On some classes of formulas in $S5$ which are pre-complete relative to existential expressibility

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Abstract

Existential expressibility for all k-valued functions was proposed by A. V. Kuznetsov and later was investigated in more details by S. S. Marchenkov. In the present paper, we consider existential expressibility in the case of formulas defined by a logical calculus and find out some conditions for a system of formulas to be closed relative to existential expressibility. As a consequence, it has been established some pre-complete as to existential expressibility classes of formulas in some finite extensions of the paraconsistent modal logic $S5$.

Keywords: Paraconsistent logic, existential expressibility, logical calculi.

MSC 2020: 68R99, 68Q25, 06E25, 03B53.

ACM CCS 2020: 10003752.10003790.10003793.

1 Introduction

It is a well known class of problems in logic, algebra, discrete mathematics, and cybernetics dealing with the possibility of obtaining some functions (operations, formulas) from other ones by means of a fixed set of tools. The notion of expressibility of Boolean functions through other ones by means of superpositions goes back to the works of E. Post [1], [2]. He described all closed (with respect to superpositions) classes of 2-valued Boolean functions. The problem of completeness (with respect to expressibility), which requires to determine the necessary and sufficient conditions for all formulas of the logic under investigation to be expressible via the given system of formulas, is also

doi:10.56415/csjm.v31.21

investigated. In 1956 ([3, p. 54], [4]), A. V. Kuznetsov established the theorem of completeness according to which we can build a finite set of closed with respect to expressibility classes of functions in the k-valued logics such that any system of functions of this logic is complete if and only if it is not included in any of these classes. In 1965 [5], Rosenberg I. established the criterion of completeness in the k-valued logics formulated in terms of a finite set of pre-complete classes of functions, i.e., in terms of maximal, incomplete, and closed classes of functions.

In the present paper, we investigate necessary conditions of completeness with respect to existential expressibility of the systems of formulas in some extensions of the modal logic $S5$.

The standard language of $S5$ is based on propositional variables and logical connectives: $\&$, $\vee$, $\rightarrow$, $\neg$, $\Box$, and $\Diamond$. We consider the para-consistent negation $\sim$ of $S5$ [6] as follows:

$$\sim a =_{Def} \Diamond\neg a.$$

The logic $S5$ can be considered, according to [6], as a para-consistent logic since it contains a para-consistent negation. The logic $S5$ is characterized by the axioms and rules of inference of the classical propositional logic, the following axioms (A and B are any valid formulas):

$$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B),$$
$$\Box A \rightarrow A,$$
$$\Diamond A \rightarrow \Box\Diamond A,$$

and the necessity rule of inference: from A infer $\Box A$.

Consider the set E_k of finite binary strings $(\alpha_1, \ldots, \alpha_k)$, where $\alpha_i \in \{0, 1\}$, $i = 1, \ldots, k$. Define Boolean operations $\&, \vee, \rightarrow, \neg$ over elements of E_k component-wise, and consider $\Box((1, \ldots, 1)) = (1, \ldots, 1)$, and put $\Box((\alpha_1, \ldots, \alpha_k)) = (0, \ldots, 0)$ otherwise. Also, as usual, $\Diamond x = \neg\Box\neg x$. It is known [7] that $(E_k; \&, \vee, \rightarrow, \neg, \Box, \Diamond)$ represents an algebraic model for $S5$.

Kuznetsov A. V. proposed in [8] some generalizations of the notion of expressibility of formulas in a superintuitionistic logic, namely the parametric expressibility, and the existential expressibility.

The formula F is said to be expressible in the logic L via a system of formulas Σ if F can be obtained from propositional variables, constants, and formulas of Σ applying a finite number of times: a) the rule of substitution of equivalent formulas in the logic L, and b) the rule of weak substitution, which permits, being given formulas A and B, to substitute one of them in another instead of a given corresponding propositional variable [8]–[10].

A. V. Kuznetsov [9] extended the notion of (explicit) expressibility from Boolean functions to formulas of the superintuitionistic propositional logics. He proposed the use of two rules (weak substitution and replacement by equivalent formula in the given logic) instead of the rule of superposition, and the problem of completeness with respect to (explicit) expressibility was solved for intuitionistic propositional logic and its extensions by M. F. Ratsa [10], [11].

In his work [8], A. V. Kuznetsov, among other things, extended the notion of (explicit) expressibility by modifying the tools previously used so as to obtain new formulas in superintuitionistic propositional logics and in the general k-valued logic P_k. Thus, he proposed the notions of implicit expressibility, parametric expressibility, and existential expressibility. The last one is similar to the notion of existential definability of predicates in arithmetics, examined by J. Robinson in [12].

Related to the problems of expressibility is the following one, which requires finding a tool (property) X that will permit us to separate the object A from the given system of objects Σ in the sense that if A is not expressible via the objects of Σ, then the objects of Σ possess property X, and the object A does not possess it. In this case, we speak about *separability of A from Σ by means of X*, or we say that *A is detachable from Σ by means of X*. In [8], A. V. Kuznetsov stated conditions of separability of a formula of the general k-valued logic from a given set of formulas with respect to explicit, parametric, and existential expressibility.

In the present paper, we specify the notion of existential expressibility to any algebra with a finite set of basic operations and we determine sufficient conditions for a system of term functions to be closed with respect to existential expressibility in the given algebra. As a consequence, some pre-complete relative to existential expressibility classes of formulas in some tabular extensions of the logic $S5$ are identified.

2 Basic notions

Consider the set of variables Var, whose elements will usually be denoted by small italic letters $a, b, d, p, q, \ldots$, possibly with indices. Let $\mathfrak{A} = (E; F_1, \ldots, F_n)$ be an algebra with support E and basic operations $F_1, \ldots, F_n$. The elements of the support of $\mathfrak{A}$ are denoted by small Greek letters $\alpha, \beta, \gamma, \delta, \ldots$. Terms of $\mathfrak{A}$ are defined as usual [13, p.62, Def. 10.1] and are denoted by capital letters. In order to stress that the variables $p_1, \ldots, p_n$ occur in the term A, we will write $A(p_1, \ldots, p_n)$. We will usually write the fact that some variable p is substituted in term $A(p, p_1, \ldots, p_n)$ by term B in the form $A[p/B]$ or $A[B]$ for short. The same notation $A[p/\gamma]$, or $A[\gamma]$, or $A(\gamma)$ is used to denote the fact that the variable p is evaluated on $\mathfrak{A}$ by the element γ of E.

The set of variables occurring in the term F is denoted by $Var(F)$. The set of terms of $\mathfrak{A}$ is denoted by $Term(\mathfrak{A})$ or shortly by $Term$ (if there is no danger for confusion). The equality $\mathfrak{A} \models A \approx B$ of 2 terms A and B on $\mathfrak{A}$ is defined as usual [13], i.e., for any evaluation of variables with elements from E, the values of the terms A and B coincide.

Definition 1. *(compare with [8]) The term $F \in Term(\mathfrak{A})$ is said to be* expressible via the system of terms Σ on $\mathfrak{A}$ *if it is equivalent to a term G of the algebra $(E; \Sigma)$ on $\mathfrak{A}$.*

Consider first-order formulas over $Terms$ on $\mathfrak{A}$ as usual, based on first-order connectives $\approx, \vee, \wedge, \rightarrow$, and $\neg$ (*equality, conjunction, disjunction, implication*, and *negation*) and quantifiers $\forall$ and $\exists$, respectively. Let Ψ be a first-order formula. The usual fact that Ψ is valid on $\mathfrak{A}$ will be denoted by $\mathfrak{A} \models \Psi$ or simply by $\models \Psi$.

Definition 2. *A term A is said to be* existentially expressible via the system of terms Σ on $\mathfrak{A}$ *(see [8, p. 30] and take into consideration [8, p. 25]), if there exist: a) integer positive numbers l, m, and k; b) $\pi, \pi_1, \ldots, \pi_l \in Var \setminus Var(A)$; c) $B_{ij}, C_{ij}, D_t \in Term$ (i=1,…,m; j=1,…,k; t=1,…,l) such that i) B_{ij}, C_{ij} are expressible via Σ on $\mathfrak{A}$; ii)*

$\pi, \pi_1, \ldots, \pi_l \notin Var(D_i)$ *($i = 1, \ldots, l$); and iii)*

$$\vDash (A \approx \pi) \to (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij} \approx C_{ij}))[\pi_1/D_1]\ldots[\pi_l/D_l], \tag{1}$$

$$\vDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij} \approx C_{ij})) \to (A \approx \pi). \tag{2}$$

Example 1. *(compare with [8, p. 30]) Let us consider the Boolean algebra* $< \{0,1\}; \&, \vee, \neg, 0, 1 >$*, where* $\&, \vee, \neg$ *are defined as usual. Then boolean functions* $p\&q$ *and* $\neg p$ *are existentially expressible via the constants* 0 *and* 1.

We have

$$\begin{aligned} \vDash ((p\&q) \approx r) \approx (((p \approx 0) \wedge (q \approx 0) \wedge (r \approx 0)) \\ \vee((p \approx 0) \wedge (q \approx 1) \wedge (r \approx 0)) \\ \vee((p \approx 1) \wedge (q \approx 0) \wedge (r \approx 0)) \\ \vee((p \approx 1) \wedge (q \approx 1) \wedge (r \approx 1))). \end{aligned} \tag{3}$$

According to [8, p. 30], we also have:

$$\begin{aligned} \vDash ((\neg p) \approx q) \approx (((p \approx 0) \wedge (q \approx 1)) \\ \vee((p \approx 1) \wedge (q \approx 0))). \end{aligned} \tag{4}$$

The closure of the system Σ of terms relative to (existential) expressibility is defined as usual. In the present paper, only terms on $\mathfrak{A}$ are considered.

Definition 3. *A term* $A(p_1, \ldots, p_n)$ *is said to* **conserve on** $\mathfrak{A}$ **the relation** R *(compare with [9]) if, for any elements* $\alpha_{ij} \in \mathfrak{A}$ *(*$i = 1, \ldots, n;\ j = 1, \ldots, s$*), the facts* $\vDash R(\alpha_{i1}, \ldots \alpha_{is})$ *imply* $\vDash R(F[\alpha_{11}, \ldots, \alpha_{1n}], \ldots, F[\alpha_{s1}, \ldots, \alpha_{sn}]))$*. Also, the system of terms* Σ *is said* **to conserve the relation** R **on** $\mathfrak{A}$ *if any term of* Σ *conserves* R *on* $\mathfrak{A}$.

3 Preliminary results

The next theorem provides sufficient conditions for a system of terms Σ to be closed relative to existential expressibility on $\mathfrak{A}$.

Theorem 1. *Suppose a) $\mathfrak{A}$ is an algebra with an arbitrary finite set of operations; b) $\mathfrak{A}_i$ are subalgebras of $\mathfrak{A}$, $i = 1, \dots, s$; c) Φ be any mapping $\Phi : \mathfrak{A}_i \to \mathfrak{A}$; d) K is a set of terms of $\mathfrak{A}$ that conserve on $\mathfrak{A}$ the relation $R(y, x)$ of the type*

$$y = \Phi(x). \tag{5}$$

Then K is closed with respect to existential expressibility.

Proof. Let us suppose on the contrary that there exists a term $A \notin K$ and A is existentially expressible via terms of K. Let $a_1, \dots, a_n \in Var(A)$.

Then, according to Definition 2, there exist a) terms B_{11}, C_{11}, $\dots$, B_{mk}, C_{mk}, and $D_1, \dots, D_l$; b) variables $a, d_1, \dots, d_l$ such that: i) B_{11}, C_{11}, $\dots$, B_{mk}, C_{mk} are expressible via K on $\mathfrak{A}$; ii) $a, d_1, \dots, d_l \notin Var(D_i)$ $(i = 1, \dots, l)$; and iii)

$$\vDash (A \approx a) \to (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij} \approx C_{ij}))[d_1/D_1] \dots [d_l/D_l], \tag{6}$$

$$\vDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij} \approx C_{ij})) \to (A \approx a). \tag{7}$$

We can consider in the following that $a, a_1, \dots, a_n, d_1, \dots, d_l \in \cup_{i=1}^{m} \cup_{j=1}^{k} \{Var(B_{ij}) \cup Var(C_{ij})\}$. So, since $B_{ij}, C_{ij} \in K$ $(i = 1, \dots, m$, $j = 1, \dots, k)$, i.e., they conserve relation (5), we have

$$\left.\begin{aligned} &\vDash \Phi(B_{ij}[\alpha_{11}, \dots, \alpha_{n1}, \alpha_1, \delta_{11}, \dots, \delta_{l1}]) = \\ &\qquad B_{ij}[\Phi(\alpha_{11}), \dots, \Phi(\alpha_{n1}), \Phi(\alpha_1), \Phi(\delta_{11}), \dots, \Phi(\delta_{l1})], \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} &\vDash \Phi(C_{ij}[\alpha_{11}, \dots, \alpha_{n1}, \alpha_1, \delta_{11}, \dots, \delta_{l1}]) = \\ &\qquad C_{ij}[\Phi(\alpha_{11}), \dots, \Phi(\alpha_{n1}), \Phi(\alpha_1), \Phi(\delta_{11}), \dots, \Phi(\delta_{l1})], \end{aligned}\right\} \tag{9}$$

where $\alpha_{u1}, \alpha_1, \delta_{w1} \in \mathfrak{A}_i$, $u = 1, \dots, n$; $w = 1, \dots, l$.

It follows from $A(a_1, \dots, a_n) \notin K$ that A does not conserve relation (5). This means that there exist elements β_{u1}, $u = 1, \dots, n$ such that

$$\Phi(A[\beta_{11}, \dots, \beta_{n1}],) \neq A[\Phi(\beta_{11}), \dots, \Phi(\beta_{n1})]. \tag{10}$$

Let us denote

$$A[\beta_{11}, \dots, \beta_{n1}] = \beta_1 \tag{11}$$

So, we have:

$$\vDash A[\beta_{11}, \ldots, \beta_{n1}] \approx \beta_1. \tag{12}$$

Substituting (11) in (10), we get:

$$\Phi(\beta_1) \neq A[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1})]. \tag{13}$$

Let us expand the relation (6):

$$\begin{aligned} &\vDash (A(a_1, \ldots, a_n) \approx a) \rightarrow \\ &\qquad (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}(a_1, \ldots, a_n, a, d_1, \ldots, d_l) \approx \\ &\qquad\qquad C_{ij}(a_1, \ldots, a_n, a, d_1, \ldots, d_l)))[d_1/D_1] \ldots [d_l/D_l]. \end{aligned} \tag{14}$$

The last relation takes place for any elements of $\mathfrak{A}$. In particular, we have

$$\left.\begin{aligned} &\vDash (A[\beta_{11}, \ldots, \beta_{n1}] \approx \beta_1) \rightarrow \\ &\qquad (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, d_1, \ldots, d_l] \approx \\ &\qquad\qquad C_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, d_1, \ldots, d_l])) \\ &\qquad\qquad [d_1/D_1[\beta_{11}, \ldots, \beta_{n1}]] \ldots [d_l/D_l[\beta_{11}, \ldots, \beta_{n1}]]. \end{aligned}\right\} \tag{15}$$

Since relations (12) are true, we have from (15) the following:

$$\left.\begin{aligned} &\vDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, d_1, \ldots, d_l] \approx \\ &\qquad C_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, d_1, \ldots, d_l])) \\ &\qquad [d_1/D_1[\beta_{11}, \ldots, \beta_{n1}]] \ldots [d_l/D_l[\beta_{11}, \ldots, \beta_{n1}]]. \end{aligned}\right\} \tag{16}$$

Let us denote the elements $D_w[\beta_{11}, \ldots, \beta_{n1}]$ by τ_{w1} for any $w = 1, \ldots, l$. Then from (16) we have:

$$\left.\begin{aligned} &\vDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}] \approx \\ &\qquad C_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}])). \end{aligned}\right\} \tag{17}$$

Let us look now at relation (7). For any elements $\gamma, \gamma_1, \ldots, \gamma_n \in \mathfrak{A}$, we get:

$$\left.\begin{aligned} &\vDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}[\gamma_1, \ldots, \gamma_n, \gamma, d_1, \ldots, d_l] \approx \\ &\qquad C_{ij}[\gamma_1, \ldots, \gamma_n, \gamma, d_1, \ldots, d_l])) \rightarrow (A[\gamma_1, \ldots, \gamma_n] \approx \gamma). \end{aligned}\right\} \tag{18}$$

So, (18) is also true for particular elements $\gamma, \gamma_1, \ldots, \gamma_n \in \mathfrak{A}$, where $\gamma = \Phi(\beta_1)$, $\gamma_1 = \Phi(\beta_{11}), \ldots, \gamma_n = \Phi(\beta_{n1})$, and we get:

$$\left.\begin{array}{r} \vDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), d_1, \ldots, d_l] \approx \\ C_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), d_1, \ldots, d_l])) \to \\ (A[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1})] \approx \Phi(\beta_1)). \end{array}\right\} \quad (19)$$

According to (13), we have:

$$\nvDash A[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1})] \approx \Phi(\beta_1). \quad (20)$$

Then it follows that relation (19) holds if the next one is true:

$$\left.\begin{array}{r} \nvDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), d_1, \ldots, d_l] \approx \\ C_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), d_1, \ldots, d_l])). \end{array}\right\} \quad (21)$$

Observe that the last relation (21) takes place for any variables $d_1, \ldots, d_l$. So, for any elements $\delta_1, \ldots, \delta_l \in \mathfrak{A}$, we have:

$$\left.\begin{array}{r} \nvDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (B_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), \delta_1, \ldots, \delta_l] \approx \\ C_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), \delta_1, \ldots, \delta_l])). \end{array}\right\} \quad (22)$$

Let us consider the following elements of algebra $\mathfrak{A}$:

$$\delta_w = \Phi(\tau_{w1}), \quad (23)$$

where $\tau_{w1} = D_w[\beta_{11}, \ldots, \beta_{n1}]$, $w = 1, \ldots, l$. Now, substituting (23) into (22), we also get:

$$\left.\begin{array}{l} \nvDash (\vee_{j=1}^{k} \wedge_{i=1}^{m} (\\ \qquad B_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), \Phi(\tau_{11}), \ldots, \Phi(\tau_{l1})] \approx \\ \qquad C_{ij}[\Phi(\beta_{11}), \ldots, \Phi(\beta_{n1}), \Phi(\beta_1), \Phi(\tau_{11}), \ldots, \Phi(\tau_{l1})])). \end{array}\right\} \quad (24)$$

From this last relation (24), since $B_{ij}, C_{ij} \in K$, $i = 1, \ldots, m$, $j = 1, \ldots, k$ and according to relations (8) and (9), we have that:

$$\left.\begin{array}{r} \nvDash \vee_{j=1}^{k} \wedge_{i=1}^{m} (\Phi(B_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}]) \approx \\ \Phi(C_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}])). \end{array}\right\} \quad (25)$$

This means that for any $j = 1, \ldots, k$, we have:

$$\left.\begin{array}{r}\nvDash \wedge_{i=1}^{m}(\Phi(B_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}]) \approx \\ \Phi(C_{ij}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}])).\end{array}\right\} \tag{26}$$

Further it follows that there exists i_j, $i_j \in \{1, \ldots, m\}$, such that

$$\left.\begin{array}{r}\nvDash (\Phi(B_{i_j j}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}]) \approx \\ \Phi(C_{i_j j}[\beta_{11}, \ldots, \beta_{n1}, \beta_1, \tau_{11}, \ldots, \tau_{l1}])).\end{array}\right\}$$

From the last relation it follows that there exist r_j, $r_j \in \{1, \ldots, s\}$, such that:

$$\begin{array}{r}\nvDash B_{i_j j}[\beta_{1r_j}, \ldots, \beta_{nr_j}, \beta_{r_j}, \tau_{1r_j}, \ldots, \tau_{lr_j}] \approx \\ C_{i_j j}[\beta_{1r_j}, \ldots, \beta_{nr_j}, \beta_{r_j}, \tau_{1r_j}, \ldots, \tau_{lr_j}].\end{array} \tag{27}$$

The last relation (27) implies also the following:

$$\begin{array}{r}\nvDash \wedge_{i=1}^{m} B_{ij}[\beta_{1r_j}, \ldots, \beta_{nr_j}, \beta_{r_j}, \tau_{1r_j}, \ldots, \tau_{lr_j}] \approx \\ C_{ij}[\beta_{1r_j}, \ldots, \beta_{nr_j}, \beta_{r_j}, \tau_{1r_j}, \ldots, \tau_{lr_j}].\end{array} \tag{28}$$

Let us remark that the relations (27) and (28) hold for any $j = 1, \ldots, k$. Therefore, the following relation is true:

$$\begin{array}{r}\nvDash \vee_{j=1}^{k} \wedge_{i=1}^{m} B_{ij}[\beta_{1r_j}, \ldots, \beta_{nr_j}, \beta_{r_j}, \tau_{1r_j}, \ldots, \tau_{lr_j}] \approx \\ C_{ij}[\beta_{1r_j}, \ldots, \beta_{nr_j}, \beta_{r_j}, \tau_{1r_j}, \ldots, \tau_{lr_j}].\end{array} \tag{29}$$

Comparing relations (29) and (17), we conclude that we get a contradiction.

The theorem is proved.

Theorem 2. *Suppose $\mathfrak{A}$ is an algebra and $b \in \mathfrak{A}$. Then the set K of terms of $\mathfrak{A}$ that conserve on $\mathfrak{A}$ the relation $x = b$ is closed relative to existential expressibility on $\mathfrak{A}$.*

The proof of this theorem is almost obvious if we consider in Theorem 1 the mapping $\Phi(x) = b$.

4 Main results

Consider logics $L\mathfrak{B}_i$, $i = 1, 2, 3$ of the corresponding algebras $\mathfrak{B}_i$, known also as extensions of the logic $S5$.

Consider classes of formulas Π_0, Π_1, Π_2, that conserve on algebra $\mathfrak{B}_1$ the relations $x = 0$, $x = 1$, and $\neg x = y$.

Theorem 3. *The classes of formulas* Π_0,Π_1, Π_2 *of the logic* $L\mathfrak{B}_1$ *are pre-complete relative to existential expressibility in* $L\mathfrak{B}_1$*.*

According to Theorem 1, these classes are closed as to existential expressibility. By E. Post's results [1],[2], these classes are pre-complete as to expressibility. So, they are also pre-complete as to existential expressibility, too.

Consider elements $\{(0,0),(0,1),(1,0),(1,1)\}$ of $\mathfrak{B}_2$ denoted by $0, \rho, \sigma, 1$, respectively. Consider mapping $f_{10} : \mathfrak{B}_2 \to \mathfrak{B}_2$ defined by relations: $f_{10}(x) = 0$, if $x \in \{0, \rho\}$ and $f_{10}(x) = 1$, if $x \in \{\sigma, 1\}$.

Consider classes of formulas Π_8, Π_9, Π_{10} that conserve on the algebra $\mathfrak{B}_2$ the relations $\Box x = y$, $\Diamond x = y$, $f_{10}(x) = y$. Similar to the previous theorem, we have:

Theorem 4. *The classes of formulas* Π_8, Π_9, Π_{10} *of the logic* $L\mathfrak{B}_2$ *are pre-complete relative to existential expressibility in* $L\mathfrak{B}_2$*.*

The proof is similar to the proof of the previous theorem.

Consider algebra $\mathfrak{B}_3$. Denote its elements $\{(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)\}$ by $\{0, \rho, \mu, \varepsilon, \sigma, \nu, \omega, 1\}$.

Consider mappings $f_2, f_3, f_4 : \mathfrak{B}_3 \to \mathfrak{B}_3$ defined in tabular form as in Table 1 below (see [10, p. 168]).

p	0	ρ	μ	ε	ω	ν	σ	1
f_2	0	σ	σ	σ	ρ	ρ	ρ	1
f_3	0	ρ	ν	ω	ε	μ	σ	1
f_4	0	σ	ω	ν	μ	ε	ρ	1

Table 1. Functions on $\mathfrak{B}_3$ [10, p. 168]

Consider [10] classes of formulas Π_{21}, Π_{22}, Π_{23} that conserve on the algebra $\mathfrak{B}_3$ the relations $f_2(x) = y$, $f_3(x) = y$, $f_4(x) = y$.

Theorem 5. *The classes of formulas* Π_{21}*,* Π_{22}*,* Π_{23} *of the logic* $L\mathfrak{B}_3$ *are pre-complete relative to existential expressibility in* $L\mathfrak{B}_3$*.*

Remark 1. *Only the class formulas* Π_2 *from the above-mentioned classes contain the para-consistent negation. So, if a system* Σ *of formulas containing para-consistent negation is complete as to existential expressibility in* $S5$*, it should satisfy the relation:* $\Sigma \not\subset \Pi_2$*.*

Now we can give necessary and sufficient conditions for a system of boolean functions to be complete relative to existential expressibility mentioned in [8].

Theorem 6. *Consider the Boolean algebra* $\mathfrak{B}_1 = (\{0,1\}; \&, \vee, \neg, 0, 1)$*. The system* Σ *of boolean functions is complete relative existential expressibility if and only if it conserves none of the relations on* $\mathfrak{B}_1$*:* $x = 0$*,* $x = 1$*, and* $x = \neg y$*.*

Proof. Each class of functions that conserve the corresponding relation is closed (according to Theorem 1) as to existential expressibility. It is also known that these classes are distinct [1], and according to [3], the constants 0 and 1 are expressible via Σ. By force of the Example 1, we conclude the system Σ is complete as to existential expressibility.

5 Conclusions

Conditions for a system of formulas containing para-consistent negations to be existential expressible in the logic $S5$ are only necessary conditions.

The discovery of all necessary and sufficient conditions for a system of formulas of $S5$ to be complete as to existential expressibility may follow the following procedure:

- Consider possible classes of formulas as possible candidates that comply with the conditions stated in Theorem 1.

- Apply the principle from simple to complex, i.e., start with the corresponding 2-, 4-, 8-valued algebras.
- Examine initially classes defined by 1-valued functions Φ.
- As the dimension of the algebraic model of the logic under consideration may increase (2-valued, 4-valued, 8-valued, 16-valued, 32-valued), it is useful to filter the possible Φ functions mentioned above. A relatively simple way is to consider different Φ functions and examine the relation of the formulas of the logic on the corresponding algebraic model relative to classes defined by those Φ functions. This will allow establishing a possible inclusion of the classes defined by Φ functions in each other. So, a software, for example, something similar to `http://tinyurl.com/4ut3f7em`, after some adaptation, may help in filtering unnecessary classes of formulas. The above theorems are useful to assure the closure of the corresponding classes relative to existential expressibility.

This paper is the extended and revised version of the conference paper [14] presented at WIIS 2023.

Acknowledgments. National Agency for Research and Development has supported part of the research for this paper through the research project 20.80009.5007.22 ”Intelligent information systems for solving ill-structured problems, processing knowledge and big data”.

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Received September 30, 2023
Accepted December 13, 2023

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