using various stochastic volatility models. To the best of our knowledge, there has been no research about the timer option pricing based upon CEV model in which the volatility process is considered to be the SV given by Heston [2] or Fouque et al. [3]. In fact, since the model dynamics of the underlying asset for the classical timer option is a form of the SV model, including the fast-mean reverting process given by Fouque et al. [3] or CIR process described by Heston [2], the pricing procedure of the timer options based upon CEV model is very similar to that of European options under the stochastic and local volatility (SVCEV) model introduced by Choi et al. [17], whose model is driven by the fast-mean reverting volatility process and constant elasticity of variance process. According to Choi et al. [17], the SVCEV model shows some improvements over the traditional CEV model. The SVCEV model has been widely used to evaluate various contingent claims. For example, Bock et al. [41] developed the pricing formula of a European barrier option under the SVCEV model. In addition, Kim et al. [42] applied the SVCEV model to evaluate the real options. Furthermore, several extended models based on SVCEV have been developed. For instance, Choi et al. [43] introduced a multiscale hybrid model consisting of fast and slow factors and evaluated an equity-linked annuity under the multiscale hybrid model. The previous SVCEV model [17] derived an approximation solution using up to the first order correction. Recently, an extended model has been developed using up to the second order correction term [44].

In this article, by using asymptotic analysis on the partial differential equations (PDEs) for timer options, we derive the approximation formulas of timer options with the CEV model. Later, we call the timer options under the CEV model TO-CEVs. To demonstrate that our approximated solution for the timer options has been obtained accurately and effectively, we compare the option value with a Monte-Carlo price. Moreover, in the numerical implications, we analyze the price sensitivity of the SV for the given elasticity value on the timer options. Above all else, the most remarkable result of our paper is that the standard European option under CEV may or may not be overvalued compared with timer option with CEV, depending on the elasticity value, which is contrary to the results of Sawyer [28] emphasizing that the price of the European vanilla call options can be quite overvalued compared to the price of timer call options in the region of in-the-money (ITM).

The remainder of this paper is organized as follows: In Section 2, we model stochastic differential equations (SDEs) to price a timer option under CEV models. Next, in Section 3, we derive the approximation solution of the option value taking advantage of asymptotic analysis. In Section 4, we examine the pricing accuracy of the approximated formula for the timer options through a Monte-Carlo simulation and analyze the parameter sensitivity on the first approximated option price. Finally, Section 5 provides concluding remarks.

2. Model formulation

In this section, we provide the stochastic dynamics for the price of a timer call option based on hybrid stochastic and local volatility. According to Choi et al. [17], we assume that the underlying asset price model, X_t, and its instantaneous variance process, Y_t, satisfy the following stochastic differential equation (SDE) model under a real probability measure $\mathbb{P}$:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + f(Y_t) X_t^{\theta/2} \mathrm{d}W_t^X, \tag{2.1}$$

$$\mathrm{d}Y_t = \alpha(m - Y_t)\mathrm{d}t + \beta \mathrm{d}W_t^Y, \tag{2.2}$$

where μ is the expected return rate of the underlying asset process, X_t, $f(\cdot)$ is a smooth function satisfying $0 < c_1 \leq f \leq c_2 < \infty$ for some constants c_1 and c_2, θ is an elasticity parameter, α and β are positive constants, and m is the long-run mean level of Y. In addition, W_t^X and W_t^Y are standard Brownian motions such that $\mathrm{d}\left\langle W^X, W^Y \right\rangle_t = \rho\,\mathrm{d}t$, where $|\rho| \leq 1$.

Herein, based on the fact that the volatility of the underlying asset has been demonstrated to be mean-reverting through data-driven study [45], we construct the model of the volatility as a fast mean-reverting Ornstein-Uhlenbeck process, as depicted in SDE (2.2). Since the ergodic process Y_t

is explicitly presented by $Y_t = m + (Y_0 - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} \mathrm{d}W_t^Y$, the process Y_t follows the normal distribution. i.e., $Y \sim \mathcal{N}(m + (Y_0 - m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t}))$. Interestingly, as t goes to ∞, its invariant distribution is given by $\mathcal{N}(m, \frac{\beta^2}{2\alpha})$, which is independent of the initial volatility level, Y_0. Nevertheless, empirical studies grounded in financial market data (refer to Choi et al. [17] and Choi et al. [44]) indicate that a (pure) stochastic volatility process (in the case of $\theta = 2$) may not accurately capture the geometric properties of the implied volatility, specifically, the implied-volatility-smile and implied-volatility-skew phenomena. Therefore, to overcome the aforementioned limitations observed in the SV model, we consider the timer options based upon the CEV diffusion, which is very similar to that of European options under stochastic and local volatility (in brief, SVCEV), proposed by Choi et al. [17].

The most distinctive feature of the SVCEV model proposed by Choi et al. [17] lies in the significant impact of the elasticity parameter θ. This economic parameter, when set to $\theta = 2$, represents the classical Black-Scholes market, whereas for $\theta > 2$, it depicts the phenomenon noticed in commodity markets (especially, coal, copper, or gold [46]) where an increase in commodity prices leads to a concurrent rise in volatility, known as the inverse leverage effect. On the other hand, when $\theta < 2$, it captures the leverage effect commonly observed in stock markets, where a fall in stock prices results in an increase in the underlying asset's volatility. Therefore, the SVCEV model of volatility for the underlying asset, as adopted from Choi et al. [17], can be considered as a suitable and realistic volatility model, efficiently capturing the random structures across various financial markets.

Then, by the Girsanov theorem in Øksendal [47], under the risk-neutral measure $\mathbb{Q}$, the stochastic dynamics in (2.1) and (2.2) are transformed into

$$\mathrm{d}X_t = rX_t \mathrm{d}t + f(Y_t) X_t^{\theta/2} \mathrm{d}\overline{W}_t^X, \tag{2.3}$$

$$\mathrm{d}Y_t = \left\{ \frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \Lambda(Y_t) \right\} \mathrm{d}t + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \mathrm{d}\overline{W}_t^Y, \tag{2.4}$$

where r is risk-free interest rate, ν is the standard deviation of the invariant distribution of Y, and $\Lambda(Y_t)$ is the market price of volatility risk. In addition, $\overline{W}^X$ and $\overline{W}^Y$ are transformed Brownian motions such that $\left\langle \overline{W}^X, \overline{W}^Y \right\rangle_t = \rho\, \mathrm{d}t$.

We now demonstrate the timer call option and derive its PDE. Let us define the accumulated variance process at time t as

$$I_t = \int_0^t f^2(Y_u) X_u^{\theta-2} \mathrm{d}u.$$

We note that if $\theta = 2$, then it reduces to the form of the accumulated variance process found in standard timer options. Next, if we denote a pre-determined variance budget as B, depending on the option buyer's choice, then the stopping timer or random maturity, τ_B, is given by

$$\tau_B = \inf \{t > 0, I_t = B\}. \tag{2.5}$$

In other words, the expiration of the timer option coincides with the initial moment at which the accumulated variance budget, I_t, reaches the predetermined variance budget level, B.

Then, the payoff of a timer call option is $\max(X_{\tau_B} - K, 0)$, where K is the strike price. Therefore, we obtain the price of the timer call option at time t as follows:

$$P(t \wedge \tau_B, x, y, I) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau_B - t\wedge\tau_B)} \max(X_{\tau_B} - K, 0) \middle| X_{t\wedge\tau_B} = x, Y_{t\wedge\tau_B} = y, I_{t\wedge\tau_B} = I \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau_{B-I}} \max\left(X_{\tau_{B-I}} - K, 0\right) \middle| X_0 = x, Y_0 = y \right], \tag{2.6}$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation with respect to risk-neutral measure $\mathbb{Q}$.

Now, by using the well-known Feynman-Kac formula and employing the well-known fact, as described in Li [36], that the price of timer options does not depend on the time variable t, we can transform this conditional expectation (2.6) into the following partial differential equation:

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right) P(x, y, I) = 0, \tag{2.7}$$

$$P(x, y, B) = \max(x - K, 0), \tag{2.8}$$

where

$$\begin{aligned}
\mathcal{L}_0 &= (m - y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1 &= -\nu\sqrt{2}\Lambda(y)\frac{\partial}{\partial y} + \nu\sqrt{2}\rho f(y) x \frac{\partial^2}{\partial x \partial y}, \\
\mathcal{L}_2 &= r\left(x\frac{\partial}{\partial x} - \cdot \right) + x^{\theta-2} f^2(y)\frac{\partial}{\partial I} + \frac{1}{2} f^2(y) x^{\theta} \frac{\partial^2}{\partial x^2}.
\end{aligned}$$

3. Asymptotic analysis and first-order approximated solution

In this section, we aim to build an approximate solution utilizing the asymptotic analysis proposed by Fouque et al. [45]. Referring to Fouque et al. [45], for small parameter ϵ, we expand the original solution $P(x, y, I)$ with respect to $\sqrt{\epsilon}$ in the following form:

$$P(x, y, I) = P_0(x, y, I) + \sqrt{\epsilon} P_1(x, y, I) + \epsilon P_2(x, y, I) + \cdots, \tag{3.1}$$

where $P_0, P_1, \cdots$ are functions of (x, y, I) satisfying $P_0(x, B) = (x - K)^+$ and $P_n(x, B) = 0$ for all $n \geq 1$. Now, if we substitute the expansion in (3.1) into the singular perturbed PDE presented in (2.7), we obtain

$$\frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) + \sqrt{\epsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) = O(\epsilon). \tag{3.2}$$

Now, we present two useful and fundamental results, known as the growth condition and the centering condition, for asymptotic analysis.

Lemma 3.1. *(Growth condition) As $y \to \infty$, if the leading-order price $P_0(x, y, I)$ and the correction term $P_1(x, y, I)$ do not grow as much as $\frac{\partial P_0}{\partial y} \sim e^{y^2/2}$ and $\frac{\partial P_1}{\partial y} \sim e^{y^2/2}$, respectively, then P_0 and P_1 are independent of the unobservable variable y.*

Proof. The proof is similar to that in Theorem 3.1 of Ha et al. [48] or Lemma 1 and Theorem 4.1 in the work of Choi et al. [17]. From (3.2), we have the following PDEs:

$$\begin{aligned}
\mathcal{L}_0 P_0 &= 0, \\
\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 &= 0.
\end{aligned} \tag{3.3}$$

Next, the ODE $\mathcal{L}_0 P_0 = 0$ in (3.3) yields

$$P_0(x, I) = c_1(x, I) \int_0^y e^{\frac{(m-u)^2}{2\nu^2}} \mathrm{d}u + c_2(x, I)$$

for some functions $c_1(x, I)$ and $c_2(x, I)$. Then, by using the assumption, as $y \to \infty$, if the leading-order price P_0 does not grow as much as $\frac{\partial P_0}{\partial y} \sim e^{y^2/2}$, then $c_1(x, I) = 0$. Therefore, P_0 is independent of the unobservable y. Similarly, if we apply this assumption to $\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$, the correction term P_1 is independent of y. □

Lemma 3.2. *(Centering condition) If the ODE $\mathcal{L}_0 \mathcal{G}(y) + G = 0$, a Poisson equation for the function $\mathcal{G}(y)$, has a unique solution, then the following centering condition must be satisfied:*

$$\langle G \rangle \equiv \int_{\mathbb{R}} G(y)\Phi(y)\mathrm{d}y = 0,$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}\nu} e^{-\frac{(y-m)^2}{2\nu^2}}$ is the invariant density distribution of the ergodic process Y.

Proof. Refer to Section 5.2 in Fouque et al. [45]. □

Proposition 3.1. *Under the growth condition presented in Lemma 3.1, the leading-order price $P_0(x, I)$ satisfies the following PDE:*

$$\mathcal{L}_{\text{TO-CEV}}(\widehat{\sigma}) P_0(x, I) = 0, \tag{3.4}$$

$$P_0(x, B) = (x - K)^+, \tag{3.5}$$

where $\mathcal{L}_{\text{TO-CEV}}$ is given by

$$\mathcal{L}_{\text{TO-CEV}} = \widehat{\sigma}^2 x^{\theta-2} \frac{\partial}{\partial I} + \frac{\widehat{\sigma}^2}{2} x^{\theta} \frac{\partial^2}{\partial x^2} + r\left(x \frac{\partial}{\partial x} - \cdot\right), \quad \widehat{\sigma} = \sqrt{\langle f^2 \rangle}.$$

Proof. According to Lemma 3.2, since $\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$ is a Poisson equation of P_2 with respect to y and P_0 is independent of the variable y, the following centering condition must be hold:

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0,$$

where

$$\langle \mathcal{L}_2 \rangle = \widehat{\sigma}^2 x^{\theta-2} \frac{\partial}{\partial I} + \frac{\widehat{\sigma}^2}{2} x^{\theta} \frac{\partial^2}{\partial x^2} + r\left(x \frac{\partial}{\partial x} - \cdot\right) \equiv \mathcal{L}_{\text{TO-CEV}}(\widehat{\sigma})$$

and the effective volatility $\widehat{\sigma}$ is given by

$$\widehat{\sigma} = \sqrt{\langle f^2 \rangle}.$$

Therefore, we can obtain the PDE (3.4) with (3.5). □

Now, we provide the analytic form solution to the PDE presented in Proposition 3.1.

Theorem 3.1. *The leading-order price, $P_0(x, I)$, is given by*

$$P_0(x, I) = e^{-r\frac{B-I}{\widehat{\sigma}^2 x^{\theta-2}}} x \int_{\widetilde{K}}^{+\infty} \left(\frac{\tilde{x}}{s}\right)^{\frac{1}{2(2-\theta)}} e^{-(\tilde{x}+s)} \mathcal{B}_{\frac{1}{2-\theta}}(2\sqrt{\tilde{x}s})\, \mathrm{d}s$$
$$+ e^{-r\frac{B-I}{\widehat{\sigma}^2 s^{\theta-2}}} K \int_{\widetilde{K}}^{+\infty} \left(\frac{s}{\tilde{x}}\right)^{\frac{1}{2(2-\theta)}} e^{-(\tilde{x}+s)} \mathcal{B}_{\frac{1}{2-\theta}}(2\sqrt{\tilde{x}s})\, \mathrm{d}s, \tag{3.6}$$
$$\tilde{x} = \frac{2xe^{(2-\theta)\frac{B-I}{\widehat{\sigma}^2 x^{\theta-2}}}}{(2-\theta)^2\chi},\ \chi = \frac{\widehat{\sigma}^2}{(2-\theta)r}\left(e^{r(2-\theta)\frac{B}{\widehat{\sigma}^2 s^{\theta-2}}} - e^{r(2-\theta)\frac{I}{\widehat{\sigma}^2 s^{\theta-2}}}\right),\ \widetilde{K} = \frac{2K^{2-\theta}}{(2-\theta)^2\chi},$$

where the modified Bessel function of the first kind of order q, denoted as $\mathcal{B}_q(x)$, is given by

$$\mathcal{B}_q(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+q}}{r!\Gamma(n+1+q)}.$$

Proof. If we define the new state variable ξ as $\xi = \frac{I}{\widehat{\sigma}^2 x^{\theta-2}}$, then $\frac{\partial P_0}{\partial I} = \frac{1}{\widehat{\sigma}^2 x^{\theta-2}} \frac{\partial P_0}{\partial \xi}$ holds. Therefore, the PDE problem in (3.4) and (3.5) can be transformed into

$$\frac{\partial P_0}{\partial \xi} + \frac{\widehat{\sigma}^2}{2} x^{\theta} \frac{\partial^2 P_0}{\partial x^2} + r\left(x\frac{\partial P_0}{\partial x} - P_0\right) = 0, \tag{3.7}$$
$$P_0\left(x, \frac{B}{\widehat{\sigma}^2 x^{\theta-2}}\right) = \max(x-K, 0). \tag{3.8}$$

Surprisingly, this PDE is related to the pricing formula of a European call option under the CEV diffusion, except that the current time t and expiry date T are now replaced by the initial accumulated variance budget ξ and variance budget B, respectively. Therefore, referring to Lipton [49], the solution of PDE (3.7) with boundary condition (3.8) is given by

$$P_0(x, I) = e^{-r\left(\frac{B}{\widehat{\sigma}^2 x^{\theta-2}}-\xi\right)} x \int_{\widetilde{K}}^{+\infty} \left(\frac{\tilde{x}^*}{s}\right)^{\frac{1}{2(2-\theta)}} e^{-(\tilde{x}^*+s)} \mathcal{B}_{\frac{1}{2-\theta}}(2\sqrt{\tilde{x}^* s})\, \mathrm{d}s$$
$$+ e^{-r\left(\frac{B}{\widehat{\sigma}^2 x^{\theta-2}}-\xi\right)} K \int_{\widetilde{K}}^{+\infty} \left(\frac{s}{\tilde{x}}\right)^{\frac{1}{2(2-\theta)}} e^{-(\tilde{x}^*+s)} \mathcal{B}_{\frac{1}{2-\theta}}(2\sqrt{\tilde{x}^* s})\, \mathrm{d}s,$$

where

$$\tilde{x}^* = \frac{2xe^{(2-\theta)\left(\frac{B}{\widehat{\sigma}^2 x^{\theta-2}}-\xi\right)}}{(2-\theta)^2\chi^*},\ \chi^* = \frac{\widehat{\sigma}^2}{(2-\theta)r}\left(e^{r(2-\theta)\frac{B}{\widehat{\sigma}^2 x^{\theta-2}}} - e^{r(2-\theta)\xi}\right) \text{ and } \widetilde{K} = \frac{2K^{2-\theta}}{(2-\theta)^2\chi}.$$

Finally, if we replace ξ with I and use the change of variables $\xi = \frac{I}{\widehat{\sigma}^2 x^{\theta-2}}$, then we can obtain the leading-order price P_0 as in (3.6). □

Next, we continue our asymptotic approaches to obtain an explicit expression for correction term P_1, which plays a key role in this research and will reflect the impact of stochastic volatility on timer prices.

Proposition 3.2. *Under the growth condition given in Lemma 3.1, the first-order correction term $P_1(x, I)$ satisfies the following non-homogeneous PDE:*

$$\mathcal{L}_{\text{TO-CEV}}(\widehat{\sigma})P_1(x, I) = \mathcal{H}(x, I; \theta), \tag{3.9}$$

$$P_1(x,B) = 0, \tag{3.10}$$

where the non-homogeneous term $\mathcal{H}(x,I;\theta)$ *is given by*

$$\begin{aligned}\mathcal{H}(x,I) = &- \nu\sqrt{2}\,\langle\Lambda\phi'\rangle\, x^{\theta-2}\frac{\partial P_0}{\partial I} + \nu\rho\sqrt{2}\,\langle f\phi'\rangle\, x\frac{\partial}{\partial x}\left(x^{\theta-2}\frac{\partial P_0}{\partial I}\right)\\ &+ \frac{1}{2}\left\{-\nu\sqrt{2}\,\langle\Lambda\phi'\rangle\, x^{\theta}\frac{\partial^2 P_0}{\partial x^2} + \nu\rho\sqrt{2}\,\langle f\phi'\rangle\, x\frac{\partial}{\partial x}\left(x^{\theta}\frac{\partial^2 P_0}{\partial x^2}\right)\right\}.\end{aligned}$$

Proof. In (3.2), the fourth term (or $\sqrt{\epsilon}$-order term) yields

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0. \tag{3.11}$$

Now, by using the centering condition for the Poisson equation P_3, we can obtain $\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\rangle = 0$. Then, since $\langle \mathcal{L}_2 P_0\rangle = 0$, we deduce

$$\begin{aligned}\mathcal{L}_2 P_0 &= \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0\rangle\\ &= (f^2(y) - \widehat{\sigma}^2)x^{\theta-2}\frac{\partial P_0}{\partial I} + \frac{1}{2}(f^2(y) - \widehat{\sigma}^2)x^{\theta}\frac{\partial^2 P_0}{\partial x^2}.\end{aligned}$$

In addition, from the relation $\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$, P_2 satisfies

$$\begin{aligned}P_2 &= -\mathcal{L}_0^{-1}\mathcal{L}_2 P_0\\ &= -(\phi(y) + C)x^{\theta-2}\frac{\partial P_0}{\partial I} - \frac{1}{2}(\phi(y) + C)x^{\theta}\frac{\partial^2 P_0}{\partial x^2}, \text{ for } C \in \mathbb{R},\end{aligned}$$

where the function $\phi(y)$, at most polynomially growing (refer to Lemma 3.1 in Fouque et al. [3]), is solution to the following Poisson equation:

$$\mathcal{L}_0\phi(y) = f^2(y) - \widehat{\sigma}^2.$$

Then, $\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\rangle = 0$ becomes

$$\begin{aligned}\langle \mathcal{L}_2\rangle P_1 &= -\langle \mathcal{L}_1 P_2\rangle\\ &= \langle \mathcal{L}_1\phi(y)\rangle\, x^{\theta-2}\frac{\partial P_0}{\partial I} + \frac{1}{2}\langle \mathcal{L}_1\phi(y)\rangle\, x^{\theta}\frac{\partial^2 P_0}{\partial x^2}\\ &= -\nu\sqrt{2}\,\langle\Lambda\phi'\rangle\, x^{\theta-2}\frac{\partial P_0}{\partial I} + \nu\rho\sqrt{2}\,\langle f\phi'\rangle\, x\frac{\partial}{\partial x}\left(x^{\theta-2}\frac{\partial P_0}{\partial I}\right)\\ &\quad + \frac{1}{2}\left\{-\nu\sqrt{2}\,\langle\Lambda\phi'\rangle\, x^{\theta}\frac{\partial^2 P_0}{\partial x^2} + \nu\rho\sqrt{2}\,\langle f\phi'\rangle\, x\frac{\partial}{\partial x}\left(x^{\theta}\frac{\partial^2 P_0}{\partial x^2}\right)\right\}\\ &\equiv \mathcal{H}(x,I;\theta).\end{aligned}$$

Therefore, combining the results $\mathcal{L}_2 P_1 = \mathcal{H}(x,I;\theta)$ and the boundary condition $P_1(x,B) = 0$, we complete the proof. □

Next, we present the first-order correction term $P_1(x,I)$, which is the solution to the PDE given in Proposition 3.2.

Theorem 3.2. *The first-order correction term $P_1(x,I)$ is given by*

$$P_1(x,I) = -\int_{\mathbb{R}} \int_{\frac{I}{\widetilde{\sigma}^2 x^{\theta-2}}}^{\frac{B}{\widetilde{\sigma}^2 x^{\theta-2}}} e^{-r\left(s-\frac{I}{\widetilde{\sigma}^2 x^{\theta-2}}\right)} \mathcal{H}(x^*, s;\theta)\, D(x^*, s; x, I)\, \mathrm{d}s\, \mathrm{d}x^*, \tag{3.12}$$

where the transition probability density function, $D(x^*, s; x, I)$, is described by

$$\begin{aligned} D(x^*, s; x, I) &= (2-\theta) k^{\frac{1}{2-\theta}} (u_1 u_2^{1-\theta})^{\frac{1}{2-\theta}} e^{-u_1-u_2}, \\ k &= \frac{2r}{\widetilde{\sigma}^2 (2-\theta)(e^{r(2-\theta)(s-I)}-1)}, \\ u_1 &= k x^{2-\theta} e^{r(2-\theta)(s-I)}, \quad u_2 = (x^*)^{2-\theta} k. \end{aligned}$$

Proof. As shown in Theorem 3.1, if we define the new state variable ξ as $\xi = \frac{I}{\widetilde{\sigma}^2 x^{\theta-2}}$, then $\frac{\partial P_1}{\partial I} = \frac{1}{\widetilde{\sigma}^2 x^{\theta-2}} \frac{\partial P_1}{\partial \xi}$ holds. Therefore, the PDE problem in (3.9) and (3.10) can be transformed into

$$\frac{\partial P_1}{\partial \xi} + \frac{\widetilde{\sigma}^2}{2} x^{\theta} \frac{\partial^2 P_1}{\partial x^2} + r\left(x \frac{\partial P_1}{\partial x} - P_1\right) = 0, \tag{3.13}$$

$$P_1\left(x, \frac{B}{\widetilde{\sigma}^2 x^{\theta-2}}\right) = 0. \tag{3.14}$$

Now, using the Feynman-Kac formula and referring to Choi et al. [17], the solution to the PDE (3.13) with the source term, $\mathcal{H}(x, I;\theta)$, as defined in Proposition 3.2, and with a zero boundary condition (3.14), is given by

$$P_1(x,\xi) = \mathbb{E}^{\mathbb{Q}}\left[-\int_{\xi}^{\frac{B}{\widetilde{\sigma}^2 x^{\theta-2}}} e^{-r(s-\xi)} \mathcal{H}(x,s) \mathrm{d}s \middle| X_t = x \right].$$

Finally, if we replace ξ with I, then we can obtain the analytic-form solution $P_1(x, I)$ as described in (3.12). □

4. Accuracy and numerical experiments

This section investigates the error of the approximation of the timer call option price $P(x,y,I)$ given by (2.6) under the CEV model and examines the price sensitivities on the TO-CEV with regard to the model parameters.

4.1. Accuracy of the approximation

If we combine the results of leading-order term $P_0(x, I)$ presented in Theorem 3.1 and the correction term $P_1(x, I)$ described in Theorem 3.2, then we can obtain a first-order approximation to the TO-CEV price $P(x,y,I)$, which is denoted by $\tilde{P}^{\epsilon}$, given by

$$P(x,y,I) \approx P_0(x,I) + \sqrt{\epsilon} P_1(x,I) \equiv \tilde{P}^{\epsilon}(x,I). \tag{4.1}$$

Then, one may want to examine the accuracy of the above approximation $\tilde{P}^{\epsilon}(x, I)$. In this regard, by referring to Fouque and Lorig [50], if we assume that the payoff function and its derivatives are continuously differentiable and bounded, utilizing the asymptotic expansions methods, then the accuracy of $\tilde{P}^{\epsilon}(x, I)$ is given by

$$\left|P(x, y, I) - \tilde{P}^{\epsilon}(x, I)\right| \leq O(\epsilon). \tag{4.2}$$

However, for a non-smooth payoff, that is, the payoff function is not continuously differentiable, it is required to introduce a regularization argument because of the singularity at $x = K$. In this research, we numerically investigate the accuracy of the analytic form solution to the PDE (2.7) via Monte-Carlo simulations (refer to Table 1 in Section 4.2.1), instead of providing the mathematical proof for the accuracy.

4.2. Numerical experiments

In this section, we provide numerical implications by using the asymptotic approximations (4.1) of the TO-CEV prices. In addition, we compare the option price calculated by the closed-form with that obtained from the Monte-Carlo method. The parameters we have chosen in the numerical experiments are described by $X_0 = 100, T = 1.0, r = 0.03, K = 100, B = 0.3, v = 0.05, \langle\Lambda(y)\psi'(y)\rangle = 0.0141, \langle f(y)\psi'(y)\rangle = 0.0023, \rho = -0.1, m = 0.5, u = 1.0, f(y) = e^y$ and $\langle f(y)^2\rangle = 0.0272$, referring to Ha et al. [48].

4.2.1. Accuracy and Monte-Carlo simulation

We show the accuracy of the model formula by comparing the price of the timer option, presented by the model formula and the option value through Monte Carlo (MC) simulation. Table 1 demonstrates a comparison of the results. In the table, P_{MC} represents the Monte Carlo price and $\tilde{P}^{\epsilon}$ is the approximate option price given by (4.2). In addition, we calculate the relative percentage error given by $100 \times \left|\frac{P_{\mathrm{MC}} - \tilde{P}^{\epsilon}(x,I)}{P_{\mathrm{MC}}}\right|$ for the numerical implications. According to Table 1, by performing simulations 10000, 50000, or 100000 times, we obtain stable MC simulation results. Furthermore, we confirm that the value of relative error approaches zero as ϵ goes to zero. The average CPU times (in seconds) required for the execution of P_{MC} are 27.5863, 4894.1, and 34745 for simulations conducted 10000, 50000, and 100000 times, respectively. In contrast, the CPU time for $\tilde{P}^{\epsilon}$ is 0.6485. Consequently, our options' formula verifies the excellence of the model not only with regards to the computational accuracy but also in terms of the computational efficiency.

4.2.2. Sensitivity to the parameters

Figure 1 displays the changes in the correction term price P_1 with respect to the accumulated variance under the variance budget for a given θ value. As seen in both figures, as the value of the accumulated variance is less than the variance budget (B), the impact of the correction price is significant, but when the v is greater than the B, the values of the correction term are zero and have no influence on the option price. In fact, in the case of timer options, as the accumulated variance reaches the variance budget, the options get exercised. Therefore, when the aggregated variance is larger than the budget level, the effect of the stochastic volatility (SV) must be zero, and then the correction term price remains zero. Moreover, it can be seen that the greater the variance budget level, the lower the price impact of the correction term is for the accumulated variance. It implies that as the time-to-maturity of the timer options is higher (as the budget levels get bigger), the price sensitivities become smaller, which ultimately decreases the influence of the stochastic volatility (SV) on the timer option. What's more remarkable is that, as shown in Figure 1, the more the elasticity value increases, the more the price of the correction term is sensitive to the variance budget. It means that the effect of the SV on the timer option tends to rise as the elasticity parameter gets larger, and in particular, it becomes more prominent as the expiration is longer for the larger elasticity value.

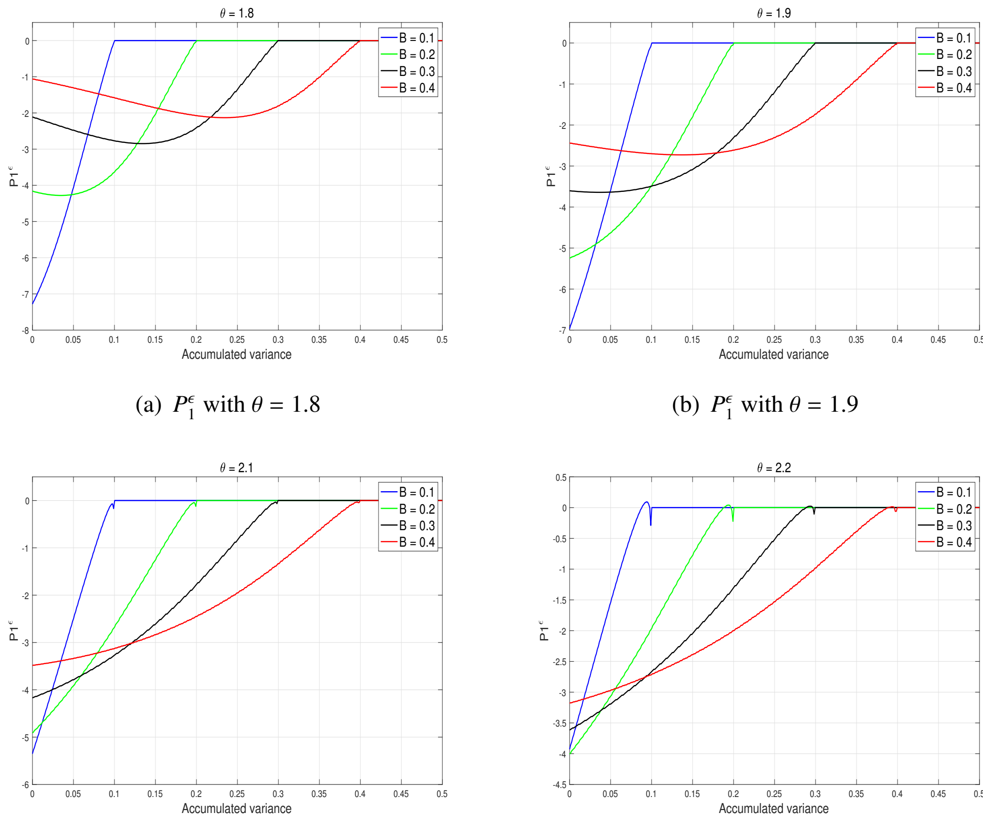
Figure 2 shows the changes in the correction term P_1 of the timer option with constant elasticity of variance (CEV) with regard to the underlying asset for each strike price and for a given elasticity parameter. As seen in Figure 2(a)–(d), the correction prices have a tendency to decrease sharply as the underlying asset is near the strike price (K) regardless of the value of the elasticity θ. It implies that the SV has a more significant impact on the time option if the value of the underlying asset is adjacent to the strike value (K) for the elasticity parameter θ, especially, exhibiting a hump phenomenon for the graph of the correction term for $\theta > 2$.

Figure 3 presents the comparison of the price of the timer option (TO) and the price of the timer CEV options (TO-CEVs). Taking account of the fact that the classical timer option has the elasticity $\theta = 2$, note that the price of TO-CEVs tends to decrease as the elasticity value gets bigger, and the sensitivities of the option value get larger when the elasticity decreases.

Table 1. For given $\theta = 1.8, 1.9, 2.1$ and $\theta = 2.2$, solutions $\tilde{P}^\epsilon$ and errors are given against ϵ, where "R.Error" is the relative percentage error. Note that the baseline parameters are given by $X_0 = 100, T = 1.0, r = 0.03, K = 100, B = 0.3, v = 0.05, \langle \Lambda(y)\psi'(y)\rangle = 0.0141, \langle f(y)\psi'(y)\rangle = 0.0023, \rho = -0.1, m = 0.5, u = 1.0, f(y) = e^y$ and $\langle f(y)^2\rangle = 0.0272$.

		$\theta = 1.8$			$\theta = 1.9$		
Simul.	ϵ	0.01	0.001	0.0001	0.01	0.001	0.0001
	$\tilde{P}^\epsilon$	17.0555	17.5684	17.7306	21.3202	21.8599	22.0306
10000		31.1461	18.5245	17.0744	25.8019	21.0699	22.6008
R.Error(%)		45.24	5.16	3.84	17.36	3.74	2.52
50000		31.8303	18.3064	18.1076	25.7835	21.1684	21.7066
R.Error(%)		46.41	4.03	2.08	17.31	3.26	1.49
100000		31.9046	18.6246	17.7975	25.8621	21.3048	22.184
R.Error(%)		46.54	5.67	0.37	17.56	2.61	0.69
		$\theta = 2.1$			$\theta = 2.2$		
Simul.	ϵ	0.01	0.001	0.0001	0.01	0.001	0.0001
	$\tilde{P}^\epsilon$	31.0636	31.9086	32.1757	35.8661	36.5744	36.7984
10000		37.2880	31.5311	33.0563	44.0128	36.9723	37.2356
R.Error(%)		16.69	1.19	2.66	18.51	1.07	1.17
50000		37.5411	32.9560	32.8798	44.2231	36.8926	37.1461
R.Error(%)		17.25	3.17	2.14	18.89	0.86	0.93
100000		37.9496	32.5929	32.4217	44.7175	37.0560	36.6976
R.Error(%)		18.14	2.09	0.75	19.79	1.29	0.27

Figure 4 exhibits the changes of the TO-CEV prices against the underlying asset for a given elasticity value, comparing them with those of the European CEV options (EO-CEVs). As shown in Figure 4(a)–(f), one can observe that the price gap between the TO-CEVs and EO-CEVs depends on the value of the elasticity. We can see that, for $\theta > 2$ or $\theta = 2$, definitely, the prices of EO-CEVs tend to be bigger than that of the TO-CEVs, and as the elasticity parameter increases, the phenomenon is more pronounced, demonstrating that our findings that TO-CEVs are noticeably underestimated compared with EO-CEVs are consistent with those of Sawyer [28]. However, in the case of $\theta < 2$, the price difference between TO-CEVs and EO-CEVs becomes smaller as the elasticity value grows down-after all, the TO-CEVs tend to be overpriced compared to the EO-CEVs in terms of underlying asset price, contrary to the results of Sawyer [28]. In particular, as seen in Figure 4(c), the EO-CEVs and the TO-CEVs have almost the same values in the region of deep ITM. In fact, referring to the empirical evidence verifying that the elasticity value calibrated from the historical data analysis of the volatility of the $S\&P$ 500 index is between 1.9 and 2, given by Choi et al. [17] and Figure 4(c),(d), it can be observed that the price of the EO-CEVs tend to get larger than that of the TO-CEVs in the real financial situation, especially, in the area of ITM. It implies that, even in the case of $\theta < 2$, if we take account of the elasticities obtained in the real situation, the value of TO-CEVs becomes undervalued compared to that of EO-CEVs in the region of ITM, similar to the findings described by Sawyer [28].



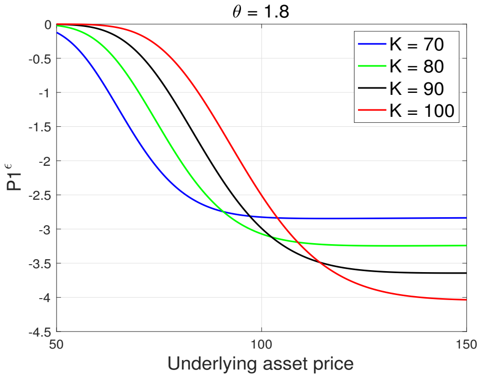
(a) P_1^ϵ with $\theta = 1.8$

(b) P_1^ϵ with $\theta = 1.9$

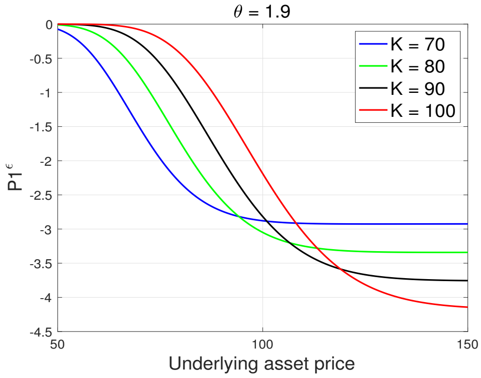
(c) P_1^ϵ with $\theta = 2.1$

(d) P_1^ϵ with $\theta = 2.2$

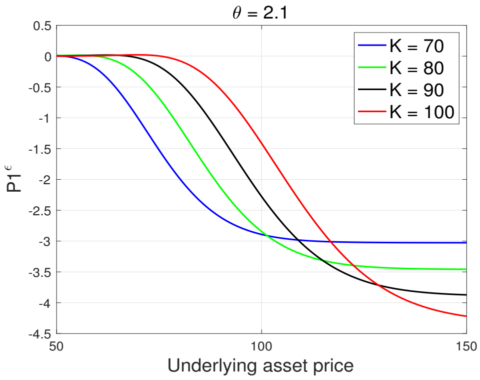
Figure 1. Value of the correction term P_1^ϵ in terms of the accumulated variance v for the given θ. Note: $X_0 = 100, T = 1.0, r = 0.03, K = 100, \langle\Lambda(y)\psi'(y)\rangle = 0.0141, \langle f(y)\psi'(y)\rangle = 0.0023, \rho = -0.1, \epsilon = 0.001, m = \log(0.1), u = 1.0, f(y) = \exp(y), \langle f(y)^2\rangle = 0.0272.$



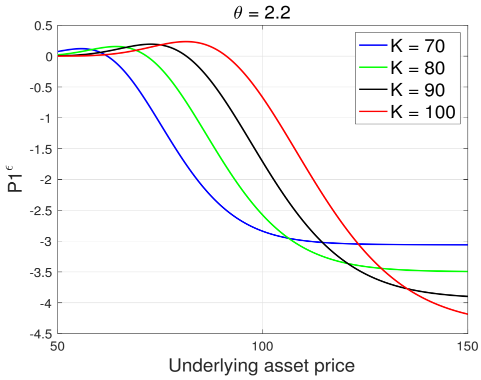
(a) P_1^ϵ with $\theta = 1.8$



(b) P_1^ϵ with $\theta = 1.9$



(c) P_1^ϵ with $\theta = 2.2$



(d) P_1^ϵ with $\theta = 2.2$

Figure 2. Value of the correction term P_1^ϵ with respect to the underlying asset x for the given θ. Note: $B = 0.03, v = 0.01$. The values of the remaining parameters are the same as in Figure 1.

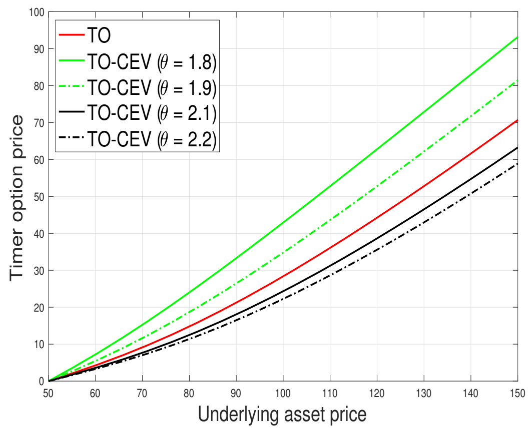
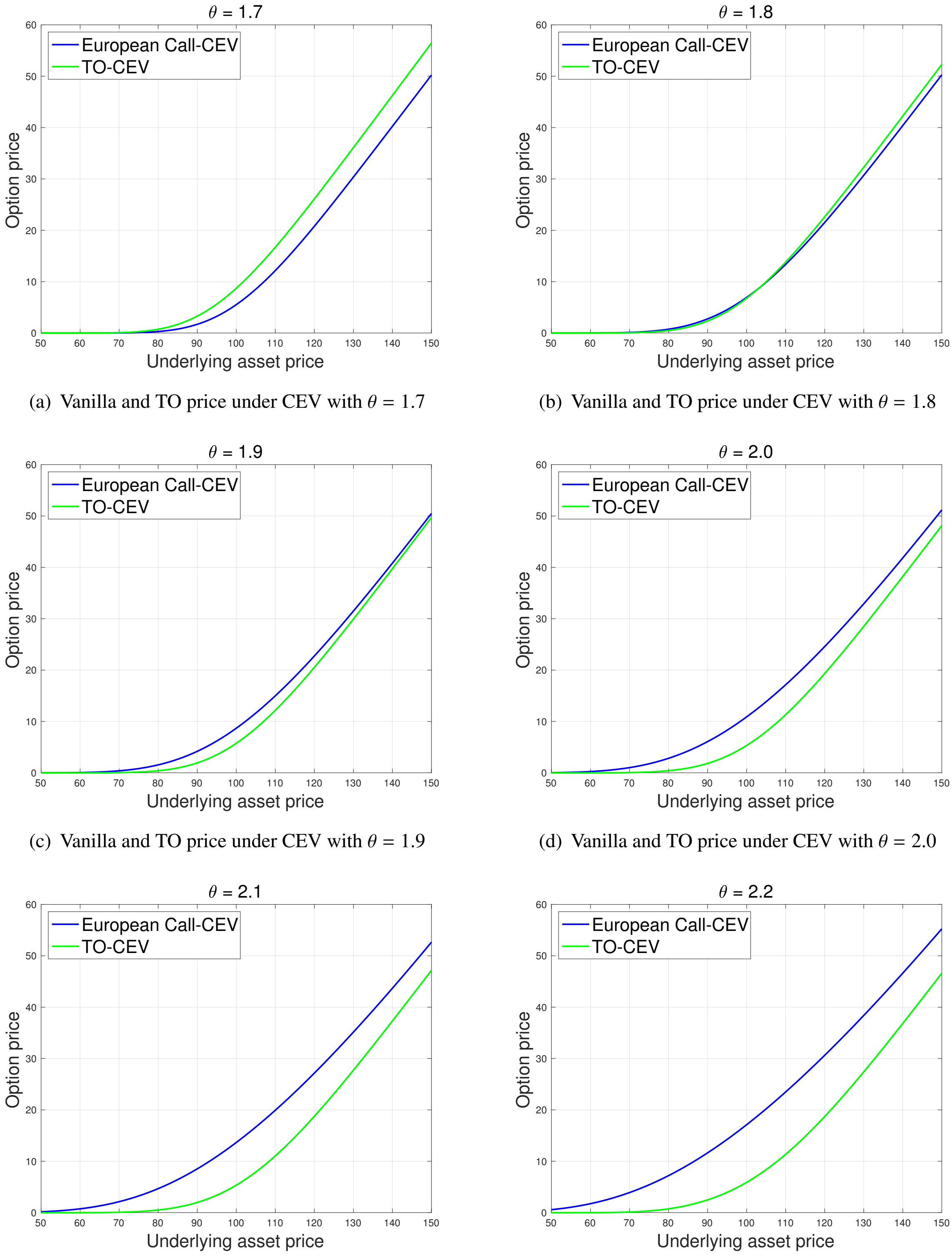


Figure 3. Comparison of the values of timer CEV option (TO-CEV) with respect to the underlying asset x for a given θ. Note: $B = 0.3, v = 0.05$. The values of the remaining parameters are the same as in Figure 1.



(a) Vanilla and TO price under CEV with $\theta = 1.7$

(b) Vanilla and TO price under CEV with $\theta = 1.8$

(c) Vanilla and TO price under CEV with $\theta = 1.9$

(d) Vanilla and TO price under CEV with $\theta = 2.0$

(e) Vanilla and TO price under CEV with $\theta = 2.1$

(f) Vanilla and TO price under CEV with $\theta = 2.2$

Figure 4. Value of Vanilla and timer option prices by CEV models with respect to the underlying asset x for the given θ. Note: $B = 0.022, v = 0.02$. The values of the remaining parameters are the same as in Figure 1.

5. Conclusions

In this paper, we investigated the valuing problem of timer options under the constant elasticity of variance (CEV). In the real financial market, timer options are one of the financial derivatives that take account of the level of volatility so that the options are expired as the realized variance arrives at the variance budget. The purpose of this research is to analyze the price sensitivities between the timer options with the CEV (TO-CEVs) and the standard European CEV options (EO-CEVs).

First, under the partial differential equations (PDEs) obtained from the underlying asset model and Feynman-Kac formula, we derive the approximated analytic solutions for the options by making use of the method of asymptotic analysis given by Fouque et al. [3]. Second, we implemented the numerical experiments to verify the pricing accuracy of our analytic-form formulas for the timer CEV options, compared to Monte Carlo prices. Third, we observed the pricing sensitivities of the options in terms of some model parameters, emphasizing the impact of the SV on the timer options in terms of the elasticity parameter. Fourth, we found that the timer CEV options tend to be underpriced compared with the standard European CEV options for $\theta > 2$ or the elasticity values that reflect the real market situation, in line with the results of [28]. Finally, our research can be more widely applied to several challengeable volatility managing problems or risk management strategies, including the problem for the timer options under diverse underlying asset models.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work of S. Y. Choi was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1F1A1046138) and the Gachon University research fund of 2023 (2023039990001). The research by J. H. Yoon was supported by the the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2022R1A5A1033624 and No. 2023R1A2C1006600).

Conflict of interest

The authors have no conflicts of interest to declare.

References

1. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.*, **81** (1973), 637–654.
2. S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financ. Stud.*, **6** (1993), 327–343. https://doi.org/10.1093/rfs/6.2.327
3. J. P. Fouque, G. Papanicolaou, R. Sircar, K. Sølna, *Multiscale stochastic volatility for equity, interest rate, and credit derivatives*, Cambridge University Press, 2011.

4. J. Hull, A. White, The pricing of options on assets with stochastic volatilities, *J. Financ.*, **42** (1987), 281–300. https://doi.org/10.1111/j.1540-6261.1987.tb02568.x
5. L. O. Scott, Option pricing when the variance changes randomly: Theory, estimation, and an application, *J. Financ. Quant. Anal.*, **22** (1987), 419–438. https://doi.org/10.2307/2330793
6. M. Chesney, L. Scott, Pricing European currency options: A comparison of the modified Black-Scholes model and a random variance model, *J. Financ. Quant. Anal.*, **24** (1989), 267–284. https://doi.org/10.2307/2330812
7. R. Schöbel, J. Zhu, Stochastic volatility with an Ornstein-Uhlenbeck process: An extension, *Rev. Financ.*, **3** (1999), 23–46. https://doi.org/10.1023/A:1009803506170
8. B. Dupire, Pricing with a smile, *Risk*, **7** (1994), 18–20.
9. E. Derman, I. Kani, Riding on a smile, *Risk*, **7** (1994), 32–39.
10. J. C. Cox, S. A. Ross, The valuation of options for alternative stochastic processes, *J. Financ. Econ.*, **3** (1976), 145–166. https://doi.org/10.1016/0304-405X(76)90023-4
11. Y. Tian, Z. Zhu, G. Lee, F. Klebaner, K. Hamza, Calibrating and pricing with a stochastic-local volatility model, *J. Deriv.*, **22** (2015), 21–39. https://doi.org/10.3905/jod.2015.22.3.021
12. E. Ghysels, A. C. Harvey, E. Renault, 5 Stochastic volatility, *Handb. Stat.*, **14** (1996), 119–191. https://doi.org/10.1016/S0169-7161(96)14007-4
13. A. W. V. Stoep, L. A. Grzelak, C. W. Oosterlee, The Heston stochastic-local volatility model: Efficient Monte Carlo simulation, *Int. J. Theor. Appl. Fin.*, **17** (2014), 1450045. https://doi.org/10.1142/S0219024914500459
14. Z. Cui, J. L. Kirkby, D. Nguyen, A general valuation framework for SABR and stochastic local volatility models, *SIAM J. Financ. Math.*, **9** (2018), 520–563. https://doi.org/10.1137/16M1106572
15. L. B. Andersen, V. V. Piterbarg, Moment explosions in stochastic volatility models, *Financ. Stoch.*, **11** (2007), 29–50. https://doi.org/10.1007/s00780-006-0011-7
16. R. Lord, R. Koekkoek, D. V. Dijk, A comparison of biased simulation schemes for stochastic volatility models, *Quant. Financ.*, **10** (2010), 177–194. https://doi.org/10.1080/14697680802392496
17. S. Y. Choi, J. P. Fouque, J. H. Kim, Option pricing under hybrid stochastic and local volatility, *Quant. Financ.*, **13** (2013), 1157–1165. https://doi.org/10.1080/14697688.2013.780209
18. Z. Cui, J. L. Kirkby, D. Nguyen, Efficient simulation of generalized SABR and stochastic local volatility models based on Markov chain approximations, *Eur. J. Oper. Res.*, **290** (2021), 1046–1062. https://doi.org/10.1016/j.ejor.2020.09.008
19. P. Carr, D. Madan, Towards a theory of volatility trading, *Volat. New Estim. Tech. Pric. Deriv.*, **29** (1998), 417–427. https://doi.org/10.1017/CBO9780511569708.013
20. A. Badescu, Z. Cui, J. P. Ortega, Closed-form variance swap prices under general affine GARCH models and their continuous-time limits, *Ann. Oper. Res.*, **282** (2019), 27–57. https://doi.org/10.1007/s10479-018-2941-9
21. K. Demeterfi, E. Derman, M. Kamal, A guide to volatility and variance swaps, *J. Deriv.*, **6** (1999), 9–32. https://doi.org/10.3905/jod.1999.319129

22. S. P. Zhu, G. H. Lian, A closed-form exact solution for pricing variance swaps with stochastic volatility, *Math. Financ.*, **21** (2011), 233–256. https://doi.org/10.1111/j.1467-9965.2010.00436.x

23. W. Zheng, Y. K. Kwok, Closed form pricing formulas for discretely sampled generalized variance swaps, *Math. Financ.*, **24** (2014), 855–881. https://doi.org/10.1111/mafi.12016

24. A. Issaka, Variance swaps, volatility swaps, hedging and bounds under multi-factor Heston stochastic volatility model, *Stoch. Anal. Appl.*, **38** (2020), 856–874. https://doi.org/10.1080/07362994.2020.1730903

25. Y. Xi, H. Y. Wong, Discrete variance swap in a rough volatility economy, *J. Futures Markets*, **41** (2021), 1640–1654. https://doi.org/10.1002/fut.22242

26. O. E. Euch, M. Rosenbaum, The characteristic function of rough Heston models, *Math. Financ.*, **29** (2019), 3–38. https://doi.org/10.1111/mafi.12173

27. A Neuberger, *Volatility trading*, Institute of Finance and Accounting: London Business School, Working Paper, 1990.

28. N. Sawyer, SG CIB launches timer options, *Risk*, **20** (2007), 6.

29. A. Bick, Quadratic-variation-based dynamic strategies, *Manag. Sci.*, **41** (1995), 722–732. https://doi.org/10.1287/mnsc.41.4.722

30. C. Li, *Managing volatility risk innovation of financial derivatives, stochastic models and their analytical implementation*, Doctoral dissertation, Columbia University, 2010.

31. D. Saunders, Pricing timer options under fast mean-reverting stochastic volatility, *Can. Appl. Math. Q.*, **17** (2009), 737–753.

32. L. Z. J. Liang, D. Lemmens, J. Tempere, Path integral approach to the pricing of timer options with the Duru-Kleinert time transformation, *Phys. Rev. E*, **83** (2011), 056112. https://doi.org/10.1103/PhysRevE.83.056112

33. C. Bernard, Z. Cui, Pricing timer options, *J. Comput. Financ.*, **15** (2011), 1–37. https://doi.org/10.21314/JCF.2011.228

34. M. Li, F. Mercurio, Closed-form approximation of perpetual timer option prices, *Int. J. Theor. Appl. Fin.*, **17** (2014), 1450026. https://doi.org/10.1142/S0219024914500265

35. J. Ma, D. Deng, Y. Lai, Explicit approximate analytic formulas for timer option pricing with stochastic interest rates, *North Am. J. Econ. Financ.*, **34** (2015), 1–21. https://doi.org/10.1016/j.najef.2015.07.002

36. C. Li, Bessel processes, stochastic volatility, and timer options, *Math. Financ.*, **26** (2016), 122–148. https://doi.org/10.1111/mafi.12041

37. J. Zhang, X. Lu, Y. Han, Pricing perpetual timer option under the stochastic volatility model of Hull-White, *ANZIAM J.*, **58** (2017), 406–416. https://doi.org/10.21914/anziamj.v58i0.11281

38. Z. Cui, J. L. Kirkby, G. Lian, D. Nguyen, Integral representation of probability density of stochastic volatility models and timer options, *Int. J. Theor. Appl. Fin.*, **20** (2017), 1750055. https://doi.org/10.1142/S0219024917500558

39. X. Wang, S. J. Wu, X. Yue, Pricing timer options: Second-order multiscale stochastic volatility asymptotics, *ANZIAM J.*, **63** (2021), 249–267. https://doi.org/10.21914/anziamj.v63.15291

40. J. L. Kirkby, J. P. Aguilar, The return barrier and return timer option with pricing under Levy processes, *Expert Syst. Appl.*, **233** (2023), 120920. https://doi.org/10.1016/j.eswa.2023.120920

41. B. Bock, S. Y. Choi, J. H. Kim, The pricing of European options under the constant elasticity of variance with stochastic volatility, *Fluct. Noise Lett.*, **12** (2013), 1350004. https://doi.org/10.1142/S0219477513500041

42. J. H. Kim, M. K. Lee, S. Y. Sohn, Investment timing under hybrid stochastic and local volatility, *Chaos Soliton. Fract.*, **67** (2014), 58–72. https://doi.org/10.1016/j.chaos.2014.06.007

43. S. Y. Choi, J. H. Kim, Equity-linked annuities with multiscale hybrid stochastic and local volatility, *Scand. Actuar. J.*, **2016** (2016), 466–487. https://doi.org/10.1080/03461238.2014.955048

44. S. Y. Choi, J. H. Kim, J. H. Yoon, Foreign exchange rate volatility smiles and smirks, *Appl. Stoch. Model. Bus.*, **37** (2021), 628–660. https://doi.org/10.1002/asmb.2602

45. J. P. Fouque, G. Papanicolaou, K. R. Sircar, Mean-reverting stochastic volatility, *Int. J. Theor. Appl. Fin.*, **3** (2000), 101–142. https://doi.org/10.1142/S0219024900000061

46. H. Geman, Y. F. Shih, Modeling commodity prices under the CEV model, *J. Altern. Invest.*, **11** (2009), 65–84. https://doi.org/10.3905/JAI.2009.11.3.065

47. B. Øksendal, *Stochastic differential equations: An introduction with applications*, 6 Eds., Springer, 2010.

48. M. Ha, D. Kim, J. H. Yoon, Valuing of timer path-dependent options, *Math. Comput. Simul.*, **215** (2024), 208–227. https://doi.org/10.1016/j.matcom.2023.08.010

49. A. Lipton, *Mathematical methods for foreign exchange: A financial engineer's approach*, World Scientific, 2001. https://doi.org/10.1142/4694

50. J. P. Fouque, M. J. Lorig, A fast mean-reverting correction to Heston's stochastic volatility model, *SIAM J. Financ. Math.*, **2** (2011), 221–254. https://doi.org/10.1137/090761458