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Statistical Distributions for Damage Modelling: A Retrospect

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Abstract

This paper reviews works on statistical damage modelling. Available literature shows that this method is flexible for modelling under-reporting and over-reporting of income, optimal replacement, accelerated test models, man-power analysis, rock compression, inventory analysis, power distribution etc . Additive and multiplicative damage models are discussed and additional literature on multivariate damage modelling are also provided. Characterizations of various distributions are discussed.

Keywords: additive damage model, multiplicative damage model, Rao-Rubin condition, Patil-Seshadri condition, characterizations.

1. Introduction

The concept of damage model, introduced by the legend of Indian Statistics Prof. C. R. Rao in Rao (1965), arose naturally in connection with the following problem in Biology. A bird lays eggs in its nest at time epochs $0 < s_1 < s_2 < ...$ and a squirrel runs around and each time, with the same probability, takes away an egg. For an individual who is unaware of timings when eggs were laid or stolen, only a 'damaged' number of eggs remain in the nest at any given point of time. It is of interest to compare the distribution of the actual number of eggs laid with the distribution of the number of eggs found in the nest or to realize conditions under which one distribution can be inferred from the other. A large number of research papers have appeared during the last two decades on damage models, since the appearance of the Rao–Rubin result in Rao (1965).

Damage models find use in the real life scenario. Generally people under report income for getting exemption from tax. If Z is the actual income of a person and Y is the reported income then Y is only a damaged part of Z. The problem is to set conclusion regarding Z through knowledge of Y.

There are also instances of over reporting of income. To get benefits on insurance policies people generally over report their income, since the benefits depends on their future earnings over a specified time period.

In statistical terminology a random variable Z reduced to some other variable by some random mechanism is called a damaged variable. The quantity Y=Z-X is the reduction in Z and is called the damaged component of Z in contrast to the undamaged part X. The random mechanism that produces the damage is represented by the survival distribution defined as

$$s(x|z) = P(X = x|Z = z) \tag{1}$$

The representation just described assumes that Z is the sum of two random variables and is therefore termed as an additive damage model.

Another well studied model to represent such phenomena is the multiplicative damage model. The reduction in Z can be assumed to materialize by a multiplicative model in the form X=RZ, where the random variable R is defined on [0, 1].

The concept of damage models has been extended to higher dimensions. Instead of considering a single random variable Z, a random vector $Z = (Z_1, Z_2, ..., Z_n)$ can be thought of as reduced to another vector $X = (X_1, X_2, ..., X_n)$. In such cases we have a multivariate damage model, which obviously can be either additive or multiplicative.

Earlier researches in damage models were centered around a necessary and sufficient condition for the distribution of a random variable Z to be Poisson, when it is assumed that the conditional distribution of X given Z has binomial distribution. This condition, viz.

$$P(X = x) = P(X = x|Y = 0) \tag{2}$$

came to be later known in literature as the Rao-Rubin (R-R) condition. Shanbhag (1974) provided a simple proof of this condition. Extensions of the R-R condition in various directions were given by several researchers such as Van der Vaart (1972), Shanbhag and Clark (1972) and Srivastava and Srivastava (1970). Possibilities of other distributions that suits to Z and conditional distribution of X given Z and their corresponding characterizations based on a different set up were proved in Patil and Seshadri (1964).

An alternative approach to analyze damage models is using the concept of regression. The characterizations of Krishnaji (1974), Revankar, Hartley, Pagano *et al.* (1974) and Korwar (1975) belong to this category. The possibility of characterizing the Pareto distribution through a multiplicative model is explored in Krishnaji (1970). For other contributions in this connection we refer to Srivastava (1971), Patil and Ratnaparkhi (1975), Srivastava and Singh (1975) and Talwalker (1970).

1.1. Relationship with other models

The damage models has intimate relationship with two concepts rarefactions and geometric compounding. Of these, the geometric compounding model has the following formulation.

Let $X_1, X_2, ..., X_N$ be independently and identically distributed random variables with a common distribution function F(x) and N be a random variable following geometric law

$$P(N = n) = p^{n-1}; n = 1, 2... \tag{3}$$

independent of the X's. If F*(x) is the distribution function of S*, defined by the equation

$$S^* = X_1 + X_2 + ... + X_N \tag{4}$$

The point of interest in geometric compounding models is the relation between F*(x) and F(x).

If

$$S_n = X_1 + X_2 + ... + X_n; n < \infty \tag{5}$$

The sequence $S_1, S_2, ...$ form a renewal process with F(x) as the interval distribution function. Each point S_n is erased with a probability q. Expanding the time scale by 1/q, we obtain a point process S^*_n, $n \geq 1$. This procedure is called rarefaction process, where the action of erasure is taken independently of the process (S_n) and also the decision on erasing distinct points is made independently of one another. Considering $F^*(x)$ to be the interval distribution corresponding to the process S_n^*, the relationship between F(x) and $F^*(x)$ is explored in rarefaction problems.

With reference to the example cited in the introductory paragraph the assumptions on the action of the squirrel imply that for a given (Z=z) the distribution of Y is binomial. If we can conceive Z as the number of eggs laid in the interval (0,t) and $S_1 < S_2 < ... < S_z$ the actual time points when eggs were laid, Y is distributed as binomial implies that each S_n has the same probability of being erased and that the actions of erasure are independent of one another. In this context the damage model of Rao and Rubin finds the additional assumption regarded to guarantee a Poisson distribution for Z. Thus a characterization of a Poisson distribution for Z implies a characterization of a Poisson process in a rarefaction process. Needless to say, therefore, that of the three models, the damage models supersedes the others generally. In view of these reasons, we focus an attention on the damage models.

As mentioned in the opening section, a convenient classification of the damage models is provided by distinguishing them as additive and multiplicative models. The additive model was explained in literature in the form of two conditions which appeared in the same year, viz. R-R and Patil-Seshadri (P-S) conditions.

In Section 2 the first two subsections are devoted for characterizations employing the R-R condition and P-S condition. Another subsection explains the multiplicative model. Moving on to Section 3, it explains the different areas in which damage models found applications. And the general conclusion of the present review has been given in Section 4.

2. Characterizations and conditions

This section concentrates on explanation of the two conditions that paved the way to the extensive study of additive damage model. And also explains the multiplicative damage model and its famous relation with income distribution.

2.1. R-R condition

The characterization available in literature on damage models can be classified into different directions. There are mainly two types of damage models, additive damage model and multiplicative damage model. The milestone literature in additive damage model was contributed by Rao and Rubin (1964). A number of characterizations has been proposed in the mentioned paper. They are

(i) Let the original variable be Z and damaged variable be Y = Z-X. If the conditional distribution of X given Z follow binomial distribution then necessary and sufficient condition for Z to follow Poisson distribution is $P(X = x) = P(X = x|Y = 0)$ where Y = Z-X.

(ii) The above theorem is also applicable for Y=1.

The statement $P(X = x) = P(X = x|Y = 0)$ is termed as the R-R condition.

Talwalker (1970) extended the R-R conditions to the bivariate case to provide a characterization of the double Poisson distribution. Another result in this connection is a characterization of the multivariate negative binomial distribution proposed by Patil and Ratnaparkhi (1975).

There is an alternative approach to characterize distributions using the concept of linear regression. Here instead of employing the R-R condition, Krishnaji (1974) has used the regression approach, on the conditional expectation of X given (Y=y) which is equal to a linear equation $a + by$.

For various values of a and b, distributions such as Poisson, binomial, negative binomial, logarithmic series and hypergeometric distributions can be obtained as the original distributions. There have been attempts to extend this approach in the case of some continuous distributions as well. Revankar *et al.* (1974) used this method to characterize the Pareto distribution. For some extended results in the same area, we can refer to Korwar (1975).

2.2. P-S condition

Almost at the same time the R-R condition was developed, another group of statisticians independently proved a characterization for the linear exponential family of distributions in the same context of damage models and have same ideology of R-R condition, which is explained in Patil and Seshadri (1964). It was based on the form of the conditional distribution of X given X+Y. It leads to further investigations in the same direction, forming a class of theorems in the framework of P-S conditions. They are

Let X and Y be independent discrete random variable and $s(x|x+y) = P(X = x|X+Y = x+y)$. Then the condition

$$\frac{s(x+y|x+y)s(0|y)}{s(x|x+y)s(y|y)} = \frac{h(x+y)}{h(x)h(y)} \tag{6}$$

where h(.) is a non-negative function is satisfied if and only if X and Y belongs to the linear exponential family having a common exponential parameter.

ie.

$$P(X = x) = f(x) = f(0)h(x)e^{ax} \tag{7}$$

and

$$P(Y = y) = g(y) = g(0)k(y)e^{ay} \tag{8}$$

where

$$k(y) = \frac{h(y)s(0|y)}{s(y|y)}, a > 0 \tag{9}$$

The equation (6) is called P-S condition. By using this condition, it immediately follow some characterizations which is fairly explained as Theorems 1, 2 and 3 and as corollaries of the same in Patil and Seshadri (1964).

2.3. Multiplicative damage model

Another major category of damage model is multiplicative damage model. This model is basically classified into univariate and bivariate multiplicative damage models. Univariate mulitiplicative damage model is defined as, X = RZ, where value of R is defined on [0,1]. It means, the reduction in Z is represented by X (damaged variable), as R is a fractional value. The bivariate multiplicative damage model is obtained through the relationship $(Y_1, Y_2) = (RX_1, RX_2)$. Here (Y_1, Y_2) represents the bivariate multiplicative damaged variable. Multiplicative damage model is most commonly used, due to its mathematical tractability.

The univariate multiplicative damage model discussed in Krishnaji (1970) has been extended to the bivariate setup by Veenus and Nair (1994). They have obtained characterization results for the bivariate Pareto distribution with survival functions

$$P(X_1 > x_1, X_2 > x_2) = (\frac{x_1}{\beta})^{-\lambda_1}(\frac{x_2}{\beta})^{-\lambda_2}$$

where $x_1, x_2 > \beta, \lambda_1, \lambda_2 > 0, \beta > 0$.

and

$$P(X_1 > x_1, X_2 > x_2) = (\frac{x_1}{\beta})^{-\lambda_1}(\frac{x_2}{\beta})^{-\lambda_2}(\frac{max(x_1, x_2)}{\beta})^{\lambda_{12}}$$

where $x_1, x_2 > \beta$, $\lambda_1, \lambda_2 > 0$, $\beta > 0$, using different set of conditions.

3. Areas of application

Damage models have wide applications in the context of science and technology. In fact, in general, whenever a random decrease is suggested in a random variable, damage models become meaningful. We point out below several areas of study where damage models provide factual representation.

3.1. Optimal replacement of machines

The concept of additive damage model was used by Taylor (1975) in optimal replacement of machines by considering cumulative damage failure model and established that damages are exponentially distributed with shocks occurring to the system according to Poisson process. A discrete time formulation for the problem and sufficient condition for optimality of a generalized control control limit rule was given in Waldmann (1983) and this was extensively studied in Posner and Zuckerman (1986) as semi-Markov shock model.

3.2. Inventory analysis

Xekalaki (1983) points out the relevance of damage models in inventory analysis by treating Z as the time demand for an item during a unit time interval and X as the item units to stock during the same time interval. She obtained a characterization of the Yule distribution in this context.

3.3. Rare event reporting

As is well known, for fear of litigation and other complications that might follow the occurrence of an accident, many of the accidents that occur are not reported. Thus the number of accidents reported usually fall short of those actually materialized. Since the frequency of accidents are generally assumed to follow Poisson pattern which is proved in Nicholson and Wong (1993), the problem lead to characterization of the Poisson distribution, through the damage model, which is in need to expand further.

3.4. Accelerated test models

Damage models also find application in accelerated test models. As in Durham and Padgett (1997) they assumed a discretized cumulative damage model for the failure of a general system under an increasing tensile load, Padgett (1998) and Owen and Padgett (1998) studied about multiplicative damage model with Birnbaum-Saunders type models for system strength, while Onar and Padgett (2000a) took a similar approach by studying inverse Gaussian distribution. This work was extended in Onar and Padgett (2000b) by considering a continuous damage model based on a Gaussian process, rather than using the discretized model.

3.5. Martin- boundary connection

In Rao, Rao, and Shanbhag (2002) we can observe that it concerns certain results leading to the connection with random walk, branching process and damage models. The Martin boundary in the environment of non-negative matrices with inherent extreme point methods is linked to damage models. And then in Raje, Sadeghi, and Rateick (2008), a damage mechanics based fatigue model to study the process of subsurface initiated spalling in bearing contacts were used which employs three parameter Weibull distribution.

3.6. Geomaterial compression

Rock is a natural engineering material, which contains a large number of internal defects (for instance, cracks, joints, voids...etc). These defects are the typical manifestations of damage. Therefore, damage-based constitutive models accord with the characteristics of rock. The existing rock damage models can be classified into several types such as, the elastic–plastic damage model, the mesoscopic damage model and the dynamic damage model. However, these models are usually complicated and inconvenient to use. For a convenient application, it is sought to establish a simple damage constitutive model to reflect the stress–strain relation of sandstone. Applications of statistical constitutive damage model in the context of different geomaterials are given below.

Compression of rocks

In the last one decade several papers have appeared in the context of application of statistical damage constitutive models. Most of the papers deals with the behavior of rocks during compression and the damage happening to the same during that compression.It is seen that Weibull distribution is used to describe strength of the rock during compression. The compression may be uniaxial, biaxial or triaxial.

The preliminary paper appeared in 2010 ie. Cao, Zhao, Li, and Zhang (2010). By using a statistical damage-based approach, the characteristics of strain softening and hardening under the influence of voids and volume changes are investigated which uses triaxial compression method and Weibull distribution to describe the strength of mesoscopic elements, which is associated with macroscopic parameters. The work has been followed in Deng and Gu (2011), with statistical mesoscopic strength theory based on maximum entropy distribution. In the same year another idea which seeks to devise non-conventional approaches to fracture and damage by means of discrete damage models accounting for the essential microscale property of a material or structural system was introduced in Rinaldi (2011).

The constitutive damage models in this overview are established from physically-based definitions of the damage parameter(s) that stem from a pervasive (bottom-up) statistical rationale aided by a vast arsenal of modeling tools (e.g. descriptive and inductive statistics, fractal theory, computer graphics, numerical methods). Later in Tian, Wang, Li, and Xu (2014), the Lade-Duncan (L-D) criterion is introduced as a new measure of rock micro-unit strength, which can reasonably consider the influence of intermediate principal stress on rock strength. Assuming that the micro-unit strength obeys the Weibull distribution, combined with the L-D failure criterion, a new statistical damage constitutive model for brittle rocks is established. The model can simulate the stress-strain relation of full process of rock failure well. Another paper which deals with the same area is Zhao, Xie, and Meng (2014).

By taking into consideration the effects of damage under a dynamic load on the dynamic loading strength of the rock, the continuous damage theory and the statistical strength theory were introduced into the development of the simplified overstress constitutive formula for the stress model. Hence, a damage-based constitutive formula for an overstress model, which can be appropriately applied to the analysis of full dynamic stress–strain curves was developed. By using the simplified damage-based constitutive formula for an overstress model, the actual measured curves are fitted, indicating that the fitted curves and those actually measured are in good agreement.

But Zhao, Zhang, Cao, and Zhao (2016) studied a challenging problem in involving the concept of damage model into the constitutive laws of rocks as the definition of the damage variable can used to characterize the mechanical behavior of rocks-sandstone. Results show that the developed constitutive law for quasi-brittle rocks with damage tolerance principle can be used to predict the post-peak behavior including the strain-softening behavior and the immobile residual strength with acceptable accuracy. Regarding the mechanical response of strain softening rock, especially in brittle failure under moderate confining pressure is explained in Zhao, Shi, Zhao, and Li (2017). Accordingly, a modified damage mechanical model with conventional triaxial compression was established. Then, the evolution equation of the damage variable was formulated based on the well-known Weibull distribution.

Laterly in Xu and Yang (2017) constitutive fracture damage model for fissured rock mass is established based on the deformation characteristics of microcracks under compression. By employing the basic theory of damage mechanics in Zhang, Wan, Wang, Ma, Zhang, and Cheng (2017), they established a statistical damage constitutive model for rocks under the PSBSS (Plane-Strain Biaxial Stress State). The constitutive model of rocks subjected to cyclic stress–temperature was proposed in Zhou, Xia, Zhao, Mei, and Zhou (2017) and model for cemented sand considering the residual strength and initial compaction phase was discussed in Tan, Yuan, Shi, Zhou, and Li (2018). The triaxial compression results are based on Weibull distribution in both. The damage created on rocks due to continuous action of high level of water and temperature were studied and a new statistical damage constitutive model based on Weibull distribution theory and Mohr- Coulomb strength criterion is established on Jiang, Jiang, Zhang, and Yang (2022).

Damage modelling in effects of cyclic drying and wetting conditions

Major work in this context is Kegang and Yuanying (2016). It studied the effects of cyclic drying and wetting conditions on the mechanical properties of rock-sandstone based on the damage constitutive theory of a continuous medium, utilizing characteristic parameters and extreme conditions indicated by a stress-strain curve. And also by considering the nonlinear relationships of the rock's mechanical parameters of drying-wetting cycles, a constitutive model for rock considering drying-wetting effect as proposed with uniaxial compression in the same paper. Similarly in Chen, He, Qin, Li, and Gong (2019) they propose a new statistical constitutive model using symmetric normal distribution. The damage variable was established using the equivalent strain principle and symmetric normal distribution, where damaging variable was defined by the elastic modulus under various dry-wet cycles.

Mechanical characteristics of saturated porous media

Gao, Xie, Xie, He, Li, Wang, and Luo (2017) deals with the void volume changes and fluid pressure of saturated porous media. A new constitutive model was developed to describe the mechanical characteristics of saturated porous media in geomaterials. On the basis of the Weibull distribution for rock-sandstone micro-strength, the damage variable was defined and a semi-analytical permeability variation model was established with triaxial compression. This work was continued in Wang, Song, Zhao, Liu, Liu, and Lai (2018) as a new damage model, which can reflect the residual deviatoric stress was established. Triaxial test results of sandstone were employed to evaluate the reasonability of the damage model established in this paper.

Energy analysis of the deformation and failure process of sandstone

Wen, Tang, Ma, and Liu (2019) is the first paper appeared in this context. Energy analysis of the deformation and failure process of sandstone with the damage variable D was redefined in light of energy dissipation and then, the damage evolution analysis was conducted based on triaxial tests. An improved rock damage constitutive model was further obtained in another expression to reflect the energy change law. Subsequently, the relationship between D and the deformation or failure process of rocks was analyzed on account of the damage evolution equation formularized by fitting to a logistic function, which can measure the influence of energy dissipation on the propagation of micro-defects. Later in 2019 establishment of damage statistical constitutive model of loaded rock and method for determining its parameters under freeze-thaw condition are discussed in Fang, Jiang, and Luo (2019).

Chemical corrosion of rocks

Regarding mechanical properties of rock, a coupled chemical-mechanical condition was discussed in Lin, Gao, Zhou, Gao, and Guo (2019a). An improved statistical damage constitutive model was established using the Drucker-Prager (D-P) strength criterion and two-parameter Weibull distribution. The damage variable correction coefficient and chemical damage variable which was determined by porosity were also considered in the model. Moreover, a series of conventional triaxial compressive tests were carried out to investigate the mechanical properties of sandstone specimens under the effect of chemical corrosion. A continuum damage mechanics model which demonstrates corrosion fatigue crack initiation prediction is discussed in Yang, Fan, and Li (2022).

In the same year Lin, Zhou, Gao, and Li (2019b) has done another work regarding damage evolution and behavior of constitutive model of sandstone. Here sandstone is subjected to chemical corrosion using uniaxial compression tests . The statistical damage constitutive model was built by combining the compaction coefficient and the chemical damage variable is proposed to describe the damage evolution of sandstones treated with chemical corrosion.

Progressive growth of damage in rock

Regarding phenomenological modelling of rocks based on the influence of damage initiation a new idea has been proposed in Zhao, Zhou, and Zhang (2019). The model addresses the progressive growth of damage that leads to the strength weakening on a macroscopic scale. Considering dramatic difference between uniaxial compression and tension strengths for rocks, the admitted Mises–Schleiche D–P strength criterion is adopted to characterize the damage initiation. On this basis, a two-parameter Weibull-type probability function is used to define the strength distribution of representative volume elements, followed by the use of damage variable for addressing the accumulated probability of failure. As a continuation, in Liu, Dan, Jia, and Zhu (2020) a statistical damage constitutive model of granite at high-temperature, which considering the damage threshold, residual stress and thermal damage is established to describe the stress–strain relationship of high-temperature granite. In this model, the same D–P criterion is used as the failure criterion and the Weibull distribution is introduced to describe the strength distribution of rock elements. The results of theoretical analysis are found to agree closely with the triaxial compression experiment of high temperature granite. The uniaxial compression-mechanical properties of rock under osmotic pressure was elaborated in Song, Wang, Wang, Xiao, and Yang (2022). A new damage constitutive model and theoretical curves of stress-strain response of limestone before its failure has been discussed.

Joint shear deformation

The first paper in 2020 regarding statistical damage model in joint shear deformation is Xie, Lin, Wang, Chen, Xiong, Zhao, and Du (2020). The phase of initial damage is determined on the assumption that, the joint shear failure results in damage evolution, according to the typical joint shear curve and the three-parameter Weibull distribution. Then a statistical damage evolution model for the whole joint shearing process is introduced to make this model to be capable of describing the residual phase of rock joints.

The next paper published immediately is Zhou, Karakus, Xu, and Shen (2020) which dealt with a new damage model accounting the effect of joint orientation for the jointed rock mass using the Weibull distribution by incorporating the Jaeger's and modified Hoek-Brown failure criteria. Therefore, it improves the prediction of rock mass response significantly. Thus, the proposed model can be used to simulate anisotropic rock mass behavior accurately.

In Chen (2020), by assuming that the rock material is able to be divided into the elastic part satisfying the Hooke's law and damaged part where rock strength follows lognormal distribution, he determined a damage variable and establishes a damage constitutive model which effectively reflects the residual strength in the of rock failure.

The most recent paper appeared in this context is Ma, Gutierrez, and Hou (2020) which explained a coupled plasticity applied damage constitutive model to describe the mechanical behavior of shales that spans the prepeak, peak and postpeak regions. Concept on damaged part of geomaterials, which possesses residual shear strength is introduced in the proposed model. A thermodynamic conjugate force is presented accounting, the residual shear strength of the damaged part.

3.7. Manpower analysis

In spite of efforts by government and other agencies the true size of labour force available in a region is never known exactly. However, it is possible to have statistics of the observed labour input in a variety of investigations. Thus Z stands for the real labour input and X, the observed labour input. This is explained well in Amirthalingam (2016).

3.8. Loading sequence effect

The relationship between rainflow counting matrix, sequence transition matrix and original loading sequence which are derived from rainflow counting sequence and transition sequence is described in Zhu, Zhang, and Ding (2022). A new modified fatigue damage model based on linear damage rule has been proposed. The damage model explaining the stress rate- dependent progressiveness of ultra high molecular weight polyethylene fiber composite laminates under impact loading is described in Mansoori and Zakeri (2022).

3.9. Predicting tensile behavior

Exponential damage evolution equation is used in forming damage model for damaged fiber and matrix/yarn it is modeled by the stiffness degradation method and a new progressive damage model for the three-dimensional woven carbon/carbon composites is developed at fiber-matrix level elaborated in Wei, Shi, Li, and Tang (2022).

3.10. Overhead power distribution

To evaluate the effectiveness of grid reliability enhancements a damage modeling framework for the overhead power distribution system under budgetary constraints is proposed in Hughes, Zhang, Bagtzoglou, Wanik, Pensado, Yuan, and Zhang (2021). Monte Carlo simulation is used to consider various uncertainties of the power distribution system.

3.11. Multiplicative damage model in the context of Income distribution

This method was first used by Krishnaji (1970) in connection with under-reporting of income. Under this framework, it has been established that the distribution of observed income suitably truncated and coincides with the true distribution if and only if the distribution is of the Pareto form and a variable having a linear regression on true income has a linear regression on observed income. From this idea Dimaki and Xekalaki (1990) established two characterizations. Characterization of Pareto as income distribution has thrown light to the World Economy. To date, economists have mostly used Pareto Type I distribution to model the upper tail of income and wealth distribution. It is a parametric distribution, with an attractive property that can be easily linked to economic theory, but the same have some disadvantages too. This is explained well in Charpentier and Flachaire (2019). But these disadvantages have been already identified by Dimaki and Xekalaki (1990) as multiplicative damage models. They are described in the next two subsections. The methodology of the size distribution of income deals with the distribution of income among individuals. A careful examination of the frequency distribution of income shows that it is very much skewed, since a large proportion of income goes to a small proportion of individuals. An earlier investigation in the analysis of income was centred on finding a suitable model for the distribution of income. Accordingly a number of distributions such as normal, Pareto, lognormal and Gamma where proposed by several researchers. A detailed discussion of the various models, their relevance and utility are available in Kakwani (1986).

Characterization in the context of under-reporting of income

It is widely believed that, the tail index estimation of Pareto I model based on surveys is biased upward, because these data are subject to topcoding, censoring and under-reporting of the rich. Regarding under-reporting of income, the Theorem 2.1, 2.2 and their corresponding corollaries in Dimaki and Xekalaki (1990), they have established results leading to the characterizations of Pareto, Yule and F distributions.

Characterization in the context of over-reporting of income

From the study of Dimaki and Xekalaki (1990) it was found that, the tail index can be biased downward and the upper tail might just as easily be over-estimated for tail index estimation of Pareto I model. Regarding over-reporting of income, Lemma 3.1, Theorem 3.1 and 2.2 and also their corresponding corollaries in Dimaki and Xekalaki (1990), have established results leading to the characterization of Pareto distribution.

The continued interest shown by researchers in identifying different probability distributions to represent the original and ruined output resulted in further areas of application and new results under simplified conditions are reported above.

4. Conclusion

In this review, we have discussed various results on damage models in the additive and multiplicative setup under a unified approach. In doing so, we have been able to identify the prominent discrete distributions suitable as potential models in varying situations. The discussions mainly relate to univariate models and slightly in bivariate cases. It is possible to have a natural generalization of most of these results to higher dimensions. However, this area has not been explored in detail.

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