FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 3 (2023), 621 - 634
https://doi.org/10.22190/FUMI230305039B
Original Scientific Paper

SOME REMARKS ON RICCI–GOLAB CONNECTIONS

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Abstract. We consider the divergence and Laplace operators defined by the Ricci–Golab connection and establish some integral properties. We provide certain results on the deformation algebras associated to pairs of Ricci–Golab connections. Almost 1-principal Golab connections are also investigated.
Keywords: Laplace operator, Ricci-Golab connection.

1. Introduction

A Riemannian manifold admits a semi-symmetric metric connection of zero curvature tensor if and only if it is conformally flat [17]. This result was generalized in [18] for the Ricci tensor. It is an interplay between the Riemannian and the semi-Riemnnian geometry with respect to semi-symmetric connections [14, 19]. The quarter-symmetric connections, introduced by Golab in [4], arose on the line of metric and semi-symmetric connections. These notions have been studied from different perspectives (e.g., [1, 3, 6, 7, 5, 12, 11, 15]).

In the present paper, we consider the Ricci–Golab connection as a natural generalization of the above. The study consists of two parts. One of the objectives is to provide properties of the divergence and the Laplace operator defined by means of the Ricci–Golab connection (Section 2). It is known that the Laplace operator, among various integral operators, is through the most widely considered in different

Received March 05, 2023. accepted May 12, 2023.
Communicated by Uday Chand De
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2010 *Mathematics Subject Classification.* 35C08, 35Q51, 53B05

branches of engineering sciences and in mathematics, and it has many applications [13, 16]. In particular, we will provide a divergence-type theorem with respect to the Ricci–Golab connection and define harmonic functions in this context. In the last part, we restrict the attention to the second objective of the paper, namely the study of the deformation algebra associated to pairs of Ricci–Golab connections, which is essential in order to characterize the geometry of the manifold (Sections 3 and 4).

2. Ricci–Golab connection

Let g be a Riemannian metric, let ∇^g be its Levi-Civita connection, and let η be a 1-form on a smooth manifold M.

The Golab connection associated to (g, η, Q) [4], where Q is the Ricci operator defined by $g(QX, Y) := \mathrm{Ric}(X, Y)$, for Ric the Ricci curvature tensor of g, will be further called the *Ricci–Golab connection* associated to (g, η, Q), and it is given by

$$\nabla = \nabla^g + Q \otimes \eta - \mathrm{Ric} \otimes \eta^\sharp, \tag{2.1}$$

with $\eta^\sharp$ the g-dual vector field of the 1-form η.

For any 1-form ω, we obtain

$$\nabla\omega = \nabla^g\omega - (\omega \circ Q) \otimes \eta + \omega(\eta^\sharp) \cdot \mathrm{Ric}, \tag{2.2}$$

hence, for $\omega = df$, with f a smooth function on M, we have

$$\nabla df = \nabla^g df - (df \circ Q) \otimes \eta + df(\eta^\sharp) \cdot \mathrm{Ric}\,. \tag{2.3}$$

We can now consider the divergence operator, the Hessian of a smooth function and the Laplace operator w.r.t. the Ricci–Golab connection. Precisely,

$$\mathrm{div}^{(\nabla, g)} = \mathrm{div} + \mathrm{scal} \cdot \eta - i_{Q\eta^\sharp} g, \tag{2.4}$$

in particular,

$$\mathrm{div}^{(\nabla, g)}(\eta^\sharp) = \mathrm{div}(\eta^\sharp) + |\eta^\sharp|^2 \cdot \mathrm{scal} - \mathrm{Ric}(\eta^\sharp, \eta^\sharp), \tag{2.5}$$

for scal the scalar curvature of (M, g). Also

$$\mathrm{Hess}^\nabla(f) = \mathrm{Hess}(f) - (df \circ Q) \otimes \eta + \eta^\sharp(f) \cdot \mathrm{Ric} \tag{2.6}$$

and consequently, the Laplace operator w.r.t. the Ricci–Golab connection

$$\Delta^{(\nabla, g)}(f) = \Delta(f) - \mathrm{Ric}(\nabla f, \eta^\sharp) + \eta^\sharp(f) \cdot \mathrm{scal}, \tag{2.7}$$

where ∇f denotes the gradient of f w.r.t. g.

Since the Ricci–Golab connection is not torsion-free, the $(0, 2)$-tensor field $\mathrm{Hess}^\nabla(f)$ is not symmetric unless $\nabla f \in \ker Q$, and for any $X, Y \in \mathcal{X}(M)$, we have

$$\mathrm{Hess}^\nabla(f)(X, Y) - \mathrm{Hess}^\nabla(f)(Y, X) = \mathrm{Ric}\left(\nabla f, \eta(X)Y - \eta(Y)X\right) =$$

$$= \Big(\eta \otimes i_{Q(\nabla f)} g - i_{Q(\nabla f)} g \otimes \eta\Big)(X, Y).$$

Proposition 2.1. *If M is a closed smooth manifold and ∇ is the Ricci–Golab connection associated to (g, η, Q), then, by integrating w.r.t. the canonical measure of g, we have:*

$$\int_M \operatorname{div}^{(\nabla,g)}(X) = \int_M \big(\operatorname{scal}\cdot\eta - i_{Q\eta^\sharp} g\big)(X),$$

for any $X \in \mathcal{X}(M)$.

In particular,

$$\int_M \operatorname{div}^{(\nabla,g)}(\eta^\sharp) = \int_M \Big(|\eta^\sharp|^2 \cdot \operatorname{scal} - \operatorname{Ric}(\eta^\sharp, \eta^\sharp)\Big).$$

We state a divergence-type theorem w.r.t. the Ricci–Golab connection.

Corollary 2.1. *Under the hypotheses of Proposition 2.1*

$$\int_M \operatorname{div}^{(\nabla,g)}(X) = 0, \quad \forall X \in \mathcal{X}(M)$$

$$\iff \quad \int_M \operatorname{scal}\cdot g(\eta^\sharp, X) = \int_M \operatorname{Ric}(\eta^\sharp, X), \quad \forall X \in \mathcal{X}(M).$$

In particular,

$$\int_M \operatorname{div}^{(\nabla,g)}(\eta^\sharp) = 0 \quad \iff \quad \int_M \operatorname{Ric}(\eta^\sharp, \eta^\sharp) = \int_M |\eta^\sharp|^2 \cdot \operatorname{scal}.$$

Proposition 2.2. *If M is a closed smooth manifold and ∇ is the Ricci–Golab connection associated to (g, η, Q), then, by integrating w.r.t. the canonical measure of g, we have:*

$$\int_M \Delta^{(\nabla,g)}(f) = \int_M \Big(\eta^\sharp(f)\cdot \operatorname{scal} - \operatorname{Ric}(\nabla f, \eta^\sharp)\Big).$$

In particular, if f is constant along the integral curves of $\eta^\sharp$, then

$$\int_M \Delta^{(\nabla,g)}(f) = -\int_M \operatorname{Ric}(\nabla f, \eta^\sharp).$$

Remark 2.1. Under the hypotheses of Proposition 2.2, we deduce that

$$\int_M \Delta^{(\nabla,g)}(f) = 0$$

if and only if $\int_M \operatorname{Ric}(\nabla f, \eta^\sharp) = 0$ (in particular, if $Q\eta^\sharp$ is orthogonal to ∇f).

Proposition 2.3. *Let M be a smooth manifold and let ∇ be the Ricci–Golab connection associated to (g, η, Q). A smooth function f on M is $\Delta^{(\nabla,g)}$-harmonic if and only if*

$$\Delta(f) = \operatorname{Ric}(\nabla f, \eta^\sharp) - \eta^\sharp(f)\cdot \operatorname{scal},$$

equivalent to

$$\Delta(f) = \eta\Big(\big(Q - \operatorname{trace}(Q)\cdot I\big)(\nabla f)\Big).$$

Remark 2.2. Under the hypotheses of Proposition 2.3, a smooth function f on M is harmonic and $\Delta^{(\nabla,g)}$-harmonic if and only if

$$\mathrm{Ric}(\nabla f,\eta^\sharp)=\eta^\sharp(f)\cdot\mathrm{scal},$$

i.e., $df(Q\eta^\sharp)=df(\eta^\sharp)\cdot\mathrm{scal}$. Also, if ∇f is an eigenvalue of the Ricci operator Q with the eigenfunction scal, then f is $\Delta^{(\nabla,g)}$-harmonic if and only if it is harmonic.

Since

$$2d(\Delta(f))+2i_{Q(\nabla f)}g=2\,\mathrm{div}(\mathrm{Hess}(f))=\Delta(df)+i_{Q(\nabla f)}g+d(\Delta(f)),$$

we get

$$\Delta(df)=d(\Delta(f))+i_{Q(\nabla f)}g. \tag{2.8}$$

Now, from the Bochner formula [11] we have

$$\frac{1}{2}\Delta(\langle df,\eta\rangle)=\langle\mathrm{Hess}(f),\nabla\eta\rangle+\mathrm{Ric}(\nabla f,\eta^\sharp)+\langle df,\Delta(\eta)\rangle,$$

which by means of (2.8) gives

$$\mathrm{Ric}(\nabla f,\eta^\sharp)=\Big(\Delta(df)-d(\Delta(f))\Big)(\eta^\sharp).$$

As a consequence, we get a characterization of $\Delta^{(\nabla,g)}$-harmonic functions, from Proposition 2.3.

Corollary 2.2. *If M is a smooth manifold and ∇ is the Ricci–Golab connection associated to (g,η,Q), then a smooth function f on M is $\Delta^{(\nabla,g)}$-harmonic if and only if*

$$\Delta(f)=\Big(\Delta(df)-d(\Delta(f))-\mathrm{scal}\cdot df\Big)(\eta^\sharp).$$

If $\eta^\sharp=\nabla f$, from (2.7), a sufficient condition for f to be $\Delta^{(\nabla,g)}$-harmonic is given by the following propositions.

Proposition 2.4. *Let M be a closed smooth manifold with* $\mathrm{scal}\geq 0$ *and let ∇ be the Ricci–Golab connection associated to (g,df,Q) with f a smooth function on M. If*

(i) $\int_M\mathrm{Ric}(\nabla f,\nabla f)\leq 0$, *and*

(ii) $\Delta^{(\nabla,g)}(f)\leq 0$,

then M has zero scalar curvature and f is a harmonic and $\Delta^{(\nabla,g)}$-harmonic function.

Proposition 2.5. *Let M be a closed smooth manifold with* scal ≤ 0 *and let ∇ be the Ricci–Golab connection associated to (g, df, Q) with f a smooth function on M. If*

(i) $\int_M \mathrm{Ric}(\nabla f, \nabla f) \geq 0$, *and*

(ii) $\Delta^{(\nabla,g)}(f) \geq 0$,

then M has zero scalar curvature and f is a harmonic and $\Delta^{(\nabla,g)}$-harmonic function.

For any two symmetric $(0,2)$-tensor fields T_1 and T_2, denote by

$$\langle T_1, T_2 \rangle := \sum_{1 \leq i,j \leq n} T_1(E_i, E_j) T_2(E_i, E_j),$$

for $\{E_i\}_{1 \leq i \leq n}$ a local orthonormal frame field on (M, g), and by $|\cdot|$ the corresponding norm.

The Bochner formula

$$\frac{1}{2}\Delta(|\nabla f|^2) = |\operatorname{Hess}(f)|^2 + \mathrm{Ric}(\nabla f, \nabla f) + (\nabla f)(\Delta(f)),$$

gives, by integration

$$\int_M \Big(|\operatorname{Hess}(f)|^2 + \mathrm{Ric}(\nabla f, \nabla f) - (\Delta(f))^2 \Big) = 0. \tag{2.9}$$

If $\eta^\sharp = \nabla f$, by using (2.7) and (2.9), we get

Lemma 2.1. *If M is a closed smooth manifold and ∇ is the Ricci–Golab connection associated to (g, df, Q), then, by integrating w.r.t. the canonical measure of g, we get:*

$$\int_M \Delta^{(\nabla,g)}(f) = \int_M |\nabla f|^2 \cdot \mathrm{scal} + \int_M |\operatorname{Hess}(f)|^2 - \int_M (\Delta(f))^2.$$

From Lemma 2.1 and by means of Schwartz's inequality, $|\operatorname{Hess}(f)|^2 \geq \frac{(\Delta(f))^2}{n}$, a sufficient condition for f to be $\Delta^{(\nabla,g)}$-harmonic is given by the following proposition.

Proposition 2.6. *Let M be a closed n-dimensional smooth manifold and let ∇ be the Ricci–Golab connection associated to (g, df, Q) with f a smooth function on M. If*

(i) $\int_M |\operatorname{Hess}(f)|^2 \leq \frac{1}{n-1} \int_M |\nabla f|^2 \cdot \mathrm{scal}$, *and*

(ii) $\Delta^{(\nabla,g)}(f) \leq 0$,

then ∇f is a concircular vector field and f is a $\Delta^{(\nabla,g)}$-harmonic function.

3. On some deformation algebras

Consider now a conformal deformation $\tilde{g} = e^{2u}g$, $u \in \mathcal{F}(M)^*$ of the metric g. Then, the Levi-Civita connections are related by

$$\nabla^{\tilde{g}}_X Y = \nabla^g_X Y + X(u)Y + Y(u)X - g(X,Y)\nabla^g u,$$

for any $X, Y \in \mathcal{X}(M)$, and the Ricci curvature tensor fields and the scalar curvatures satisfy

$$\tilde{\mathrm{Ric}} = \mathrm{Ric} - (n-2)\big(\mathrm{Hess}(u) - du \otimes du\big) + \big(\Delta(u) - (n-2)|\nabla^g u|^2\big)\, g,$$

$$\tilde{\mathrm{scal}} = e^{-2u}\,\mathrm{scal} + (n-1)e^{-2u}\left(2\Delta(u) - (n-2)|\nabla^g u|^2\right),$$

therefore, the Ricci operators are connected by

$$\tilde{Q}X = e^{-2u}\Big(QX - (n-2)\big(\nabla^g_X \nabla^g u - du(X)\nabla^g u\big) + \big(\Delta(u) - (n-2)|\nabla^g u|^2\big)X\Big).$$

Since $\nabla^{\tilde{g}} f = e^{-2u}\nabla^g f$, the Laplace operators are related by

$$\tilde{\Delta} = e^{-2u}\Delta - (n-2)e^{-2u}\nabla^g u.$$

If we consider the particular case $n = 2$, then the Ricci curvature tensors, the Ricci operators, the scalar curvatures and the Laplace operators satisfy

$$\tilde{\mathrm{Ric}} = \mathrm{Ric} + \Delta(u)g,$$

$$\tilde{Q} = e^{-2u}\big(Q + \Delta(u)\cdot I\big),$$

$$\tilde{\mathrm{scal}} = e^{-2u}\big(\mathrm{scal} + 2\Delta(u)\big),$$

$$\tilde{\Delta} = e^{-2u}\Delta.$$

Let A be a $(1,2)$-tensor field on M. If one defines the multiplication of two vector fields $X, Y \in \mathcal{X}(M)$ by the rule $X \circ Y = A(X,Y)$, then the $\mathcal{F}(M)$-module $\mathcal{X}(M)$ becomes an $\mathcal{F}(M)$-algebra. This algebra, denoted by $\mathcal{U}(M,A)$ [6, 8], is called the *deformation algebra associated to A*.

If we consider $A = \overline{\nabla} - \nabla$, then $\mathcal{U}(M,A)$ is called the deformation algebra associated to the pair of linear connections $(\overline{\nabla}, \nabla)$ on M.

An element $Y \in \mathcal{U}(M,A)$ is called an *almost 1-principal vector field* if there exists a map $f \in \mathcal{F}(M)$ and a 1-form $\sigma \in \bigwedge^1(M)$ such that

$$A(X,Y) = fX + \sigma(X)Y,$$

for any $X \in \mathcal{X}(M)$. If $\sigma = 0$, then Y is called a *characteristic vector field*.

Let $M^2 \subset \mathbb{R}^3$ be a surface in the Euclidean space, such that the Ricci tensor field is positive definite. Let g be the first fundamental form. Therefore M^2 is an

Einstein space and there exists $u \in \mathcal{F}(M)^*$, such that $\mathrm{Ric} = e^{2u}g$, where $e^{2u} = K$ is the Gauss curvature.

Let us denote by $\tilde{\nabla}$ and ∇ the Ricci–Golab connections associated to $(\tilde{g} := \mathrm{Ric}, \eta, \tilde{Q})$ and (g, η, Q). Then we have

$$\tilde{\nabla}_X Y = \nabla^{\tilde{g}}_X Y + \big(1 + e^{-2u}\Delta(u)\big)\eta(Y)X - \big(e^{2u} + \Delta(u)\big)g(X,Y)\eta^\sharp,$$

$$\nabla_X Y = \nabla^g_X Y + e^{2u}\eta(Y)X - e^{2u}g(X,Y)\eta^\sharp.$$

One gets

$$\tilde{\nabla}_X Y - \nabla^g_X Y = \Big(du(Y) + \big(1 + e^{-2u}\Delta(u)\big)\eta(Y)\Big)X + du(X)Y$$

$$-g(X,Y)\Big(du + \big(e^{2u} + \Delta(u)\big)\eta\Big)^\sharp,$$

$$\nabla_X Y - \nabla^g_X Y = e^{2u}\eta(Y)X - g(X,Y)\big(e^{2u}\eta\big)^\sharp.$$

Theorem 3.1. *The deformation algebras $\mathcal{U}(M, \tilde{\nabla} - \nabla^g)$ and $\mathcal{U}(M, \nabla - \nabla^g)$ have the same almost 1-principal vector fields if*

(i) the function u is constant

or

(ii) $\Delta(u)(x) \neq 0$, for any $x \in M$, and $\eta = -\frac{1}{\Delta(u)}du$.

Proof. Let $Y \in \mathcal{U}(M, \tilde{\nabla} - \nabla^g)$ be an almost 1-principal vector field. Therefore there exists a map $f \in \mathcal{F}(M)$ and a 1-form $\sigma \in \bigwedge^1(M)$ such that

$$\tilde{\nabla}_X Y - \nabla^g_X Y = fX + \sigma(X)Y,$$

for any $X \in \mathcal{X}(M)$.

(i) If the function u is constant, then the previous remark leads to

$$\tilde{\nabla}_X Y - \nabla^g_X Y = \eta(Y)X - e^{2u}g(X,Y)\eta^\sharp,$$

$$\nabla_X Y - \nabla^g_X Y = e^{2u}\eta(Y)X - e^{2u}g(X,Y)\eta^\sharp.$$

From our assumption and the last two relations one gets

$$\nabla_X Y - \nabla^g_X Y = \Big(f + (e^{2u} - 1)\eta(Y)\Big)X + \sigma(X)Y$$

and $Y \in \mathcal{U}(M, \nabla - \nabla^g)$ is an almost 1-principal vector field.

(ii) If $\Delta(u)(x) \neq 0$, for any $x \in M$, and $\eta = -\frac{1}{\Delta(u)}du$, then the previous remark leads to

$$\tilde{\nabla}_X Y - \nabla^g_X Y = -\frac{1 + (e^{-2u} - 1)\Delta(u)}{\Delta(u)}du(Y)X + du(X)Y + \frac{e^{2u}}{\Delta(u)}g(X,Y)\nabla^g u,$$

$$\nabla_X Y - \nabla^g_X Y = -\frac{e^{2u}}{\Delta(u)} du(Y)X + \frac{e^{2u}}{\Delta(u)} g(X,Y)\nabla^g u.$$

From our assumption and the last two relations one gets

$$\nabla_X Y - \nabla^g_X Y = \Big(f + \Big(\frac{1 - e^{2u} + (e^{-2u}-1)\Delta(u)}{\Delta(u)}\Big) du(Y)\Big)X$$

$$+\big(\sigma(X) - du(X)\big)Y$$

and $Y \in \mathcal{U}(M, \nabla - \nabla^g)$ is an almost 1-principal vector field.

Similarly we deduce that if $Y \in \mathcal{U}(M, \nabla - \nabla^g)$ is an almost 1-principal vector field, then $Y \in \mathcal{U}(M, \tilde{\nabla} - \nabla^g)$ is an almost 1-principal vector field, too. □

Let $\xi \in \mathcal{X}(M)$. Two linear connections ∇^1 and ∇^2 on a Riemannian manifold (M, g) are called *ξ-subgeodesically related* if they satisfy the Yano formula

$$\nabla^2_X Y = \nabla^1_X Y + \psi(X)Y + \psi(Y)X - g(X,Y)\xi,$$

for any $X, Y \in \mathcal{X}(M)$, where $\psi \in \bigwedge^1(M)$.

Theorem 3.2. *If the function u satisfies $\Delta(u) = e^{2u}(e^{2u}-1)$, then the Ricci–Golab connections ∇ and $\tilde{\nabla}$ are $\theta^\sharp$-subgeodesically related, where*

$$\theta := du + e^{2u}(e^{2u}-1)\eta.$$

Proof. The Ricci–Golab connections are related by

$$\tilde{\nabla}_X Y - \nabla_X Y = du(X)Y + \Big(du(Y) + \big(1 + e^{-2u}\Delta(u) - e^{2u}\big)\eta(Y)\Big)X$$

$$-g(X,Y)\big(du + \Delta(u)\eta\big)^\sharp.$$

If we impose $\Delta(u) = e^{2u}(e^{2u}-1)$, then

$$\tilde{\nabla}_X Y - \nabla_X Y = du(X)Y + du(Y)X - g(X,Y)\theta^\sharp,$$

where $\theta = du + e^{2u}(e^{2u}-1)\eta$. □

Proposition 3.1. *Let $\Delta(u) = e^{2u}(e^{2u}-1)$ and $\eta = e^{-2u}du$. If Y is a characteristic vector field of the algebra $\mathcal{U}(M, \tilde{\nabla} - \nabla)$, then*

$$|Y|^2 du = Y(u)Y^\flat. \tag{3.1}$$

Proof. We get

$$\tilde{\nabla}_X Y - \nabla_X Y = du(X)Y + du(Y)X - e^{2u}g(X,Y)\nabla^g u.$$

Then $Y \in \mathcal{U}(M, \tilde{\nabla} - \nabla)$ is a characteristic vector field if and only if there exists $\lambda \in \mathcal{F}(M)$ such that $\tilde{\nabla}_X Y - \nabla_X Y = \lambda X$, for any $X \in \mathcal{X}(M)$. It follows

$$du(X)Y + du(Y)X - e^{2u} g(X,Y)\nabla^g u = \lambda X, \tag{3.2}$$

for any $X \in \mathcal{X}(M)$. If one considers $X = Y$, then (3.2) becomes

$$(2 - e^{2u})du(Y)|Y|^2 = \lambda |Y|^2.$$

Therefore, if $|Y| \neq 0$, we obtain

$$\lambda = (2 - e^{2u})du(Y). \tag{3.3}$$

Using (3.2) and (3.3), we get

$$du(X)Y + (e^{2u} - 1)du(Y)X = e^{2u} g(X,Y)\nabla^g u.$$

This formula leads to

$$du(X)g(Y,Y) + (e^{2u} - 1)du(Y)g(X,Y) = e^{2u} du(Y)g(X,Y). \tag{3.4}$$

Therefore

$$du(X)g(Y,Y) = du(Y)g(X,Y), \tag{3.5}$$

for any $X \in \mathcal{X}(M)$ and formula (3.5) implies (3.1). $\square$

The previous result leads to

Corollary 3.1. *Let $\Delta(u) = e^{2u}(e^{2u} - 1)$, $\eta = e^{-2u} du$ and $Y \in \mathcal{U}(M, A = \tilde{\nabla} - \nabla)$ a characteristic vector field.*

(i) If $Y_x(u) = 0$, for any $x \in M$, then $A = 0$.

(ii) If $A_x = 0$, for any $x \in M$, then $g(Y, \nabla^g u) = 0$.

Corollary 3.2. *Let $\Delta(u) = e^{2u}(e^{2u} - 1)$ and $\eta = e^{-2u} du$. If all the elements of the algebra $\mathcal{U}(M, A = \tilde{\nabla} - \nabla)$ are characteristic vector fields, then $\tilde{\nabla} = \nabla$.*

Proof. Let $x \in M$ and $Y \in \mathcal{X}(M)$ be a characteristic vector field such that $Y_x \neq 0$. There exists $X \in \mathcal{U}(M, \tilde{\nabla} - \nabla)$ such that $X_x \neq 0$ and $g_x(X_x, Y_x) = 0$.

From (3.5) we get $du_x(X_x)g_x(Y_x, Y_x) = du_x(Y_x)g_x(X_x, Y_x)$ which leads to $du_x(X_x) = 0$ and since X is a characteristic vector field, we get $\lambda_x = 0$ and $A_x = 0$. $\square$

Proposition 3.2. *Let $\Delta(u) = e^{2u}(e^{2u} - 1)$ and $\eta = e^{-2u} du$. If $\lambda \in \mathcal{F}(M)$ such that $\lambda(x) \neq 0$, for any $x \in M$, then there exists a nonzero characteristic vector field $Y \in \mathcal{X}(M)$ such that $A(Y,Y) = \lambda Y$, where $A = \tilde{\nabla} - \nabla$.*

Proof. Let $x_0 \in M$. We need to show that there exists a vector field Y such that $Y_{x_0} \neq 0$ and

$$\lambda Y = 2g(\nabla^g u, Y)Y - e^{2u}|Y|^2\nabla^g u.$$

In this case, by taking the scalar product with Y, we get $\lambda = (2 - e^{2u})g(\nabla^g u, Y)$, hence

$$g(\nabla^g u, Y) = \frac{\lambda}{2 - e^{2u}}$$

(since $\lambda(x) \neq 0$, for any $x \in M$), which replaced in the previous relation gives

$$Y = \frac{2 - e^{2u}}{\lambda}|Y|^2\nabla^g u.$$

Now, by taking the scalar product with $\nabla^g u$, we obtain

$$|Y|^2 = \frac{\lambda^2}{(2 - e^{2u})^2|\nabla^g u|^2},$$

hence

$$Y = \frac{\lambda}{2 - e^{2u}} \cdot \frac{\nabla^g u}{|\nabla^g u|^2}. \tag{3.6}$$

Therefore, Y defined by (3.6) is a nonzero characteristic vector field such that $A(Y, Y) = \lambda Y$. □

Corollary 3.3. *Let* $\Delta(u) = e^{2u}(e^{2u} - 1)$ *and* $\eta = e^{-2u}du$. *Then the following assertions are equivalent:*

(i) there exists nonzero characteristic vector fields in any $x \in M$, *in the algebra* $\mathcal{U}(M, A)$;

(ii) $A_x \neq 0$, *for any* $x \in M$.

Proof. (i) ⇒ (ii) It is obvious.

(ii) ⇒ (i) It is a consequence of the previous result. If $A_x \neq 0$, for any $x \in M$, then $\lambda_x \neq 0$ and a nonzero characteristic vector field is given by (3.6). □

4. Almost 1-principal Golab connections

Proposition 4.1. *[8] Let* A *be a fixed* $(1, 2)$*-tensor field. Then the following assertions are equivalent:*

(i) all the elements of the algebra $\mathcal{U}(M, A)$ *are almost 1-principal vector fields;*

(ii) there exist two 1-forms ω *and* ν *on* M *such that*

$$A(X, Y) = \omega(X)Y + \nu(Y)X,$$

for any $X, Y \in \mathcal{X}(M)$.

Let g be a Riemannian metric, let η be a 1-form on a smooth manifold M and let ∇ be the Ricci–Golab connection associated to (g, η, Q).

A linear connection $\overline{\nabla}$ is called an *almost 1-principal Golab connection* if all the elements of the deformation algebra $\mathcal{U}(M, \overline{\nabla} - \nabla)$ are almost 1-principal vector fields, therefore if there exist two 1-forms ω and ν on M such that

$$\overline{\nabla}_X Y - \nabla_X Y = \omega(X)Y + \nu(Y)X,$$

for any $X, Y \in \mathcal{X}(M)$. One has

$$\nabla_X Y - \nabla^g_X Y = \eta(Y)QX - \mathrm{Ric}(X,Y)\eta^\sharp,$$

$$\overline{\nabla}_X Y - \nabla^g_X Y = \omega(X)Y + \nu(Y)X + \eta(Y)QX - \mathrm{Ric}(X,Y)\eta^\sharp.$$

Theorem 4.1. *The deformation algebras $\mathcal{U}(M, \overline{\nabla} - \nabla^g)$ and $\mathcal{U}(M, \nabla - \nabla^g)$ have the same almost 1-principal vector fields.*

Proof. Let $Y \in \mathcal{U}(M, \overline{\nabla} - \nabla^g)$ be an almost 1-principal vector field. Therefore there exists a map $f \in \mathcal{F}(M)$ and a 1-form $\sigma \in \bigwedge^1(M)$ such that

$$\overline{\nabla}_X Y - \nabla^g_X Y = fX + \sigma(X)Y,$$

for any $X \in \mathcal{X}(M)$. Then

$$\nabla_X Y - \nabla^g_X Y = \big(f - \nu(Y)\big)X + (\sigma - \omega)(X)Y,$$

for any $X \in \mathcal{X}(M)$, hence $Y \in \mathcal{U}(M, \nabla - \nabla^g)$ is an almost 1-principal vector field. Similarly we deduce that if $Y \in \mathcal{U}(M, \nabla - \nabla^g)$ is an almost 1-principal vector field, then $Y \in \mathcal{U}(M, \overline{\nabla} - \nabla^g)$ is an almost 1-principal vector field. $\square$

Let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface in the Euclidean space, $n \geqslant 2$. Let g and $II =: h$ be the first and the second fundamental forms. We suppose that the second fundamental form is positive definite, thus it can be regarded as a Riemannian metric. In particular, for $n = 2$, such surfaces will be called *ovaloids*. This condition means that the Gaussian curvature is strictly positive and that the unit normal vector is chosen such as to make both principal curvatures positive.

Theorem 4.2. *The following assertions are equivalent:*

(i) all the elements of the deformation algebra $\mathcal{U}(M^n, \nabla^h - \nabla^g)$ are almost 1-principal vector fields;

(ii) M^n is a spherical hypersurface;

(iii) $\mathrm{scal} \geqslant K > 0$ *and the lowest eigenvalue of Δ verifies $\lambda_1 = \frac{n}{n-1}K$.*

Proof. (i)$\Longleftrightarrow$ (ii) [8]; (ii)$\Longleftrightarrow$ (iii) [9]. $\square$

For $n = 2$, let ∇ be the Ricci–Golab connection associated to (g, η, Q), where $Q = KI, K$ is the Gauss curvature (since M^2 is Einstein). Therefore

$$\nabla_X Y = \nabla^g_X Y + K\eta(Y)X - Kg(X,Y)\eta^\sharp.$$

By analogy, we consider ∇^{II} to be the Ricci–Golab connection associated to (h, η, Q^{II}), where $Q^{II} = K^{II} I$. Hence

$$\nabla^{II}_X Y = \nabla^h_X Y + K^{II}\eta(Y)X - K^{II} g(S(X), Y)\eta^\sharp,$$

where $g(S(X), Y) = h(X, Y)$ with S the shape operator.

Corollary 4.1. *We denote by* $A := \nabla^{II} - \nabla$ *and we have*

$$A(X,Y) = S^{-1}(\nabla_X(S(Y));$$

$$S(\nabla^{II} f) = \nabla^g f;$$

$$\text{trace}(A) = \text{trace}\{X \mapsto A(X,\cdot)\} = \frac{1}{2} d \log K;$$

$$\text{trace}^{II}(A) = \text{trace}^{II}\{(X,Y) \mapsto A(X,Y)\} = \frac{1}{2}\nabla^{II} \log K;$$

$$\text{trace}(A) = \text{trace}\{(X,Y) \mapsto A(X,Y)\} = \nabla^{II} H;$$

$$\text{div}^{II}(X) = \text{div}(X) + \big(\text{trace}(A)\big)(X);$$

$$\Delta^{II}(f) = \text{trace}^{II}\left(\text{Hess}(f)\right) - h\big(\nabla^{II} f, \text{trace}^{II}(A)\big),$$

where H *and* K *are the mean and Gauss curvatures.*

Conclusions

In Section 2, we studied the divergence operator, the Hessian of a smooth function and the Laplace operator with respect to the Ricci–Golab connection. We found necessary and sufficient conditions for a smooth function f on M to be harmonic and $\Delta^{(\nabla,g)}$-harmonic, characterizing, in this terms, the manifolds of zero scalar curvature. In the following sections we considered a conformal class of metrics and we focused on dimension 2, on surfaces in the Euclidean space, such that the Ricci tensor field is positively definite and the metric is the first fundamental form. We characterized the conformal class of metrics in terms of the Laplace operator. Then we explored the deformation algebras of the Ricci–Golab connections having the same almost 1-principal vector fields, the deformation algebras with characteristic vector fields and showed that the Ricci–Golab connections are subgeodesically related. Finally, we investigated in Section 4 the deformation algebras of 1-principal Golab connections. A next level of a future study would be to characterize the deformation algebras of ovaloids.

REFERENCES

1. A. Barman, *On a type of quarter-symmetric non-recurrent metric connection on a P-Sasakian manifold.* Publ. l'Institut Math. N.S. **110**(124), (2021), 91–102.
2. E. O. Canfes, I. Gul, *Weyl manifold with a Ricci quarter-symmetric connection.* Iran. J. Sci. Technol. Trans. Sci. **40**, (2016), 171–175.
3. S. K. Chaubey, J. W. Lee, S.K. Yadav, *Riemannian manifolds with a semi-symmetric metric P-connection.* J. Korean Math. Soc. **56**(4), (2019), 1113–1129.
4. S. Golab, *On semi-symmetric and quarter-symmetric connections.* Tensor. N.S. **29**(3), (1975), 249–254.
5. Y. Han, T. Ho, P. Zhao, *Some invariants of quarter-symmetric metric connections under the projective transformation.* Filomat. **27**(4), (2013), 679–691.
6. I. E. Hirică, *On some Golab connections.* Demonstratio Mathematica. **39**(4), (2006), 919–926.
7. I. E. Hirică, L. Nicolescu, *On quarter-symmetric metric connections on pseudo-Riemannian manifolds.* Balkan J. Geom. Appl. **16**(1), (2011), 56–65.
8. L. Nicolescu, *Sur les courbes presque m-principales associees à un champ tensoriel du type (1,2).* An. Univ. Bucureşti, Matematica. **53**(1), (2004) 93–104.
9. M. Obata, *Certain conditions for a Riemannian manifold to be isometric to the sphere.* J. Math. Soc. Japan. **14**, (1962) 333–340.
10. P. Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, (2006).
11. M. Z. Petrovic, M. S. Stankovic, *A note on F-planar mappings of manifolds with non-symmetric linear connection.* Int. J. Geom. Meth. Modern Phys. **16**(5), (2019).
12. K. T. Pradeep Kumar, B. M. Roopa, K. H. Arun Kumar, *On* W_0 *and* W_2 *φ-symmetric contact manifold admitting quarter-symmetric metric connection.* J. Phys. Conf. Ser., (2021), 2070–2075.
13. M. Reuter, S. Biasotti, D. Georgi, G. Patane, M. Spagnuolo, *Discrete Laplace-Beltrami operators for shape analysis and segmentation.* Computers and Graphics. **33**(3), (2009), 381–390.
14. I. Suhendro, *A New Semi-symmetric unified field theory of the classical fields of gravity and electromagnetism.* Progress in Physics. **4**, (2007), 47–62.
15. W. Tang, J. Yong, T. Ho, G. He, P. Zhao, *Geometries of manifolds equipped with a Ricci (projection-Ricci) quarter-symmetric connection.* Filomat. **33**(16), (2019), 5237–5248.
16. H. Volkmer, *The Laplace-Beltrami operator on the embedded torus.* Journal of Differential Equations. **271**, (2021), 821–848.
17. K. Yano, *On semi-symmetric metric connection.* Rev. Roum. Math. Pures Appl. **15**, (1970), 1570–1586.
18. K. Yano, J. Imai, *Quarter-symmetric connection and their curvature tensors.* Tensor. N. S. **38**, (1982), 13–18.
19. P. Zhao, T. Ho, A. J. Hyon, *Geometries for a mutual connection of semi-symmetric metric recurrent connections.* Filomat. **34**(13), (2020) 4367–4374.

20. P. Zhao, H. Song, X. Yang, *Some invariant properties of semi-symmetric recurrent connections and curvature tensor expression.* Chin. Quart. J. Math. **14**(4), (2004), 355–361.