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Arithmetic Mean Derivative-Based Quartet Midpoint Rule

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Abstract

A definite integral that is difficult to solve analytically can be calculated using the numerical integration methods. The midpoint rule is a prominent rule for approximating definite integrals. This article discusses a version of the quartet midpoint rule that includes the derivative of the arithmetic mean (M_{qa}). *The proposed rule increases precision over the previous rules. Furthermore, the error term is obtained by using the concept of precision between quadrature and exact values. Finally, the proposed rule is more effective than the present rule, according to numerical simulation results.*

Keywords: *Midpoint Rule; Arithmetic Mean; Derivative-Based Quadrature; Numerical Integration; Definite Integral*

1. Introduction

Numerical integration is used to determine the numerical value of a definite integral. This method is applied because many definite integrals are difficult to solve analytically, or the value of the integral can be computed analytically, but the solution process is exceedingly complicated and time-consuming. The quadrature rule is a prominent method for evaluating numerical integration, which is given by Germun [2]:

$$\int_s^t f(x)dx = \sum_{i=0}^{k} v_i f(x_i) + E_n \tag{1}$$

where weights v_i and the nodes represented as $x_i \in [s, t]$ have to be calculated and E_n represents the formula error. The formula is named by the closed Newton-Cotes method if the interval integration's end points are included as node points, if not, it is commonly referred to as the open Newton-Cotes method.

Deghan et al. [3] improved the closed Newton-Cotes method by introducing the end of interval integration as a new variable to be estimated. Furthermore, Ramachandran *et al.* [7] - [13] enhanced the closed Newton-Cotes method by including the derivative of the function evaluated at the centroidal mean, geometric mean, harmonic mean, heronian mean, root mean square, and contra-harmonic mean. Ramachandran et al. conducted a comparison of their previously proposed methods in [14].

As with the closed newton-cotes method, the open Newton-Cote method is also modified to improve accuracy. Burg [1] modified open-newton cotes open-newton cotes using function examination at the interval's midpoint with odd derivatives at the endpoints. Ramachandran et al. [11] was also modified the midpoint rule by adding derivative evaluation. In the meantime, Zafar et. al. [15] enhanced the modified midpoint rule by determining a linear combination of both the function

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with the derivative of it values at the node points. Lately, we proposed double midpoint rule [4] and corrected closed Newton-Cotes [5] by adding arithmetic mean derivative in their endpoints.

By changing $n = 2$, the formula midpoint rule can be adjusted to acquire a quartet midpoint rule (M_q)[6]:

$$M_q(f) = \frac{t-s}{3}\left[f\left(\frac{5s+t}{6}\right) + f\left(\frac{s+t}{2}\right) + f\left(\frac{s+5t}{6}\right)\right] + \frac{t-s}{216} f''(\xi), \quad \xi \in [s,t] \tag{2}$$

The same technique as Ramachandran [8] was used by the author derive the quartet midpoint rule (2) by including a derivative that evaluates at the arithmetic mean of the nodes. The precison of the trapezoidal rule based on the geometric means generated by Ramachandran [8] is 2, but the resulting error is still quite large. According to Germun [2], the quartet midpoint rule is a more accurate rule than the trapezoidal rule because it divides the integration interval from a to b into four times at the midpoint of each interval. Therefore, in order to obtain a method that produces better accuracy in approximating the definite integral, the author derived the quartet midpoint rule. The proposed rule's error analysis is then performed in the next section. The discussion is concluded with some numerical computation aimed at assessing the efficiency of the proposed rule.

2. Researchs Methods

Ramachandran et.al [8] have modified the closed Newton-Cotes method, one of which is the trapezoidal rule. The general form of the trapezoidal rule is as follows:

$$\int_s^t f(x)dx = \frac{(t-s)}{2}[f(s)+f(t)] - \frac{(t-s)^2}{12} f''(\xi), \ \ \xi \in [s,t]$$

This rule is modified by including the geometric mean's derived value, yielding

$$\int_s^t f(x)dx = \frac{(t-s)}{2}[f(s)+f(t)] - \frac{(t-s)^2}{12} f''\left(\sqrt{st}\right) - \frac{(t-s)^3}{24}\left(\sqrt{t}-\sqrt{s}\right)^2 f'''(\xi), \ \ \xi \in [s,t].$$

Similarly, the author would like to add the arithmetic mean derivative to the quartet midpoint rule in equation (2).

3. Results and Discussion

This section discusses the derivation of the quartet midpoint rule based on the arithmetic mean derivative $\left(M_{qa}\right)$, including its precision, order, and error term.

3.1 Arithmetic Mean Derivative-Based Quartet Midpoint Rule

In this part, the error in equation (2) will be modified by adding the arithmetic mean to obtain.

Teorema 1. If $f \in C^2[s,t]$ then the arithmetic mean derivative-based quartet midpoint rule to estimate $\int_s^t f(x)dx$ provided by

$$M_{qa}(f) = \frac{t-s}{3}\left[f\left(\frac{5s+t}{6}\right) + f\left(\frac{s+t}{2}\right) + f\left(\frac{s+5t}{6}\right)\right] + \frac{t-s}{216} f''\left(\frac{s+t}{2}\right). \tag{3}$$

This rule has a degree of precision of 3.

Proof. Equation (3) is verified with $f(x) = x^3$ which has the exact value $\int_s^t x^3 dx = \frac{1}{4}(t^4 - s^4)$ and hence we obtain

$$M_{qa}(f) = \frac{t-s}{3}\left[\left(\frac{5s+t}{6}\right)^3 + \left(\frac{s+t}{2}\right)^3 + \left(\frac{s+5t}{6}\right)^3\right] + \frac{t-s}{216}\left(6\left(\frac{s+t}{2}\right)\right)$$
$$= \frac{17t^4 + 2t^3s - 2ts^3 - 17s^4}{72} + \frac{t^4 - 2t^3s + 2ts^3 - 1s^4}{72}$$
$$= \frac{1}{4}(t^4 - s^4)$$

This proved the theorem.■

3.2 Error Analysis

The error term of the M_{qa} is derived using the monomial quadrature formula $\frac{x^{p+1}}{(p+1)!}$ and the exact value of $\frac{1}{(p+1)!}\int_s^t x^{p+1}dx = \frac{t^{p+2}-s^{p+2}}{(p+2)!}$, where p is the formula's precision.

Teorema 2. Let $f[s,t] \in \mathbb{R} \to \mathbb{R}$ is a continuous function and has a continuous derivative. The quartet midpoint rule based on the arithmetic mean derivative has the error term

$$EM_{qa}(f) = \frac{83}{466560}(t-s)^5 f^{(4)}(\xi) \ \ with \ \ \xi \in [s,t]. \tag{4}$$

The accuracy of this rule is fifth order.

Proof. *Let that $f(x)$ in equation (4) is verified by $f(x) = \frac{x^4}{4!}$, with the exact solution being $\int_s^t \frac{x^4}{4!}dx = \frac{1}{120}(t^5 - s^5)$. Furthermore, using the M_{qa} formula (3) obtained*

$$M_{qa}(f) = \left(\frac{1}{4!}\right)\frac{t-s}{3}\left[\left(\frac{5s+t}{6}\right)^4 + \left(\frac{s+t}{2}\right)^4 + \left(\frac{s+5t}{6}\right)^4\right] + \frac{t-s}{216}\left(\frac{1}{2}\right)\left(\frac{s+t}{2}\right)^2$$
$$M_{qa}(f) = \frac{3805b^5 + 415b^4a - 830b^3a^2 + 830b^3a^2 - 415ba^4 - 3805a^5}{466560}$$

the error term is obtained

$$E_n[f] = \text{exact value} - M_{qa}(f) = \frac{83}{466560}(t-s)^5 f^{(4)}(\xi), \text{with } \xi \in [s,t]. \ ■$$

Table 1 shows the precision, ordering, and error terms for the quartet midpoint rule (*Mq*) [6], double midpoint rule based on the arithmetic mean derivative (M_{da}) [4], midpoint rule based on the arithmetic mean derivative (M_m) [14], Simpson's $\frac{1}{3}^{rd}$ rule [11], and the suggested rule M_{qa}.

Tabel 1. Comparison of error terms

Rules	Precision	Order	Error terms
Mq	1	3	$\frac{1}{216}(t-s)^3 f''(\xi)$
Mda	3	5	$\frac{11}{30720}(t-s)^5 f^{(4)}(\xi)$
S	3	5	$\frac{1}{1920}(t-s)^5 f^{(4)}(\xi)$
M_m	3	5	$\frac{1}{2880}(t-s)^5 f^{(4)}(\xi)$
M_{qa}	3	5	$\frac{83}{466560}(t-s)^5 f^{(4)}(\xi)$

3.3 Numerical Example

Table 2 shows the values of $\int_0^1 e^x dx$, $\int_0^{\frac{\pi}{4}} sin^4 x\ dx$, dan $\int_0^1 (1+x^7)dx$ are approximated by M_q[6], M_{da} [4], S [11], M_m [14], and M_{qa}. The results of this simulation were obtained using Octave software

Table 2. Comparison of M_q, M_{da}, S, M_m, and M_{qa}

Rules	$\int_0^1 e^x dx$	$\int_0^{\frac{\pi}{4}} sin^4 x\ dx$	$\int_0^1 (1+x^7)dx$
	error	error	error
Mq	0.00792930350	0.00287878998	0.02936742700
Mda	0.00059493209	0.00059392417	0.04455566400
S	0.00057932440	0.00056996938	0.04687500000
M_m	0.00086383700	0.00086766600	0.06250000000
M_{qa}	0.00029633450	0.00029319331	0.02329103800

The computational results shown in Table 2 show that the error M_{qa} is smaller for the three examples than the comparison rules. For example, for $\int_0^1 e^x dx$, error that are resulted by M_q and M_m is about fourfold that of M_{qa}, whereas M_{da} and S are double than eror's M_{qa}. The remaining two examples likewise fit this description. This demonstrates that M_{qa} is more accurate than previous methods

4. Conclusions

The quartet midpoint rule based on the arithmetic mean derivative is obtained by modifying the quartet midpoint rule by adding the arithmetic mean derivative to the error. This rule has higher accuracy than the quartet midpoint rule. The precision of the rule increase two precisions. Resulting from numerical computation shows that the proposed rule more accurate than the previous rules. In conclusion, Arithmetic mean derived-based quartet midpoint rule can be one of the alternative rules to determine definite integral. This work focuses solely on the arithmetic mean derivative-based adaptation of the quartet midpoint rule. Future study can be conducted by adjusting other numerical integration methods and employing other average variations.

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