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# Stability of Finite Difference Schemes to Pseudo-Hyperbolic Telegraph Equation

**Mahmut Modanlı[1] and Fatih Özbağ[1*]**

[1]*Department of Mathematics, Faculty of Science and Arts, Harran University, Şanlıurfa, Turkey*
[*]*Corresponding author*

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### Abstract

Hyperbolic partial differential equations are frequently referenced in modeling real-world problems in mathematics and engineering. Therefore, in this study an initial-boundary value issue is proposed for the pseudo-hyperbolic telegraph equation. By operator method, converting the PDE to an ODE provides an exact answer to this problem. After that, finite difference method is applied to construct first order finite difference schemes to calculate approximate numerical solutions. The stability estimations of finite difference schemes are shown, as well as some numerical tests to check the correctness in comparison to the precise solution. The numerical solution is subjected to error analysis. As a result of the error analysis, the maximum norm errors tend to decrease as we increase the grid points. It can be drawn that the established scheme is accurate and effective.

## 1. Introduction

Mathematical models in physics, chemistry, engineering, biology and economics are often expressed using partial and ordinary differential equations [1]–[4]. Analytical and numerical solutions of partial differential equations play a significant role in understanding the underlying phenomena [5, 6]. Partial differential equations are classified into three types and in this study, hyperbolic type partial differential equations are considered.

Pseudo-hyperbolic equations are hyperbolic partial differential equations which contain mixed time and space partial derivatives. For these types of equations, there are several technics to calculate exact and approximate numerical solutions. In [7], the conformable double Laplace transform decomposition approach was used to solve linear and nonlinear singular pseudo-hyperbolic equations. In [8], by the regularization method and the method of continuation parameter the regular solvability of the boundary value problem for pseudo-hyperbolic equations with variable time direction were proved. In [9, 10], numerical schemes based on $H^1$-Galerkin mixed finite element method were constructed for pseudo-hyperbolic equations. Some other works on pseudo-hyperbolic equations are [11]–[15].

There has been a wide range of research on finite difference schemes for approximate solutions to telegraph equations and there are considerable approximation for stability of these difference schemes. The stability estimates of these difference schemes are constructed applying the operator splitting approach and some energy inequalities using certain assumptions on the grid step sizes $\tau$ and $h$ [16].

In the present work we consider pseudo-hyperbolic telegraph equation. Telegraph equation is mostly used for modeling of wave propagation of electric signals in a cable transmission line. To get accurate and approximate numerical solutions of hyperbolic telegraph equations, a variety of numerical and analytical approaches are applied. In [17], exact solution of the telegraph equation was solved by $(G'/G)$-expansion method. In [18], a numerical technique was developed for the one-dimensional telegraph equation with purely integral conditions. Daftardar-Gejji-Jafaris (DGJ) method was used to obtain approximate solution of the hyperbolic telegraph equation [19]. In [20], Differential Transformation Method (DTM) has been utilized to obtain the exact solutions of the one-space-dimensional hyperbolic telegraph equation. In [21], a new numerical scheme was constructed to solve the second-order hyperbolic telegraph equation using the collocation method. Laplace transform homotopy perturbation method was used in [22] to solve the telegraph equation with the initial and boundary conditions. An operator method was studied in [23] for the difference equations and partial differential telegraph equation. In [24], a numerical method for the first and second-order of accuracy for telegraph equations was discussed and the stability of difference schemes were obtained. Numerical solutions were computed for the telegraph equations arising in transmission lines [25].

***Email addresses and ORCID numbers***: **mmodanli@harran.edu.tr, 0000-0002-7743-3512 (M. Modanlı), fozbag@harran.edu.tr, 0000-0002-5456-4261 (F. Özbağ)**

In this work, numerical solution of the third-order pseudo-hyperbolic telegraph equation is investigated using the first-order finite difference technique. The numerical solution is subjected to an error analysis. The stability inequalities of finite difference schemes are presented, as well as some numerical experiments to verify the correctness in terms of precise solution.

## 2. Introducing Problem

We define the pseudo-hyperbolic telegraph equation below

$$\begin{cases} v_{\eta\eta}(\eta,\theta)+\lambda v_\eta(\eta,\theta)+\mu v(\eta,\theta)=v_{\eta\theta\theta}(\eta,\theta)+v_{\theta\theta}(\eta,\theta)+g(\eta,\theta), \\ v(0,\theta)=\varphi_1(\theta), \quad v_\eta(0,\theta)=\varphi_2(\theta), \quad 0<\theta<L, \\ v(\eta,0)=v(\eta,L)=0, \quad 0<\eta<T, \\ 0<\lambda, \quad 0<\mu. \end{cases} \tag{2.1}$$

We can rewrite equation (2.1) as

$$\begin{cases} v_{\eta\eta}(\eta)+\lambda v_\eta(\eta)+Av_\eta(\eta)+Av(\eta)+\mu v(\eta)=g(\eta) \quad 0\le\eta\le T, \\ v(0)=\varphi_1, \quad v'(0)=\varphi_2, \end{cases} \tag{2.2}$$

where $A\ge\delta I$ and A is positive definite self-adjoint operator. For positive $\delta$ and $\lambda$, the following restriction is requried

$$\delta+\mu\ge\frac{(\lambda+\delta)^2}{4}.$$

One can obtain equation (2.2) from (2.1) under the following conditions. In a Hilbert space $\mathscr{L}_2[0,L]$ define

$$Av(\theta)=-v_{\theta\theta}+\delta v(\theta), \tag{2.3}$$

then

$$(Av)_\eta=-v_{\theta\theta\eta}=-v_{\eta\theta\theta}$$

with the domain

$$D(A)=\{v(\theta):v,v_\theta,v_{\theta\theta}\in\mathscr{L}_2, v(0)=v(L), v'(0)=v'(L)\}.$$

Here we set $g(\eta)=g(\eta,\theta)$ and $v(\eta)=v(\eta,\theta)$ which are given and will be determined abstract functions in $\mathscr{L}_2[0,L]$. $v(\eta)$ is a solution of (2.2) if it is three times continuously differentiable on $[0,T]$, $v(\eta)\in D(A)$ and $Av(\eta)$ is continuous on $[0,T]$. Also $v(\eta)$ must satisfy equation (2.2) and the initial conditions. If the operator A satisfies the properties given above, then the partial differential equation (2.1) turns into the ordinary differential equation (2.2). Thus the method used is know as the operator method [23]–[26].
Next we introduce the Hilbert space $\mathscr{L}_2(\overline{\Omega})$, where $\overline{\Omega}=\Omega\cup S$ and $\Omega\subset R^n$ is a bounded open domain with smooth boundary $S$, with the norm

$$\|g\|_{\mathscr{L}_2(\overline{\Omega})}=\left\{\int\cdots\int_{\theta\in\overline{\Omega}}|g(\theta)|^2dx_1...dx_n\right\}^{\frac{1}{2}}.$$

The problem (2.2) can easily be converted to a system of first order differential equation with initial conditions. Therefore we obtain

$$\begin{cases} v'(\eta)+\dfrac{\lambda+A}{2}v(\eta)+iK^{\frac{1}{2}}v(\eta)=z(\eta), \quad 0\le\eta\le T, \\ v(0)=\varphi_1, \quad v'(0)=\varphi_2, \\ \\ z'(\eta)+\frac{\lambda+A}{2}z(\eta)-iK^{\frac{1}{2}}z(\eta)=g(\eta), \\ z(0)=v'(0)+\left(\frac{\lambda+A}{2}+iK^{\frac{1}{2}}\right)v(0). \end{cases} \tag{2.4}$$

where $K=A+\mu-\frac{(\lambda+A)^2}{4}$. If we integrate (2.4), we obtain

$$v(\eta)=e^{-\left(\frac{\lambda+A}{2}+iK^{\frac{1}{2}}\right)\eta}v(0)+\int_0^\eta e^{-\left(\frac{\lambda+A}{2}+iK^{\frac{1}{2}}\right)(\eta-z)}z(s)ds, \tag{2.5}$$

$$z(\eta)=e^{-\left(\frac{\lambda+A}{2}-iK^{\frac{1}{2}}\right)\eta}z(0)+\int_0^\eta e^{-\left(\frac{\lambda+A}{2}-iK^{\frac{1}{2}}\right)\eta}g(s)ds. \tag{2.6}$$

Applying the method in [24] with the equations (2.5) and (2.6), we have the solution of the problem (2.2)

$$v(\eta)=e^{-\left(\frac{\lambda+A}{2}\right)\eta}c(\eta)\varphi_1+\frac{\lambda+A}{2}e^{-\left(\frac{\lambda+A}{2}\right)\eta}s(\eta)\varphi_1+e^{-\left(\frac{\lambda+A}{2}\right)\eta}s(\eta)\varphi_2+\int_0^t e^{-\left(\frac{\lambda+A}{2}+iK^{\frac{1}{2}}\right)(\eta-z)}\delta(\eta-z)g(z)dz, \tag{2.7}$$

where $c(\eta)=\frac{e^{i\eta K^{1/2}}+e^{-i\eta K^{1/2}}}{2}$ and $s(\eta)=K^{-1/2}\frac{e^{i\eta K^{1/2}}-e^{-i\eta K^{1/2}}}{2i}$.

**Lemma 2.1.** *Following estimates hold*

- $\|e^{-\left(\frac{\lambda+A}{2}\right)\eta}\|_{\mathscr{L}_2}\le 1$.
- $\|c(\eta)\|_{\mathscr{L}_2\to\mathscr{L}_2}\le 1$.
- $\|K^{1/2}s(\eta)\|_{\mathscr{L}_2\to\mathscr{L}_2}\le 1$.

- $\|A^{1/2}K^{-1/2}\|_{\mathscr{L}_2\to\mathscr{L}_2} \leq M(\delta)$.
- $\|K^{-1/2}\varphi_1\|_{\mathscr{L}_2} \leq \frac{1}{\sqrt{\delta}}\|\varphi_1\|_{\mathscr{L}_2}$.

For easy proof, see [27, 28]. Now we give the main theorem and prove it.

**Theorem 2.2.** *Let $\varphi_1 \in D(A)$, $\varphi_2 \in D(A^{1/2})$, $g(\eta)$ be a continuous differentiable function on [0,T] and $\delta+\mu \geq \frac{(\lambda+\delta)^2}{4}$. Then there exist a unique solution of (2.2) and its stability estimate is*

$$\max_{0\leq\eta\leq T}\|v(\eta)\|_{\mathscr{L}_2} \leq M\left[\|\varphi_1\|_{\mathscr{L}_2} + \|A^{-1/2}\varphi_2\|_{\mathscr{L}_2} + \max_{0\leq\eta\leq T}\|A^{-1/2}g(\eta)\|_{\mathscr{L}_2}\right]$$

*where $M$ is independent from $\varphi_1, \varphi_2$ and $g(\eta)$.*

*Proof.* By Lemma 2.1 with $A \geq \delta I$ and using the formula (2.7), we have the following inequalities

$$\begin{aligned}
\|v(\eta)\|_{\mathscr{L}_2} \leq\ & \|c(\eta)\|_{\mathscr{L}_2\to\mathscr{L}_2}\|e^{-(\frac{\lambda+A}{2})\eta}\|_{\mathscr{L}_2}\|\varphi_1\|_{\mathscr{L}_2} + \|K^{1/2}s(\eta)\|_{\mathscr{L}_2\to\mathscr{L}_2}\|A^{1/2}K^{-1/2}\|_{\mathscr{L}_2\to\mathscr{L}_2}|\frac{\lambda+A}{2}e^{-(\frac{\lambda+A}{2})\eta}| \\
& \|A^{-1/2}\varphi_1\|_{\mathscr{L}_2} + \|K^{1/2}s(\eta)\|_{\mathscr{L}_2\to\mathscr{L}_2}\|A^{1/2}K^{-1/2}\|_{\mathscr{L}_2\to\mathscr{L}_2}|e^{-(\frac{\lambda+A}{2})\eta}|\|A^{-1/2}\varphi_2\|_{\mathscr{L}_2} \\
& + \int_0^\eta \|K^{1/2}s(\eta-z)\|_{\mathscr{L}_2\to\mathscr{L}_2}\|A^{1/2}K^{-1/2}\|_{\mathscr{L}_2\to\mathscr{L}_2}\|A^{-1/2}g(s)\|_{\mathscr{L}_2}ds \\
\leq\ & M(\delta,\lambda)\left[\|\varphi_1\|\|_{\mathscr{L}_2} + \|A^{-1/2}\varphi_2\|_{\mathscr{L}_2} + \max_{0\leq\eta\leq T}\|A^{-1/2}g(\eta)\|_{\mathscr{L}_2}\right].
\end{aligned}$$

□

## 3. Stability for First Order Difference Scheme

We investigate the approximation in $\eta$ of first order difference scheme

$$\begin{cases}
\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2} + \lambda\frac{v_k-v_{k-1}}{\tau} + A\frac{v_k-v_{k-1}}{\tau} + \mu v_k + Av_k = g_k, \\
g_k = g(\eta_k), \quad 1\leq k\leq N-1, \quad N\tau = T, \\
v_0 = \varphi_1, \quad \frac{v_1-v_0}{\tau} + (\mu+A)\tau v_0 = (1-(\lambda+A)\tau)\varphi_2
\end{cases} \tag{3.1}$$

for the numerical solution of the initial value problem (2.1). We can write (3.1) into the following difference problem

$$\begin{cases}
(1-(\lambda+A)\tau)v_{k-1} - (2-(\lambda+A)\tau-(\mu+A)\tau^2)v_k + v_{k+1} = \tau^2 g_k, \\
1\leq k\leq N-1, \\
v_0 = \varphi_1, \quad v_1 = (1-(\mu+A)\tau^2)v_0 + (1-(\lambda+A)\tau)\tau\varphi_2.
\end{cases} \tag{3.2}$$

The formula (3.2) can be simplified as

$$av_{k-1} - cv_k + bv_{k+1} = \varphi_k,$$

where $a = 1-(\lambda+A)\tau$, $\quad c = 2-((\lambda+A)\tau+(\mu+A)\tau^2)$, $\quad b = 1$ and $\varphi_k = \tau^2 g_k$.

**Theorem 3.1.** *Let $\varphi_1 \in D(A)$, $\varphi_2 \in D(A^{1/2})$ and $\delta+\mu \geq \frac{(\lambda+\delta)^2}{4}$. The following stability inequality for (3.1)*

$$\max_{0\leq k\leq 1}\|v_k\|_{\mathscr{L}_2} \leq M(\lambda,\mu,\delta)\left[\|\varphi_1\|_{\mathscr{L}_2} + \|A^{-1/2}\varphi_2\|_{\mathscr{L}_2} + \max_{1\leq k\leq N-1}\|A^{-1/2}g_k\|_{\mathscr{L}_2}\right] \tag{3.3}$$

*holds where $M(\lambda,\mu,\delta)$ is independent from $\tau, \varphi_1, \varphi_2$ and $g_k$, $1\leq s\leq N-1$.*

The proof of Theorem 3.1 follows from [26] by using the formula for the solution of the difference scheme (3.2).
Now, we consider applications of Theorem 3.1. We begin by discretizing the problem (2.1). First we define the grid space

$$[0,L]_h = \{\theta = \theta_n : \theta_n = nh, 0\leq n\leq M, Mh = L\}.$$

$\mathscr{L}_{2h}$ with the grid space $[0,L]_h$ is defined as $\mathscr{L}_2([0,L]_h)$. The following $\mathscr{L}_{2h}$ norm

$$\|\varphi^h\|_{\mathscr{L}_{2h}} = \left(\sum_{\theta\in[0,L]_h}|\varphi(\theta)|^2 h\right)^{1/2}$$

is used for the grid functions $\varphi^h(\theta) = \{\varphi_n\}_0^M$. Then we define the difference operator $A_h$ for the differential operator $A$ in equation (2.3)

$$A_h\varphi^h(\theta) = \{-(\varphi_{\theta\theta})_n + \delta\varphi_n\}_1^{M-1}.$$

$A_h$ is a positive definite and self-adjoint operator in $\mathscr{L}_{2h}$ and satifies the conditions $\varphi_0 = \varphi_M$ and $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. Now we can write

$$\begin{cases} v^h_{\eta\eta}(\eta,\theta)+\lambda v^h_{\eta}(\eta,\theta)+A^h v^h_{\eta}(\eta,\theta)+A^h v^h(\eta,\theta)+\mu v(\eta,\theta)=g^h(\eta,\theta), \\ v^h(0,\theta)=\varphi^h_1(\theta), \quad v^h_{\eta}(0,\theta)=\varphi^h_2(\theta), \quad 0<\eta<T, \theta\in[0,L]_h. \end{cases} \tag{3.4}$$

After replacing (3.4) with the difference scheme (3.1), we obtain

$$\begin{cases} \dfrac{v^h_{k+1}(\theta)-2v^h_k(\theta)+v^h_{k-1}(\theta)}{\tau^2}+\lambda\dfrac{v^h_k(\theta)-v^h_{k-1}(\theta)}{\tau}+A^h\dfrac{v^h_k(\theta)-v^h_{k-1}(\theta)}{\tau} \\ +\mu v^h_k(\theta)+A^h v^h_k(\theta)=g^h_k(\theta), \\ g^h_k(\theta)=g^h(\eta_k,\theta), \quad 1\le k\le N-1, \theta\in[0,L]_h, \eta_k=k\tau, N\tau=T, \\ v^h_0(\theta)=\varphi^h_1(\theta), \quad \frac{v^h_1(\theta)-v^h_0(\theta)}{\tau}+(\mu I_h+A^h)\tau v^h_0(\theta)=(I_h-(\lambda I_h+A^h)\tau)\varphi^h_2(\theta). \end{cases} \tag{3.5}$$

**Theorem 3.2.** *The stability estimate of the solution $\{v^h_k(\theta)\}^N_0$ of the discretized problem (3.5)*

$$\max_{1\le k\le N}\|v^h_k\|_{\mathscr{L}_{2h}} \le M\left[\|\varphi^h_1\|_{\mathscr{L}_{2h}}+\|\varphi^h_2\|_{\mathscr{L}_{2h}}+\max_{1\le k\le N-1}\|g^h_k\|_{\mathscr{L}_{2h}}\right]$$

*holds where M is independent from $\varphi^h_1(\theta),\varphi^h_2(\theta)$ and $g^h_k(\theta)$, $1\le k\le N-1$.*

Proof of Theorem 3.2 follows from stability estimate (3.3).

## 4. Simulations

In this chapter, a numerical example is provided to support the theoretical statements. Some numerical results are given as an application of the Theorem 3.1. Consider the following problem

$$\begin{cases} v_{\eta\eta}(\eta,\theta)+v_{\eta}(\eta,\theta)+v(\eta,\theta)=v_{\eta\theta\theta}(\eta,\theta)+v_{\theta\theta}(\eta,\theta)+g(\eta,\theta), \\ g(\eta,\theta)=e^{-\eta}(\theta-\theta^2), 0<\theta<1, 0<\eta<1, \\ v(0,\theta)=\theta-\theta^2, \quad v_{\eta}(0,\theta)=-(\theta-\theta^2), \quad 0\le\theta\le 1, \\ v(\eta,0)=v(\eta,1)=0, \quad 0\le\eta\le 1. \end{cases} \tag{4.1}$$

By using Modified Double Laplace Decomposition method the exact solution of the problem (4.1) is $v(\eta,\theta)=e^{-\eta}(\theta-\theta^2)$. See [29, 30, 31] for similar examples.
The first order difference scheme of the problem (4.1) is as follows

$$\begin{cases} \frac{v^{k+1}_n-2v^k_n+v^{k-1}_n}{\tau^2}+\frac{v^k_n-v^{k-1}_n}{\tau}+v^k_n=\frac{1}{\tau}\left(\frac{v^k_{n-1}-2v^{k+1}_n+v^k_{n+1}}{h^2}-\frac{v^{k-1}_{n-1}-2v^k_n+v^{k-1}_{n+1}}{h^2}\right)+\frac{v^k_{n+1}-2v^k_n+v^k_{n-1}}{h^2}+g^k_n, \\ \theta_n=nh, \quad \eta_k=k\tau, 1\le k\le N-1, \quad 1\le n\le M-1, \\ v^0_n=\theta-\theta^2, \quad \frac{v^1_n-v^0_n}{\tau}=-(\theta-\theta^2), \quad 0\le n\le M, \\ v^k_0=v^k_M=0, \quad 0\le k\le N. \end{cases} \tag{4.2}$$

Next, we consider the following matrix equation

$$\mathscr{A}v_{n+1}+\mathscr{B}v_n+\mathscr{C}v_{n-1}=I\varphi_n$$

where $v_n=[v^1_n,v^2_n,...,v^{N-1}_n]$, $\varphi_n=[\varphi^1_n,\varphi^2_n,...,\varphi^{N-1}_n]^T$. We have $(N+1)\times(N+1)$ system of equation with the coefficient matrices $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$. Following the same approach as in [23], we compute the maximum difference between the approximate and exact solution by

$$\varepsilon=\max_{\substack{1\le k\le N-1\\1\le n\le M-1}}|v(\eta,\theta)-v(\eta_k,\theta_n)|.$$

Calculated results are presented in Table 1 for (4.2).

| $\tau=\frac{1}{N}, h=\frac{1}{M}$ | $\varepsilon$ |
|---|---|
| $N=25, M=5$ | 0.1600 |
| $N=100, M=10$ | 0.0900 |
| $N=225, M=15$ | 0.0622 |
| $N=400, M=20$ | 0.0475 |
| $N=625, M=25$ | 0.0384 |
| $N=900, M=30$ | 0.0322 |
| $N=1600, M=40$ | 0.0244 |
| $N=2500, M=50$ | 0.0196 |

**Table 1:** Error Analysis

From Table 1, one can observe that maximum norm errors tend to decrease as we increase the grid points. This shows the established scheme's precision. Moreover numerical results in Table 1 are calculated by taking $\tau=h^2$. To show the precision of the numerical results, we calculate the error of the difference scheme (4.2) by taking $\tau = h$. For example, for $N = M = 20$ maximum norm error is 0.0811 which is greater than for $N = 400$, $M = 20$. Also this maximum norm error is increasing as N and M are increasing.

Figures 4.1 and 4.2 show how the solutions to the example (4.1) look very similar. In addition, in Figure 4.3, 2d-line plot is given to see how the solutions fit together.

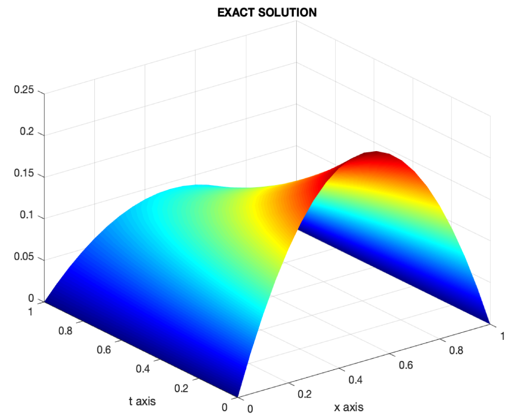


**Figure 4.1:** For $N = 400$ and $M = 20$, graph of the exact solution of the problem (4.1).

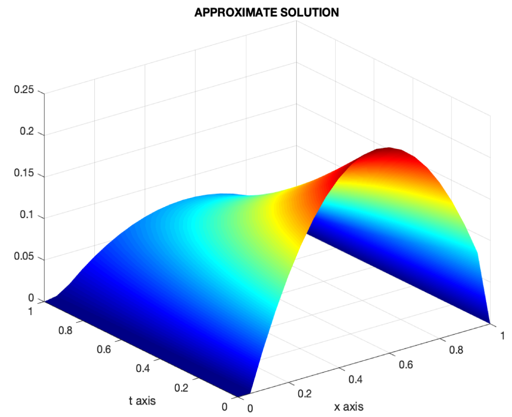


**Figure 4.2:** For $N = 400$ and $M = 20$, the graph with maximum error 0.0475.

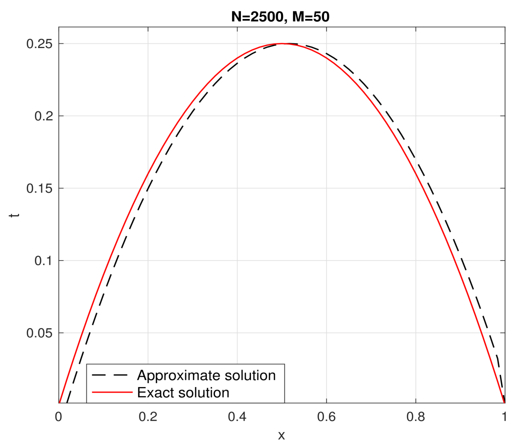


**Figure 4.3:** For $N = 2500$ and $M = 50$, comparison of the approximate and exact solutions.

## 5. Conclusion

In this paper, the pseudo-hyperbolic telegraph equation was addressed and its stability estimates were calculated. In the literature, finite difference technic has not been applied for the numerical solution of this equation. Although the Modified Double Laplace Decomposition method gave the exact solution of this problem, we also constructed the first order finite difference scheme. Besides, stability estimates of the scheme were obtained. Then, we performed the difference scheme technique on the considered numerical example to confirm the correctness. Error calculations showed that the established scheme has good results and is effective for this equation. Also, some simulations were plotted to see it clearly and assist with the results. MATLAB programming was used to calculate the numerical solutions for the test example.

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## Competing interests

The authors declare that they have no competing interests.

## Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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