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MATRIX TRANSFORMS OF SPEED-SUMMABLE AND SPEED-BOUNDED SEQUENCES

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Abstract. First we recall the notions of A^{λ}-boundedness, A^{λ}-summability and the absolute A^{λ}-summability of sequences, and the notion of λ-reversibility of A, where A is a matrix with real or complex entries and λ is the speed of convergence, i.e.; a monotonically increasing positive sequence. Let B be a lower triangular matrix with real or complex entries, and $\mu = (\mu_n)$ be another speed of convergence. We find necessary and sufficient conditions for a matrix M (with real or complex entries) to map the set of all A^{λ}-bounded sequences (for a normal matrix A) into the set of all absolutely B^{μ}-summable sequences, and the set of all A^{λ}-summable sequences (for a λ-reversible matrix A) into the set of all absolutely B^{μ}-summable sequences.
Keywords: bounded sequences, summable sequences, λ-reversibility.

1. Introduction

Let X, Y be two sequence spaces and $M = (m_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$M_n x = \sum_k m_{nk} x_k$$

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converge and the sequence $Mx = (M_n x)$ belongs to Y, we say that the matrix M transforms X into Y. By (X, Y) we denote the set of all matrices which transform X into Y. Let ω be the set of all real or complex valued sequences. Further we need the following well-known subspaces of ω: c - the space of all convergent sequences, c_0 - the space of all sequences converging to zero, m - the space of all bounded sequences, and

$$l := \{x = (x_n) \, : \, \sum_n |x_n| < \infty\}.$$

Let throughout this paper $\lambda = (\lambda_k)$ be a positive monotonically increasing sequence, i.e.; the speed of convergence. Following Kangro [17], [18] a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := s(x) \text{ and } v_k(x) = \lambda_k (x_k - s(x)) \tag{1.1}$$

is called bounded with the speed λ (shortly, λ-bounded) if $v_k(x) = O_x(1)$ (or $(v_k(x)) \in m$), and convergent with the speed λ (shortly, λ-convergent) if there the finite limit

$$\lim_k v_k(x) := b(x)$$

exists (or $(v_k(x)) \in c$). Following the authors of the present paper [3], a convergent sequence $x = (x_k)$ with the finite limit $s(x)$ is called absolutely convergent with the speed λ (shortly, absolutely λ-convergent) if $(v_k(x)) \in l$.

We denote the set of all λ-bounded sequences by m^λ, and the set of all λ-convergent sequences by c^λ, and the set of all absolutely λ-convergent sequences by l^λ. Moreover, let

$$c_0^\lambda := \{x = (x_k) \, : \, x \in c^\lambda \text{ and } \lim_k \lambda_k (x_k - s(x)) = 0\}$$

and

$$m_0^\lambda = \{x = (x_k) \, : \, x \in m^\lambda \cap c_0\}.$$

It is not difficult to see that

$$l^\lambda \subset c_0^\lambda \subset c^\lambda \subset m^\lambda \subset c, \; m_0^\lambda \subset m^\lambda \subset c.$$

In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$, we get $c^\lambda = m^\lambda = c$.

Let $A = (a_{nk})$ be a matrix with real or complex entries. A sequence $x = (x_k)$ is said to be A^λ-bounded (A^λ-summable), if $Ax \in m^\lambda$ ($Ax \in c^\lambda$, respectively). Following the authors of the present paper [1], a sequence $x = (x_k)$ is absolutely A^λ-summable if $Ax \in l^\lambda$. The set of all A^λ-bounded sequences will be denoted by m_A^λ, the set of all A^λ-summable sequences by c_A^λ, and the set of all absolutely A^λ-summable sequences by l_A^λ. Let

$$(c_0)_A^\lambda := \{x \in c_A^\lambda \, : \, Ax \in c_0^\lambda\},$$

and c_A be the summability domain of A, i.e.; the set of sequences x (with real or complex entries), for which the finite limit $\lim_n A_n x$ exists. It is easy to see that

$$l_A^\lambda \subset (c_0)_A^\lambda \subset c_A^\lambda \subset m_A^\lambda \subset c_A,$$

and, if λ is a bounded sequence, then

$$m_A^\lambda = c_A^\lambda = (c_0)_A^\lambda = c_A.$$

Let $e = (1, 1, ...)$, $e^k = (0, ..., 0, 1, 0, ...)$, where 1 is in the k-th position, and $\lambda^{-1} = (1/\lambda_k)$. We note that

$$e, e^k, \lambda^{-1} \in c^\lambda; \quad e, e^k \in l_1^\lambda.$$

Let $\mu = (\mu_n)$ be another speed of convergence and $B = (b_{nk})$ a lower triangular matrix, A matrix A is said to be normal if it is lower triangular and $a_{nn} \neq 0$ for every n, and λ-reversible, if the infinite system of equations $z_n = A_n x$ has a unique solution, for each sequence $(z_n) \in c^\lambda$. It is not difficult to see that every normal matrix is λ-reversible.

The sets (m^λ, m^μ) and (c^λ, c^μ) have been described correspondingly in [18] and [17], and the set (c^λ, m^μ) in [16] and [19]. The sets $\left(m^\lambda, c^\mu\right)$, $\left(m^\lambda, m_0^\mu\right)$, $\left(m^\lambda, c_0^\mu\right)$, $\left(c^\lambda, m_0^\mu\right)$, $\left(c^\lambda, c_0^\mu\right)$, $\left(m_0^\lambda, m^\mu\right)$, $\left(m_0^\lambda, m_0^\mu\right)$, $\left(m_0^\lambda, c^\mu\right)$, $\left(m_0^\lambda, c_0^\mu\right)$, $\left(c_0^\lambda, m^\mu\right)$, $\left(c_0^\lambda, m_0^\mu\right)$, $\left(c_0^\lambda, c^\mu\right)$ and $\left(c_0^\lambda, c_0^\mu\right)$ have been characterized in [4]. The sets (l^λ, l^μ), (l^λ, c_0^μ), (l^λ, c^μ), (l^λ, m^μ), (m^λ, l^μ), (c^λ, l^μ) and (c_0^λ, l^μ) have been studied in [3].

For a normal matrix A necessary and sufficient conditions for $M \in \left(m_A^\lambda, m_B^\mu\right)$ have been proved in [9], and for $M \in \left(m_A^\lambda, c_B^\mu\right)$ or $M \in \left(m_A^\lambda, (c_0)_B^\mu\right)$ in [5]. For a λ-reversible matrix A necessary and sufficient conditions for $M \in \left(c_A^\lambda, c_B^\mu\right)$ and $M \in \left(c_A^\lambda, m_B^\mu\right)$ have been found in [10], and for $M \in \left(l_A^\lambda, l_B^\mu\right)$, $M \in \left(l_A^\lambda, m_B^\mu\right)$, $M \in \left(l_A^\lambda, c_B^\mu\right)$ and $M \in \left(l_A^\lambda, (c_0)_B^\mu\right)$ in [1]. A short overview on matrix transforms of sequence spaces and subspaces of summability domains of matrices determined by speeds of convergence has been presented in [6] and [19].

We note that the results connected with convergence, absolute convergence, boundedness, A^λ-boundedness and A^λ-summability with speed can be used in several applications, for example in the approximation theory. Besides, one author of the present paper used such results for the estimation of the order of approximation of Fourier expansions in Banach spaces ([7] - [9], [11]).

In this paper we continue the studies started in [1], [2], [5], [9] and [10]. We prove necessary and sufficient conditions for $M \in \left(m_A^\lambda, l_B^\mu\right)$, $M \in \left((c_0)_A^\lambda, l_B^\mu\right)$ and $M \in \left(c_A^\lambda, l_B^\mu\right)$, if A is a normal or λ-reversible matrix.

2. Auxiliary results

For the proof of the main results we need some auxiliary results.

Lemma 2.1. ([13], p. 44, see also [24], Proposition 12). *A matrix $A = (a_{nk}) \in (c_0, c)$ if and only if conditions*

(I) $\lim_n a_{nk} := a_k$ *for all* k,

(II) $\sum_k |a_{nk}| = O(1)$

are satisfied. Moreover,

$$\lim_n A_n x = \sum_k a_k x_k. \tag{2.1}$$

Lemma 2.2. ([13], p. 51, see also [20], p. 8, Theorem 1.2 or [24], Proposition 10). *The following statements are equivalent:*

(a) $A = (a_{nk}) \in (m, c)$.

(b) The conditions (I), (II) are satisfied and

$$\lim_n \sum_k |a_{nk} - a_k| = 0. \tag{2.2}$$

(c) The condition (I) holds and

$$\text{the series } \sum_k |a_{nk}| \text{ converges uniformly in } n.$$

Moreover, if one of the statements (a)-(c) is satisfied, then the equation (2.1) *holds.*

Lemma 2.3. ([12], p. 38-40 or [24], Proposition 72). *A matrix* $A = (a_{nk}) \in (c_0, l) = (m, l)$ *if and only if*

(III) $\sum_{n \in I} \sum_{k \in J} a_{nk} = O(1)$

for all finite subsets I *and* J *of* $\mathbf{N}$.

Let $A = (a_{nk})$ be a λ-reversible matrix. For $x \in c_A^{\lambda}$, let

$$\phi := \lim_n A_n x, \; d_n = \lambda_n (A_n x - \phi), \; d := \lim_n d_n,$$

and $\eta := (\eta_k)$, $\varphi := (\varphi_k)$ and $\eta := (\eta_{kj})$, for each fixed j, are the solutions of the system $y = Ax$ corresponding to $y = (\delta_{nn})$, $y = (\delta_{nn}/\lambda_n)$ and $y = (y_n) = (\delta_{nj})$ (where $\delta_{nj} = 1$ if $n = j$, and $\delta_{nj} = 0$ if $n \neq j$).

Lemma 2.4. ([6], Corollary 9.1). *Let* $A = (a_{nk})$ *be a* λ*-reversible matrix. Then every coordinate* x_k *of a sequence* $x := (x_k) \in c_A^{\lambda}$ *can be represented in the form*

$$x_k = \phi \eta_k + d\varphi_k + \sum_n \frac{\eta_{kn}}{\lambda_n}(d_n - d), \; \sum_n \left| \frac{\eta_{kn}}{\lambda_n} \right| < \infty \; for\, every\, fixed\; k. \tag{2.3}$$

Remark 2.1. If $x \in (c_0)_A^{\lambda}$, then $d = 0$. Hence every coordinate x_k of a sequence $x = (x_k) \in (c_0)_A^{\lambda}$ can be represented in the form

$$x_k = \phi \eta_k + \sum_n \frac{\eta_{kn}}{\lambda_n} d_n, \; \sum_n \left| \frac{\eta_{kn}}{\lambda_n} \right| < \infty \; for\, every\, fixed\; k. \tag{2.4}$$

3. The set $\left(m_A^\lambda, l_B^\mu\right)$

First we introduce necessary and sufficient conditions for existence of the transformation $y = Mx$ for every $x \in m_A^\lambda$. Let $A^{-1} := (\eta_{nk})$ be the inverse matrix of a normal matrix A. Then

$$\sum_{k=0}^{j} m_{nk}x_k = \sum_{k=0}^{j} m_{nk} \sum_{l=0}^{k} \eta_{kl} y_l = \sum_{l=0}^{j} h_{jl}^n y_l$$

for each $x := (x_k) \in m_A^\lambda$, where $y_l := A_l x$ and $H^n := \left(h_{jl}^n\right)$ is the lower triangular matrix for every fixed n, with

$$h_{jl}^n := \sum_{k=l}^{j} m_{nk}\eta_{kl}, \ l \le j.$$

Hence the transformation $y = Mx$ exists for every $x \in m_A^\lambda$ if and only if the matrix $H^n := \left(h_{jl}^n\right) \in \left(m^\lambda, c\right)$ for every fixed n. Thus we can formulate the following result (see [6], Proposition 8.1 or [9], Lemma 1).

Proposition 3.1. *Let $A = (a_{nk})$ be a normal matrix and $M = (m_{nk})$ an arbitrary matrix. Then the transformation $y = Mx$ exists for every $x \in m_A^\lambda$ if and only if*

(IV) there exist finite limits $\lim_j h_{jl}^n := h_{nl}$ for every fixed l and n,

(V) $\lim_j \sum_{l=0}^{j} h_{jl}^n$ exists and is finite for every fixed n,

(VI) $\sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l} = O_n(1)$ for every fixed n,

(VII) $\lim_j \sum_{l=0}^{j} \frac{\left|h_{jl}^n - h_{nl}\right|}{\lambda_l} = 0$ for every fixed n.

Also, condition (VI) can be replaced by the condition

(VIII) $\sum_l \frac{|h_{nl}|}{\lambda_l} = O_n(1)$ for every fixed n.

Remark 3.1. Using Lemma 2.2 c) it is possible to show that conditions (VI) and (VII) can be replaced by the condition

(IX) the series $\sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l}$ converges uniformly in j for every fixed n.

Now we can prove the main results. Let $G = (g_{nk}) = BM$; i.e.,

$$g_{nk} := \sum_{l=0}^{n} b_{nl} m_{lk}.$$

Theorem 3.1. *Let $A = (a_{nk})$ be a normal matrix, $B = (b_{nk})$ a triangular matrix, $M = (m_{nk})$ an arbitrary matrix and $\lambda_n \neq O(1)$. Then $M \in \left(m_A^\lambda, l_B^\mu\right)$ if and only if conditions (IV)-(VII) are satisfied and*

(X) there exist the finite limits $\lim_n \gamma_{nl} := \gamma_l$,

(XI) $\lim_n \sum_l \frac{|\gamma_{nl}-\gamma_l|}{\lambda_l} = 0$,

(XII) $\sum_l \frac{|\gamma_{nl}|}{\lambda_l} = O(1)$,

(XIII) $\sum_{n\in I} \mu_n \sum_{l\in J} \frac{\gamma_{nl}-\gamma_l}{\lambda_l} = O\,(1)$,

where

$$\gamma_{nl} := \lim_j \gamma_{nl}^j,$$

and

(XIV) $(\rho_n) \in l^\mu$, $\rho_n := \lim_j \sum_{l=0}^j \gamma_{nl}^j$,

where $\Gamma^n := \left(\gamma_{nl}^j\right)$ *is the lower triangular matrix for every fixed n with*

$$\gamma_{nl}^j := \sum_{k=l}^j g_{nk}\eta_{kl},\ l \leq j.$$

Also, condition (XII) can be replaced by the condition

(XV) $\sum_l \frac{|\gamma_l|}{\lambda_l} < \infty$.

Proof. Necessity. Assume that $M \in \left(m_A^\lambda, l_B^\mu\right)$. Then the transformation $y = Mx$ exists for every $x \in m_A^\lambda$. Hence conditions (IV) - (VII) hold by Proposition 3.1, and

$$B_n y = G_n x \tag{3.1}$$

for every $x \in m_A^\lambda$ because the change of the order of summation is allowed by the lower triangularity of B. From (3.1) we can conclude that $G \in \left(m_A^\lambda, l^\mu\right)$. In addition,

$$\sum_{k=0}^j g_{nk}x_k = \sum_{l=0}^j \gamma_{nl}^j A_l x \tag{3.2}$$

for every $x \in m_A^\lambda$. By the normality of A, there exists an $x \in m_A^\lambda$, such that $(A_l x) = e$. Consequently condition (XIV) is satisfied by (3.2).

By the normality of A, for each bounded sequence (β_n) there exists an $x \in m_A^\lambda$, such that

$$\lim_n A_n x := \delta \text{ and } \beta_n = \lambda_n (A_n x - \delta) . \tag{3.3}$$

As from (3.3) we have

$$A_n x = \delta + \frac{\beta_n}{\lambda_n},$$

then, using (3.2) and (3.2), we obtain

$$\sum_{k=0}^{j} g_{nk} x_k = \delta \sum_{l=0}^{j} \gamma_{nl}^j + \sum_{l=0}^{j} \frac{\gamma_{nl}^j}{\lambda_l} \beta_l \tag{3.4}$$

for every $x \in m_A^\lambda$. As the series $G_n x$ are convergent for every $x \in m_A^\lambda$, and the finite limits ρ_n exist by (XIV), then the matrix $\Gamma_\lambda^n := \left(\gamma_{nl}^j / \lambda_l\right) \in (m, c)$ for every n. Therefore, from (XIV), we obtain, using Lemma 2.2 that

$$G_n x = \delta \rho_n + \sum_l \frac{\gamma_{nl}}{\lambda_l} \beta_l \tag{3.5}$$

for every $x \in m_A^\lambda$. In addition, the finite limit $\lim_n \rho_n := \rho$ exists by (XIV). Therefore, from (3.5), we can conclude that the matrix $\Gamma_\lambda := (\gamma_{nl}/\lambda_l) \in (m, c)$. Consequently conditions (X) - (XII) hold and

$$\lim_n G_n x = \delta \rho_n + \sum_l \frac{\gamma_l}{\lambda_l} \beta_l \tag{3.6}$$

for every $x \in m_A^\lambda$ by Lemma 2.2.

Writing

$$\mu_n (G_n x - \lim_n G_n x) = \delta \mu_n (\rho_n - \rho) + \mu_n \sum_l \frac{\gamma_{nl} - \gamma_l}{\lambda_l} \beta_l \tag{3.7}$$

for every $x \in m_A^\lambda$, we can conclude by (XIV) that the matrix $\Gamma_{\lambda,\mu} := (\mu_n (\gamma_{nl} - \gamma_l)/\lambda_l) \in (m, l)$. Hence, using Lemma 2.3, we conclude that condition (XIII) holds.

Finally, we note that the necessity of condition (XV) follows from the validity of conditions (XI) and (XII).

Sufficiency. Let all of the conditions of the present theorem be fulfilled. Then the transformation $y = Mx$ exists for every $x \in m_A^\lambda$ by Proposition 3.1, and equations (3.1) - (3.4) hold for every $x \in m_A^\lambda$. As in the proof of the necessity of the present theorem, we get, using (XIV) and Lemma 2.2, that, from (3.4), follows the validity of (3.5) for every $x \in m_A^\lambda$. Hence $\Gamma_\lambda^n \in (m, c)$ for every n by (X) - (XII); i.e., $M \in \left(m_A^\lambda, c_B\right)$.

Relation (3.6) holds for every $x \in m_A^\lambda$ by virtue of Lemma 2.2, and therefore relation (3.7) holds for every $x \in m^\lambda$. Hence $\Gamma_{\lambda,\mu} \in (m, l)$ by (XIII). Consequently, $M \in \left(m_A^\lambda, l_B^\mu\right)$ by (XIV).

Condition (XII) can be replaced by (XV) because the validity of (XII) follows from the validity of (XI) and (XV). $\square$

If $\lambda_n = O(1)$, then $\beta_n = o(1)$ in (3.3). Therefore instead of $\Gamma_\lambda^n \in (m, c)$ we get $\Gamma_\lambda^n \in (c_0, c)$ for every n. Hence, handling Γ_λ^n, instead of Lemma 2.2 it is necessary to use Lemma 2.3. Thus we can immediately to formulate the following result.

Theorem 3.2. *Let $A = (a_{nk})$ be a normal matrix, $B = (b_{nk})$ a triangular matrix, $M = (m_{nk})$ an arbitrary matrix and $\lambda_n = O(1)$. Then $M \in \left(m_A^\lambda, l_B^\mu\right)$ if and only if conditions (IV)-(VII), (X) and (XII) - (XIV) are satisfied.*

Remark 3.2. Using (c) in Lemma 2.2 it is possible to show that conditions (XI) and (XII) in Theorem 3.1 can be replaced by the condition

(XVI) the series $\sum\limits_l \frac{|\gamma_{nl}|}{\lambda_l}$ converges uniformly in n.

4. The sets $\left(c_A^\lambda, l_B^\mu\right)$ and $\left((c_0)_A^\lambda, l_B^\mu\right)$

In this section we characterize the sets $\left(c_A^\lambda, l_B^\mu\right)$ and $\left((c_0)_A^\lambda, l_B^\mu\right)$, if A is a λ-reversible matrix. For the beginning we present necessary and sufficient conditions for existence of the transformation $y = Mx$ for every $x \in (c_0)_A^\lambda$ (first proved in [2]) and for every $x \in c_A^\lambda$.

Proposition 4.1. *Let $A = (a_{nk})$ be a λ-reversible matrix and $M = (m_{nk})$ an arbitrary matrix. Then the transformation $y = Mx$ exists for every $x \in (c_0)_A^\lambda$ if and only if conditions (IV), (VI) are satisfied and*

(XVII) $\sum_k m_{nk}\eta_k < \infty$ *for every fixed* n.

Proof. Necessity. Assume that the transformation $y = Mx$ exists for every $x \in (c_0)_A^\lambda$. By Remark 2.1, every coordinate x_k of a sequence $x := (x_k) \in (c_0)_A^\lambda$ can be represented in the form (2.4) for every fixed k. Hence we can write

$$\sum_{k=0}^{j} m_{nk}x_k = \phi \sum_{k=0}^{j} m_{nk}\eta_k + \sum_l \frac{h_{jl}^n}{\lambda_l} d_l \tag{4.1}$$

for every sequence $x := (x_k) \in (c_0)_A^\lambda$. It is easy to see that $\eta \in (c_0)_A^\lambda$ since $e \in (c_0)_A^\lambda$ and A is λ-reversible. Consequently condition (XVII) holds.

Using (4.1), we obtain that the matrix $H_\lambda^n := \left(h_{jl}^n/\lambda_l\right)$ for every n transforms this sequence $(d_l) \in c_0$ into c. We show that H_λ^n transforms every sequence $(d_l) \in c_0$ into c. Indeed, for every sequence $(d_l) \in c_0$, the sequence $(d_l/\lambda_l) \in c_0$. But, for

(d_l/λ_l), there exists a convergent sequence $z := (z_l)$ with $\phi := \lim_l z_l$, such that $d_l/\lambda_l = z_l - \phi$. Due to λ-reversibility of A for every convergent sequence $z := (z_l)$ with $\phi := \lim_l z_l$ there exists a convergent sequence x, such that $z_l = A_l x$. Thus, we have proved that, for every sequence $(d_l) \in c_0$ there exists a sequence $(x_k) \in (c_0)^\lambda_A$ such that $d_l = \lambda_l (A_l x - \phi)$. Hence $H^n_\lambda \in (c_0, c)$. Therefore, by Lemma 2.1, conditions (IV) and (VI) are satisfied.

Sufficiency. Let all conditions of the present proposition be satisfied. Then conditions (IV) and (VI) imply, by Lemma 2.1, that $H^n \in (c_0, c)$. Consequently, from (4.1), we can conclude, by (XVII) that the transformation $y = Mx$ exists for every $x \in (c_0)^\lambda_A$. $\square$

As the validity of (IV) and (VI) implies the validity of (VIII), then from Proposition 4.1 we immediately obtain the following result.

Corollary 4.1. *Let $A = (a_{nk})$ be a λ-reversible matrix and $M = (m_{nk})$ an arbitrary matrix. If the transformation $y = Mx$ exists for every $x \in (c_0)^\lambda_A$, then condition (VIII) holds.*

Proposition 4.2. ([10], Lemma 2, see also [6], Proposition 9.5). *Let $A = (a_{nk})$ be a λ-reversible matrix and $M = (m_{nk})$ an arbitrary matrix. Then the transformation $y = Mx$ exists for every $x \in c^\lambda_A$ if and only if conditions (IV), (VI) and (XVII) are satisfied and*

(XVIII) $\sum_k m_{nk}\varphi_k < \infty$ *for every fixed* n.

Now we are able to prove the main results. Let $\Gamma^n := \left(\gamma^j_{nl}\right)$ is the lower triangular matrix for every fixed n with

$$\gamma^j_{nl} := \sum_{k=0}^{j} g_{nk}\eta_{kl}.$$

If the matrix transform $y = Mx$ exists for every $x \in c^\lambda_A$ or $x \in (c_0)^\lambda_A$, then the finite limits

$$\gamma_{nl} := \lim_j \gamma^j_{nl}$$

exist.

Theorem 4.1. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in (c^\lambda_A, l^\mu_B)$ if and only if conditions (IV), (VI), (X), (XII), (XIII), (XVII), (XVIII) are satisfied, and*

(XIX) $\eta, \varphi \in l^\mu_G$.

Proof. Necessity. Assume that $M \in (c_A^\lambda, l_B^\mu)$. Then the transformation $y = Mx$ exists for every $x \in c_A^\lambda$. Hence conditions (IV), (VI), (XVII) and (XVIII) hold by Proposition 4.2, and relation (3.1) holds for every $x \in c_A^\lambda$ because the change of the order of summation is allowed by the lower triangularity of B. This implies that $c_A^\lambda \subset l_G^\mu$ and series $G_n x$ are convergent for every $x \in c_A^\lambda$. Hence condition (XIX) is satisfied because $\eta, \varphi \in c_A^\lambda$. As every element x_k of a sequence $x := (x_k) \in c_A^\lambda$ may be presented in the form (2.3) by Lemma 2.4, we can write

$$\sum_{k=0}^{j} g_{nk} x_k = \phi \sum_{k=0}^{j} g_{nk} \eta_k + d \sum_{k=0}^{j} g_{nk} \varphi_k + \sum_l \frac{\gamma_{nl}^j}{\lambda_l} (d_l - d) \tag{4.2}$$

for every $x \in c_A^\lambda$. As series $G_n x$ are convergent for every $x \in c_A^\lambda$, then from (4.2) it follows, by (XIX), that $\Gamma_\lambda^n := \left(\gamma_{nl}^j / \lambda_l\right) \in (c_0, c)$ for every n because A is λ-reversible (see a proof of the necessity of Proposition 4.1). Therefore from (4.2) we obtain by Lemma 2.1 that

$$G_n x = \phi G_n \eta + d G_n \varphi + \sum_l \frac{\gamma_{nl}}{\lambda_l} (d_l - d) \tag{4.3}$$

for every $x \in c_A^\lambda$. From (4.3) we see, with the help of (XIX) that $\Gamma_\lambda := (\gamma_{nl}/\lambda_l) \in (c_0, c)$. Consequently, conditions (X) and (XII) hold, and for every $x \in c_A^\lambda$ we obtain by Lemma 2.1 that

$$u(x) = \phi u(\eta) + d u(\varphi) + \sum_l \frac{\gamma_l}{\lambda_l} (d_l - d)\,, \tag{4.4}$$

where

$$u(x) := \lim_n G_n(x).$$

Therefore we can write

$$\begin{aligned} \mu_n(G_n x - u(x)) = \phi \mu_n (G_n \eta - u(\eta)) + d \mu_n (G_n \varphi - u(\varphi)) + \\ \mu_n \sum_l \frac{\gamma_{nl} - \gamma_l}{\lambda_l} (d_l - d) \end{aligned} \tag{4.5}$$

for every $x \in c_A^\lambda$. With the help of (XIX), it follows from (4.5) that the matrix $\Gamma_{\lambda,\mu} := (\mu_n(\gamma_{nl} - \gamma_l)/\lambda_l) \in (c_0, l)$. Hence, using Lemma 2.3, we conclude that condition (XIII) is satisfied.

Sufficiency. We assume that all of the conditions of the present theorem hold. Then the matrix transformation $y = Mx$ exists for every $x \in c_A^\lambda$ by Proposition 4.2. This implies that relations (3.1) and (4.2) hold for every $x \in c_A^\lambda$ (see the proof of the necessity). Using (X) and (XII), we conclude, with the help of Lemma 2.1, that $\Gamma_\lambda^n \in (c_0, c)$ for every n, one can take the limit under the summation sign in the last summand of (4.2). Then, from (4.2), we obtain by (XIX), the validity of (4.3) for every $x \in c_A^\lambda$. Conditions (X), (XII) and (XIX) imply that (4.4) holds for every $x \in c_A^\lambda$, due to Lemma 2.1. Then relation (4.5) also holds for every $x \in c_A^\lambda$. Moreover, $\Gamma_{\lambda,\mu} \in (c_0, l)$, by Lemma 2.3. Therefore $M \in (c_A^\lambda, l_B^\mu)$ by (XIX). □

If instead of c_A^λ to take $(c_0)_A^\lambda$ in Theorem 4.1, then $d = 0$ in the proof of this theorem. Therefore we can immediately formulate the following result.

Theorem 4.2. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in ((c_0)_A^\lambda, l_B^\mu)$ if and only if conditions (IV), (VI), (X), (XII), (XIII), (XVII) are satisfied, and $\eta, \varphi \in l_G^\mu$.*

5. Conclusions

In this paper, we continued the investigations started in [1], [2], [5], [9] and [10] (see also [6]), where we studied the matrix transformations of subspaces of summability domains of matrices with real or complex entries defined by speeds of convergence, i.e.; by monotonically increasing positive sequences λ and μ. Now we found necessary and sufficient conditions for a matrix M (with real or complex entries) to map the λ-boundedness domain of a normal matrix A or the λ-convergence domain (or the specific subdomain of λ-convergence domain) of a reversible matrix A into the absolute μ-convergence domain of lower triangular matrix B.

In the future, we can apply the obtained results in several directions. Several interesting papers have been published recently, for example, in [14] and [23] the strong Riesz and Nörlund summability and the rate of convergence of Fourier series have been studied, in [21] the absolute index Nörlund summability of improper integrals has been investigated, in [15] the relatively equi-statistical convergence via deferred Nörlund mean and related approximation theorems have been handled, in [22] generalised Riesz and Nörlund type means of sequences of fuzzy numbers have been considered. Further we can apply our results for acceleration of the converging processes handled in these papers.

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