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Meromorphic Solutions to Certain Differential-Difference Equations

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Abstract

The aim of this paper is to investigate the growth and constructions of meromorphic solutions of the nonlinear differential-difference equation

$$f^n(z) + h(z)\Delta_c f^{(k)}(z) = A_0(z) + A_1(z)e^{\alpha_1 z^q} + \cdots + A_m(z)e^{\alpha_m z^q},$$

where $n, m, q \in \mathbb{N}^+$, $\alpha_1, \cdots, \alpha_m$ are distinct nonzero complex numbers, $h(z)$ is a nonzero entire function and $A_j(z)$ $(0 \le j \le m)$ are meromorphic functions. In particular, for $A_0(z) \equiv 0$, we give the exact form of meromorphic solutions of the above equation under certain conditions. In addition, our results are shown to be sharp.

Keywords

Nevanlinna Theory; Meromorphic Solutions; Nonlinear; Differential-Difference Equations.

1. Introduction and Main Results

Nevanlinna theory is an important tool in this paper, and we assume that the reader is familiar with its standard notations and terms such as $T(r,f)$, $m(r,f)$, etc. (see, [5, 7]). The order $\rho(f)$ and the convergence exponent of zero-sequence $\lambda(f)$ of meromorphic function $f(z)$ are respectively defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r,\frac{1}{f})}{\log r}.$$

A meromorphic function $\alpha(z)$ on $\mathbb{C}$ is called a small function with respect to f if $T(r,\alpha) = S(r,f)$, where $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set of r of finite linear measure.

One important aspect of the studies for complex differential equations is to investigate the properties of their meromorphic solutions (see, e.g. [5, 7, 11, 14, 18]). In recent decades, the Tumura-Clunie type differential equations have attracted much attention and various interesting results have been derived. Among them are the following results.

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Theorem 1. ([19]) *The differential equation* $4f^3(z) + 3f''(z) = -\sin 3z$ *has exactly three entire solutions*

$$f_1(z) = \sin z, f_2(z) = (\sqrt{3}\cos z)/2 - (\sin z)/2, f_3(z) = -(\sqrt{3}\cos z)/2 - (\sin z)/2.$$

Theorem 2. ([8, Theorem 4]) *Let* a, p_1, p_2 *and* λ *be nonzero constants. Then the differential equation*

$$f^3(z) + af''(z) = p_1e^{\lambda z} + p_2e^{-\lambda z}$$

has transcendental entire solutions if and only if $p_1p_2 + (a\lambda^2/27)^3 = 0$*. Further, these entire solutions are*

$$f(z) = \alpha_j e^{\frac{\lambda z}{3}} - \left(\frac{a\lambda^2}{27\alpha_j}\right) e^{\frac{-\lambda z}{3}}, \quad j = 1, 2, 3,$$

where $\alpha_j^3 = p_1$*.*

Along this direction, researchers have extensively studied the properties of solutions to differential equations

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$$

(see, e.g. [9, 10, 11, 14]), where $P_d(z, f)$ is a polynomial in f and its derivatives with small coefficients, $p_1(z)$, $p_2(z)$ are small functions of f, and $\alpha_1(z)$, $\alpha_2(z)$ are nonzero polynomials. With the help of the difference version of Nevanlinna theory, many scholars also considered the difference analogue of the above equation, and obtained some related results (see, e.g. [12, 13, 15, 17, 21]). Let $f(z)$ be a nonconstant meromorphic function and c be a nonzero constant. We define the difference operator as $\Delta_c f(z) := f(z + c) - f(z)$. In 2014, Liu et al. [12] studied expression of entire solutions of the difference equation

$$f^n(z) + q(z)\Delta_1 f(z) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}, \tag{1}$$

where $q(z)$ is a polynomial and $p_1, p_2, \alpha_1, \alpha_2(\neq \alpha_1)$ are nonzero constants. We rewrite their result as follows:

Theorem 3. ([12]) *Let* $n \geq 4$ *be an integer. If there exists some finite order entire solution* f *of* (1)*, then* $q(z)$ *is a constant, and one of the following relations holds:*
(i) $f(z) = c_1e^{\alpha_1 z/n}$*, and* $c_1(e^{\alpha_1/n} - 1)q = p_2, \alpha_1 = n\alpha_2$*,*
(ii) $f(z) = c_2e^{\alpha_2 z/n}$*, and* $c_2(e^{\alpha_2/n} - 1)q = p_1, \alpha_2 = n\alpha_1$*,*
where c_1, c_2 *are constants satisfying* $c_1^n = p_1, c_2^n = p_1$*.*

Latreuch [13] and Zhang et al. [21] further studied the structure and growth of entire solutions of (1) independently for $n = 3$, and obtained some similar results as in Theorem C. Recently, Li, Hao and Yi [16] used Cartan's version of the second main theorem to consider the growth of solutions to a difference equation

$$f^n(z) + p(z)\Delta_c f(z) = H_1(z)e^{\alpha_1 z^q} + \cdots + H_m(z)e^{\alpha_m z^q}, \tag{2}$$

and obtained the following result.

Theorem 4. ([16, Theorem 1.6]) *Let* m, n *be two positive integers satisfying* $n \geq m + 2$*,* $p(z)$ *be a nonzero polynomial,* c *be a nonzero complex number such that* $\Delta_c f(z) \not\equiv 0$*,* $\omega_1, \cdots, \omega_m$ *be* m *distinct nonzero complex numbers, and let* H_j $(1 \leq j \leq m)$ *be either exponential polynomials of degree less than* q*, or polynomials in* z *such that* $H_j \not\equiv 0$ $(1 \leq j \leq m)$*. If* (2) *admits a nonconstant meromorphic solution, then* $m \geq 2$ *and* f *reduces to a transcendental entire function such that* $\rho(f) = \infty$*, or satisfies* $\rho(f) = q$ *with* $m = 2$*, while* f *can be expressed as either*

$$f(z) = A_1(z)e^{\omega_1 z^q}, \quad \text{with} \quad A_1(z) = \frac{H_1 f}{p\Delta_c f} \quad \text{and} \quad n\omega_1 - \omega_2 = 0,$$

or

$$f(z) = A_2(z)e^{\omega_2 z^q}, \quad \text{with} \quad A_2(z) = \frac{H_2 f}{p\Delta_c f} \quad \text{and} \quad n\omega_2 - \omega_1 = 0,$$

where $A_1(z)$ *and* $A_2(z)$ *are small entire functions with respect to* f*.*

We note that, in Theorem 4, the authors only considered the case when $n \geq m+2$ ($m \in \mathbb{N}^+$) and the coefficients $H_j(z)$ $(1 \leq j \leq m)$ on the right side of (2) are entire functions. It is natural to pose the following question: what can be said for meromorphic solutions of (2) when $n \leq m+1$ and the entire coefficients $H_j(z)$ $(1 \leq j \leq m)$ are replaced by meromorphic functions of order less than q?

The aim of this paper is to answer the above questions. Besides that, we consider the properties of meromorphic solutions to a more general nonlinear differential-difference equation

$$f^n(z) + h(z)\Delta_c f^{(k)}(z) = A_0(z) + A_1(z)e^{\alpha_1 z^q} + \cdots + A_m(z)e^{\alpha_m z^q}, \tag{3}$$

where $n, m \in \mathbb{N}^+, k \in \mathbb{N}$, $\alpha_1, \cdots, \alpha_m$ are distinct nonzero complex numbers, $h(z)$ is an entire function, and $A_j(z)(0 \leq j \leq m)$ are meromorphic functions. Our main results are as follows:

Theorem 5. *Let $n \geq 2, m, k$ be positive integers, and let c be a constant such that $\Delta_c f^{(k)}(z) \not\equiv 0$. Suppose that $h(z)$ is a nonzero entire function with $\rho(h) < q$, and that $A_0(z), A_1(z), \cdots, A_m(z)$ are meromorphic functions with finitely many poles satisfying $A_i(z) \not\equiv 0 (1 \leq i \leq m)$ and $\rho(A_j) < q (0 \leq j \leq m)$. If (3) admits a meromorphic solution f such that $N(r, f) = S(r, f)$, then $\rho(f) = \infty$, or $\rho(f) = q$ and the following facts hold:*

(i) *When $A_0(z) \equiv 0$, we have two possibilities:*
 (1) *$m = 2$ and $f(z) = \tau_0(z)e^{\alpha_t z^q/n}$, where $\tau_0^n(z) = A_t(z)$, $\alpha_t = n\alpha_{t'} (t, t' \in \{1, 2\}, t \neq t')$.*
 (2) *$\lambda(f) = \rho(f) = q$ and $n \leq m+1$.*

(ii) *When $A_0(z) \not\equiv 0$, we have $\lambda(f) = \rho(f) = q$ and $n \leq m+2$.*

We now give some examples such that the conditions in Theorem 5 hold.

Example 1. *The meromorphic function $f(z) = e^{iz}/z$ satisfies the nonlinear differential-difference equation*

$$f^4(z) + z^2(z+4\pi)^2(f'(z+4\pi) - f'(z)) = \frac{1}{z^4}e^{4iz} + (-4\pi i z^2 + 8\pi z - 16\pi^2 iz + 16\pi^2)e^{iz}.$$

Here $n = m+2$, $\tau_0(z) = 1/z$. Set $A_1(z) = 1/z^4$, then $A_1(z) = \tau_0^4(z)$ and $\alpha_1 = 4\alpha_2$.

Example 2. *The meromorphic function $f(z) = e^{iz}/z + z$ satisfies the equation*

$$f^2(z) - \frac{z^2}{2\pi}(f(z+2\pi) - f(z)) = \frac{1}{z^2}e^{2iz} + \frac{3z+4\pi}{z+2\pi}e^{iz}.$$

Here $A_0(z) \equiv 0$, $m = 2$, $n = 2 < m+1$ and $\lambda(f) = \rho(f) = 1$.

Example 3. *The meromorphic function $f(z) = 1/z + e^{2\pi z}$ satisfies the differential-difference equation*

$$f^3(z) + (z+i)^2(f'(z+i) - f'(z)) = \frac{2z^2 i - z + 1}{z^3} + \frac{3}{z^2}e^{2\pi z} + \frac{3}{z}e^{4\pi z} + e^{6\pi z}.$$

Here $A_0(z) = (2z^2 i - z + 1)/z^3 \not\equiv 0$, $m = 3$, $n < m+2$ and $\lambda(f) = \rho(f) = 1$.

The following corollary, which can be derived immediately from Theorem 5, is an extension of Theorem 4.

Corollary 1. *Under the conditions of Theorem 5, let f be a finite order meromorphic solution of the difference equation*

$$f^n(z) + h(z)\Delta_c f(z) = A_0(z) + A_1(z)e^{\alpha_1 z^q} + \cdots + A_m(z)e^{\alpha_m z^q}.$$

If $N(r, f) = S(r, f)$, then $\rho(f) = q$ and the following assertions hold.

(i) *When $A_0(z) \equiv 0$, we have two possibilities:* (1) *$m = 2$ and $f(z) = \tau_0(z)e^{\frac{\alpha_t z^q}{n}}$, where $\tau_0^n(z) = A_t(z)$, $\alpha_t = n\alpha_{t'} (t, t' \in \{1, 2\}, t \neq t')$;* (2) *$\lambda(f) = \rho(f) = q$ and $n \leq m+1$.*

(ii) *When $A_0(z) \not\equiv 0$, we have $\lambda(f) = \rho(f) = q$ and $n \leq m+2$.*

Note that in Theorem 5 the entire function $h(z)$ satisfies the condition $\rho(h) < q$. Next, we continue to consider the case of $\rho(h) \geq q$, and obtain the following result.

Theorem 6. *Let n, q be positive integers, let $A_0(z), \cdots, A_m(z)$ be meromorphic functions of order less than q such that $A_i(z) \not\equiv 0 (1 \leq i \leq m)$. Suppose that $\Delta_c f^{(k)}(z) \not\equiv 0$, and that $h(z)$ is a nonzero entire function satisfying $\rho(h) \geq q$ and $\lambda(h) < \rho(h)$. Then for any finite order transcendental meromorphic function solution f of (3) satisfying $N(r, f) = S(r, f)$, we have*

$$\rho(f) \geq \rho(h).$$

In particular, we have $\rho(f) = \rho(h)$ provided that $n \geq 2$.

We will give two examples that the conditions of Theorem 6 hold.

Example 4. *The differential-difference equation*

$$f^n(z) + e^z \Delta_c f^{(k)}(z) = e^{nz} + (e^c - 1)e^{2z}$$

has a solution $f(z) = e^z$, where $h(z) = e^z$ and $\rho(f) = \rho(h) = 1$.

Example 5. *The equation*

$$f^3(z) + e^z(f(z + 2\pi i) - f(z)) = \frac{1}{z^3}e^{3z} + (\frac{1}{z + 2\pi i} - \frac{1}{z})e^{2z}$$

has a solution $f(z) = e^z/z$, where $h(z) = e^z$ and $\rho(f) = \rho(h) = 1$.

By Theorem 6, we can also deduce the following corollary.

Corollary 2. *Under the conditions of Theorem 6, let f be a finite order meromorphic solution of the difference equation*

$$f^n(z) + h(z)\Delta_c f(z) = A_0(z) + A_1(z)e^{\alpha_1 z^q} + \cdots + A_m(z)e^{\alpha_m z^q}.$$

If $N(r, f) = S(r, f)$, then we have $\rho(f) \geq \rho(h) \geq q$.

The remainder of this paper is organized as follows: in Section 2 we state several results that will be used in our proofs. The details of the proofs of Theorems 5 and 6 are shown in Sections 3 and 4, respectively.

2. Auxiliary Lemmas

In the following, let E be a set of finite linear measure, respectively, not necessarily the same at each occurrence.

Lemma 1. *[2] Let $f(z)$ be a meromorphic function of finite order ρ, and let η be a fixed nonzero complex number. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\rho-1+\varepsilon}),$$

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r),$$

and

$$N(r, f(z+\eta)) = N(r, f) + O(r^{\lambda(1/f)-1+\varepsilon}) + O(\log r),$$

where the symbol $\lambda(1/f)$ here represents the exponent of convergence of poles of f.

Remark 1. *By [3, Theorem 1.3.1′] and the lemma of the logarithmic derivative [7], we can also get*

$$N\left(r, \frac{1}{f(z+\eta)}\right) = N\left(r, \frac{1}{f}\right) + S(r,f) \text{ and } m\left(r, \frac{f^{(k)}(z+\eta)}{f(z)}\right) = S(r,f),$$

as $r \to \infty$ outside a possible exceptional set E.

Lemma 2. *[17, Lemma 2.5] Let $m, q \in \mathbb{N}^+$, $\alpha_1, \cdots, \alpha_m$ be distinct nonzero complex numbers, and $A_0(z), \cdots, A_m(z)$ be nonzero meromorphic functions of order less than q. Set $\varphi(z) = A_0(z) + \sum_{i=1}^m A_i(z)e^{\alpha_i z^q}$, then the following results hold.*

1. *There exist two positive numbers $d_1 < d_2$, such that*
$$d_1 r^q \le T(r,\varphi) \le d_2 r^q, \quad (r \to \infty).$$
2. *If $A_0 \not\equiv 0$, then $m(r, 1/\varphi) = o(r^q)$ as $r \to \infty$.*

Lemma 3. *Under the conditions of Theorem 5, if f is a finite order meromorphic solution of (3) satisfying $N(r,f) = S(r,f)$, then $\rho(f) = q$. Specially, if $A_0 \not\equiv 0$, then*

$$N\left(r, \frac{1}{f}\right) = T(r,f) + S(r,f).$$

Proof. Applying Lemmas 2 to equation (3), one can deduce that

$$\begin{aligned} d_1 r^q &\le T(r, f^n(z) + h(z)(f^{(k)}(z+c) - f^{(k)}(z))) \\ &\le m(r, f^n) + m(r, h(z)(f^{(k)}(z+c) - f^{(k)}(z))) + \sum_{j=0}^m N(r, A_j) + O(1) \\ &\le (n+1)m(r,f) + m(r,h) + m\left(r, \frac{f^{(k)}(z+c) - f^{(k)}(z)}{f(z)}\right) + \sum_{j=0}^m N(r, A_j) + O(1), \end{aligned}$$

where d_1 is a positive constant. With Remark 1, $\rho(h) < q$ and $\rho(A_j) < q (0 \le j \le m)$, we have

$$d_1 r^q \le (n+1)T(r,f) + S(r,f) + o(r^q), \tag{4}$$

as $r \to \infty, r \notin E$. Now, we rewrite (3) as

$$f^n(z) = A_0(z) + \sum_{i=1}^m A_i(z)e^{\alpha_i z^q} - h(z)f(z)\frac{f^{(k)}(z+c) - f^{(k)}(z)}{f(z)}.$$

By Lemma 2, there exist $d_2 > d_1$, such that for sufficiently large r,

$$\begin{aligned} (n-1)m(r,f) &\le m(r, A_0) + m(r, \sum_{i=1}^m A_i(z)e^{\alpha_i z^q}) + m(r, h(z)) + S(r,f) \\ &\le d_2 r^q + S(r,f) + o(r^q). \end{aligned} \tag{5}$$

Note that $n \ge 2$ and $N(r,f) = S(r,f)$. It follows from (4) and (5) that

$$C_1 r^q \le T(r,f) \le C_2 r^q, \ (r \to \infty, r \notin E),$$

where C_1, C_2 are two positive numbers. This implies that $\rho(f) = q$.

If $A_0(z) \not\equiv 0$, we can also rewrite (3) as follows:

$$\frac{1}{A_0(z) + \sum_{i=1}^m A_i(z)e^{\alpha_i z^q}} + \frac{h(z)}{A_0(z) + \sum_{i=1}^m A_i(z)e^{\alpha_i z^q}} \cdot \frac{\Delta_c f^{(k)}(z)}{f^n(z)} = \frac{1}{f^n(z)}.$$

By the second conclusion of Lemma 2 and Remark 1, we get $m(r, 1/f) = S(r,f)$. Together with this fact and the first main theorem, we get $N(r, 1/f) = T(r,f) + S(r,f)$. This completes the proof of Lemma 3. $\square$

Lemma 4. *[18] Let $f(z)$ be a nonconstant meromorphic function, and k be a positive integer. Then, for $r \to \infty, r \notin E$,*

$$N\left(r, \frac{1}{f^{(k)}(z)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + k\overline{N}(r, f(z)) + S(r, f).$$

Furthermore, if f is a transcendental meromorphic function, we have

$$T(r, f^{(k)}) = T(r, f) + k\overline{N}(r, f) + S(r, f).$$

Let $f(z)$ be a nonconstant meromorphic function and let p be a positive integer. We denote by $n_p(r, 1/f)$ the number of zeros of f in $\{z : |z| \leq r\}$, counted in the following manner: a zero of f of multiplicity m is counted exactly $k = \min\{m, p\}$ times, and its corresponding integrated counting function is denoted by $N_p(r, 1/f)$.

Lemma 5. *[1, 4] Let $f_1, f_2, \cdots, f_p$ be linearly independent entire functions. Suppose that for each complex number z, we have $\max\{|f_1(z)|, \cdots, |f_p(z)|\} > 0$. Set*

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta - u(0), \text{ for } r > 0,$$

where $u(z) = \sup_{1\leq j\leq p} \log |f_j(z)|$. Let $f_{p+1} = f_1 + \cdots + f_p$. Then

$$T(r) \leq \sum_{j=1}^{p+1} N_{p-1}\left(r, \frac{1}{f_j}\right) + S(r) \leq (p-1)\sum_{j=1}^{p+1} \overline{N}\left(r, \frac{1}{f_j}\right) + S(r),$$

where $S(r)$ is a quantity satisfying $S(r) = O(\log(rT(r)))$ as $r \to \infty, r \notin E$. Furthermore, for any j and m, $1 \leq j \neq m \leq p + 1$, we have

$$T\left(r, \frac{f_j}{f_m}\right) = T(r) + O(1) \ (r \to \infty),$$

and

$$N\left(r, \frac{1}{f_j}\right) = T(r) + O(1) \ (r \to \infty).$$

Remark 2. *[1, 4] If at least one of the quotients f_j/f_m is a transcendental function, then $S(r) = o(T(r))$ $(r \to \infty, r \notin E)$, while if all the quotients f_j/f_m are rational functions, then $S(r) \leq -\frac{1}{2}p(p-1)\log r + O(1)$ $(r \to \infty, r \notin E)$.*

Lemma 6. *[17, 18] Let $f_1, f_2, \cdots, f_p$ be linearly independent meromorphic functions such that $\sum_{j=1}^{p} f_j = 1$. Then for $1 \leq j \leq p$, we have*

$$T(r, f_j) \leq \sum_{k=1}^{p} N\left(r, \frac{1}{f_k}\right) + (p-1)\sum_{k=1}^{p} \overline{N}(r, f_k) - N\left(r, \frac{1}{D}\right) + o\left(\max_{1\leq k\leq p}\{T(r, f_k)\}\right)$$

as $r \to \infty$ and $r \notin E$, where D is the Wronskian determinant $W(f_1, f_2, \cdots, f_p)$.

Lemma 7. *[6, Theorem 2.4] Let c be a nonzero complex number, let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Assume that $q \geq 2 (\in \mathbb{N}^+)$, and that $a_1(z), \cdots, a_q(z)$ are distinct meromorphic periodic functions with period c such that $T(r, a_k) = S(r, f)$ for $1 \leq k \leq q$. Then*

$$m(r, f) + \sum_{k=1}^{q} m\left(r, \frac{1}{f - a_k}\right) \leq 2T(r, f) - N_{pair}(r, f) + S(r, f),$$

where

$$N_{pair}(r, f) := 2N(r, f) - N(r, \Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right)$$

and the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

Lemma 8. *[18, Theorem 1.51] Let $f_i(z)$ $(i=1,2,\cdots,n(n\geq 2))$ be meromorphic functions, and $g_i(z)$ $(i=1,2,\cdots,n)$ be entire functions satisfying*
(1)$\sum_{i=1}^{n} f_i(z)e^{g_i(z)} \equiv 0$;
(2)$g_j(z)-g_m(z)$ are not constants for $1\leq j<m\leq n$;
(3)For $1\leq i\leq n, 1\leq t<k\leq n$, $T(r,f_i)=o(T(r,e^{g_t-g_k}))$ as $r\to\infty, r\notin E$.
Then $f_i(z)\equiv 0$ for all $i=1,\cdots,n$.

Lemma 9. *Let $f(z)$ be a nonconstant meromorphic function of $\rho(f)=q$ and $N(r,f)=S(r,f)$. Then, for $r\to\infty, r\notin E$,*

$$N\left(r,\frac{1}{\Delta_c f^{(k)}(z)}\right)\leq N(r,\frac{1}{f(z)})+S(r,f).$$

Proof. Since $\rho(f^{(k)})=\rho(f)$, it follows from $N(r,f)=S(r,f)$ and Lemma 1 that

$$N(r,f^{(k)}(z))\leq (k+1)N(r,f(z))=S(r,f) \tag{6}$$

and

$$\begin{aligned}N(r,\Delta_c f^{(k)}(z))&\leq N(r,f^{(k)}(z+c))+N(r,f^{(k)}(z))\\&=2N(r,f^{(k)}(z))+S(r,f^{(k)})=S(r,f).\end{aligned} \tag{7}$$

Then by (6), (7) and Lemma 7, we have

$$\begin{aligned}T(r,f^{(k)})\leq{}& N(r,f^{(k)})+N\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)\\&-N\left(r,\frac{1}{\Delta_c f^{(k)}}\right)+N(r,\Delta_c f^{(k)})-2N(r,f^{(k)})+S(r,f^{(k)})\\\leq{}& N\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)-N\left(r,\frac{1}{\Delta_c f^{(k)}}\right)+S(r,f).\end{aligned} \tag{8}$$

In view of $\rho(f)=q$, (8) and Lemma 4, we obtain that

$$\begin{aligned}N\left(r,\frac{1}{\Delta_c f^{(k)}}\right)&\leq N\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)-T(r,f^{(k)})+S(r,f)\\&\leq N\left(r,\frac{1}{f}\right)+o(r^q),\quad (r\to\infty, r\notin E).\end{aligned}$$

□

3. The proof of Theorem 5

Let f be a meromorphic solution of (3) satisfying $N(r,f)=S(r,f)$. Suppose that $\rho(f)<\infty$. Then, it follows from Lemma 3 that

$$\rho(f)=q \quad\text{and}\quad S(r,f)=o(r^q). \tag{9}$$

By Hadamard's factorization theorem, there exists an entire function $g_1(z)$ such that $f(z)g_1(z)$ is an entire function and

$$N\left(r,\frac{1}{g_1}\right)=N(r,f)=S(r,f).$$

Note that z_0 is a pole of order $l+k(\leq l(1+k))$ of $f^{(k)}(z)$ provided that $z_0\in\mathbb{C}$ is a pole of order l of $f(z)$. This implies that $g_1^{k+1}(z)f^{(k)}(z)$ is also an entire function.

Set $g(z) = g_1^{n+k+1}(z)g_2(z)$, where $g_2(z)$ consists of the poles of meromorphic functions $A_0(z), A_1(z), \cdots, A_m(z)$ (The poles of $A_0(z), A_1(z), \cdots, A_m(z)$ correspond to the zeros of $g_2(z)$). Then

$$N\left(r, \frac{1}{g}\right) \leq (n+k+1)N\left(r, \frac{1}{g_1}\right) + \sum_{j=0}^{m} N(r, A_j) = o(r^q), \tag{10}$$

and both $h(z)\Delta_c f^{(k)}(z)g(z)$ and $h(z)f^{(k)}(z+c)g(z)$ are entire functions.

Now we discuss the following two cases:

Case 1: $A_0(z) \equiv 0$. By $\frac{1}{f^n(z)} = \frac{h(z)\Delta_c f^{(k)}(z)}{f^n(z)} \cdot \frac{1}{h(z)\Delta_c f^{(k)}(z)}$, $\rho(h) < q$ and the first main theorem, we have

$$T\left(r, \frac{h(z)\Delta_c f^{(k)}(z)}{f^n(z)}\right) \geq nT(r, f) - T(r, \Delta_c f^{(k)}(z)) - o(r^q). \tag{11}$$

Since $h(z)\Delta_c f^{(k)}(z)g(z)$ is entire, it follows from $\rho(h) < q$ and (10) that

$$N\left(r, \Delta_c f^{(k)}(z)\right) \leq N\left(r, \frac{1}{h(z)}\right) + N\left(r, \frac{1}{g(z)}\right) = o(r^q). \tag{12}$$

Then by (11), (12) and Remark 1, there exists a positive constant D_1 such that

$$\begin{aligned} T\left(r, \frac{h(z)\Delta_c f^{(k)}(z)}{f^n(z)}\right) &\geq nT(r, f) - m\left(r, \frac{f^{(k)}(z+c) - f^{(k)}(z)}{f(z)} \cdot f(z)\right) - o(r^q) \\ &\geq (n-1)T(r, f) - o(r^q) \\ &\geq D_1 r^q, \qquad (r \to \infty, r \notin E). \end{aligned} \tag{13}$$

• First, we consider the case of $n \geq m+2$.

Subcase 1.1. Assume $h(z)f^{(k)}(z+c), h(z)f^{(k)}(z), A_1(z)e^{\alpha_1 z^q}, \cdots, A_m(z)e^{\alpha_m z^q}$ are linearly independent. Then $\Delta_c f^{(k)}(z) \not\equiv 0$ and

$$h(z)\Delta_c f^{(k)}(z), A_1(z)e^{\alpha_1 z^q}, \cdots, A_m(z)e^{\alpha_m z^q}$$

are $m+1$ linearly independent meromorphic functions.

By the definition of $g(z)$, we know $f^n(z)g(z), h(z)\Delta_c f^{(k)}(z)g(z), A_1(z)g(z), \cdots, A_m(z)g(z)$ are entire functions.

Let $\{a_{1,k}\}_{k=1}^{u}$ be the common zeros of $A_1(z)g(z)$, $\cdots$, $A_m(z)g(z)$, $f^n(z)g(z)$, $h(z)\Delta_c f^{(k)}(z)g(z)$, and $H_1(z) = \prod_{k=1}^{u}(z - a_{1,k})^{v_k}$, where v_k is the minimum of all the multiplicities of $a_{1,k}$ as the zero of $f^n(z)g(z)$, $h(z)\Delta_c f^{(k)}(z)g(z)$, $A_1(z)g(z)$, $\cdots$, $A_m(z)g(z)$, $u = \infty$ or finite integer. (If $f^n(z)g(z)$, $h(z)\Delta_c f^{(k)}(z)g(z)$, $A_1(z)g(z)$, $\cdots$, $A_m(z)g(z)$ have no common zeros, we set $H_1(z)$ is a nonzero constant.)

By $\rho(A_1) < q$ and (10), we obtain

$$N\left(r, \frac{1}{H_1(z)}\right) \leq N\left(r, \frac{1}{A_1(z)g(z)}\right) = o(r^q). \tag{14}$$

Rewrite (3) in the form

$$\frac{f^n(z)g(z)}{H_1(z)} = \sum_{i=1}^{m} \frac{A_i(z)e^{\alpha_i z^q}g(z)}{H_1(z)} - \frac{h(z)\Delta_c f^{(k)}(z)g(z)}{H_1(z)}, \tag{15}$$

where $\frac{f^n(z)g(z)}{H_1(z)}, \frac{h(z)\Delta_c f^{(k)}(z)g(z)}{H_1(z)}, \frac{A_1(z)e^{\alpha_1 z^q}g(z)}{H_1(z)}, \cdots, \frac{A_m(z)e^{\alpha_m z^q}g(z)}{H_1(z)}$ are entire functions without common zeros.

Since $n \geq m+2$, it follows from (13) that $\frac{h(z)\Delta_c f^{(k)}(z)g(z)}{H_1(z)} / \frac{f^n(z)g(z)}{H_1(z)}$ is transcendental. Then by (14), (15), Lemma 5 and Remark 2, we have

$$
\begin{aligned}
nN\left(r, \frac{1}{f(z)}\right) &\leq N\left(r, \frac{H_1(z)}{f^n(z)g(z)}\right) + N\left(r, \frac{g(z)}{H_1(z)}\right)\\
&= T_1(r) + o(r^q)\\
&\leq \sum_{i=1}^{m} N_m\left(r, \frac{H_1(z)}{A_i(z)e^{\alpha_i z^q}g(z)}\right) + N_m\left(r, \frac{H_1(z)}{h(z)\Delta_c f^{(k)}(z)g(z)}\right)\\
&\quad + N_m\left(r, \frac{H_1(z)}{f^n(z)g(z)}\right) + o(T_1(r)) + o(r^q) \qquad (16)\\
&\leq N\left(r, \frac{1}{\Delta_c f^{(k)}(z)}\right) + m\overline{N}\left(r, \frac{1}{f(z)}\right) + o(T_1(r)) + o(r^q),
\end{aligned}
$$

as $r \to \infty, r \notin E$, where $T_1(r) = \frac{1}{2\pi}\int_0^{2\pi} u_1(re^{i\theta})d\theta - u_1(0)$ with

$$
u_1(z) = \sup\left\{\log\left|\frac{h(z)\Delta_c f^{(k)}(z)g(z)}{H_1(z)}\right|, \log\left|\frac{A_i(z)e^{\alpha_i z^q}g(z)}{H_1(z)}\right| : \ 1 \leq i \leq m\right\}.
$$

By Lemma 9 and (16), we get

$$
(n-m-1)N\left(r, \frac{1}{f(z)}\right) \leq o(T_1(r)) + o(r^q)
$$

and

$$
\begin{aligned}
T_1(r) &\leq (m+1)N\left(r, \frac{1}{f(z)}\right) + o(T_1(r)) + o(r^q)\\
&\leq (m+1)T(r, f(z) + o(T_1(r)) + o(r^q),
\end{aligned}
$$

as $r \to \infty, r \notin E$. By (9) and the assumption that $n \geq m+2$, one deduce

$$
N\left(r, \frac{1}{f(z)}\right) = o(r^q), \ (r \to \infty, r \notin E). \qquad (17)
$$

By dividing $f^n(z)$ on both sides of (3), we obtain

$$
\sum_{i=1}^{m} \frac{A_i(z)e^{\alpha_i z^q}}{f^n(z)} - \frac{h(z)\Delta_c f^{(k)}(z)}{f^n(z)} = 1.
$$

Set $\frac{A_i(z)e^{\alpha_i z^q}}{f^n(z)} = f_{1,i}$ $(1 \leq i \leq m)$ and $\frac{h(z)\Delta_c f^{(k)}(z)}{-f^n(z)} = f_{1,m+1}$. Applying Lemma 6 to above equation, we have for $1 \leq j \leq m+1$

$$
\begin{aligned}
T(r, f_j) \leq& \sum_{i=1}^{m} N\left(r, \frac{1}{f_{1,i}}\right) + N\left(r, \frac{1}{f_{1,m+1}}\right) + m\sum_{i=1}^{m}\overline{N}\left(r, f_{1,i}\right)\\
&+ m\overline{N}\left(r, f_{1,m+1}\right) + o\left(\max_{1 \leq i \leq m+1}\{T(r, f_{1,i})\}\right) \qquad (18)
\end{aligned}
$$

as $r \to \infty$, $r \notin E$. Let $j = i$ and $T_f(r) = \max\{T(r, f_{1,i}) : 1 \leq i \leq m+1\}$. By (9), (17), (18), Lemma 9 and the assumption that $N(r, f) = S(r, f)$, we deduce that

$$
(1 - o(1))T_f(r) = o(r^q), \ (r \to \infty, r \notin E).
$$

It follows that

$$
T\left(r, \frac{h(z)\Delta_c f^{(k)}(z)}{f^n(z)}\right) = o(r^q), \ (r \to \infty, r \notin E),
$$

which contradicts (13).

Subcase 1.2. Suppose that

$$h(z)f^{(k)}(z+c), h(z)f^{(k)}(z), A_1(z)e^{\alpha_1 z^q}, \cdots, A_m(z)e^{\alpha_m z^q}$$

are linearly dependent. From the fact that $A_1(z)e^{\alpha_1 z^q}, \cdots, A_m(z)e^{\alpha_m z^q}$ are linearly independent, we consider the following two subcases:

Subcase 1.2.1. Suppose that $h(z)f^{(k)}(z)$ and $A_1(z)e^{\alpha_1 z^q}, \cdots, A_m(z)e^{\alpha_m z^q}$ are linearly dependent. This means that there exist m complex constants $l_{1,i}, (1 \leq i \leq m)$, at least one of them is not zero, such that

$$h(z)f^{(k)}(z) = \sum_{i=1}^{m} l_{1,i} A_i(z)e^{\alpha_i z^q}. \tag{19}$$

Substituting (19) into (3), we get

$$f^n(z) = -h(z)f^{(k)}(z+c) + \sum_{i=1}^{m} (1 + l_{1,i})A_i(z)e^{\alpha_i z^q}. \tag{20}$$

Next, on basis of (19) and (20), we consider the following two situations:

$\diamond$ If $-h(z)f^{(k)}(z+c)$, $(1+l_{1,i})A_i(z)e^{\alpha_i z^q} (1 \leq i \leq m)$ are linearly independent. By the definition of $g(z)$ and (20), one knows that $h(z)f^{(k)}(z+c)g(z)$ is an entire function. Notice that $\frac{1}{f^n(z)} = \frac{h(z)f^{(k)}(z+c)}{f^n(z)} \cdot \frac{1}{h(z)f^{(k)}(z+c)}$. Then we can obtain

$$\begin{aligned} T\left(r, \frac{h(z)f^{(k)}(z+c)}{f^n(z)}\right) &\geq nT(r,f) - m\left(r, \frac{f^{(k)}(z+c)}{f(z)} \cdot f(z)\right) - o(r^q) \\ &\geq (n-1)T(r,f) - o(r^q) \\ &\geq D_2 r^q, \end{aligned} \tag{21}$$

as $r \to \infty, r \notin E$, where D_2 is a positive constant. Now, we rewrite (20) in the form

$$\frac{f^n(z)g(z)}{H_2(z)} = \frac{-h(z)f^{(k)}(z+c)g(z)}{H_2(z)} + \sum_{i=1}^{m} \frac{(1+l_{1,i})A_i(z)e^{\alpha_i z^q} g(z)}{H_2(z)}, \tag{22}$$

where $H_2(z)$ is defined as $H_1(z)$ such that $\frac{-h(z)f^{(k)}(z+c)g(z)}{H_2(z)}$, $\frac{(1+l_{1,i})A_i(z)e^{\alpha_i z^q} g(z)}{H_2(z)}$, $\frac{f^n(z)g(z)}{H_2(z)}$ are all entire functions without common zeros, and

$$N\left(r, \frac{1}{H_2(z)}\right) \leq N\left(r, \frac{1}{A_1(z)g(z)}\right) = o(r^q). \tag{23}$$

By Remark 1, Lemma 4 and $N(r,f) = S(r,f)$, we have $N\left(r, \frac{1}{f^{(k)}(z+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r,f)$. Then using the similar manner as (15)-(17) to (22), we deduce

$$N\left(r, \frac{1}{f}\right) = o(r^q) \quad (r \to \infty, r \notin E). \tag{24}$$

Next, we rewrite (20) as

$$-\frac{h(z)f^{(k)}(z+c)}{f^n(z)} + \sum_{i=1}^{m} \frac{(1+l_{1,i})A_i(z)e^{\alpha_i z^q}}{f^n(z)} = 1.$$

By Lemma 6 we can get an inequality similar to (18), it follows from (24), Lemma 4 and Remark 1 that

$$T\left(r, \frac{h(z)f^{(k)}(z+c)}{f^n(z)}\right) = o(r^q),$$

as $r(\notin E) \to \infty$, which contradicts to (21).

$\diamond$ If $-h(z)f^{(k)}(z+c)$, $(1+l_{1,i})A_i(z)e^{\alpha_i z^q}$ are linearly dependent, there exists a finite nonzero constant $k_{1,i}$ such that

$$-h(z)f^{(k)}(z+c) = \sum_{i=1}^{m} k_{1,i}(1+l_{1,i})A_i(z)e^{\alpha_i z^q}. \tag{25}$$

Substituting the above equation into (20), we get

$$f^n(z) = \sum_{i=1}^{m}(1+k_{1,i})(1+l_{1,i})A_i(z)e^{\alpha_i z^q}. \tag{26}$$

Next, on basis of (19), (25) and (26), we will prove the following fact:

Claim (a) There exists only one nonzero element among each set $\{k_{1,1}(1+l_{1,1}), \cdots, k_{1,m}(1+l_{1,m})\}$, $\{(1+k_{1,1})(1+l_{1,1}), \cdots, (1+k_{1,m})(1+l_{1,m})\}$ and $\{l_{1,1}, \cdots, l_{1,m}\}$.

Proof. If at least two nonzero elements exist among the set $\{(1+k_{1,1})(1+l_{1,1}), \cdots, (1+k_{1,m})(1+l_{1,m})\}$, without loss of generality, we assume that $(1+k_{1,1})(1+l_{1,1}) \neq 0$. From this, we can rewrite (26) as the form

$$f^n(z)e^{-\alpha_1 z^q} = (1+k_{1,1})(1+l_{1,1})A_1(z) + \sum_{i=2}^{s}(1+k_{1,i})(1+l_{1,i})A_i(z)e^{(\alpha_i-\alpha_1)z^q}, \tag{27}$$

where $2 \le s \le m$. By (27) and Lemma 2, there exists a positive constant d_1, such that for sufficiently large r,

$$N\left(r, \frac{1}{f^n(z)}\right) \ge d_1 r^q. \tag{28}$$

On the other hand, rewrite (26) in the form

$$\frac{f^n(z)g(z)}{H_3(z)} = \sum_{i=1}^{s} \frac{(1+k_{1,i})(1+l_{1,i})A_i(z)g(z)e^{\alpha_i z^q}}{H_3(z)}, \tag{29}$$

where $H_3(z)$ is defined similarly to $H_1(z)$, such that there are no common zeros for each term in the above equation, and $N(r, 1/H_3(z)) = o(r^q)$ $(r \to \infty)$. Then using the similar manner as (15) and (16) to (29), we have

$$(n-s+1)N\left(r, \frac{1}{f}\right) \le S_3(r) + o(r^q) = O(\log T_3(r)) + o(r^q)$$

and

$$T_3(r) \le (s-1)T(r, f(z)) + O(\log T_3(r)) + o(r^q),$$

where $T_3(r) = (\int_0^{2\pi} u_3(re^{i\theta})d\theta - u_3(0))/(2\pi)$ with

$$u_3(z) = \sup\left\{\log\left|\frac{(1+k_{1,i})(1+l_{1,i})A_i(z)g(z)e^{\alpha_i z^q}}{H_3(z)}\right|, 1 \le i \le s\right\}.$$

By the above facts and (9), one can deduce that $N(r, 1/f(z)) = o(r^q)$, which contradicts (28). Thus the set $\{(1+k_{1,1})(1+l_{1,1}), \cdots, (1+k_{1,m})(1+l_{1,m})\}$ has only one nonzero element. In what follows, without loss of generality, we let $(1+k_{1,1})(1+l_{1,1}) \neq 0$, i.e.,

$$f^n(z) = (1+k_{1,1})(1+l_{1,1})A_1(z)e^{\alpha_1 z^q}. \tag{30}$$

Suppose at least two elements exist among the set $\{l_{1,1}, \cdots, l_{1,m}\}$. By (19) and (26), using the same argument as (27), we can also obtain a positive constant d_2, such that for sufficiently large r,

$$N\left(r, \frac{1}{h(z)f^{(k)}(z)}\right) \ge d_2 r^q. \tag{31}$$

Next, by Lemma 4, (30), $\rho(h) < q$, and $N(r, f) = S(r, f)$, we have

$$N\left(r, \frac{1}{h(z)f^{(k)}(z)}\right) = o(r^q). \tag{32}$$

We get a contraction, and hence the set $\{l_{1,1}, \cdots, l_{1,m}\}$ has only one nonzero element.

If at least two elements exist among the set $\{k_{1,1}(1+l_{1,1}), \cdots, k_{1,m}(1+l_{1,m})\}$, then by (19) and (25), using the same argument as in above, we can also obtain a contradiction. Here, we omit the details for the proof. □

By Claim (a), without loss of generality, we assume that $l_{1,t_1} \neq 0$, $k_{1,t_2}(1+l_{1,t_2}) \neq 0$, $(1 \leq t_1, t_2 \leq m)$. It follows from (19) and (25) that

$$h(z)f^{(k)}(z) = l_{1,t_1}A_{t_1}(z)e^{\alpha_{t_1}z^q} \tag{33}$$

and

$$-h(z)f^{(k)}(z+c) = k_{1,t_2}(1+l_{1,t_2})A_{t_2}(z)e^{\alpha_{t_2}z^q}. \tag{34}$$

Obviously, $1 + k_{1,1} \neq 0$. If $1 + k_{1,1} = 0$, it follows that $f(z) \equiv 0$. This is impossible. By (30), we get

$$f(z) = \widetilde{\tau}_0(z)e^{\frac{\alpha_1 z^q}{n}},\ f^{(k)}(z) = \widetilde{\tau}_k(z)e^{\frac{\alpha_1 z^q}{n}}, \tag{35}$$

where $\widetilde{\tau}_0(z), \widetilde{\tau}_k(z)$ satisfy $\widetilde{\tau}_0^n(z) = (1+k_{1,1})(1+l_{1,1})A_1(z)$ and $\widetilde{\tau}_i(z) = \widetilde{\tau}'_{i-1}(z) + (\alpha_1 q z^{q-1})\widetilde{\tau}_{i-1}(z)/n$ $(1 \leq i \leq k)$. Together with (33), (34) and (35), we obtain

$$h(z)f^{(k)}(z) = h(z)\widetilde{\tau}_k(z)e^{\frac{\alpha_1 z^q}{n}} = l_{1,t_1}A_{t_1}(z)e^{\alpha_{t_1}z^q}$$

and

$$-h(z)f^{(k)}(z+c) = h(z)\widetilde{\tau}_k(z+c)e^{\frac{\alpha_1 (z+c)^q}{n}} = k_{1,t_2}(1+l_{1,t_2})A_{t_2}(z)e^{\alpha_{t_2}z^q}.$$

Since the order of $h(z)$, $\widetilde{\tau}_k(z)$, $A_{t_i}(z)$ are less than q, it follows that $t_i \neq 1$, $\alpha_1 = n\alpha_{t_i}$. According to (3), (30), (33) and (34), we get

$$m = 2, t_1 = t_2 = 2, l_{1,1} = 0, k_{1,1} = 0, k_{1,2} = -1, l_{1,2} \neq -1, 0.$$

So we have

$$h(z)f^{(k)}(z) = h(z)\widetilde{\tau}_k(z)e^{\frac{\alpha_1 z^q}{n}} = l_{1,2}A_2(z)e^{\alpha_2 z^q},$$

$$h(z)f^{(k)}(z+c) = h(z)\widetilde{\tau}_k(z+c)e^{\frac{\alpha_1 (z+c)^q}{n}} = (1+l_{1,2})A_2(z)e^{\alpha_2 z^q},$$

then

$$\frac{1+l_{1,2}}{l_{1,2}} = \frac{\widetilde{\tau}_k(z+c)}{\widetilde{\tau}_k(z)}e^{\alpha_2((z+c)^q - z^q)}.$$

Obviously, $q > 1$ is impossible, now we consider $q = 1$. This means that

$$\frac{\widetilde{\tau}_k(z+c)}{\widetilde{\tau}_k(z)} = b,$$

where b is a nonzero constant and $(1+l_{1,2})/l_{1,2} = be^{\alpha_2 c}$. Then,

$$m = 2, q = 1, f(z) = \widetilde{\tau}_0(z)e^{\alpha_2 z}, \alpha_1 = n\alpha_2, \widetilde{\tau}_0^n(z) = A_1(z). \tag{36}$$

Subcase 1.2.2. Suppose that $h(z)f^{(k)}(z)$ and $A_1(z)e^{\alpha_1 z^q}, \cdots, A_m(z)e^{\alpha_m z^q}$ are linearly independent. Then, by the assumption of Subcase 1.2, we can see that $-h(z)f^{(k)}(z+c)$ can be linearly expressed by $h(z)f^{(k)}(z)$, $A_1(z)e^{\alpha_1 z^q}$, $\cdots$, $A_m(z)e^{\alpha_m z^q}$. This means that there exist $m+1$ finite complex constants $l_{2,i}, (0 \leq i \leq m)$, at least one of them is not zero, such that

$$-h(z)f^{(k)}(z+c) = l_{2,0}h(z)f^{(k)}(z) + \sum_{i=1}^{m} l_{2,i}A_i(z)e^{\alpha_i z^q}.$$

Substituting the above equality into (3), we get

$$f^n(z) = (1 + l_{2,0})h(z)f^{(k)}(z) + \sum_{i=1}^{m}(1 + l_{2,i})A_i(z)e^{\alpha_i z^q}. \tag{37}$$

In view of (37), we consider the following situations.

Subcase 1.2.2.1. If $1 + l_{2,0} = 0$, (37) can be rewritten as

$$f^n(z) = \sum_{i=1}^{m}(1 + l_{2,i})A_i(z)e^{\alpha_i z^q}. \tag{38}$$

It follows from (3) that

$$-h(z)\Delta_c f^{(k)}(z) = h(z)f^{(k)}(z) - h(z)f^{(k)}(z + c) = \sum_{i=1}^{m} l_{2,i}A_i(z)e^{\alpha_i z^q}. \tag{39}$$

On basis of (38), (39) and Lemma 9, using the same manner as Claim(a), we also have the following fact: The set $\{1 + l_{2,1}, \cdots, 1 + l_{2,m}\}$ has only one nonzero element, and the set $\{l_{2,1}, \cdots, l_{2,m}\}$ has also only one nonzero element.

Without loss of generality, we set

$$f^n(z) = (1 + l_{2,1})A_1(z)e^{\alpha_1 z^q} \text{ and } h(z)\Delta_c f^{(k)}(z) = -l_{2,t_3}A_{t_3}(z)e^{\alpha_{t_3} z^q}, \tag{40}$$

where $1 + l_{2,1} \neq 0$ and $l_{2,t_3} \neq 0,\ (1 \leq t_3 \leq m)$.

In view of (3) and (40), we only need to consider $m = 1$ or 2.

$\diamond$ Suppose that $m = 1$, then it follows from (3) and (40) that $t_3 = 1$ and

$$f^n(z) = -\frac{(1 + l_{2,1})}{l_{2,1}}h(z)\Delta_c f^{(k)}(z).$$

By the above equality, (9) and Remark 1, we get

$$\begin{aligned} nm(r, f(z)) = m(r, f^n(z)) &= m\left(r, \frac{(1 + l_{2,1})}{l_{2,1}}h(z)\Delta_c f^{(k)}(z)\right) \\ &\leq m(r, h(z)) + m\left(r, \frac{f^{(k)}(z + c) - f^{(k)}(z)}{f(z)}f(z)\right) + O(1) \\ &= m(r, f(z)) + o(r^q), \quad (r \to \infty, r \notin E). \end{aligned} \tag{41}$$

Therefore, according to (41) and $N(r, f) = S(r, f)$, we get

$$T(r, f) = o(r^q),$$

which contradicts the fact that $\rho(f) = q$.

$\diamond$ Suppose that $m = 2$. By the first equality of (40), and Hadamard's factorization theorem, we get

$$f(z) = \tau_0(z)e^{\frac{\alpha_1 z^q}{n}},\ f^{(k)}(z) = \tau_k(z)e^{\frac{\alpha_1 z^q}{n}} \tag{42}$$

where $\tau_0^n(z) = (1 + l_{2,1})A_1(z)$, and $\tau_i(z) = \tau_{i-1}'(z) + (\alpha_1 q z^{q-1})\tau_{i-1}(z)/n\ (1 \leq i \leq k)$. Substituting (42) into the second equality of (40), we get

$$\begin{aligned} h(z)\Delta_c f^{(k)}(z) &= h(z)(\tau_k(z + c)e^{\frac{\alpha_1 (z+c)^q}{n}} - \tau_k(z)e^{\frac{\alpha_1 z^q}{n}}) \\ &= -l_{2,t_3}A_{t_3}(z)e^{\alpha_{t_3} z^q}. \end{aligned}$$

Since the order of $h(z)$ and $\tau_k(z)$ are less than q, we obtain $t_3 \neq 1$, $\alpha_1 = n\alpha_{t_3}$. By (3) and (40), one can obtain

$$m = 2, t_3 = 2, l_{2,1} = 0, l_{2,t_3} = l_{2,2} = -1.$$

So we have

$$m = 2, f(z) = \tau_0(z)e^{\frac{\alpha_1 z^q}{n}}, \tau_0^n(z) = A_1(z), \alpha_1 = n\alpha_2. \tag{43}$$

Subcase 1.2.2.2. Suppose that $1 + l_{2,0} \neq 0$. If $1 + l_{2,i}(1 \le i \le m)$ are not all 0, then by the assumption of Subcase 1.2.2, we can see that $h(z)f^{(k)}(z)$, $A_1(z)e^{\alpha_1 z^q}$, $\cdots$, $A_m(z)e^{\alpha_m z^q}$ are linearly independent. Next, we can get a contradiction in the same manner as in Subcase 1.1. If $1 + l_{2,i} = 0(1 \le i \le m)$, we can use the similar manner as (41) to get a contradiction.

• Now we consider the case of $2 \le n \le m + 1$.

By the definition of $\lambda(f)$ and (9), we have $\lambda(f) \le \rho(f) = q$. If $\lambda(f) < \rho(f) = q$, we can get

$$N\left(r, \frac{1}{f}\right) = o(r^q).$$

Now, we consider the two cases: $h(z)f^{(k)}(z + c)$, $h(z)f^{(k)}(z), A_1(z)e^{\alpha_1 z^q}, \cdots,$ $A_m(z)e^{\alpha_m z^q}$ are linearly independent or not. Using the similar argument as in Subcases 1.1 and 1.2, we can also obtain (36) and (43). Thus, the result (i) of Theorem 5 is proved.

Case 2: $A_0(z) \not\equiv 0$. By Lemma 3, we have

$$N\left(r, \frac{1}{f}\right) = T(r, f) + o(T(r, f)), \quad \lambda(f) = \rho(f) = q \tag{44}$$

as $r \to \infty$, $r \notin E$. Assume that $n \ge m + 3$, and use the similar approach to Case 1 to consider whether $h(z)f^{(k)}(z + c)$, $h(z)f^{(k)}(z)$, $A_0(z)$, $A_1(z)e^{\alpha_1 z^q}$, $\cdots$, $A_m(z)e^{\alpha_m z^q}$ are linearly dependent or not, then we have $N(r, 1/f) = o(r^q)$. From this and (44), we get $T(r, f) = o(r^q)$. This is impossible. So we have $\lambda(f) = \rho(f) = q$ and $n \le m + 2$. The result (ii) is proved. Then we complete the proof of Theorem 5.

4. The proof of Theorem 6

Let f be a finite order meromorphic solution of (3) satisfying $N(r, f) = S(r, f)$. We rewrite (3) in the form

$$\frac{f^n(z) - A_0(z) - \sum_{i=1}^m A_i(z)e^{\alpha_i z^q}}{\Delta_c f^{(k)}(z)} = -h(z). \tag{45}$$

By the fact that $N(r, f) = S(r, f)$ and Lemmas 1, 4, we obtain

$$\begin{aligned} T(r, \Delta_c f^{(k)}(z)) &\le T(r, f^{(k)}(z + c)) + T(r, f^{(k)}(z)) \\ &\le T(r, f(z + c)) + T(r, f) + 2kN(r, f) + S(r, f) \\ &\le 2T(r, f) + S(r, f). \end{aligned} \tag{46}$$

On basis of (45) and (46), one can deduce that

$$\rho(h) \le \max\{\rho(f), \rho(\Delta_c f^{(k)}), q\} = \max\{\rho(f), q\}. \tag{47}$$

If $\rho(f) \ge q$, it follows from the above equality that $\rho(f) \ge \rho(h) \ge q$. Now, we consider the case of $\rho(f) < q$. From (47) and the condition that $\rho(h) \ge q$, we get $\rho(h) = q$. Note that $h(z)$ is an entire function and $\lambda(h) < \rho(h)$. Then, by Hadamard's factorization theorem, we have

$$h(z) = p(z)e^{Q(z)}, \tag{48}$$

where $p(z)$ is an entire function and $Q(z)$ is a polynomial such that $\rho(p) < q,\ \deg(Q(z)) = q$.

Set $Q(z) = a_q z^q + \cdots + a_1 z + a_0$, where $a_q \in \mathbb{C}\backslash\{0\}$ and $a_{q-1}, \cdots, a_0 \in \mathbb{C}$. Substituting (48) into (3), we obtain

$$A_0(z) + \sum_{i=1}^{m} A_i(z) e^{\alpha_i z^q} - p(z)\Delta_c f^{(k)}(z) e^{a_q z^q + \cdots + a_1 z + a_0} - f^n(z) = 0, \tag{49}$$

where $\rho(p\Delta_c f^{(k)}(z)) < q$ and $\rho(f^n) < q$ since $\rho(f) < q$.

In view of (49), we consider the following situations:

$\diamond$ Suppose that $a_q \neq \alpha_i,\ 1 \leq i \leq m$. Since $\alpha_1, \alpha_2, \cdots, \alpha_m$ are distinct nonzero complex numbers, it follows from (49) and Lemma 8 that $A_i(z) \equiv 0,\ 1 \leq i \leq m$. A contradiction.

$\diamond$ If $a_q = \alpha_i, 1 \leq i \leq m$, then (49) can be rewritten as

$$\sum_{j \neq i, j=1}^{m} A_j(z) e^{\alpha_j z^q} + \left(A_i(z) - Q_1(z)\Delta_c f^{(k)}(z)\right) e^{\alpha_i z^q} + A_0(z) - f^n(z) = 0,$$

where $Q_1(z) = p(z) e^{a_{q-1} z^{q-1} + \cdots + a_1 z + a_0}$. Applying Lemma 8 to the above equation, we can also obtain

$$A_j(z) \equiv 0, (1 \leq j \leq m, j \neq i),$$

which is a contradiction. Therefore, we prove that $\rho(f) \geq \rho(h) \geq q$.

Next, when $n \geq 2$, applying Remark 1 to (3), we obtain

$$\begin{aligned} nT(r,f) &= nm(r,f) + nN(r,f) \\ &= m\left(r, A_0(z) + \sum_{i=1}^{m} A_i(z) e^{\alpha_i z^q} - h(z)\Delta_c f^{(k)}(z)\right) + S(r,f) \\ &\leq O(r^q) + m(r,h) + m(r,f) + S(r,f) \\ &\leq T(r,f) + T(r,h) + O(r^q) + S(r,f). \end{aligned}$$

Then, we get

$$(n-1)T(r,f) \leq T(r,h) + O(r^q) + S(r,f).$$

Since $n \geq 2$ and $\rho(h) \geq q$, it follows that $\rho(f) \leq \rho(h)$. By the fact that $\rho(f) \geq \rho(h) \geq q$, one can deduce $\rho(f) = \rho(h)$. Then we complete the proof of Theorem 6.

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