CONDITIONS FOR GRAPHS ON *n* VERTICES WITH THE SUM OF DEGREES OF ANY TWO NONADJACENT VERTICES EQUAL TO *n-2* TO BE A HAMILTONIAN GRAPH

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Abstract

Let G be an undirected simple graph on $n \geq 3$ vertices with the degree sum of any two nonadjacent vertices in G equal to $n-2$. We determine the condition for G to be a Hamiltonian graph.

Keywords: Connected graph; Hamiltonian graph; Independent set; Regular graph; t-tough graph.

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1. INTRODUCTION

The concepts and symbols in this article are referenced from the *Handbook of Combinatorics* (Graham et al., 1995). Let $G=(V,E)$ be an undirected and single graph on n vertices, where V or $V(G)$ is the vertex set and E or $E(G)$ is the edge set of G. We use $|V|$ and $|E|$ to denote the number of vertices and the number of edges of G. In G, the edge connecting two vertices u and v is denoted by (u,v), the degree of vertex v is denoted by $\deg(v)$, and the minimum degree of the vertices is denoted by δ or $\delta(G)$. A graph on n vertices is called *complete* and denoted by K_n if its vertices have degree $n-1$. A graph is called a k*-regular graph* if its vertices have degree k. A subset of the vertices in a graph is called *independent* if no two vertices in the set are adjacent. *A maximum independent set* is an independent set that is not a subset of any other independent set. The cardinality of a maximum independent set in G is denoted by $\alpha(G)$. A subset of the vertices in a graph is called a *clique* if any two of its vertices are adjacent.

The graph $H=(W,F)$ is called a *subgraph* of $G=(V,E)$ if $W\subseteq V$ and $F\subseteq E$. Let v be a vertex of G; we use $G-v$ to denote the subgraph obtained by deleting vertex v and edges attached to v from G. Likewise, if $B\subseteq V$ then $G-B$ is a subgraph of G obtained by deleting B from G. A graph is *connected* if any two of its vertices are connected by a path. A *component* of G is a maximal connected subgraph of G. The number of components of G is denoted by $\omega(G)$. In G, vertex v is called a *cut vertex* if $\omega(G)<\omega(G-v)$, set $D\subset V$ is called *disconnecting* if $\omega(G)<\omega(G-D)$, and the smallest size of a disconnecting set in G is denoted by $\kappa(G)$. Graph G is called *1-tough* if $\omega(G-B)\leq|B|$ for every nonempty subset $B\subset V$. A path or circuit that includes every vertex of a graph is called a *Hamiltonian path* or *circuit*. A *Hamiltonian graph* is one that contains a Hamiltonian circuit.

In the general case, recognizing whether a given graph is Hamiltonian or not is a difficult problem. Currently, several sufficient conditions are known for a graph to be Hamiltonian. Dirac (1952) has shown that if $\delta(G)\geq n/2$, then G is a Hamiltonian graph. Ore (1960) has proved a more general result. If in G the degree sum of any two nonadjacent vertices is at least n, denoted by $\sigma_2(G)\geq n$, then G is a Hamiltonian graph. Jung (1978) has shown that, if G is a 1-tough graph, $n\geq 11$, and $\sigma_2(G)\geq n-4$, then G is a Hamiltonian graph. An (2008, 2019) has proved that, if $n\geq 7$ is an odd number, $\sigma_2(G)=n-1$, and $2<\alpha(G)<(n-1)/2$, then G is a Hamiltonian graph.

In this paper, we extend results to graphs G on n vertices satisfying the condition $\sigma_2(G)=n-2$. First, we add the following concepts and notation. In G, a vertex of degree $(n-1)$ is called a *total vertex,* and the set of total vertices in G is denoted by

$T(G)$, and $G(n) = \{G: |V(G)| = n, \sigma_2(G) = n-2\}$ is the set of graphs G on $n \geq 3$ vertices with $\sigma_2(G) = n-2$.

An (2021) has proved the following results about the structure of graphs G in $G(n)$.

Proposition 1. Let $n \geq 3$ be an odd number and $G \in G(n)$. Then G is a disconnected graph (Figure 1).

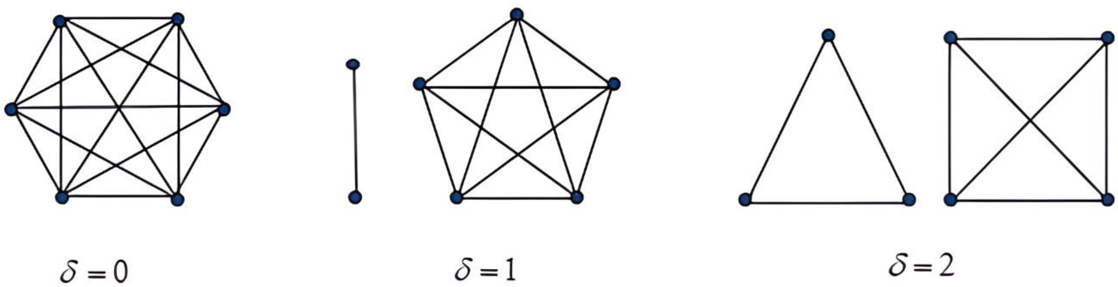

Figure 1. Disconnected graphs corresponding to $\delta = 0, 1, 2$ in G(7)

Proposition 2. Let $n \geq 4$ be an even number, $G \in G(n)$, and S be an independent set in G.

a) If $|S| \geq 3$, the vertices of S have degree $(n-2)/2$ in G.

b) If G is a disconnected graph, G has exactly two components.

Proposition 3. Let $n \geq 4$ be an even number and $G \in G(n)$. Then,

a) $0 \leq |T(G)| \leq \delta \leq (n-2)/2$.

b) $2 \leq \alpha(G) \leq (n+2)/2$.

c) $\alpha(G) = (n+2)/2 \Leftrightarrow |T(G)| = (n-2)/2$.

d) $\alpha(G) = n/2 \Rightarrow |T(G)| = 0$ and G is an $(n-2)/2$-regular graph.

Proposition 4. Let $n \geq 6$ be an even number and $G \in G(n)$. Then,

a) $\alpha(G) = 2$ if and only if G is a disconnected graph.

b) If $3 \leq \alpha(G) \leq (n+2)/2$, G is a connected graph that contains k total vertices and $n-k$ vertices of degree $\delta = (n-2)/2$, where $0 \leq k = |T(G)| \leq (n-2)/2$ (Figures 2 and 3).

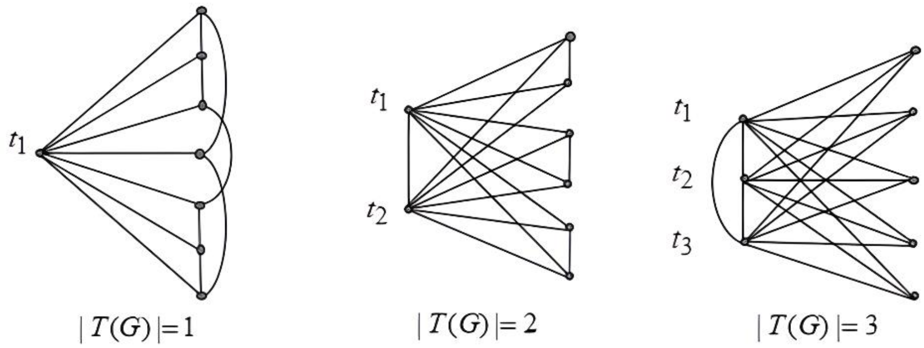

Figure 2. Connected graphs for $\delta = 3$ and $|T(G)| = 1,2,3$ in $G(8)$

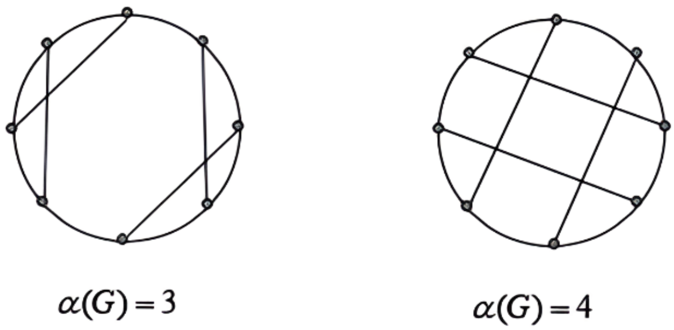

Figure 3. 3-regular graphs with $\alpha(G) = 3$ and $\alpha(G) = 4$ in $G(8)$

The following results of Bondy and Chvátal (1976), Erdös and Hobbs (1978), and Nash and Williams (1971) can be used to prove the results of this paper.

Corollary 1 (Bondy-Chvátal). *If G is not a 1-tough graph, then G is a non-Hamiltonian graph.*

Corollary 2 (Nash-Williams). *If $\alpha(G) \le \delta(G)$, $\kappa(G) \ge 2$, and $\delta(G) \ge (n+2)/3$, then G is Hamiltonian.*

Corollary 3 (Gordon-Erdös-Hobbs). *If $n \ge 4$, $\kappa(G) \ge 2$, $|V| = 2n$, and G is an $(n-1)$-regular graph, then G is a Hamiltonian graph.*

2. RESULTS

From Corollaries 1, 2, and 3 and Propositions 1, 2, 3, and 4, we get the following results.

Proposition 5. *Let $n \ge 3$ be an odd number and $G \in G(n)$. Then G is a non-Hamiltonian graph.*

Proof.

For every odd number $n \geq 3$ and by Proposition 1, G is a disconnected graph, and therefore G is a non-Hamiltonian graph.

Theorem 1. *Let* $n \geq 4$ *be an even number and* $G \in G(n)$. *If* $\alpha(G) = 2$ *or* $\alpha(G) = (n+2)/2$, *then* G *is a non-Hamiltonian graph.*

Proof.

We consider each of the following cases. For $\alpha(G) = 2$ then by Proposition 4a, G is disconnected, and therefore G is a non-Hamiltonian graph. For $\alpha(G) = (n+2)/2$ then by Proposition 3c, $|T(G)| = (n-2)/2$, and by Proposition 4b, the vertices of $V \setminus T(G)$ have degree $\delta = (n-2)/2$ in G. In addition, each vertex of $V \setminus T(G)$ must be adjacent to $(n-2)/2$ total vertices in G; therefore, $V \setminus T(G)$ is a maximum independent set in G and $\alpha(G) = |V \setminus T(G)| = |V| - |T(G)| = n - (n-2)/2 = (n+2)/2$. Now, we have $\omega(G - T(G)) = \alpha(G) = (n+2)/2 > (n-2)/2 = |T(G)|$. This shows that G is not a 1-tough graph, and by Corollary 1, G is a non-Hamiltonian graph. Theorem 1 is proved.

Theorem 2. *Let* $n \geq 6$ *be an even number,* $G \in G(n)$, *and* $\alpha(G) = n/2$, *then* G *is an* $(n-2)/2$*-regular Hamiltonian graph.*

Proof.

By Proposition 3d and $\alpha(G) = n/2$, G is an $(n-2)/2$-regular graph. (1)

Now we will prove that if G is an $(n-2)/2$-regular graph, then $\kappa(G) \geq 2$. (2)

Indeed, suppose otherwise; let $\kappa(G) = 1$ and $w \in V$ be a cut vertex of G. Then, $G - w$ is a disconnected graph by the same argument as in the proof of Proposition 3b, $G - w$ has two components G_1 and G_2, and is denoted by $G - w = G_1 \oplus G_2$, where G_1 and G_2 are two disjoint subgraphs of G, and vertex w must be adjacent to the vertices of G_1 and G_2 in G. Note that the number of vertices of graph $G - w$ is $|V(G - w)| = n - 1$, which is an odd number. Without loss of generality, we may suppose that $|V(G_1)| < n/2 \leq |V(G_2)|$. However, each vertex in G_1 has degree no less than $\delta - 1$, so $(n-2)/2 = \delta \leq |V(G_1)|$. It follows that $(n-2)/2 \leq |V(G_1)| < n/2$, and so $|V(G_1)| = (n-2)/2 = \delta$, i.e., G_1 is a complete graph K_δ and all vertices in G_1 must be adjacent to w. But vertex w must also be adjacent to the vertices of G_2, and so $\deg(w) > |V(G_1)| = \delta$, which contradicts the fact that G is an $(n-2)/2$-regular graph, therefore $\kappa(G) \geq 2$. We get (2).

From (1) and (2) it is shown that graph G satisfies the condition of Corollary 3, and therefore G is a Hamiltonian graph. Hence, Theorem 2 is proved.

Theorem 3. *Let* $n \geq 8$ *be an even number,* $G \in G(n)$, *and* $3 \leq \alpha(G) \leq (n-2)/2$.

If $n \bmod 4 \neq 0$, *then* G *is a Hamiltonian graph.*

If $n \bmod 4 = 0$, *then* G *is Hamiltonian if and only if* G *does not contain* $K_{n/2}$.

Proof.

Since $3 \leq \alpha(G) \leq (n-2)/2$, by Proposition 4b graph G is connected, G contains k total vertices, and the remaining ($n-k$) vertices have degree $\delta = (n-2)/2$, where $0 \leq k = |T(G)| \leq (n-2)/2$.

Since $\delta = (n-2)/2$ and $3 \leq \alpha(G) \leq (n-2)/2$, we have $\alpha(G) \leq \delta$. (3)

For $n \bmod 4 \neq 0$.

First, if $n \geq 10$, $(n-2)/2 \geq (n+2)/3$ and so $\delta \geq (n+2)/3$. (4)

Note that n is an even number and $n \bmod 4 \neq 0$; hence $\delta = (n-2)/2$ is an even number, and the total vertex has degree $n-1$, an odd number. Therefore, $k = |T(G)|$ must be an even number in G. We consider the following cases: $|T(G)| = 0$ and $|T(G)| \geq 2$.

Case where $|T(G)| = 0$. By Proposition 4b, G is an $(n-2)/2-$regular connected graph, and hence $\kappa(G) \geq 2$ by (2). This shows that graph G satisfies the condition of Corollary 3; therefore, G is a Hamiltonian graph.

Case where $|T(G)| \geq 2$.

Obviously, $\kappa(G) \geq |T(G)|$ and so $\kappa(G) \geq 2$. From (3) and (4), it follows that graph G satisfies the condition of Corollary 2, and therefore G is a Hamiltonian graph.

Thus, Theorem 3a is proved.

For $n \bmod 4 = 0$.

We will consider each of the following cases: $|T(G)| = 0$, $|T(G)| = 1$, and $|T(G)| \geq 2$.

First, similar to the case $n \bmod 4 \neq 0$, we can easily prove that if $n \bmod 4 = 0$ and $|T(G)| = 0$, then $(n-2)/2$-regular graph G satisfies the condition of Corollary 3. If

$n \bmod 4 = 0$ and $|T(G)| \geq 2$, then graph G satisfies the condition of Corollary 2. In both cases above, we also get the result that G is a Hamiltonian graph.

Now, we are interested in the last case $|T(G)| = 1$ and suppose $T(G) = \{t\}$.

By Proposition 4b and $|T(G)| = 1$, G contains a total vertex t and $(n-1)$ vertices of degree $\delta = (n-2)/2$. The two possibilities for the graph are $\kappa(G) = 1$ and $\kappa(G) \geq 2$.

- For $\kappa(G) \geq 2$, and by (3) and (4), it follows that graph G satisfies the condition of Corollary 2, and therefore G is a Hamiltonian graph.
- For $\kappa(G) = 1$, the total vertex t is a cut vertex of G, and so G is not a 1-tough graph, and by Corollary 1, G is a non-Hamiltonian graph.

Finally, to complete the proof of Theorem 3b, we will show that for $n \bmod 4 = 0$ and $|T(G)| = 1$, $\kappa(G) = 1$ if and only if G contains $K_{n/2}$.

Suppose that $\kappa(G) = 1$. Then, the unique total vertex t is a cut vertex in G, and $G - t$ is a $(\delta - 1)$-regular and disconnected graph. By Proposition 2b, $G - t$ has two components, $G - t = G_1 \oplus G_2$, where G_1 and G_2 are two disjoint subgraphs of G. Moreover, since $|V(G - t)| = n - 1$ is an odd number, without loss of generality, we may assume that $|V(G_1)| < n/2 \leq |V(G_2)|$. However, the vertices of G_1 have degree $(\delta - 1)$ in $G - t$, so $|V(G_1)| \geq \delta = (n-2)/2$. Hence, we get $(n-2)/2 \leq |V(G_1)| < n/2$ or $|V(G_1)| = (n-2)/2$. This shows that each pair of vertices in G_1 must be adjacent. In other words, G_1 is a complete graph K_δ in $G - t$, and therefore G contains $K_{\delta+1}$ (or $K_{n/2}$).

Conversely, suppose that G contains $K_{\delta+1}$; we will prove that $\kappa(G) = 1$.

By G contains $K_{\delta+1}$, the total vertex t must be a vertex of $K_{\delta+1}$. Let G_2 be a subgraph of G obtained from the vertices of $V(G) \setminus V(K_{\delta+1})$ in G. Obviously, $G - t$ is a $(\delta - 1)$-regular graph and $G - t$ contains K_δ. It follows that the vertices of K_δ are not adjacent to the vertices of G_2 in G; therefore, $G - t$ is a disconnected graph and $G - t = K_\delta \oplus G_2$. This implies that $\kappa(G) = 1$ where the total vertex t is a cut vertex of G.

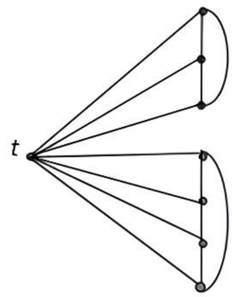

Figure 4. A non-Hamiltonian graph $G \in G(8)$ for $|T(G)| = 1$ and $\kappa(G) = 1$

This completes the proof of Theorem 3.

Note that it is not difficult to show that $\kappa(G) = 1$ for $n \bmod 4 = 0$. Theorem 3b is also true when $n = 8$. Figure 4 illustrates a non-Hamiltonian graph $G \in G(8)$ for $|T(G)| = 1$ and $\kappa(G) = 1$.

3. CONCLUSION

From Proposition 5 and Theorems 1, 2, and 3, we have shown the condition that a simple graph on $n \geq 3$ vertices with the degree sum of any two nonadjacent vertices in G equal to an $n-2$ graph is a Hamiltonian graph.

For $G(n) = \{G: |V(G)| = n, \sigma_2(G) = n-2\}$ and $G \in G(n)$, if $n \geq 3$ is an odd number, then G is a family of disconnected non-Hamiltonian graphs $K_{\delta+1} \oplus K_{n-1-\delta}$, $\delta = 0, 1, 2, ..., (n-3)/2$. If $n \geq 4$ is an even number and $\alpha(G) = 2$ or $\alpha(G) = (n+2)/2$, G is a non-Hamiltonian graph. Otherwise, if $n \geq 6$ is an even number and $\alpha(G) = n/2$, then G is a Hamiltonian graph. If $n \geq 8$ is an even number and $3 \leq \alpha(G) \leq (n-2)/2$, then G is a Hamiltonian graph for; otherwise, graph G is Hamiltonian if and only if G does not contain the complete graph $K_{n/2}$.

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