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# Generalized Sum-Asymmetry Model and Orthogonality of Test Statistic for Square Contingency Tables

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**Abstract**

For analyzing contingency tables, we are usually interested in whether or not the independence model holds. On the other hand, for the analysis of square contingency tables, we are usually interested in whether or not the model having the structure of symmetry or asymmetry with respect to the main diagonals cells holds. This study proposes a generalized sum-asymmetry model including the exponential and relative exponential sum-symmetry models. This generalized model indicates that the cumulative probability that the sum of classes for row and column variables is $s$ within the upper right cell of the table, is exponentially higher than the cumulative probability that the sum of classes for row and column variables is $s$ within the lower left cell. Additionally, this study gives a separation of the sum-symmetry model using the proposed model, and reveals that the new separation satisfies the asymptotic equivalence for the test statistic. The utilities of the proposed methods are demonstrated through the real data analysis.

*Keywords*: asymmetry, asymptotic equivalence, exponential sum-symmetry, separation, sum-symmetry.

## 1. Introduction

For analyzing of contingency tables, we are usually interested in whether or not the independence model holds. On the other hand, for the analysis of square contingency tables having same row and column classifications, we are usually interested in whether or not the model having the structure of symmetry or asymmetry with regard to the main diagonals cells holds, instead of the independence. This is because, there is a strong association between row and column variables in square contingency tables.

Let $\pi_{ij}$ denote the cell probability that an observation will fall in the cell of the row class $i$ and column class $j$ for the square contingency table ($i = 1, \ldots, C; j = 1, \ldots, C$). The symmetry (S) model (Bowker 1948) has the structure of symmetry between the $\pi_{ij}$ and $\pi_{ji}$ with regard to the main-diagonals cell of the square contingency table. The S model is defined by

$$\pi_{ij} = \pi_{ji} \quad \text{for } i < j.$$

The linear diagonals-parameter symmetry (LDPS) model (Agresti 1983) has the structure of asymmetry between the $\pi_{ij}$ and $\pi_{ji}$. The LDPS model is defined by

$$\pi_{ij} = \theta^{j-i}\pi_{ji} \quad \text{for } i < j.$$

When the $\theta = 1$, the LDPS model is identical to the S model. Under the LDPS model, the $\pi_{ij}/\pi_{ji}$, for all $i < j$, exponentially varies relying on the absolute gap $j - i$ (i.e., the difference between the column and row classes). Additionally, the $\pi_{ij}/\pi_{ji}$, for all $i < j$, exponentially varies relying on the relative gap between the difference $j - i$ and its minimum 1.

As a generalization of the LDPS model, the generalized exponential symmetry (GES) model (Kurakami, Yamamoto, and Tomizawa 2011) is defined by

$$\pi_{ij} = \theta^{w_{ij}}\pi_{ji} \quad \text{for } i < j,$$

where $\{w_{ij}\}$ are the specified positive values. When $\{w_{ij} = 1\}$, $\{w_{ij} = j - i\}$, $\{w_{ij} = R-(j-i)\}$, and $\{w_{ij} = j-1\}$, the GES models are identical to the conditional symmetry (CS) model (McCullagh 1978; Read 1977), the LDPS model, the another LDPS (ALDPS) model (Tomizawa 1990), and the linear columns-parameter symmetry (LCPS) model (Tomizawa, Miyamoto, and Iwamoto 2006), respectively. Moreover, Kurakami *et al.* (2011) gave the separation of the S model using the GES model, and revealed that this separation satisfies the asymptotic equivalence for the test statistic, that is, the likelihood ratio (LR) statistic for testing goodness-of-fit of the S model is asymptotic equivalent to the sum of LR statistics of the separated models. The separation of the model $\mathrm{M}_1$ implies that the model $\mathrm{M}_1$ holds if and only if both the models $\mathrm{M}_2$ and $\mathrm{M}_3$ hold. We refer to a separation that satisfies the asymptotic equivalence for the test statistic as "orthogonal separation".

Let

$$U_s = \sum_{(i,j)\in u_s}\sum \pi_{ij} \quad \text{and} \quad L_s = \sum_{(i,j)\in l_s}\sum \pi_{ij} \quad \text{for } s = 3, 4, \ldots, 2C-1,$$

where

$$u_s = \{(i,j) \mid i+j = s, i < j\} \quad \text{and} \quad l_s = \{(i,j) \mid i+j = s, i > j\}.$$

The $U_s$ is the cumulative probability that the sum of classes for row and column variables is $s$ within the upper right cell of the table. Additionally, the $L_s$ is the cumulative probability that the sum of calsses for row and column variables is $s$ within the lower left cell. Such as vision data, when we want to evaluate an individual's acuity of vision, we may be interested in whether or not the model having the structure of symmetry or asymmetry between the cumulative probabilities $U_s$ and $L_s$ holds.

The sum-symmetry (SS) model (Yamamoto, Tanaka, and Tomizawa 2013) indicates the structure of symmetry between the cumulative probabilities $U_s$ and $L_s$ for $s = 3, 4, \ldots, 2C-1$. The SS model is defined by

$$U_s = L_s \quad \text{for } s = 3, 4, \ldots, 2C-1.$$

The exponential sum-symmetry (ESS) model (Yamamoto, Aizawa, and Tomizawa 2016; Ando 2021a) and relative ESS (RESS) model (Ando 2022) indicate the structure of asymmetry between the cumulative probabilities $U_s$ and $L_s$ for $s = 3, 4, \ldots, 2C-1$. The ESS model is defined by

$$U_s = \Theta^{s-2}L_s \quad \text{for } s = 3, 4, \ldots, 2C-1.$$

The RESS model is defined by

$$U_s = \Theta^{s/3}L_s \quad \text{for } s = 3, 4, \ldots, 2C-1.$$

When the $\Theta = 1$, the ESS and RESS models are identical to the SS model. Under the ESS model, the $U_s/L_s$ exponentially varies relying on the absolute gap $s - 2$, and the $U_3/L_3$ is $\Theta$. On the other hand, under the RESS model, the $U_s/L_s$ exponentially varies relying on the relative gap between the $s$ and its minimun 3, and the $U_3/L_3$ is $\Theta$.

Ando (2021a) gave the separation of the SS model using the ESS model, and revealed that this separation is the orthogonal separation. Moreover, Ando (2022) gave the separation of the SS model using the RESS model, and revealed that this separation is the orthogonal separation.

In line with the ralation between the LDPS and GES models, this study proposes a generalized model including the ESS and RESS models. Additionally, this study gives a new separation of the SS model using the proposed model, and reveals that the proposed separation is the orthogonal separation. This advance would introduce new models and give orthogonal separations of the SS model using those models.

The remainder of this paper is organized as follows. Section 2 inducts new models and new separations of the SS model using proposed models. Section 3 shows that the proposed separation is the orthogonal separation. Section 4 demonstrates the utilities of the proposed model and separation through the data analysis. Sections 5 closes with the concluding remarks.

## 2. Proposed model and separation of sum-symmetry

In this section, we propose a generalized model including the ESS and RESS models. The generalized ESS (GESS) model defined by

$$U_s = \Theta^{W_s} L_s \quad \text{for } s = 3, 4, \ldots, 2C - 1,$$

where $\{W_s\}$ are the specified positive values. Under the GESS model, the $U_s/L_s$ exponentially varies relying on $W_s$ for $s = 3, 4, \ldots, 2C - 1$. The advantage of the GESS model is that the $U_s/L_s$ can be resresented a variety of exponential changes, including absolute and relative gaps. When $\{W_s = 1\}$, $\{W_s = s - 2\}$, and $\{W_s = s/3\}$, the GESS models are identical to the conditional sum-symmetry (CSS) model (Yamamoto *et al.* 2013), the ESS model, and the RESS model, respectively. In line with the relation between the LDPS and ALDPS models, we can consider new models, that is the GESS models with $\{W_s = (2C - 2) - (s - 2)\}$ and $\{W_s = (2C - 1)/s\}$.

Additionally, to give a separation of the SS model using the GESS model, we introduce the generalized weighted global-sum-symmetry (GWGSS) model defined by

$$\sum_{s=3}^{2C-1} W_s U_s = \sum_{s=3}^{2C-1} W_s L_s.$$

The GWGSS model indicates that the weighted (i.e., $W_s$) average of $U_s$ is equal to the weighted average of $L_s$. The restriction of the GWGSS model is weaker than the SS model. When $\{W_s = 1\}$, the GWGSS model is identical to the global symmetry model (Read 1977).

The numbers of degrees of freedom (DF) for testing the goodness-of-fit of the SS, GESS, and GWGSS models are $2C - 3$, $2C - 4$, and 1, respectively. It must be noted that the number of DF for the SS model is equal to the sum of those for the GESS and GWGSS models.

We obtain the following separation of the SS model.

**Theorem 1.** *The SS model holds if and only if both the GESS and GWGSS models hold.*

*Proof.* The necessary condition obviously holds. Therefore, we need to show that the sufficient condition also holds.

Assume that the both GESS and GWGSS models hold, we obtain the following equality:

$$\sum_{s=3}^{2C-1} W_s \Theta^{W_s} L_s = \sum_{s=3}^{2C-1} W_s L_s \Longleftrightarrow \sum_{s=3}^{2C-1} (\Theta^{W_s} - 1) W_s L_s = 0. \tag{1}$$

From the expression (1), we obtain $\Theta^{W_s} = 1$ for all $s = 3, 4, \ldots, 2C-1$ because the $W_s$ and $L_s$ are always positive. Therefore, under the both GESS and GWGSS models hold, the $\Theta$ is equal to one, that is $U_s = L_s$ for $s = 3, 4, \ldots, 2C-1$. The proof is completed. □

It must be noted that Theorem 1 includes the former separations (Yamamoto *et al.* 2013; Ando 2021a, 2022). For the GESS model with $\{W_s\}$ and the GWGSS model with $\{W_s^* \ (\neq W_s)\}$, Theorem 1 holds. However, they are not the orthogonal separation.

## 3. Asymptotic equivalence for test statistic

Let $n_{ij}$ denote the observed frequency in the cell of the row class $i$ and column class $j$ for the square contingency table ($i = 1, \ldots, C; j = 1, \ldots, C$), with a sample size $N \ (= \sum\sum n_{ij})$. Assume multinomial sampling over the cells of the square contingency table.

Each model can be tested for the goodness-of-fit using the LR chi-squared statistic (denoted by $G^2$) with the corresponding DF. The $G^2$ of the model M is given by

$$G^2(\mathrm{M}) = 2\sum_{i=1}^{C}\sum_{j=1}^{C} n_{ij} \log\left(\frac{n_{ij}}{\hat{e}_{ij}}\right),$$

where $\hat{e}_{ij}$ is the maximum likelihood estimate (MLE) of the expected frequency $e_{ij}$ under the model M.

Suppose that the model $\mathrm{M}_1$ holds if and only if both the models $\mathrm{M}_2$ and $\mathrm{M}_3$ hold and the following asymptotic equivalence holds:

$$G^2(\mathrm{M}_1) \simeq G^2(\mathrm{M}_2) + G^2(\mathrm{M}_3), \tag{2}$$

where the number of DF for the model $\mathrm{M}_1$ is equal to the sum of those for the models $\mathrm{M}_2$ and $\mathrm{M}_3$. Darroch and Silvey (1963) pointed out that (i) under the expression (2), if both the models $\mathrm{M}_2$ and $\mathrm{M}_3$ are accepted with high probability, then the model $\mathrm{M}_1$ would be accepted, and (ii) when the expression (2) does not hold, it will most likely give rise to an incompatible situation wherein the $\mathrm{M}_1$ model is rejected with high probability although both the models $\mathrm{M}_2$ and $\mathrm{M}_3$ are accepted with high probability. This incompatible situation has been exemplified by Darroch and Silvey (1963) and Tahata, Ando, and Tomizawa (2011). Therefore, it is preferred that the separation satisfies the expression (2). The separation that satisfies the expression (2) is the orthogonal separation.

We reveal that the separation of Theorem 1 is the orthogonal separation.

**Theorem 2.** *Under the SS model,*

$$G^2(\mathrm{SS}) \simeq G^2(\mathrm{GESS}) + G^2(\mathrm{GWGSS}).$$

*Proof.* Let $\boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \ldots, \pi_{1C}, \ldots, \pi_{C1}, \pi_{C2}, \ldots, \pi_{CC})^\top$, and let $\boldsymbol{A}^\top$ denote the transpose of the matrix (or vector) $\boldsymbol{A}$. The GESS model is expressed as follows:

$$\boldsymbol{h}_1(\boldsymbol{\pi}) = \boldsymbol{0}_{2C-4},$$

where

$$\boldsymbol{h}_1(\boldsymbol{\pi}) = (h_{1,4}(\boldsymbol{\pi}), h_{1,5}(\boldsymbol{\pi}), \ldots, h_{1,2C-1}(\boldsymbol{\pi}))^\top$$

with

$$h_{1,s}(\boldsymbol{\pi}) = (L_3)^{\frac{W_s}{W_3}} U_s - (U_3)^{\frac{W_s}{W_3}} L_s \quad \text{for } s = 4, 5, \ldots, 2C-1,$$

and $\mathbf{0}_d$ is an $d \times 1$ zero-vector. This is because, under the GESS model,

$$U_3 = \Theta^{W_3} L_3 \Longleftrightarrow \Theta = \left(\frac{U_3}{L_3}\right)^{\frac{1}{W_3}}.$$

The GWGSS model is expressed as

$$\boldsymbol{h}_2(\boldsymbol{\pi}) = \mathbf{0}_1,$$

where

$$\boldsymbol{h}_2(\boldsymbol{\pi}) = \sum_{s=3}^{2C-1} W_s U_s - \sum_{s=3}^{2C-1} W_s L_s.$$

From Theorem 1, the SS model is expressed as

$$\boldsymbol{h}_3(\boldsymbol{\pi}) = \left(\boldsymbol{h}_1(\boldsymbol{\pi})^\top, \boldsymbol{h}_2(\boldsymbol{\pi})^\top\right)^\top = \mathbf{0}_{2C-3}.$$

Let $\boldsymbol{H}_t(\boldsymbol{\pi}) = \partial \boldsymbol{h}_t(\boldsymbol{\pi})/\partial \boldsymbol{\pi}^\top$ $(t = 1, 2, 3)$. Let $\boldsymbol{\Sigma}$ be $\boldsymbol{D} - \boldsymbol{\pi}\boldsymbol{\pi}^\top$, where $\boldsymbol{D}$ is a diagonal matrix with the $i$th component of $\boldsymbol{\pi}$ as the $i$th diagonal element. Let denote $\boldsymbol{p}$ as $\boldsymbol{\pi}$ with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$, where $p_{ij} = n_{ij}/N$. Using the delta method (Agresti 2013, p. 591), $\sqrt{N}(\boldsymbol{h}_3(\boldsymbol{p}) - \boldsymbol{h}_3(\boldsymbol{\pi}))$ is asymptotically $(N \to \infty)$ distributed as an normal distribution, with mean vector $\mathbf{0}_{2C-3}$ and covariance matrix

$$\boldsymbol{H}_3(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_3(\boldsymbol{\pi})^\top = \begin{bmatrix} \boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_1(\boldsymbol{\pi})^\top & \boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_2(\boldsymbol{\pi})^\top \\ \boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_1(\boldsymbol{\pi})^\top & \boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_2(\boldsymbol{\pi})^\top \end{bmatrix}.$$

Under the SS model, all elements of $\boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_2(\boldsymbol{\pi})^\top$ are zero. This is because, the following equalities hold:

$$\frac{\partial h_{1,s}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}^\top} \boldsymbol{D} \frac{\partial \boldsymbol{h}_2(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = 0 \quad \text{for } s = 4, 5, \ldots, 2C-1,$$

$$\boldsymbol{\pi}^\top \frac{\partial \boldsymbol{h}_2(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_{t=3}^{2C-1} W_t U_t - \sum_{t=3}^{2C-1} W_t L_t.$$

Therefore, under the SS model, we obtain the following equality:

$$\begin{aligned} &\boldsymbol{h}_3(\boldsymbol{\pi})^\top [\boldsymbol{H}_3(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_3(\boldsymbol{\pi})^\top]^{-1} \boldsymbol{h}_3(\boldsymbol{\pi}) \\ &= \boldsymbol{h}_1(\boldsymbol{\pi})^\top [\boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_1(\boldsymbol{\pi})^\top]^{-1} \boldsymbol{h}_1(\boldsymbol{\pi}) + \boldsymbol{h}_2(\boldsymbol{\pi})^\top [\boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}\boldsymbol{H}_2(\boldsymbol{\pi})^\top]^{-1} \boldsymbol{h}_2(\boldsymbol{\pi}). \end{aligned}$$

We obtain $G^2(\text{SS}) \simeq G^2(\text{GESS}) + G^2(\text{GWGSS})$ from the asymptotic equivalence of the Wald statistic and the LR statistic, see, for example, Rao (1973, Sec. 6e. 3), Darroch and Silvey (1963), and Aitchison (1962). The proof is completed. □

Theorem 2 includes the former orthoginal separations (Yamamoto *et al.* 2013; Ando 2021a, 2022). It must be noted that, when $\{W_s = 1\}$, the former orthoginal separation (Yamamoto *et al.* 2013) satisfies that the LR statistic for testing goodness-of-fit of the SS model is exactly equal to the sum of LR statistics of the separated models.

## 4. Real data analysis

The real dataset in Table 1 is analyzed to illustrate the proposed models that is competitive or even better than other models. This real dataset is taken from Tan (2017, p. 78). This dataset is recorded the distance vision of patients in an eye clinic. An individual's acuity of vision is usually evalueted as the sum of right and left eye grades (Yamamoto *et al.* 2013, 2016; Ando 2021a,b, 2022). Thus, for data such as those in Table 1, we are interested in applying the GESS model.

In the GESS and GWGSS models, we set $\{W_s = 1\}$, $\{W_s = s - 2\}$, $\{W_s = s/3\}$, $\{W_s = (2C-2) - (s-2) = 6 - (s-2)\}$, and $\{W_s = (2C-1)/s = 7/s\}$. The GESS and GWGSS models with $\{W_s = 6-(s-2)\}$ and $\{W_s = 7/s\}$ are innovated in this study. Table 2 gives the values of $G^2$, for each the SS, GESS, GWGSS model. From Table 2, we see that the GESS models with $\{W_s = 1\}$, $\{W_s = s/3\}$, $\{W_s = 6-(s-2)\}$ and $\{W_s = 7/s\}$ fit well, but the other models fit poorly. Additonally, the GESS model with $\{W_s = 1\}$ is the best-fitting model among applied models, and the GESS model with $\{W_s = 7/s\}$ is the second best-fitting model.

Table 1: Dataset of distance vision of patients in an eye clinic; source Tan (2017, p. 78)

| | Left eye grades | | | | |
|---|---|---|---|---|---|
| Right eye grades | (1) | (2) | (3) | (4) | Total |
| Highest grade (1) | 50 | 21 | 24 | 35 | 130 |
| | (50) | (21.047) | (26.726) | (36.322) | |
| | (50) | (16.465) | (25.159) | (37.946) | |
| (2) | 42 | 22 | 12 | 42 | 118 |
| | (41.953) | (22) | (12.453) | (41.425) | |
| | (46.535) | (22) | (13.010) | (45.249) | |
| (3) | 56 | 32 | 52 | 16 | 156 |
| | (53.274) | (31.426) | (52) | (12.027) | |
| | (54.841) | (30.721) | (52) | (14.058) | |
| Lowest grade (4) | 67 | 82 | 20 | 62 | 231 |
| | (65.799) | (82.575) | (23.973) | (62) | |
| | (64.323) | (77.751) | (21.943) | (62) | |
| Total | 215 | 157 | 108 | 155 | 635 |

Note: MLEs of expected frequencies under the GESS models with $\{W_s = 1\}$ and $\{W_s = (2C-1)/s = 7/s\}$ are shown in parentheses in the second and third lines.

Under the GESS model with $\{W_s = 1\}$ (i.e., the CSS model), the MLE of the parameter $\Theta$ is 0.502, it is less than 1. Therefore, the patient's acuity of vision in which the left eye grade is higher than the right eye grade tends to be better than the patient's acuity of vision in which the right eye grade is higher than the left eye grade.

Under the GESS model with $\{W_s = 7/s\}$, the MLE of the parameter $\Theta$ is 0.641, Thus, the MLEs of $\Theta^{7/s}$, for all $s = 3, 4, 5, 6, 7$, are 0.354, 0.459, 0.536, 0.595, and 0.641, respectively, and they are less than 1. Therefore, in line with the CSS model, the patient's acuity of vision in which the left eye grade is higher than the right eye grade tends to be better than the patient's acuity of vision in which the right eye grade is higher than the left eye grade.

Using Theorem 1, we consider the causality that the SS model fits poorly for the dataset in Table 1. From Theorem 1, we see that the SS model does not hold because of the GWGSS model rather than the GESS model.

The value of $G^2$ of the SS model (i.e., 52.817) is almost equal to the sum of $G^2$ of the GESS and GWGSS models with corresponding $\{W_s\}$, see Table 2. When $\{W_s = 1\}$, the value of $G^2$ of the SS model (i.e., 52.817) is exactly equal to the sum of $G^2$ of the GESS and GWGSS

Table 2: Values of the $G^2$ for the models applied to the data of Table 1

| Models | $W_s$ | DF | $G^2$ |
|---|---|---|---|
| SS | - | 5 | 52.817* |
| GESS | 1 | 4 | 2.421 |
| | $s-2$ | 4 | 13.288* |
| | $s/3$ | 4 | 7.829 |
| | $6-(s-2)^\dagger$ | 4 | 3.894 |
| | $7/s^\dagger$ | 4 | 3.226 |
| GWGSS | 1 | 1 | 50.396* |
| | $s-2$ | 1 | 38.709* |
| | $s/3$ | 1 | 44.443* |
| | $6-(s-2)^\dagger$ | 1 | 48.150* |
| | $7/s^\dagger$ | 1 | 49.010* |

Note: The symbol $*$ indicates significance at the 5% level, and † indicates a new model. The number of category $C$ is 4 in the data of Table 1.

models (i.e, $2.421 + 50.396 = 52.817$). On the other hand, we see that the value of $G^2$ of the SS model is not nearly equal to the sum of $G^2$ of the GESS and GWGSS models with conflicting $\{W_s\}$. In fact, the sum of $G^2$ of the GESS model with $\{W_s = 1\}$ and the GWGSS model with $\{W_s = s-2\}$ is 41.130 (i.e., $2.421 + 38.709$).

Next, we analyze the real dataset in Table 3, taken from Tomizawa (1984). This real dataset is recorded the distance vision of students aged 18 to approximately 25 including approximately 10% women in Tokyo University of Science.

Table 4 gives the values of $G^2$, for each the SS, GESS, GWGSS model. From Table 4, we see that the GESS models with $\{W_s = 1\}$, $\{W_s = s-2\}$, $\{W_s = s/3\}$, and $\{W_s = 7/s\}$ fit well, but the other models fit poorly. Additonally, the GESS model with $\{W_s = s/3\}$ is the best-fitting model among applied models, and the GESS model with $\{W_s = 1\}$ is the second best-fitting model.

Table 3: Dataset of distance vision data of students in Tokyo University of Science; source Tomizawa (1984)

| | Left eye grades | | | | |
|---|---|---|---|---|---|
| Right eye grades | (1) | (2) | (3) | (4) | Total |
| Highest grade (1) | 1291 | 130 | 40 | 22 | 1483 |
| | (1291) | (131.473) | (48.014) | (20.479) | |
| | (1291) | (125.245) | (46.687) | (20.333) | |
| (2) | 149 | 221 | 114 | 23 | 507 |
| | (147.527) | (221) | (106.120) | (21.247) | |
| | (153.755) | (221) | (105.361) | (21.548) | |
| (3) | 64 | 124 | 660 | 185 | 1033 |
| | (55.986) | (132.095) | (660) | (188.006) | |
| | (57.314) | (132.874) | (660) | (194.826) | |
| Lowest grade (4) | 20 | 25 | 249 | 1429 | 1723 |
| | (21.306) | (26.753) | (245.994) | (1429) | |
| | (21.431) | (26.452) | (239.174) | (1429) | |
| Total | 1524 | 500 | 1063 | 1659 | 4746 |

Note: Note: MLEs of expected frequencies under the GESS models with $\{W_s = s/3\}$ and $\{W_s = 1\}$ are shown in parentheses in the second and third lines.

Table 4: Values of the $G^2$ for the models applied to the dataset of Table 3

| Models | $W_s$ | DF | $G^2$ |
|---|---|---|---|
| SS | - | 5 | 16.668* |
| GESS | 1 | 4 | 4.692 |
| | $s-2$ | 4 | 4.799 |
| | $s/3$ | 4 | 4.159 |
| | $6-(s-2)^\dagger$ | 4 | 9.860* |
| | $7/s^\dagger$ | 4 | 7.343 |
| GWGSS | 1 | 1 | 11.976* |
| | $s-2$ | 1 | 11.867* |
| | $s/3$ | 1 | 12.509* |
| | $6-(s-2)^\dagger$ | 1 | 6.797* |
| | $7/s^\dagger$ | 1 | 9.305* |

Note: The symbol $*$ indicates significance at the 5% level, and † indicates a new model. The number of category $C$ is 4 in the dataset of Table 3.

Under the GESS model with $\{W_s = s/3\}$, the MLE of the parameter $\Theta$ is 0.891. Thus, the MLEs of $\Theta^{s/3}$, for all $s = 3, 4, 5, 6, 7$, are 0.891, 0.858, 0.825, 0.794, and 0.764, respectively, and they are less than 1. Therefore, the sutudent's acuity of vision in which the left eye grade is higher than the right eye grade tends to be better than the student's acuity of vision in which the right eye grade is higher than the left eye grade.

Under the GESS model with $\{W_s = 1\}$ (i.e., the CSS model), the MLE of the parameter $\Theta$ is 0.815, Therefore, in line with the CSS model, the student's acuity of vision in which the left eye grade is higher than the right eye grade tends to be better than the student's acuity of vision in which the right eye grade is higher than the left eye grade.

Using Theorem 1, we consider the causality that the SS model fits poorly for the dataset in Table 3. From Theorem 1, we see that the SS model does not hold because of the GWGSS model rather than the GESS model except when $\{W_s = 6-(s-2)\}$. When $\{W_s = 6-(s-2)\}$, we see that the SS model does not hold because the both GWGSS and GESS models do not fit.

From Table 4, we see that the value of $G^2$ of the SS model (i.e., 16.668) is almost equal to the sum of $G^2$ of the GESS and GWGSS models with corresponding $\{W_s\}$.

## 5. Concluding remarks

This study proposed a generalized model (i.e., the GESS model) including the ESS and RESS models, and established a separation in which the SS model holds if and only if both the GESS and GWGSS models hold (i.e., Theorem 1), Moreover, this study revealed that the proposed separation satisfied the expression (2) (i.e., Theorem 2), that is, the $G^2$(SS) is asymptotically equivalent to the $G^2$(GESS) + $G^2$(GWGSS). Theorems 1 and 2 are the generalization of Ando's results (Ando 2021a, 2022).

This study showed the utilities of the GESS model, Theorems 1 and 2 through the real data analysis. In the real data analysis, we introduced that the GESS model is useful to right and left eye distance vision data. However, when the expression (2) holds, it will not most likely give rise to this conflicted situation. Therefore, it is preferred that the separation satisfies the expression (2).

A two way contingency table with $I$ row-categories and $J$ column-categories classifies $N$ individuals in $I \times J$ cells, with $n_{ij}$ individuals assigned to the $(i, j)$th cell. Assuming the sampling was multinomial, each of the $(i, j)$th cells has some probability, $\pi_{ij} > 0$, of an individual to be assigned to it, with $\sum\sum \pi_{ij} = 1$. Probabilities of a multinomial distribution add to 1 and, consequently, they are an element of the $D = (I \times J)$-part simplex, the set of real vectors with strictly positive components adding to a constant. The simplex has been proven to be a $(D-1)$-dimensional Euclidean space, which particular algebraic-geometric structure is called Aitchison geometry (Fačevicová, Hron, Todorov, Guo, and Templ 2014). In two-way contingency tables, Egozcue, Pawlowsky-Glahn, Templ, and Hron (2015) discussed the independence between the row and column variables using Aitchison geometry. In future study, we will discuss the proposed models in this study using Aitchison geometry.

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