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A Note on Gershgorin Discs in the Elliptic Plane

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Abstract

In this study, we derive Gershgorin discs of elliptic complex matrices in the elliptic plane. Also, we investigate the location of the zeros of an elliptic complex-valued polynomial with the help of Gershgorin discs of elliptic complex matrices. To prove the authenticity of our results and to distinguish them from existing ones, some illustrative examples are also given. Elliptic complex numbers are a generalized form of complex and so real numbers. Thus, the obtained results extend, generalize and complement some known Gershgorin discs results from the literature.

1. Introduction

The Gershgorin set that is composed of a union of discs, is a region that is inclusive of eigenvalues of matrices. This result is stated in the following theorem, where the deleted row sum R_i' of a complex matrix A with elements a_{ij} is defined as;

$$R_i' = \sum_{j=1,\, j\neq i}^{n} |a_{ij}|.$$

Theorem 1.1. *(Gershgorin's Theorem) All the eigenvalues of the $n \times n$ complex matrix A are located in the union of the discs n*

$$\bigcup_{i=1}^{n} \Gamma_i^R = \Gamma^R$$

where

$$\Gamma_i^R = \left\{ z \in C : \ |z - a_{ii}| \leq {R'}_i \right\}.$$

The Gershgorin set in matrix theory has important applications in modeling human faces, size reduction and data compression, signal and image processing-restoration, computational mathematics, some fields of pure and applied mathematics and so on [1]−[10]. With the rapid development of these fields, more and more researchers are interested in the Gershgorin set and have obtained many valuable results. For the Gershgorin set, they mainly consider real, complex and real quaternion matrices.

On the other hand, elliptic complex numbers are defined as

$$z = x + uy$$

where $x,\ y \in R$ and $u^2 = p < 0 \in R$. Since many physical systems have elliptical behaviors, elliptic complex number systems have many applications in science and technology, [11]−[18]. Thus, it is getting more and more necessary for us to further study the theoretical properties and numerical computations of elliptic complex numbers and their matrices.

In this study, we introduce concepts of the Gershgorin sets of the elliptic complex matrices and investigate the location of the zeros of an elliptic complex-valued polynomial with the help of this theory. To prove the authenticity of our results and to distinguish them from existing ones, some illustrative examples are also given. Elliptic numbers are a generalized form of complex and so real numbers. Thus, the obtained results extend, generalize and complement some known Gershgorin set results from the literature.

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2. Algebraic Properties of Elliptic Complex Numbers

The set of elliptic complex numbers is denoted by

$$C_p = \left\{ z = x + uy : \ x, y \in R, \ u^2 = p < 0 \right\}.$$

For a elliptic complex number $z = x + uy \in C_p$, the real number $\mathrm{Re}(z) = x$ is called the real part of z and $\mathrm{Im}(z) = y$ is called the imaginary part of z.
The conjugate and norm of elliptic complex number $z = x + uy$ are defined as

$$\bar{z} = x - uy \ \text{ and } \ \|z\|_p = \sqrt{z\bar{z}} = \sqrt{x^2 - py^2},$$

respectively.
Addition, multiplication and scalar multiplication of the elliptic complex numbers $z_1 = x_1 + uy_1, \ z_2 = x_2 + uy_2 \in C_p$ are defined by

$$z_1 + z_2 = (x_1 + uy_1) + (x_2 + uy_2) = x_1 + x_2 + u(y_1 + y_2),$$

$$z_1 z_2 = (x_1 + uy_1)(x_2 + uy_2) = (x_1 x_2 + p y_1 y_2) + u(x_1 y_2 + x_2 y_1),$$

$$\lambda z_1 = \lambda (x_1 + uy_1) = \lambda x_1 + u \lambda y_1, \quad \lambda \in R,$$

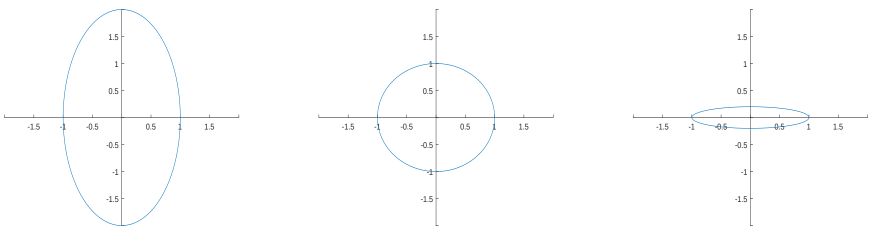
respectively, [19].

Theorem 2.1. *C_p is 2D vector space over a field R according to addition and scalar multiplication, [20].*

Also, each elliptic complex number can be represented in a single form in an elliptic plane. In the elliptic plane, the distance between of the elliptic complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined as

$$\|z_1 - z_2\|_p = \sqrt{(x_1 - x_2)^2 - p(y_1 - y_2)^2},$$

[19].
Unit circles are defined by requiring $\|z\|_p = \sqrt{x^2 - py^2} = 1$ as in Figure 2.1 In special case $p = -1$, the elliptic plane corresponds to the Euclidean plane.



(a) **(b)** **(c)**

Figure 2.1: Unit circles in $C_{-0.5}, C_{-1}, C_{-5}$.

Definition 2.2. *Let $z = x + uy \in C_p$ be given. C_p is algebraically isomorphic to complex numbers*

$$C = \left\{ x + iy \mid x, \ y \in R \ \text{ and } i^2 = -1 \right\}$$

through the bijective map

$$\begin{gathered} \alpha_p : C_p \to C \\ z = x + uy \to \alpha_p(z) = x + i\sqrt{|p|}y, \end{gathered}$$

[19].

The fundamental theorem of algebra for complex-valued polynomials: Let us consider the polynomial f with real coefficient and degree $N > 0$. So, thanks to the polynomial f, two algebraic curves are defined by

$$\mathrm{Re}(f(z)) = 0 \quad \text{and} \quad \mathrm{Im}(f(z)) = 0.$$

Each of these two algebraic curves consists of different continuous branches and these curves intersect the circle $\|z\| = r$ at $2N$ points. In addition, the crossing points of these two algebraic curves remain within this circle. This shows that the polynomial f has at least one real root, [21].
Now let's give the fundamental theorem of algebra for complex numbers that Gauss proved given above for elliptic complex-valued polynomials.

Theorem 2.3. *A $n-th$ degree polynomial function with elliptic complex coefficients and elliptic complex-valued*

$$f_p(z) = \sum_{i=0}^{n} a_i z^i = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

has exactly n zeros in the set of the elliptic complex numbers, counting repeated zeros.

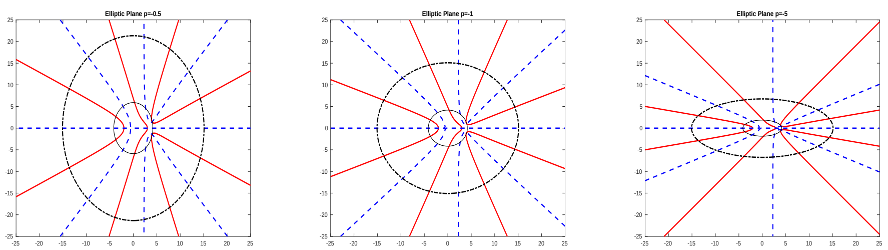
Proof. Let $z = x + uy \in C_p$ be given. Considering the Definition 2.2 we can express each elliptic complex number in terms of the complex number. Then we can take a complex number instead of an elliptic complex number using the equation $u = i\sqrt{|p|}$ and the equivalence $z = x + uy \equiv x + i\sqrt{|p|}y$ exists. Let $f_p(z)$ be the $n-th$ order monic polynomial with the elliptic complex-valued real coefficient

$$f_p(z) = z^N + c_{N-1}z^{N-1} + \ldots + c_1 z + c_0 = z^N + \sum_{n=0}^{N-1} c_n z^n,$$

where $z = x + uy \in C_p$ and $c_0, c_1, \ldots, c_{N-1} \in R$.
Considering the equation $z = x + uy \equiv x + i\sqrt{|p|}y$, the polynomial $f_p(z)$ becomes a monic polynomial with $n-th$ order complex-valued real coefficients. Therefore, according to the fundamental theorem of algebra for complex valued and complex coefficients, the polynomial f has at least one root. Let $z_0 = x_0 + iy_0$ be one root of $f_p(z)$. In this case, taking into account the equality $u = i\sqrt{|p|}$, the equality $z_0 = x_0 + \frac{u}{\sqrt{|p|}}y_0$ can be written and z_0 becomes a root of the polynomial f_p. As a result, every $n-th$ order monic polynomial with elliptic complex-valued and real coefficient has at least one root. □

This theorem is true for all polynomials with elliptic complex coefficients. To see why this is true, suppose the theorem holds for elliptic complex-valued polynomials with real coefficients, and let $f_p(z) = z^N + c_{N-1}z^{N-1} + \ldots + c_1 z + c_0$, be an elliptic complex valued polynomial with elliptic complex coefficients. Let $\overline{f_p}(z) = z^N + \overline{c_{N-1}}z^{N-1} + \ldots + \overline{c_0}$ be the polynomial whose coefficients are the elliptic conjugates of the coefficients of f, and let $g_p(z) = f_p(z)\overline{f_p}(z) = f_p(z)\overline{f_p(z)}$. Then g is a polynomial with real coefficients, so by assumption it has a root z_0. This means that $g_p(z_0) = f_p(z)\overline{f_p}(\overline{z})$, so either z_0 or $\overline{z_0}$ is a root of f.

Also, according to our theorem, the two curves $\mathrm{Re}\left(f_p(z)\right) = 0$ and $\mathrm{Im}\left(f_p(z)\right) = 0$ must intersect at some point in the interior of the elliptic disc. At this intersection point, the real and imaginary parts of $f_p(z)$ are both 0, so $f_p(z) = 0$; in other words, the intersection point is a root of f.



(a) **(b)** **(c)**

Figure 2.2: The solid red lines are the points where $\mathrm{Re}(f_p(z)) = 0$ and the dashed blue lines are the points where $\mathrm{Im}(f_p(z)) = 0$ in elliptic plane for the elliptic complex valued polynomial $f_p(z) = z^4 - 9z^3 + 19z^2 + 31z - 102$. In here intersection points of algebraic curves $\mathrm{Re}(f_p(z)) = 0$ and $\mathrm{Im}(f_p(z)) = 0$ in elliptic plane are root of $f_p(z)$. Also, the large, solid ellipse is $|z| = r^*$, and the smaller, dotted ellipse is $|z| = r_0$.

The set $C_p^{m\times n}$ denotes all $m\times n$ type matrices with elliptic complex number entries. For $A = A_1 + uA_2$, $B = B_1 + uB_2 \in C_p^{m\times n}$, $C = C_1 + uC_2 \in C_p^{n\times l}$ the ordinary matrix addition, scalar multiplication and multiplication are defined by

$$A + B = (A_1 + uA_2) + (B_1 + uB_2) = (A_1 + B_1) + u(A_2 + B_2) \in C_p^{m\times n},$$

$$\lambda A = \lambda(A_1 + uA_2) = \lambda A_1 + u(\lambda A_2) \in C_p^{m\times n}$$

and

$$AC = (A_1 + uA_2)(C_1 + uC_2) = (A_1C_2 + pA_2C_2) + u(A_1C_2 + A_1C_1) \in C_p^{m\times l},$$

respectively.

Theorem 2.4. *[22] Let A and B be elliptic complex matrices of appropriate sizes. Then the following are satisfied:*

1. $\left(A^{-1}\right)^{-1} = A$,
2. $(AB)^{-1} = B^{-1}A^{-1}$,
3. $\left(A^k\right)^{-1} = \left(A^{-1}\right)^k$, $k \in Z^+$,
4. $\left(A^T\right)^T = A$,
5. $(\lambda A)^T = \lambda A^T$,

6. $(AB)^T = B^T A^T$,
7. $\left(A^k\right)^T = \left(A^T\right)^k$, $k \in Z^+$,
8. $\overline{(\overline{A})} = A$, $(A^*)^* = A$,
9. $\overline{(A+B)} = \overline{A} + \overline{B}$, $(A+B)^* = A^* + B^*$,
10. $\overline{(AB)} = \overline{A}\,\overline{B}$, $(AB)^* = B^* A^*$.

Definition 2.5. *Let $A \in C_p^{n\times n}$, $\lambda \in C_p$. If there exists $0 \neq x \in C_p^{n\times 1}$ such that*

$$Ax = \lambda x$$

then λ is called a eigenvalues of A and x is called a eigenvector of A associate with λ. The set of eigenvalues of elliptic complex matrix A is defined as

$$\sigma_p(A) = \left\{\lambda \in C_p : Ax = \lambda x, \ \exists x \neq 0\right\}.$$

Theorem 2.6. *Elliptic complex matrix $A \in C_p^{n\times n}$ has exactly n elliptic eigenvalues.*

Proof. Since the characteristic polynomial $f_A(s) = \det(A - sI)$ of matrix $A \in C_p^{n\times n}$ will be an $n-th$ order polynomial, from the fundamental theorem of algebra for elliptic complex numbers, matrix A has at most n eigenvalues. □

Example 2.7. *Let find the eigenvalues of the elliptic complex matrix*

$$A = \begin{pmatrix} 1+u & 0 & 1 \\ 0 & u & 0 \\ 1 & 0 & 1-u \end{pmatrix} \in C_p^{3\times 3}.$$

Characteristic polynomial of the elliptic complex matrix A is $f_A(s) = \det(A - sI_{2\times 2}) = (s-u)(s^2 - 2s - p)$. Zeros of f_A are

$$s_1 = u, \quad s_2 = 1 + \sqrt{1-p}, \quad s_3 = 1 - \sqrt{1-p}.$$

These roots are also the eigenvalues of the matrix A. The eigenvalues of the matrix A according to the values p are given in the table below.

	λ_1	λ_2	λ_3
$p = -0.5$	1.7071	0.2929	u
$p = -1$	1	1	u
$p = -5$	$1 + 0.8944u$	$1 - 0.8944u$	u

Table 1: The eigenvalues of the matrix A according to the values p.

Theorem 2.8. *Each eigenvalue of the elliptic complex matrix $A \in C_p^{n\times n}$ is inside at least one of the ellipses $D_i(A)$ in the elliptic plane,*

$$D_i(A) = \left\{z : \|z - a_{ii}\|_p \leq R_i, \ 1 \leq i \leq n\right\}$$

in here $R_i = \sum\limits_{j\neq i} \left\|a_{ij}\right\|_p$. In other words, all the eigenvalues of matrix A are in the region $D(A)$,

$$D(A) = \bigcup_{i=1}^{n} D_i(A).$$

Proof. Let's admit that λ is an eigenvalue of matrix A. In this case, there is a non-zero vector $x = (x_1, x_2, ..., x_n)^T \in C_p^{n\times 1}$ such that $Ax = \lambda x$. Let's say x_k is the largest component of x, so

$$\|x_k\|_p = \max\left\{\|x_i\|_p, \ 1 \leq i \leq n\right\} > 0$$

is. In this case

$$\sum_{j\neq k} a_{kj} x_j = (\lambda - a_{kk})$$

can be written from the equation $a_{k1}x_1 + a_{k2}x_2 + ... + a_{kn}x_n = \lambda x_k$. In the last equation, the norm of both sides of the equation is taken and if the triangle inequality is used,

$$\begin{aligned} \|\lambda - a_{kk}\|_p \|x_k\|_p &= \left\|\sum_{j\neq k} a_{kj} x_j\right\|_p \\ &\leq \sum_{j\neq k} \left\|a_{kj}\right\|_p \left\|x_j\right\|_p \\ &\leq \left(\sum_{j\neq k} \left\|a_{kj}\right\|_p\right) \|x_k\|_p \end{aligned}$$

is obtained. Here,

$$\|\lambda - a_{kk}\|_p \le \sum_{j\neq k} \|a_{kj}\|_p$$

inequality is obtained.
If it is called

$$R_i = \sum_{j\neq i} \|a_{ij}\|_p = \|a_{i1}\|_p + \|a_{i2}\|_p + \ldots + \left\|a_{i(i-1)}\right\|_p + \left\|a_{i(i+1)}\right\|_p + \ldots + \|a_{in}\|_p \ \ (i = 1,2,3,\ldots,n),$$

it is proved that each eigenvalue of the elliptic matrix $A \in C_p^{n\times n}$ is inside at least one of ellipses in the elliptic plane

$$D_i(A) = \left\{ z : \|z - a_{ii}\|_p \le R_i,\ 1 \le i \le n \right\}.$$

□

Example 2.9. *Let*

$$A = \begin{pmatrix} 4-3u & u & 2 & -2 \\ u & -1+u & 0 & 0 \\ 1+u & -u & 5+6u & 2u \\ 1 & -2u & 2u & -5-5u \end{pmatrix} \in C_p^{4\times 4}.$$

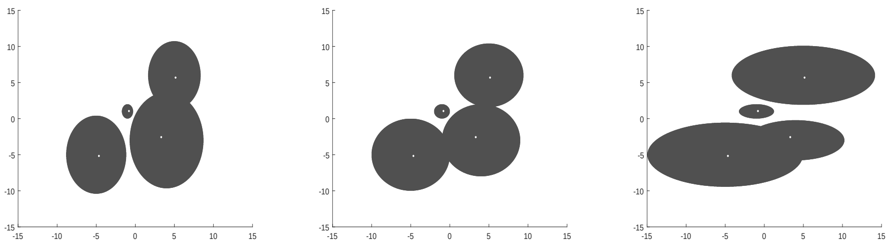
According to the Theorem 2.8 we have

$$R_1 = 4+\sqrt{-p},\ R_2 = \sqrt{-p},\ R_3 = \sqrt{1-p}+3\sqrt{-p} \text{ and } R_4 = 1+4\sqrt{-p}.$$

For elliptic complex matrix A there are Gershgorin disc:

$$\begin{aligned} &D_1 : (x-4)^2 - p(y+3)^2 \le (4+\sqrt{-p})^2 \\ &D_2 : (x+1)^2 - p(y-1)^2 \le (\sqrt{-p})^2 \\ &D_3 : (x-5)^2 - p(y-6)^2 \le \left(\sqrt{1-p}+3\sqrt{-p}\right)^2 \\ &D_4 : (x+5)^2 - p(y+5)^2 \le (1+4\sqrt{-p})^2. \end{aligned}$$

In the elliptic plane, regions of the eigenvalues of the matrix A according to the state of p are as shown in the lower graph.



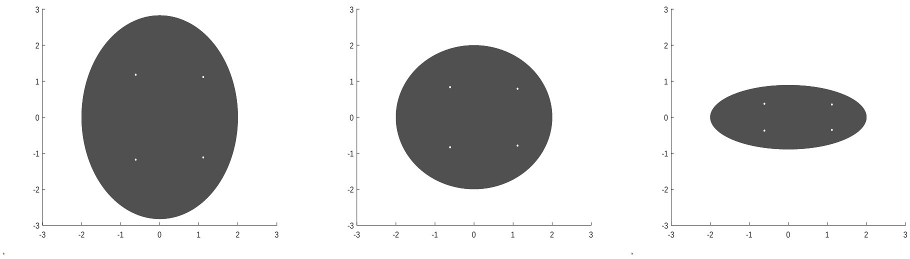
(a) (b) (c)

Figure 2.3: In elliptic plane Gershgorin discs for $p=-0.5, p=-1$ and $p=-5$, *respectively.*

Locate Zeros of Polynomials: Eigenvalue inclusion sets can be used to locate zeros of elliptic valued polynomials by using the polynomial's companion matrix, whose characteristic polynomial is the given polynomial, [23]−[28]. Thus, its eigenvalues are the zeros of the polynomial. The companion matrix of elliptic valued monic polynomial $f_p(z) = \sum\limits_{i=0}^{n} a_i z^i = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ is

$$C_p(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

Figure 2.4 shows regions of the zeros of the elliptic valued polynomial $f_p(z) = z^4 - z^3 + 0.2z^2 - 0.1z + 2$ according to the state of p. The zeros are indicated by the white dots.



(a) (b) (c)

Figure 2.4: Regions of the zeros of the elliptic valued polynomial $f_p(z)$ for $p=-0.5$, $p=-1$ and $p=-5$, respectively.

3. Conclusions

In this study, we derive Gershgorin discs of elliptic complex matrices in the elliptic plane. Eigenvalues of matrices have important applications in modeling human faces, gene analysis, information retrieval and extraction, size reduction and data compression, signal and image processing-enhancement processes. The use of elliptic matrices in these application areas will enable the previously known definitions and theorems to be interpreted with a wider perspective, and by selecting the ideal space for the problems, great flexibility and efficiency will be brought to existing techniques.

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