The Traveling Salesman Problem with Stochastic and Correlated Customers

Wissink, Pascal

published in
Transportation Science
2023

DOI (link to publisher)
10.1287/trsc.2022.0005

document version
Publisher's PDF, also known as Version of record

document license
Article 25fa Dutch Copyright Act

Transportation Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

The Traveling Salesman Problem with Stochastic and Correlated Customers

Pascal L. J. Wissink

informs
https://pubsonline.informs.org/journal/trsc

TRANSPORTATION SCIENCE
Vol. 57, No. 5, September–October 2023, pp. 1321–1339
ISSN 0041-1655 (print), ISSN 1526-5447 (online)

The Traveling Salesman Problem with Stochastic and Correlated Customers

Pascal L. J. Wissink[a]

[a] School of Business and Economics, Vrije Universiteit Amsterdam, Amsterdam 1081 HV, Netherlands
Contact: pascal.wissink@vu.nl, https://orcid.org/0000-0002-6031-6627 (PLJW)

Received: January 6, 2022
Revised: January 10, 2023; May 25, 2023
Accepted: June 15, 2023
Published Online in Articles in Advance: July 28, 2023

https://doi.org/10.1287/trsc.2022.0005

Abstract. It is well-known that the cost of parcel delivery can be reduced by designing routes that take into account the uncertainty surrounding customers' presences. Thus far, routing problems with stochastic customer presences have relied on the assumption that all customer presences are independent from each other. However, the notion that demographic factors retain predictive power for parcel-delivery efficiency suggests that shared characteristics can be exploited to map dependencies between customer presences. This paper introduces the correlated probabilistic traveling salesman problem (CPTSP). The CPTSP generalizes the traveling salesman problem with stochastic customer presences, also known as the probabilistic traveling salesman problem (PTSP), to account for potential correlations between customer presences. I propose a generic and flexible model formulation for the CPTSP using copulas that maintains computational and mathematical tractability in high-dimensional settings. I also present several adaptations of existing exact and heuristic frameworks to solve the CPTSP effectively. Computational experiments on real-world parcel-delivery data reveal that correlations between stochastic customer presences do not always affect route decisions, but could have a considerable impact on route cost estimates.

Supplemental Material: The online appendix is available at https://doi.org/10.1287/trsc.2022.0005.

Keywords: traveling salesman problem • stochastic vehicle routing • correlation

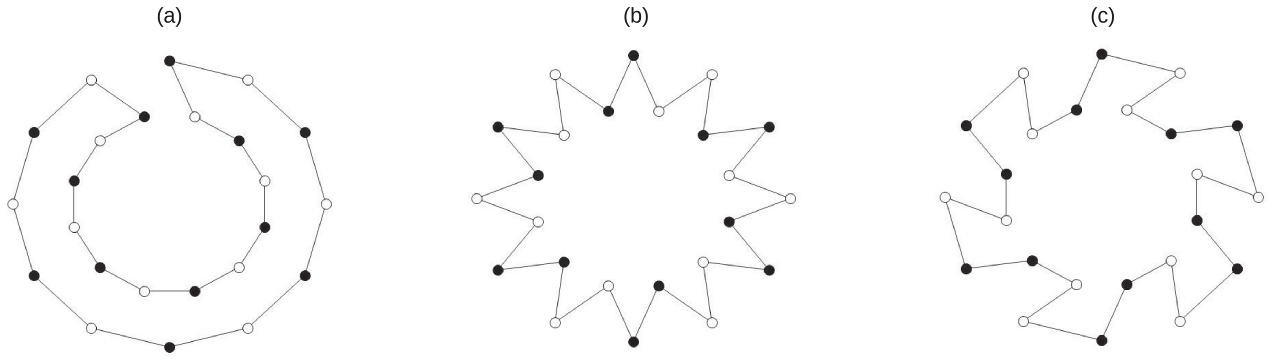
1. Introduction

The surge of e-commerce has led to a tremendous increase in the number of business-to-consumer deliveries, with domestic parcel-delivery volumes growing 54% across the European Union between 2016 and 2020 (European Commission 2021). In the fulfilment of online orders, the so-called "last mile"—the delivery of the order from the carrier to the customer's doorstep—is arguably the least efficient and most expensive stage in the delivery process (Macioszek 2017). One of the most difficult challenges carriers are facing in this stage of the delivery process is the high number of failed deliveries due to absent customers (Mangiaracina et al. 2019). With failure rates as high as 3%–50% (Rai, Verlinde, and Macharis 2021), failed deliveries give rise to enormous amounts of extra emissions (Van Loon et al. 2015), a deterioration of service levels (Mangiaracina et al. 2019), collection and delivery-point expenses (Liu, Wang, and Susilo 2019), and costs associated with redelivery (IMRG 2018).

Jaillet (1988) was among the first to recognize that efficiency can be gained from accounting for uncertainty among customer presences in the design of a route. Jaillet (1988) proposes the traveling salesman problem with stochastic customers, more commonly known as the probabilistic traveling salesman problem (PTSP), a version of the traveling salesman problem (TSP) in which every customer is only present with some predefined probability. He uses a PTSP instance where all customers have a 0.5 probability of being present to illustrate that a good tour that exploits this information to skip absent customers during execution (Figure 1(b)) does not need to coincide with a good tour that neglects information about uncertainty (Figure 1(a)).

Although the probability estimates for every customer's presence can be hard to obtain in the real world, evidence shows that demographic characteristics are reasonable predictors of daily customer presences on a joint level (Boyer, Prud'homme, and Chung 2009, Van Duin et al. 2016). This suggests that the presence patterns of customers who possess similar demographic characteristics are likely to follow a similar trend and, thus, correlate over time. Because residents in the same neighborhood often exhibit similar characteristics (e.g., Schaefer and Figliozzi 2021), this correlation could also manifest spatially. For example, the positive interaction between the delivery success rates of neighboring customers who belong to the same age group (Kübler et al. 2022) implies that their presence probabilities could be correlated. Gendreau, Jabali, and Rei (2016) argue that accounting for

Figure 1. Differences Between Good TSP, PTSP, and CPTSP Tours



Notes. (a) Good TSP tour. (b) Good PTSP tour. (c) Good CPTSP tour.

correlations in the setup of probabilistic routing problems should warrant more efficient solutions than, for example, the ones presented in Figure 1(a) and (b).

In spite of this promising hypothesis, no attempts have been made so far to investigate the claim by Jaillet (1985, p. 45) that the PTSP can be generalized to a version that considers dependencies between customer presences. Instead, researchers have always worked under the strong assumption that stochastic customer presences are independent from each other. As a consequence, (1) how a traveling salesman problem with stochastic and correlated customer presences should be modeled; (2) how a traveling salesman problem with stochastic and correlated customer presences can be solved; and (3) what impact correlation between stochastic customers in a traveling salesman problem has on route costs and route decisions remain open research questions.

To answer these questions, this paper introduces a novel stochastic optimization problem in which the uncertain customer presences of a PTSP are dependent. I refer to this new stochastic optimization problem as the correlated probabilistic traveling salesman problem (CPTSP). The CPTSP is a generalization of the PTSP, such that their overall objectives and two-stage setups are the same. More specifically, the objective of the CPTSP is to find the a priori tour along the complete set of stochastic customers in the first stage that minimizes the expected length of the a posteriori tour along only those customers who are present in the second stage, while skipping the absent customers. Unlike the PTSP, however, the CPTSP also takes into account the correlation between the presences of customers in the evaluation of the length of the tour besides only their marginal presence probabilities. Hence, the objective function of the CPTSP explicitly captures potential interactions between the presences of customers.

The discrete nature of the stochastic customer presences (either present or absent) elicits several dependence modeling challenges. To address the first research question (1), this research focuses on deriving a generic and flexible objective function that can model dependencies between stochastic customers with different dependence structures. To this end, the objective function of the CPTSP relies on discrete-vine (D-Vine) pair copula constructions (PCCs) for the estimation of probabilities of jointly correlated customer presences (Panagiotelis, Czado, and Joe 2012). I also derive a special form of the CPTSP objective function under homogeneous (i.e., equiprobable and equicorrelated) conditions, which allows one to study limiting behavior in a controlled environment.

To address the second research question (2), I demonstrate how the CPTSP can be solved by adapting existing solution techniques that were originally developed for the PTSP. First, I modify an adaptation of the integer L-shaped method for the PTSP (Laporte, Louveaux, and Mercure 1994) to solve the CPTSP to optimality. Second, I present two different approximation procedures that allow one to evaluate the expected length of a CPTSP tour with less computational effort than the quartic complexity required to evaluate the full function. These approximation procedures can be directly embedded into many existing heuristic methods for the PTSP. I consider probabilistic ant-colony optimization (Bianchi, Gambardella, and Dorigo 2002) and 2.5-opt-empirical estimation and speedup (2.5-opt-EEs) (Birattari et al. 2008) to illustrate this notion.

I subsequently use these solution methods to revisit the problem instance in Figure 1, while adding a pairwise correlation of 0.9 between the presences of customers who share different node colors (see Online Appendix A for the full details). This experiment reveals that a good solution for the CPTSP (Figure 1(c)) admits the shape of a gear that coincides neither with the star-shaped solution of the PTSP (Figure 1(b)) nor with the C-shaped solution of the TSP (Figure 1(a)). Instead, the CPTSP solution pulls the PTSP solution toward the TSP solution whenever adjacent customers express a higher probability of being jointly present (or absent) than they would have under the assumption of independence.

Analogous to the conjecture posed by the star-shaped PTSP solution, the uniqueness of the gear-shaped CPTSP solution illustrates that neglecting dependencies could lead to different, suboptimal outcomes. This evidence in support of the hypothesis by Gendreau, Jabali, and Rei (2016) also raises the question of how sensitive routes are to correlation in different settings.

To answer the last research question (3), I therefore conduct a number of computational experiments that highlight the impact of various degrees of dependence on the expected length of a tour, algorithmic performance, and route decisions. I find that the marginal effect of correlation on the expected length of a tour tends to grow with increasing correlation, reflecting the construct that correlation induces uncertainty about the need to visit multiple stochastic customers jointly. The integer L-shaped method generally yields the best solutions within 600 CPU seconds, regardless of the dependence structure and intensity of dependence. In many cases, however, a CPTSP adaptation of probabilistic ant-colony optimization delivers only marginally inferior results in just a fraction of the computation time. A last computational experiment illustrates the practical value of the CPTSP in a last-mile parcel-delivery problem with delivery cancellations. Using real-world parcel-delivery data for an area in Amsterdam, Netherlands, I show that correlation has a noticeable effect on route cost estimates, but does not lead to different route decisions.

To the best of my knowledge, the research presented in this paper is the first to integrate and study dependencies between discrete random variables in a stochastic routing problem. Specifically, the main contributions that result from addressing the research questions are as follows. (1) This research derives a novel generic and computationally efficient expression for the expected length of a Hamiltonian tour, where the presences of the nodes (i.e., customers) in a graph are both stochastic and correlated. The CPTSP is proposed by embedding this expression in the objective function of the two-stage stochastic program with recourse for the PTSP. This paper also proves a number of properties about the behavior of stochastic customer dependencies for the CPTSP in homogeneous settings. (2) This study proposes several new adaptations of solution approaches, both exact and heuristic in nature, to solve the CPTSP. (3) This paper applies these solution approaches to demonstrate the impact and practical value of modeling dependencies between stochastic customer presences in a CPTSP under both stylized and practical conditions.

The remainder of this paper is organized as follows. Section 2 surveys the relevant literature in the field. Section 3 discusses the mathematical setup of the CPTSP. I also derive an expression for its objective function in this section—that is, the expected length of a tour along stochastic and correlated customers. Section 4 describes and characterizes the generic probability mass function (pmf) incorporated by the expected length. It is followed by a discussion of a computationally more attractive method to compute the pmf of the CPTSP using D-Vine PCCs in Section 5. Section 6 formalizes the homogeneous CPTSP and derives several properties. Sections 7 and 8 propose exact and approximate methods, respectively, to solve the CPTSP. Section 9 describes a number of computational experiments that demonstrate the impact of different dependence structures and intensities on the route costs and route decisions produced by different solution approaches, along with a more practical application of the CPTSP. Finally, Section 10 concludes.

2. Literature Review

The CPTSP has its roots in the stochastic vehicle-routing literature. Stochastic vehicle routing is a relatively young area of research that can be traced back to the introduction of the vehicle-routing problem (VRP) with stochastic demands in 1969. The field was developed as a straightforward extension from deterministic routing problems, most notably the TSP (Dantzig, Fulkerson, and Johnson 1954) and the VRP (Dantzig and Ramser 1959), nourished by the simultaneous development of stochastic programming and stochastic-dynamic programming.

Stochastic routing problems are among the most studied topics of the overarching optimization branch known as stochastic mixed-integer programming. The most commonly encountered stochastic components in the routing literature are stochastic demands (Tillman 1969), stochastic travel or service times (Kao 1978, Leipälä 1978), and stochastic customer presences (Jaillet 1985, 1988). Comprehensive overviews of the body of literature devoted to stochastic routing appeared in Gendreau, Jabali, and Rei (2014), Berhan et al. (2014), and Oyola, Arntzen, and Woodruff (2017, 2018). Driven by the increasing availability of bigger data sets, more computational power, and the realization that many real-world routing problems are inherently stochastic in nature, the body of literature devoted to stochastic routing problems is rapidly growing.

Not surprisingly, extensions of the TSP with uncorrelated stochastic customers are abundant. For example, Beraldi et al. (2005) consider the probabilistic pickup and delivery TSP, in which some customer requests have to be fulfilled prior to serving other customers. Tang and Miller-Hooks (2007) study the probabilistic generalized TSP, where each stochastic customer is assigned to a cluster, and each cluster must be traversed at least once. Campbell and Thomas (2008) develop the PTSP with deadlines, in which late arrivals are penalized. Voccia, Campbell, and Thomas (2013) subsequently enhance this problem to also penalize early arrivals and name it the PTSP with time windows accordingly. Zhang et al. (2018) propose a version of the PTSP where a profit is made from every customer visit, such that both profit maximization and travel-time minimization are part of the objective.

In contrast to extensions of the PTSP, generalizations of the PTSP are scarce. Undoubtedly the best-known generalization of the PTSP is the case with multiple vehicles, referred to as the VRP with stochastic customers (Bertsimas 1988). Other, lesser-related variations on the PTSP that deserved considerable attention include the VRP with both stochastic demands and customers (Bertsimas 1992); the dynamic VRP with time windows and stochastic customers (Bent and Van Hentenryck 2004); the courier delivery problem with uncertain customer presence and service times (Sungur et al. 2010); and the vehicle-routing and districting problem with stochastic customers (Lei, Laporte, and Guo 2012). None of these related problems, nor their extensions, accommodate dependencies within the stochastic customer component.

In an invited review for the 50th anniversary of *Transportation Science*, Gendreau, Jabali, and Rei (2016) raise the concern that only a few researchers consider dependence between the stochastic components of routing problems—whereas, in reality, the uncertain components of stochastic routing problems often maintain complex relationships. That is, only a small number of specific cases have been investigated in the literature prior to 2016. Most notably, Golden and Yee (1979) and Stewart and Golden (1983) study correlations between stochastic demands in a chance-constrained setup of the VRP under a set of special conditions. Furthermore, Toriello, Haskell, and Poremba (2014) consider stochastically correlated travel times in a dynamic version of the TSP, and Letchford and Nasiri (2015) study a Steiner TSP with stochastic and correlated road-traversal cost.

Partially inspired by the observation from Gendreau, Jabali, and Rei (2016), a few notable papers recently emerged that do address generic cases of correlation between stochastic components. As far as stochastic demands are concerned, Gounaris et al. (2016) explore correlations between the uncertain demands of a robust capacitated VRP; Dell'Amico et al. (2018) investigate a bike-sharing rebalancing problem with correlations among stochastic demands; and Dinh, Fukasawa, and Luedtke (2018) formulate another, more general chance-constrained version of the VRP with stochastic demands that features correlations. Among the papers in the stochastic travel times domain, Köster et al. (2018) extend the dynamic TSP with stochastic and correlated travel times from Toriello, Haskell, and Poremba (2014) to the dynamic VRP; Rajabi-Bahaabadi et al. (2021) analyze a VRP with correlated stochastic travel times and soft time windows; Rostami et al. (2021) propose branch-price-and-cut algorithms for another variant of the VRP with stochastic and correlated travel times; and Bakach et al. (2021) examine a VRP with correlated stochastic travel times and makespan objectives.

It is striking that dependence in the last remaining major stochastic component, customer presences, has not yet been investigated, even though many real-world problems in logistics and transportation are characterized by dependencies between uncertain customer presences (Gendreau, Jabali, and Rei 2016). Instead, the PTSP—along with the rest of the literature concerning stochastic customers—has always been investigated while assuming independently distributed Bernoulli variables associated with each individual customer's presence (success) or absence (failure).

Recent advances in distributionally robust optimization (DRO) provide several practical tools to solve routing problems with potential dependencies between uncertain components (Delage and Ye 2010), including stochastic customer presences (Carlsson, Behroozi, and Mihic 2018). In contrast to stochastic programming, DRO does not rely on an exact analytical representation of the objective function. Instead, DRO relies on possibly correlated samples from an unknown empirical distribution. It uses these samples to target chance-constrained versions of stochastic problems (Gendreau, Laporte, and Séguin 1996, Rahimian and Mehrotra 2019). Unlike stochastic programs with recourse, chance-constrained programs do not allow recourse actions to take place once the stochastic outcomes are observed (Ghosal and Wiesemann 2020), such as skipping absent customers in the second stage of a PTSP. Although DRO forms a promising direction to address modeling challenges involving stochastic dependence in routing problems, redesigning some of DRO's core elements to facilitate recourse actions is beyond the scope of this research.

Yet, the alternative challenges that arise if one seeks to model dependence between discrete valued variables explicitly (Molenberghs and Verbeke 2005, Nikoloulopoulos 2013) are also far from trivial. Because stochastic travel times and stochastic demands are traditionally modeled as continuous variables, rather than discrete variables, these challenges did not emerge previously in the routing domain. As a consequence, the concepts from other routing problems with stochastically correlated components cannot simply be adapted to routing problems with stochastic customer presences. This may have contributed to the persistence of the research gap described above. To fill this gap, the next sections derive a novel analytical framework to model dependence between stochastic customer presences in a two-stage stochastic program with recourse for the TSP.

3. Problem Formulation

Because the CPTSP is a generalization of the PTSP, it also shares most of its problem definition (Jaillet 1988). Consider a routing problem in which a Hamiltonian tour needs to be determined through a set of customers. Every customer in the set is only present with some known probability, which can be correlated with the probabilities that other customers are present. Hence, only a subset of the customers are present in any given instance of the problem, whereas the remaining customers are absent. Only present customers require a visit. As in the TSP, all customers who require a visit should be visited no more than once.

The problem is set up according to the following two-stage stochastic program with recourse (Laporte and Louveaux 1993). In the first stage of the problem, an *a priori tour* τ along all n customers in the set N needs to be constructed, where $n := |N|$. In the second stage, the tour is modified to account for only those customers who are present and require a visit. More specifically, when the subset $S \subseteq N$ of customers along τ who are present is revealed in this stage, the customers in $N \setminus S$ who are absent must be skipped. Those in S must be visited in the same order as they appear in the a priori tour. The resulting tour is referred to as the *a posteriori tour*. The objective of the CPTSP is to find the a priori tour τ along all customers in N that minimizes the expected length of the a posteriori tour along the customers in S, given the joint probabilities of the customers' presences and their dependence structure.

The problem can be mathematically formulated as follows. Following common assumptions in the PTSP (e.g., Jaillet 1985, 1988, and Laporte, Louveaux, and Mercure 1994), assume a symmetric CPTSP where the triangle inequality holds. Given a complete, symmetric, and undirected graph $G = (N, E)$, an instance of the CPTSP is defined on G with the following elements:

N is a set of nodes or customers with cardinality $n := |N|$;

E is the set of edges $\{(i,j) : i,j \in N, i \neq j\}$, where (i, j) denotes the edge connecting customers i and j;

$\mathbf{D}$ denotes a symmetric distance matrix with elements $d_{ij} \in \mathbf{D}, 1 \le i,j \le n$, where d_{ij} corresponds to the distance of the edge $(i,j) \in E$ between customers i and j;

$\mathbf{Y} = (Y_1, \ldots, Y_n)$ is a vector of Bernoulli random variables taking realizations $\mathbf{y} = (Y_1 = y_1, \ldots, Y_n = y_n)$, where $Y_i = 1$ if customer i is present (and requires a visit) and $Y_i = 0$ if customer i is absent (and does not require a visit);

$S = \{i : Y_i = 1, i \in N\}$ is the subset of customers in N who are present and require a visit;

$\tau = (1, \ldots, n, 1)$ is a tour along all customers in N, and $L(\tau)$ denotes its length;

x is a set of binary variables $x = (x_{ij})$, assuming $x_{ij} = 1$ if (i, j) is on τ and $x_{ij} = 0$ otherwise.

The stochastic program of the CPTSP can then be written as:

$$\min_{\tau} \quad \mathbb{E}[L(\tau)], \tag{1a}$$

$$\text{s.t.} \quad \sum_{i<j} x_{ij} + \sum_{j>i} x_{ij} = 2, \qquad i \in N, \tag{1b}$$

$$\sum_{\substack{i,j \in S \\ i<j}} x_{ij} \le |S| - 1, \quad S \subset N; 3 \le |S| \le n-3, \tag{1c}$$

$$x_{ij} \in \{0,1\}, \qquad i,j \in N. \tag{1d}$$

The Objective Function (1(a)) seeks the a priori tour τ along all customers in N whose expected length $\mathbb{E}[L(\tau)]$ with respect to the probability of any random subset $S \subseteq N$ occurring is minimal. It is subject to the same degree Equations (1(b)), Subtour Elimination Constraints (1(c)), and Integrality Constraints (1(d)) as the TSP and PTSP to ensure that each customer is visited exactly once (Laporte, Louveaux, and Mercure 1994).

The challenge in the formulation of the CPTSP lies in finding a suitable expression for the expected length, $\mathbb{E}[L(\tau)]$. Given a pmf $p(\mathbf{y}) = \Pr(Y_1 = y_1, \ldots, Y_n = y_n)$ of all customers in N, I derive in Online Appendix B.1 that the general expression for the expected length of a CPTSP tour τ is given by

$$\mathbb{E}[L(\tau)] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij} p(\mathbf{y}_{ij}^*) + \sum_{j=2}^{n} \sum_{i=1}^{j-1} d_{ji} p(\mathbf{y}_{ji}^*). \tag{2}$$

Here, $p(\mathbf{y}_{ij}) = \Pr(Y_i = y_i, \ldots, Y_j = y_j)$ denotes the pmf associated with the random variables $\mathbf{Y}_{ij} = (Y_i, \ldots, Y_j)$ of the customers on path $(i, \ldots, j)$ of τ, and $\mathbf{y}_{ij}^*$ is a specific realization of $\mathbf{Y}_{ij}$ equal to

$$\mathbf{y}_{ij}^* = \begin{cases} (1,1) & \text{if } m = 2, \\ (1,0,1) & \text{if } m = 3, \\ (1,0,0,1) & \text{if } m = 4, \\ \ldots & \ldots \\ (1,0,0,\ldots,0,1) & \text{if } m = n. \end{cases} \tag{3}$$

That is, $p(\mathbf{y}_{ij}^*)$ is the probability that customer j needs to be visited immediately after i, while all the customers $i+1, \ldots, j-1$ that lie in between i and j on τ are skipped. The cardinality m of the path $(i, \ldots, j)$ is given by $m = (j - i) \bmod n + 1$, with "mod" the modulo operator. For ease of notation, I assume throughout the rest of this paper that $k := k \mod n$ for any $k \in (i+1, \ldots, j+m-2)$ if $k > n$. The Equation (2) applies the law of total expectation to sum over all possible combinations of $\mathbf{y}_{ij}^*$, such that each distance term is multiplied by its probability of occurring (cf. Bertsimas and Howell 1993).

The pmf $p(\mathbf{y}_{ij})$ can be obtained from $p(\mathbf{y})$ by marginalizing over every $y_r \in \mathbf{y} : r \notin (i, \ldots, j)$ corresponding to the presences of all customers $1, \ldots, i-1, j+1, \ldots, n$ beyond those on path $(i, \ldots, j)$. That is,

$$p_{ij}(\mathbf{y}_{ij}) = \sum_{y_1=0,1} \cdots \sum_{y_{i-1}=0,1} \sum_{y_{j+1}=0,1} \cdots \sum_{y_n=0,1} p(\mathbf{y}) = p(\mathbf{y}_{ij}),$$

where $p_{ij}(\mathbf{y}_{ij})$ denotes the pmf of the presences of customers $i, \ldots, j$. For the second equation to hold, $p(\mathbf{y})$ must be "reproducible" (Fitzmaurice, Laird, and Rotnitzky 1993): Subsets $\mathbf{y}_{ij}$ of $\mathbf{y}$ share the same distribution as $\mathbf{y}$, now defined on $\mathbf{y}_{ij}$. Consequently, $p_{ij}(\mathbf{y}_{ij})$ does not depend on customers beyond path $(i, \ldots, j)$. Every joint probability associated with a particular tour segment can therefore be treated separately and with less computational effort. It also allows us to express Equation (2) as a sum of $n(n-1)$ independent terms.

4. The Probability Mass Function of a CPTSP

The joint pmf $p(\mathbf{y})$ used by the CPTSP Objective Function (2) should be appealing from both a theoretical and a practical perspective. That is, the pmf should not only be reproducible (i), but it should ideally also take marginal probabilities from Bernoulli distributed customer presences as input (ii), reduce to the same pmf as the pmf embedded by the PTSP in the case of independence (iii), and be scalable to higher dimensions without the need to compromise on estimation accuracy (iv).

All four properties are satisfied by only a small number of multivariate discrete distributions, which belong to a category known as marginal models—see Agresti (2013) for an overview. Marginal models that satisfy (i)–(iv) include the well-researched Bahadur model (Bahadur 1961) and several copula models (Nelsen 2006). Unfortunately, many of these remaining candidates, including the Bahadur model, suffer from tight bounds on the range of permitted correlations in higher dimensions (Declerck, Aerts, and Molenberghs 1998, Nikoloulopoulos 2013). Consequently, many CPTSP instances, even of small size, cannot be described by such models.

Not surprisingly, the select remaining group that does not suffer from such limitations—namely, copulas—has become increasingly popular in a number of fields over the last few decades, most notably in mathematical finance (Cherubini, Luciano, and Vecchiato 2004). Even though copulas were originally intended to be used with continuous variables only, many of their properties still remain valid when the marginal variables, like in our case (ii), describe discrete-valued events (Genest and Neálehová 2007). Consequently, applying copulas to discrete variables has become increasingly common (see Joe 1997, Song 2007, and Nikoloulopoulos 2013 for details).

Copulas can be used by the CPTSP in the following way. Sklar's (1959) theorem states that every multivariate cumulative density function (cdf) can be written in terms of its marginal probabilities and a copula function. Let us consider the vector of random variables $\mathbf{Y}_{ij}$ with Bernoulli-distributed marginal cdfs $F_r(y_r) = \Pr(Y_r \le y_r)$ —that is,

$$F_r(y_r) = \begin{cases} 0 & \text{if } y_r < 0, \\ 1 - p_r & \text{if } 0 \le y_r < 1, \\ 1 & \text{if } y_r \ge 1, \end{cases} \tag{4}$$

where $p_r = \Pr(Y_r = 1)$ denotes the marginal probability that customer r is present and $1 - p_r = \Pr(Y_r = 0)$ describes the probability that the customer is absent. Applying the probability integral transformation to each $Y_r \in \mathbf{Y}_{ij}$ yields the random vector $U = (U_i, \ldots, U_j) = (F_i(Y_i), \ldots, F_j(Y_j))$ with uniformly distributed marginals $u_r \sim U_r(0,1)$. A copula describes the joint cumulative distribution of $\mathbf{Y}_{ij}$ in terms of its probability integral transformed set U:

$$\begin{aligned} C(u_i, \ldots, u_j) &= \Pr(U_i \le u_i, \ldots, U_j \le u_j) \\ &= \Pr(Y_i \le F^{-1}(u_i), \ldots, Y_j \le F^{-1}(u_j)), \end{aligned} \tag{5}$$

where the second equation follows from applying the transformation $Y_r = F^{-1}(U_r)$ on each U_r.

A common way to derive a joint pmf—for example, $p(\mathbf{y}_{ij})$—from the cdf above is by taking finite differences (Panagiotelis, Czado, and Joe 2012):

$$\begin{aligned} p(\mathbf{y}_{ij}) &= \sum_{k_i=0,1} \cdots \sum_{k_j=0,1} (-1)^{k_i+\cdots+k_j} \Pr(Y_i \le y_i - k_i, \ldots, Y_j \le y_j - k_j) \\ &= \sum_{k_i=0,1} \cdots \sum_{k_j=0,1} (-1)^{k_i+\cdots+k_j} C(F_i(y_i - k_i), \ldots, F_j(y_j - k_j)). \end{aligned}$$

Because this expression grows exponentially in m, it clearly does not meet property (iv). Yet, in our special case, where $\mathbf{y}_{ij} = \mathbf{y}^*_{ij}$, the computational burden can be relaxed by using that, by definition, $C(u_i, \ldots, u_j) = 0$ if $u_r = 0 : r \in \{i, \ldots, j\}$; and $C(1, u_i, \ldots, u_j) = C(u_i, \ldots, u_j, 1) = C(u_i, \ldots, u_j)$. Therefore,

$$p(\mathbf{y}^*_{ij}) = \begin{cases} p_i + p_j - 1 + C(1-p_i, 1-p_j) & \text{if } m = 2, \\ 1 - p_{i+1} - C(1-p_i, 1-p_{i+1}) - C(1-p_{i+1}, 1-p_j) & \\ \quad + C(1-p_i, 1-p_{i+1}, 1-p_j) & \text{if } m = 3, \\ C(1-p_{i+1}, \ldots, 1-p_{j-1}) - C(1-p_i, \ldots, 1-p_{j-1}) & \\ \quad - C(1-p_{i+1}, \ldots, 1-p_j) + C(1-p_i, \ldots, 1-p_j) & \text{otherwise.} \end{cases} \tag{6}$$

The case $m = 2$ reduces to $C(p_i, p_j)$ for copulas that maintain the survival copula link $\hat{C}(p_i, p_j) = p_i + p_j - 1 + C(1-p_i, 1-p_j)$ (Nelsen 2006). One can easily verify that (6) also satisfies property (iii)—see Online Appendix B.2.

5. Discrete-Vine Pair Copula Constructions

The computation of the right-hand side (rhs) of (6) is straightforward as long as the copulas attain a closed-form expression. This is indeed the case for a distinct number of multivariate copulas that belong to the Archimedean copula family (Nikoloulopoulos 2013). The expected length of a Tour (2) can, in such case, be computed in $4n(n-1)$ steps. The downside to the members of this copula family, excluding the copulas that only admit narrow ranges of dependence, is that these permit only a single homogeneous measure of association to be specified (Denuit and Lambert 2005, Nešlehová 2007, Nikoloulopoulos 2013). Even though the CPTSP under such equicorrelated conditions allows us to derive some useful properties (see Section 6), it has limited practical value.

Elliptical copulas, like the Gaussian copula used for the example of the good CPTSP tour in Figure 1(c) (see Online Appendix A), do allow for the specification of dependence structures beyond equicorrelated conditions alone. However, elliptical copulas do not bear a closed-form expression. Consequently, (6) would require the evaluation of an m-dimensional integral. This task becomes especially challenging when the size n of a CPTSP instance, and, therefore, the dimensions $m \leq n$ of the copulas in (6), grows (Smith and Khaled 2012). Property (iv) would soon be violated.

In order to avoid the repeated evaluation of high-dimensional integrals for elliptical copulas, we must resort to a strategy that decomposes $p(\mathbf{y}^*_{ij})$ into a series of smaller conditional probabilities. Panagiotelis, Czado, and Joe (2012) propose a decomposition that, when applied to $p(\mathbf{y}^*_{ij})$, takes the form

$$\begin{aligned} p(\mathbf{y}^*_{ij}) = {} & \Pr(Y_i = 1 \mid Y_{i+1} = 0, \ldots, Y_{j-1} = 0, Y_j = 1) \\ & \times \Pr(Y_j = 1 \mid Y_{i+1} = 0, \ldots, Y_{j-1} = 0) \\ & \times \prod_{r=i+1}^{j-2} \Pr(Y_r = 0 \mid Y_{r+1} = 0, \ldots, Y_{j-1} = 0) \\ & \times \Pr(Y_{j-1} = 0). \end{aligned} \tag{7}$$

The structure on the rhs, which is effectively the result of repeatedly applying Bayes' theorem to $p(\mathbf{y}^*_{ij})$, is known as a D-vine. The first two terms correspond to the conditional probabilities of the present customers on either two ends of $(i, \ldots, j)$, whereas the last two terms correspond to the probabilities of every absent customer $r \in \{i+1, \ldots, j-1\}$, conditional on a collapsing series of all the other absent customers until none remain. The conditional probability terms $\Pr(Y_a = y_a \mid \mathbf{V} = \mathbf{v})$ on the rhs of (7), with $\mathbf{V}$ indicating the set of conditioned variables and $\mathbf{v}$ its corresponding set of realizations, can be calculated with PCCs. Specifically, Panagiotelis, Czado, and Joe (2012) show that

$$\begin{aligned} \Pr(Y_a = y_a \mid \mathbf{V} = \mathbf{v}) = & \left[\sum_{k_a = 0,1} \sum_{k_b = 0,1} (-1)^{k_a + k_b} C_{Y_a, V_b \mid \mathbf{V}_{\setminus b}} \right. \\ & \left. (F_{Y_a \mid \mathbf{V}_{\setminus b}}(y_a - k_a \mid \mathbf{v}_{\setminus b}), F_{V_b \mid \mathbf{V}_{\setminus b}}(v_b - k_b \mid \mathbf{v}_{\setminus b})) \right] \\ & / \Pr(V_b = v_b \mid \mathbf{V}_{\setminus b} = \mathbf{v}_{\setminus b}), \end{aligned} \tag{8}$$

where $F_{A\mid B}(a \mid b) = \Pr(A \leq a \mid B \leq b)$, V_b denotes an element of $\mathbf{V}$, $\mathbf{V}_{\setminus b}$ denotes its complement, and $C_{A,B\mid C}(u_1, u_2)$ denotes a bivariate copula density for variables A and B conditional on C, evaluated at u_1 and u_2. Every $F_{A\mid B}(a \mid b)$ can be computed as

$$\begin{aligned} & F_{Y_a \mid V_b, \mathbf{V}_{\setminus b}}(y_a \mid v_b, \mathbf{v}_{\setminus b}) \\ & = [C_{Y_a, V_b \mid \mathbf{V}_{\setminus b}}(F_{Y_a \mid \mathbf{V}_{\setminus b}}(y_a \mid \mathbf{v}_{\setminus b}), F_{V_b \mid \mathbf{V}_{\setminus b}}(v_b \mid \mathbf{v}_{\setminus b})) \\ & \quad - C_{Y_a, V_b \mid \mathbf{V}_{\setminus b}}(F_{Y_a \mid \mathbf{V}_{\setminus b}}(y_a \mid \mathbf{v}_{\setminus b}), F_{V_b \mid \mathbf{V}_{\setminus b}}(v_b - 1 \mid \mathbf{v}_{\setminus b}))] \\ & \quad / \Pr(Z_k = z_k \mid \mathbf{Z}_{\setminus k} = \mathbf{z}_{\setminus k}). \end{aligned} \tag{9}$$

For computational details and implementation, see Panagiotelis, Czado, and Joe (2012).

The evaluation of a d-dimensional pmf using (8) requires the evaluation of $d(d-1)/2$ bivariate copulas at four different points, resulting in a total of $2d(d-1)$ evaluations. However, because the random variables Y_r in (7) are Bernoulli distributed according to (4), the number of evaluations can be significantly reduced by observing that many terms in (8) and (9) vanish when $Y_r = 0$. Therefore, (7) only requires $m(m-1)/2 + 5$ bivariate copula evaluations in total. The evaluation of the expected length of a Tour (2) with D-Vine PCCs then takes a total of $(n^4 + 59n^2 - 60n)/12$ bivariate copula evaluations.

To the best of my knowledge, D-Vine PCCs are the only discrete model that satisfies properties (i)–(iv) without imposing strong restrictions on the dependence structure. In fact, D-Vine PCCs offer substantial flexibility in the dependence structure by admitting a plethora of different copulas. Moreover, D-Vine PCCs allow a different bivariate copula to be specified for every relationship (Panagiotelis et al. 2017). This also enables multiparameter specifications of Archimedean copulas through a combination of heterogeneous single-parameter copulas as the building blocks of the D-Vine structure. Despite their $O(n^2)$ computational complexity, D-Vine PCCs therefore provide an attractive alternative to estimating (7) with finite differences or numerical approaches.

6. The Homogeneous CPTSP

Analogous to the *homogeneous PTSP* (Jaillet 1988), referring to the PTSP under equiprobable conditions, I introduce the term *homogeneous CPTSP* to refer the CPTSP under both equiprobable and equicorrelated conditions. Let $\mathbf{R} \in [-1, 1]^{n \times n}$ denote an $n \times n$ symmetric and positive semidefinite correlation matrix containing zero-order pairwise correlation coefficients $\rho_{ij} \in \mathbf{R}$, $1 \leq i, j \leq n$, which describe the dependence between the presences of customers i and j. If the marginal variables $Y_i \in \mathbf{Y}$ associated with the customer presences are dependent and identically distributed (d.i.d.), then the CPTSP is said to be homogeneous. This means that all customers share the same probability of occurrence, say, $p_i := p$, for all $i = \{1, \ldots, n\}$, and all off-diagonal elements of $\mathbf{R}$ assume the same value, say, $\rho_{ij} := \rho$, for all $i \neq j$.

Under such homogeneous conditions, the expression for the expected length (2) of a tour τ reduces to

$$\mathbb{E}_\rho[L(\tau)] = \sum_{r=0}^{n-2} \alpha(r, p, \rho) \sum_{j=1}^{n} d_{j,\,(j+r) \bmod n+1}, \tag{10}$$

where

$$\alpha(r, p, \rho) = \begin{cases} 2p - 1 + C_\rho(1-p, 1-p) & \text{if } r = 0, \\ C_\rho(p, p) \times \Pr(Y_1 = \cdots = Y_r = 0 \mid Y_{r+1} & \\ = Y_{r+2} = 1, \rho) & \text{if } 1 \leq r \leq n-2, \end{cases} \tag{11}$$

and $C_\rho(u_i, u_j)$ denotes a bivariate copula evaluated at u_i and u_j assuming correlation ρ. The proof for (10) can be found in Online Appendix C.1.

The probability on the rhs of $\alpha(r, p, \rho)$ can be computed efficiently with (6) for copulas with a closed-form expression or with (7)–(9) otherwise. In particular, if we choose the popular bivariate Gaussian copula $C(u_i, u_j) = \Phi_{\rho_{ij}}(\Phi^{-1}(u_i), \Phi^{-1}(u_j))$ to describe dependence between customer presences in the homogeneous CPTSP, then

$$\alpha(r,p,\rho) = \int_{-\infty}^{\infty} \left[\Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)\right]^2 \left[1 - \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)\right]^r \phi(M)dM,$$

for all $0 \le r \le n-2$, where $\phi(x)$ denotes the probability density function of the univariate standard normal distribution, $\Phi(\cdot)$ the cdf of a univariate standard normal distribution, and $\Phi_\rho(\cdot,\cdot)$ its bivariate extension with correlation coefficient ρ. The generic structure of this result is known as a Bernoulli mixture model. These models are commonly employed by financial risk managers to estimate credit defaults (Gordy 2000, Frey and McNeil 2003). The specific result above follows from the observation that the expected number of customer presences, as the expected number of credit defaults, admits a Binomial representation (see Laurent and Gregory 2005 for a proof).

The expected length of the Homogeneous CPTSP (10) has a structure that is similar to the weight-form notation of the homogeneous PTSP. In fact, the expected length of the tour τ according to the homogeneous PTSP, where all random variables $Y_i \in \mathbf{Y}$ are independently and identically distributed (i.i.d.), can be obtained by setting $\rho = 0$ in $\alpha(r, p, \rho)$. Then, it follows that $\alpha(r, p, 0) = p^2(1-p)^r$ for all $0 \le r \le n-2$, $0 \le p \le 1$. This matches exactly with the results of the weights for the homogeneous PTSP (Jaillet 1988). Consequently, the expected length of the homogeneous CPTSP is identical to the expected length of the homogeneous PTSP when there is no correlation ($\rho = 0$):

$$\mathbb{E}_0[L(\tau)] = \sum_{r=0}^{n-2} p^2(1-p)^r \sum_{j=1}^{n} d_{j,(j+r)\text{mod } n+1}.$$

This result is not surprising, given the equivalence of the heterogeneous PTSP and the heterogeneous CPTSP under independence (see Section 4 and Online Appendix C.1 for a proof). The homogeneous (C)PTSP is, after all, a special case of the heterogeneous (C)PTSP.

Further, to $\rho = 0$, the limiting cases of ρ also provide useful insights on the behavior of the expected length under homogeneous conditions (see Online Appendix C.2 for proofs). First, as $\rho \to 1$, we find $\lim_{\rho\to 1}\alpha(0, p, \rho) = p$ and $\lim_{\rho\to 1}\alpha(r, p, \rho) = 0$ for all $1 \le r \le n-2$. As a result,

$$\lim_{\rho\to 1} \mathbb{E}_\rho[L(\tau)] = pL(\tau),$$

where $L(\tau) = \sum_{j=1}^{n} d_{j,j\text{mod } n+1}$ denotes the deterministic length of tour τ. Intuitively, there are only two possible outcomes when all customer presences share perfect positive correlation ($\rho = 1$): Either a customer is present, and, consequently, all other customers must also be present; or a customer is absent, and all other customers must also be absent. The former event occurs with probability p and results in a tour of length $L(\tau)$ along all customers. The latter event occurs with a probability $1-p$ and results in a tour along no customers with length zero. Combining these two events yields the expression above.

Along similar lines, as $\rho \to -1$, we find

$$\lim_{\rho\to -1} \mathbb{E}_\rho[L(\tau)] = (p - 1 \bmod p)\sum_{j=1}^{n} d_{j,(j+\lfloor\frac{1-p}{p}\rfloor) \bmod n+1} + (1 \bmod p)\sum_{j=1}^{n} d_{j,(j+\lfloor\frac{1}{p}\rfloor) \bmod n+1}. \tag{12}$$

This result can be interpreted as follows. Consider a homogeneous customer presence probability $p = z^{-1}$, $z \in \mathbb{N}^+$, and a subset $Z \subset N$ of $z = |Z|$ ordered customers on τ whose presences are perfectly negatively correlated. Then, there is a $z^{-1} = z^{-1} - 1 \mod z^{-1}$ probability that the first customer is present, whereas the perfect negative correlation mandates that the subsequent $z - 1 = \lfloor z(1-z^{-1})\rfloor$ remaining customers are absent. This logic is expressed through the first term on the rhs of (12) for any possible subset Z over τ by setting $p = z^{-1}$. The second term on the rhs then vanishes. The same relationship can also be obtained through the second term on the rhs of (12) by setting $p = \lim_{\gamma\uparrow 1}(z - 1 + \gamma)^{-1}$. Then, the first term on the rhs vanishes. For more general probabilities of p, say, $z'-1$ with $z' \in \mathbb{R}$, (12) prescribes that the expected length can be expressed as a linear combination of the expected length for $p = \lfloor z'\rfloor^{-1}$ and the expected length for $p = \lceil z'\rceil^{-1}$. These expected lengths are associated with all possible subsets Z that contain exactly $\lfloor z'\rfloor$ and $\lceil z'\rceil$ consecutive and perfectly negatively correlated customers on τ, respectively.

As a final result on limiting cases, observe that $\alpha(0, 1, \rho) = 1$ and $\alpha(r, 1, \rho) = 0$ for all $1 \le r \le n-2$. Therefore, the expected length of τ according to the homogeneous CPTSP is equal to its length according to the TSP whenever $p = 1$:

$$\mathbb{E}_\rho[L(\tau)|p = 1] = L(\tau).$$

This obviously matches the result of the heterogeneous CPTSP when $p_i = 1$ for all $i \in \{1, \ldots, n\}$.

Unfortunately, the limiting cases do not automatically induce boundaries with respect to $\mathbb{E}_\rho[L(\tau)]$. Because $\alpha(r, p, \rho)$ is neither strictly increasing in ρ nor strictly decreasing in ρ for any given combination of $r \ge 1$ and p, the expected length does not naturally gravitate toward the results for $\rho \to \pm 1$. Instead, $\mathbb{E}_\rho[L(\tau)]$ satisfies the boundaries derived by Jaillet (1985) for general pmfs to describe stochastic customer presences:

$$\mathbb{E}_\rho[L(\tau)] \leq L(\tau) \times (1 - \Pr(Y_1 = \cdots = Y_n = 0) - n\Pr(Y_1 = 1, Y_2 = \cdots = Y_n = 0)),$$

$$\mathbb{E}_\rho[L(\tau)] \geq L(\tau^*_{\mathrm{TSP}}) \times (p - \Pr(Y_1 = 1, Y_2 = \cdots = Y_n = 0)),$$

where τ^*_{TSP} denotes the optimal TSP tour. For the special case of the Gaussian copula, the joint probabilities on the rhs are given by

$$\Pr(Y_1 = \cdots = Y_n = 0) = \int_{-\infty}^{\infty} \left[1 - \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)\right]^n \phi(M)dM,$$

and

$$\Pr(Y_1 = 1, Y_2 = \cdots = Y_n = 0) = \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \left[1 - \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)\right]^{n-1} \phi(M)dM,$$

by virtue of the same binomial properties that are explored for $\alpha(r, p, \rho)$, $0 \leq r \leq n-2$ (Laurent and Gregory 2005).

7. An Exact Algorithm for the CPTSP

The integer L-shaped method (Laporte and Louveaux 1993) is a stochastic variant of the branch-and-cut algorithm developed to solve two-stage stochastic mixed-integer programs to optimality. The version of the integer L-shaped method for the PTSP proposed by Laporte, Louveaux, and Mercure (1994) can be adapted to the CPTSP as follows.

In line with Laporte, Louveaux, and Mercure (1994), let us rewrite the expected length in (1(a)) as

$$\mathbb{E}[L(\tau)] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij}x_{ij} - \mathbb{E}[Q(x, \mathbf{y})], \tag{13}$$

where $Q(x) = \mathbb{E}[Q(x, \mathbf{y})]$ denotes the recourse function. That is, for any given a priori tour of length $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij}x_{ij}$, the function $-Q(x)$ corresponds to the distance reduction that results from skipping customers in the a posteriori tour. To solve (13) efficiently, its expected value $-\mathbb{E}[Q(x, \mathbf{y})]$ is replaced with an approximation θ, the lower bound of which is gradually updated through optimality cuts in the branching process. The entire procedure is described in greater detail by the following algorithm.

Step 1. Set the current iteration index $\nu := 0$ and the objective value of the best solution found thus far to $\bar{z} = \infty$. The list of subproblems only contains the initial problem: $\min_{x^\nu, \theta^\nu} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij}x^\nu_{ij} + \theta^\nu$, subject to (1(b)), $0 \leq x^\nu_{ij} \leq 1$, and $\theta^\nu \geq L$. The initial upper bound L on the distance reduction approximation θ^ν can be obtained by solving the auxiliary mixed-integer problem (L.9)–(L.13) in Laporte, Louveaux, and Mercure (1994). This auxiliary problem remains unchanged for the CPTSP.

Step 2. Select a subproblem from the list. If none exists, stop.

Step 3. Update $\nu := \nu + 1$. Solve the νth subproblem and denote its optimal solution by (x^ν, θ^ν).

Step 4. If $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij}x^\nu_{ij} + \theta^\nu \geq \bar{z}$, fathom the current problem and return to step 2.

Step 5. Check if a Subtour Elimination Constraint (1(c)) is violated. If a subtour can be identified, augment the problem accordingly with a subtour elimination constraint and return to step 3.

Step 6. Check if x^ν is integral. If one or more Integrality Constraints (1(d)) are violated, select the most fractional variable (Tang and Miller-Hooks 2007). Create two new subproblems by branching on the selected variable and add these subproblems to the list. Then, return to step 2.

Step 7. Use (2) to compute the expected length $z^\nu := \mathbb{E}[L(\tau)]$ of the current solution x^ν. If $z^\nu < \bar{z}$, set $\bar{z} := z^\nu$.

Step 8. Calculate the true expected recourse costs $Q(x^\nu)$ using (13). If $\theta^\nu \geq Q(x^\nu)$, fathom the current subproblem and return to step 2.

Step 9. Impose the optimality cut (Laporte, Louveaux, and Mercure 1994):

$$\theta \geq 1/2(Q(x^\nu) - L)\left(\sum_{(i,j)\in E^\nu} x_{ij} - n\right) + Q(x^\nu), \tag{14}$$

where $E^\nu = \{(i, j) \in E : x^\nu_{ij} = 1\}$. Return to step 3.

The proofs on the validity of the upper bound L on θ in step 1 and the optimality cut in step 9 remain unchanged from the ones proposed by Laporte and Louveaux (1993) and Laporte, Louveaux, and Mercure (1994). The same holds for the logic surrounding the derivation of additional lower bounding functionals (L.14)–(L.20) on $Q(x^\nu)$, presented by Laporte, Louveaux, and Mercure (1994). However, these lower-bounding functionals need to be adapted to take potential dependencies between customer presences into account. Online Appendix D presents the updated versions of these valid inequalities that can be used in conjunction with the optimality cuts presented in step 9 to speed up convergence. The algorithm may possibly converge even faster by exploiting some of the callback routines featured by modern solvers to separate optimality cuts on the fly and, thus, makes a manual implementation of every step above redundant.

8. Heuristic Procedures for the CPTSP

The $O(n^4)$ computational effort required to evaluate the objective function of the CPTSP can be reduced by sacrificing some of its estimation accuracy (iv). Embedding an approximation of the "real" objective function in a solution method enables faster computation times, albeit a global optimum to the original problem can no longer be guaranteed. As a consequence, the solution method becomes heuristic by design. The generic approximation methods proposed below can be used in combination with several existing heuristic methods for the PTSP, of which I discuss two well-studied cases: the probabilistic ant-colony system (Bianchi, Gambardella, and Dorigo 2002) and 2.5-opt-EEs (Birattari et al. 2008).

8.1. A Priori Approximation

A first approach to reduce the computational burden of repeatedly evaluating the expected length of the CPTSP targets the Length Expression (2) directly. The reduction is accomplished by relaxing the contributions of both stochastic terms and dependence terms to the objective function.

The stochastic contributions to the expected length are formed by the weighted distance terms between any two consecutive present customers on the a priori tour in (2) for the heterogeneous CPTSP and (10) for the homogeneous CPTSP. As the number of intermediary absent customers between two present customers in the a priori customer sequence increase—that is, if the dimensionality m of $\mathbf{y}^*_{ij}$ in (3) grows—the associated probability $p(\mathbf{y}^*_{ij})$ tends to decline until it eventually becomes negligible. A common approach to reduce the $O(n^2)$ computational complexity of the double summation in (2) and (10) is therefore to restrict its attention to only those probability terms $p(\mathbf{y}^*_{ij})$ for which $m \le \kappa$, where $\kappa \le n$ denotes some predefined truncation level (see, e.g., Tang and Miller-Hooks 2004 and Campbell and Thomas 2009). Computational experiments on the generalized PTSP show that accurate approximations of the objective function can be achieved for truncation levels as little as $\kappa = 4$ (Tang and Miller-Hooks 2004), while decreasing the computational burden of the double summation from $O(n^2)$ to $O(\kappa n)$.

The contributions of dependencies between stochastic presences to the Objective Function (2) are given by the D-Vine (7). Each conditional probability term $\Pr(Y_a = y_a | \mathbf{V} = \mathbf{v})$ in the product represents another order of dependence, also known as a *tree*. Truncating trees beyond λ implies that independence copulas are assumed for all terms $|\mathbf{V}| > \lambda$, where $1 \le \lambda \le m - 2$ (Brechmann, Czado, and Aas 2012). Panagiotelis, Czado, and Joe (2012) suggest that accurate approximations can already be achieved for truncation levels as low as $\lambda = 2$, while decreasing the computational complexity of the D-vine from $O(m^2)$ to $O(\lambda m)$.

Combining the two forms of truncation reduces the total number of required computations for (2) to $(\kappa + 1)(\lambda^2(1-2n) + 24n + \lambda(2n + \kappa + 1) - 12)/2$, which scales according to $O((\kappa\lambda^2 + \kappa + \lambda)n)$ in the total number of customers n. The expected length embedded in the Objective Function (1(a)) can thus be approximated by

$$\mathbb{E}[L(\tau)] \approx \sum_{i=1}^{n-1} \sum_{j=i+1}^{\min\{i+\kappa-1,n\}} d_{ij}\tilde{p}(\mathbf{y}^*_{ij}) + \sum_{j=2}^{n} \sum_{i=\max\{1,j-\kappa+1\}}^{j-1} d_{ji}\tilde{p}(\mathbf{y}^*_{ji}), \tag{15}$$

where each probability approximation $\tilde{p}(\mathbf{y}^*_{ij})$ is calculated in the usual way through (7)–(9), while assuming independence copulas for all terms where $|\mathbf{V}| > \lambda$.

The approximate form of the expected length can be embedded by solution methods that directly target the objective function. Ant-Colony Systems (ACS) are one such flexible and competitive class of metaheuristic solution methods that rely on the repeated evaluation of the objective function (Dorigo and Gambardella 1997). ACS, like most metaheuristics, can be used to handle a wide variety of problems with only minor adjustments to their design. The probabilistic ant-colony system (pACS) metaheuristic (Bianchi, Gambardella, and Dorigo 2002) is a well-known example of such an adaptation of ACS for the PTSP. The pACS metaheuristic is known to produce competitive results relative to other state-of-the-art methods that are capable of handling stochastic settings, including particle-swarm optimization (Marinakis and Marinaki 2010, Marinakis, Marinaki, and Migdalas 2015), memetic algorithms (Liu 2008, Balaprakash et al. 2010), and methods relying on empirical estimation heuristics (see below). Replacing the expected length from the PTSP in the pACS heuristic by the exact objective function of the CPTSP (1(a)) or its Approximate Form (15) straightforwardly extends ACS to the CPTSP. No further modifications to the original setup of pACS by Bianchi, Gambardella, and Dorigo (2002) need to be made for the CPTSP. I refer to the metaheuristic that results as the correlated probabilistic Ant-Colony System (cpACS) metaheuristic. Note that the cpACS heuristic is a generalization of the pACS heuristic, as it reduces to the pACS heuristic under the assumption of independent customer presences.

8.2. A Posteriori Approximation

An alternative way to reduce the computational burden of the CPTSP objective function is to approach the problem from an a posteriori, rather than a priori, perspective. That is, rather than targeting the objective function directly in the search for the optimal a priori tour, a good a priori tour may also be reconstructed from a sample of a posteriori tour realizations. (Note that these tour realizations can be equally derived from customer realizations and a given a priori tour because the two-stage setup of the (C)PTSP implies that all customers must be visited in the same order.) A posteriori realizations can be obtained by simulating directly from the objective function or by simply drawing from an empirical distribution of already observed (real-world) realizations. The latter procedure, like DRO, eliminates the need for any knowledge or assumptions about the stochastic process that drives the customer presences altogether (Carlsson, Behroozi, and Mihic 2018).

The empirical estimation and speedup feature of the 2.5-opt local search framework proposed in Birattari et al. (2008) is such a well-known estimation-based algorithm for the PTSP. EEs involves the deterministic evaluation of so-called "deltas" for a sample of customer realizations. That is, EEs essentially performs sample average approximation (Kleywegt, Shapiro, and Homem-de Mello 2002), as each delta refers to the average difference in the lengths of the implied a posteriori tour realizations that results from manipulating the order of customers on the a priori tour. A candidate perturbation that manipulates the order of the customers on the a priori tour—for example, a *k*-opt candidate move—is accepted whenever the delta evaluation yields a reduction in the average length of the sampled a posteriori tours. By only relying on samples, EEs avoid the need to repeatedly evaluate the computationally expensive CPTSP objective function.

EEs can also be applied to the CPTSP—for example, by simulating correlated Bernoulli distributed customer presences and absences from the copula that describes their relationship. Sampling from most copulas does not require great computational effort. For example, a sample from a Gaussian copula can easily be obtained by drawing from a multivariate normal cumulative distribution and transforming the results to Bernoulli-distributed outcomes using the inverse of (4), given by $F_r^{-1}(u_r) = 0$ if $0 \leq u_r < 1 - p_r$, and $F_r^{-1}(u_r) = 1$ otherwise. For details on generic copula sampling procedures and a specific D-Vine PCC sampling procedure, I refer to Cherubini, Luciano, and Vecchiato (2004) and Panagiotelis, Czado, and Joe (2012), respectively. Even though a single sample can often be quickly generated, beware that the total number of a posteriori samples required to obtain an accurate estimate of the expected a priori tour length could still be substantial. Therefore, even for medium-sized problem instances and seemingly simple dependence structures, the total computational burden may rapidly increase. The same observation applies to samples bootstrapped from an empirical distribution.

Besides adapting the sampling procedure for multivariate, rather than univariate, distributions, no further modifications to the EEs framework from Birattari et al. (2008)—nor the rest of their 2.5-opt-EEs algorithm—need to be made for the CPTSP. On the one hand, 2.5-opt-EEs produce competitive results relative to other methods that involve settings with stochastic customer presences (see, e.g., Birattari et al. 2008, Balaprakash et al. 2010, and Li 2017), including pACS. On the other hand, its combination with the EEs feature ensures computationally efficient evaluation of potentially improving 2.5-opt candidate moves without the need to compute the objective function.

9. Computational Experiments

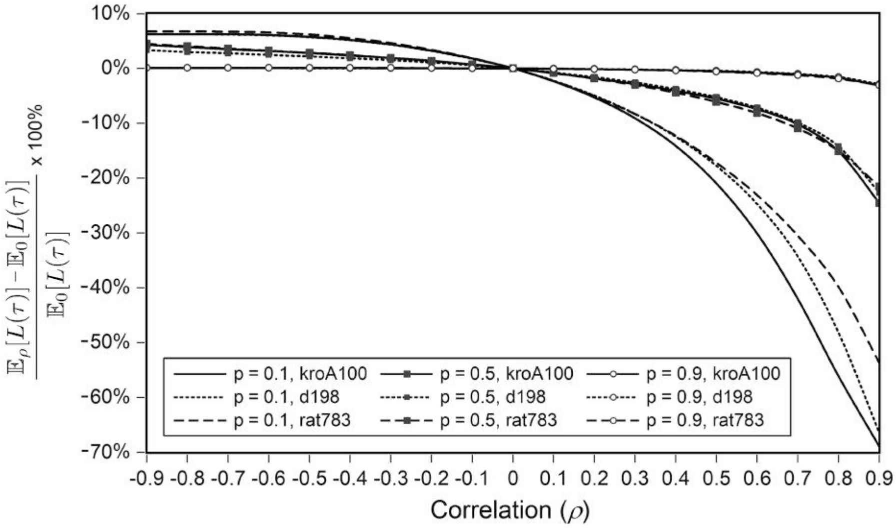
The computational experiments in this section illustrate the value and the significance of modeling dependencies between stochastic customers. Section 9.1 shows the general behavior of expected tour-length estimates versus varying degrees of correlation between stochastic customers in a controlled environment. A second series of experiments in Section 9.2 illustrate the impact of different dependence intensities and structures on algorithmic performance. Section 9.3 demonstrates how the CPTSP can be applied in practical industry applications where potential dependencies between customer presences play a role.

9.1. The Impact of Dependence on the Expected Length of a Tour

The uniqueness of the CPTSP solution (see Figure 1) implies that the expected length estimates of a tour under the assumption of independent customer presences can be significantly different from the same estimates under correlated customer presences. The results in Figure 2 strengthen this notion. This figure shows the expected length of a tour under independence relative to the expected length of that tour under customer dependencies (vertical axis) for various homogeneous correlations ρ of the Gaussian copula (horizontal axis). The names kroA100, d198, and rat783 refer to three well-known instances obtained from TSPLIB (http://comopt.ifi.uni-heidelberg.de/software/TSPLIB95/) of varying degrees of customer dispersion and sizes, loosely based on a common selection for PTSP experiments (e.g., Balaprakash et al. 2010, Marinakis and Marinaki 2010, and Gambardella, Montemanni, and Weyland 2012). A good tour for each of these instances is generated with the Concorde algorithm (Applegate 2006) and retained throughout the calculations of the expected length, such that all differences in the expected length can be attributed to the impact of correlation, rather than algorithmic performance. The expected length of each tour and for every instance is calculated over a homogeneous set of probabilities equal to $p = 0.1$, $p = 0.5$, and $p = 0.9$, respectively.

The results in Figure 2 show that large positive correlations have a bigger impact on the relative expected length than large negative correlations. The marginal effect of correlation on expected length dissipates toward the negative end of the scale, whereas the marginal effect for positive correlations tends to increase. This trend reflects the expected behavior of intensifying dependence under most copulas (Plackett 1954, Xiao and Zhou 2019). Overall, the expected lengths of the tested tours show deviations ranging from −69.96% to +6.78% versus their expected lengths under independence, with an

Figure 2. Expected Tour Length of the Homogeneous CPTSP Relative to the Homogeneous PTSP for Different Dependence Levels



average difference of −6.82%. Tests to evaluate the significance of the impact of correlation on the expected length (not reported here) reveal that dependence has a significant impact on the expected length at the 1% confidence level for each of the tested scenarios.

The generally negative relationship between the weights $\alpha(r,p,\rho)$ and $\alpha(r,p,0)$ of each distance term in (10) results in a diminishing expected length with intensifying correlation. Intuitively, this negative trend can also be explained by the previously derived results on the limiting cases: Correlation induces uncertainty with respect to the joint outcome of customers' presences. For example, the proportional impact of a 0.4 higher correlation on the expected tour length for kroA100, $p=0.1$ at $\rho = 0.38$, can be negated by a 0.4 higher probability of customer presence. A lower probability of customer presence generally results in a lower expected tour length, as the number of customers that one expects to visit decreases. Similarly, higher correlation results in a lower expected tour length as the joint probability to visit a group of correlated customers generally decreases.

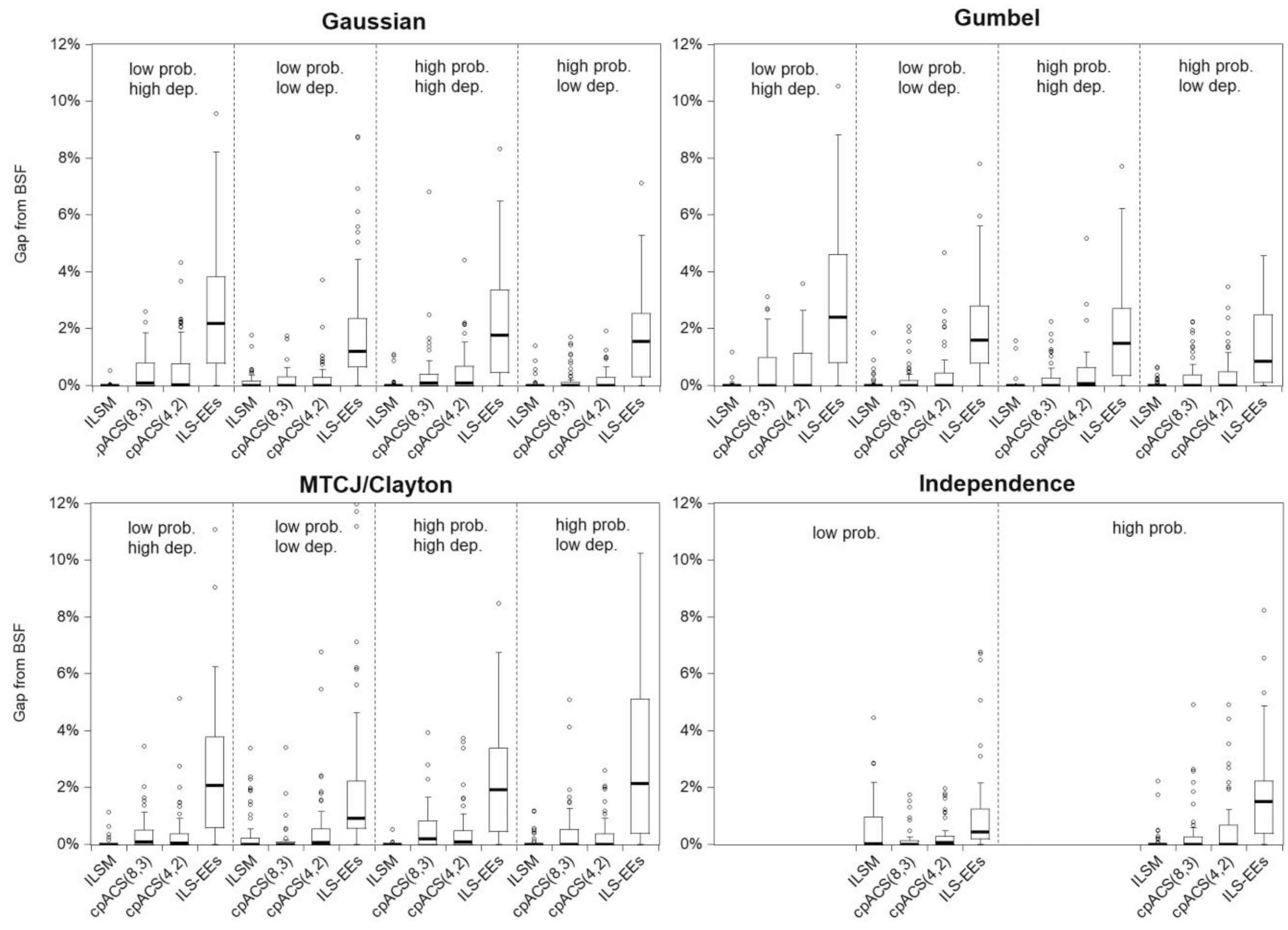
9.2. The Impact of Dependence on Algorithmic Performance

To assess the impact of dependence on algorithmic performance, let us consider the following experimental setup inspired by a combination of the setup from Panagiotelis, Czado, and Joe (2012) for D-Vine PCCs, the setup from Tang and Miller-Hooks (2007) for the generalized PTSP, and the solution procedures proposed in Sections 7 and 8. Ten different problem instances of 10, 20, 30, 50, and 100 nodes are randomly generated using the DIMACS TSP instance generator (http://dimacs.rutgers.edu/programs/challenge/). Each instance is subjected to low and high regimes of customer-presence probabilities and dependence, where low probabilities refer to $p_i = 0.3\ \forall i = \{1,\dots,n\}$, high probabilities refer to $p_i = 0.7\ \forall i = \{1,\dots,n\}$, low dependence to Kendall's $\tau = \{0.3, 0.2, 0.1, 0.05\}$, and high dependence to Kendall's $\tau = \{0.7, 0.4, 0.3, 0.2\}$ for all pair copulas corresponding to trees 1, 2, 3, and 4, respectively. Each combination is tested under three different copulas used to describe the dependence structure between customers in the D-Vine—namely, Gaussian, Mardia-Takahasi-Cook-Johnson (MTCJ)/Clayton, and Gumbel (see Nelsen 2006 for details).

The relaxed subproblems considered by the integer L-shaped method (ILSM) are solved using IBM ILOG CPLEX 12.9. Two different versions of the cpACS algorithm are implemented, with truncation levels at $(\kappa,\lambda) = (8,3)$ and $(\kappa,\lambda) = (4,2)$, and referred to as cpACS(8,3) and cpACS(4,2), respectively. The 2.5-opt-EEs algorithm is implemented with 1,000 a posteriori samples as its input for the delta evaluations (2.5-opt-EEs-1000) and embedded in an iterated local search (ILS) procedure to perturb the incumbent local optimum and explore neighboring solutions for a total of 10 iterations (Balaprakash et al. 2010). The resulting ILS-2.5-opt-EEs-1000 algorithm is referred to as ILS-EEs for the sake of brevity. The remaining parameter settings are identical to the ones used in Birattari et al. (2008) for 2.5-opt-EEs; Balaprakash et al. (2010) for the ILS framework that embeds it; and Bianchi, Gambardella, and Dorigo (2002) for both implementations of cpACS. The algorithms are implemented in C# and run on an Intel Core i5-6600 (3.3 GHz) machine with 16 GB of RAM, each with a maximum computation time of 600 CPU seconds.

Figure 3 presents box plots that show the gap between the expected length of the solutions found by each algorithm and the best solution found (BSF) for every

Figure 3. Gap in Expected Length Between Solutions Found by Each Algorithm and the BSF for that Instance



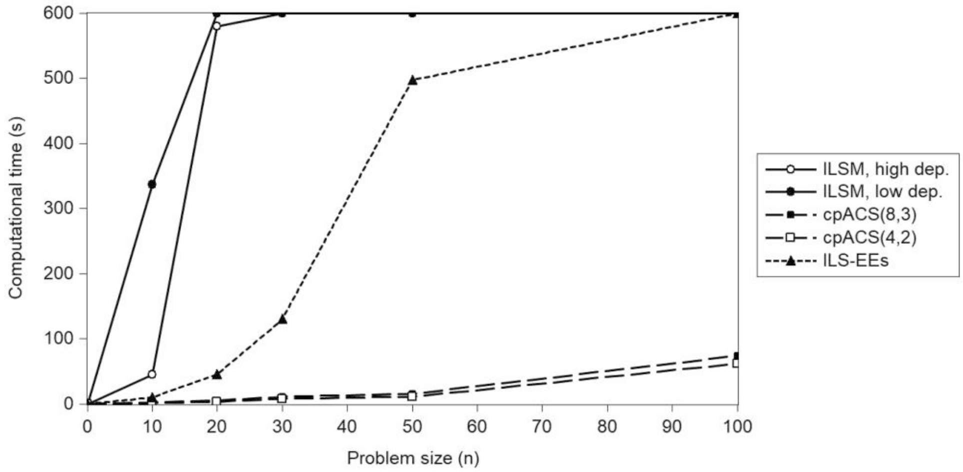
problem instance. The figure is split into different combinations of copulas, probability settings, and dependence settings to highlight specific impacts. (See Online Appendix E for the full set of results.) The combinations are sorted in declining order of uncertainty for each copula, based on the observation that dependence induces uncertainty with respect to the joint presence of customers.

Overall, the performance of the different algorithms across different copulas exhibit similar trends. The ILSM manages to produce the BSF for every problem instance within the maximum runtime in most cases. It should be noted, however, that computation times for the ILSM increase sharply with problem size (see Figure 4), following similar trends observed by Laporte, Louveaux, and Mercure (1994) and Tang and Miller-Hooks (2007) for the PTSP and generalized PTSP. Figure 4 also shows that computation times increase even faster for low dependence or, similarly, low probability (see Online Appendix E) as a function of the problem size.

Furthermore, Figures 3 and 4 show that the median performance of both cpACS implementations does not substantially differ from the ILSM, whereas the computation times required for the cpACS implementations are substantially lower, regardless of the selected truncation levels and probability and dependence settings. However, the variance in the performance of cpACS(8,3) is substantially larger than that of the ILSM, and outliers occur more often. The same phenomena slightly exacerbate for cpACS(4,2)—that is, when the length estimates are truncated earlier. ILS-EEs demonstrates weaker performance across all tested cases. An exception is formed by the moderately competitive performance of ILS-EEs for low probabilities under the independence copula, in line with the results by Balaprakash et al. (2010).

To reveal discrepancies in route decisions, Table 1 reports differences in the customer sequence on the a priori tour found by each algorithm versus the customer sequence in the BSF ($\Delta\tau^*$, left panel) and also versus the tour sequence found under the independence copula ($\Delta\tau_{PTSP}$, right panel). Compared with an optimal alignment (Kruskal 1983) of every customer sequence with the customer sequence of the BSF, the tours produced by the ILSM differ between 0.44% and 8.69%, on average. The tour sequences differ more from the BSF in each instance under the combination of low levels of probability and dependence or in the case of the PTSP—that is, when an independence copula (Ind.) is adopted. Differences in tour sequence become more pronounced when

Figure 4. Average Run Time Development Plots for the Gaussian, MTCJ/Clayton, and Gumbel Copulas Across Problem Sizes



heuristic methods are employed, with sequences differing between 3.28% and 10.01%, on average, for both cpACS implementations and between 16.11% and 19.66% for ILS-EEs. Regardless of the algorithm, copula, or uncertainty conditions, all tour sequences differ substantially from the tours that would have been found by the same algorithms under the assumption of independence (right panel). On average, neglecting dependence results in a 11.16% different tour for the tested algorithms and scenarios. Although the differences tend to increase for higher levels of uncertainty, even low levels of dependence can have a considerable impact on the tour sequence.

In sum, the results suggest that the performance of algorithms benefits from using length definitions that incorporate dependencies between stochastic customers in addition to marginal presence probabilities only, regardless of the chosen copula or probability and dependence settings. Although the expected length estimates for the tours found by the ILSM may only suggest marginal improvement over the estimates found by some heuristic methods, their customer sequences can differ substantially. The average computational times required by the latter methods are, however, considerably lower than those of the ILSM. Stricter levels of truncation in cpACS do not greatly affect median performance or computation time, but do tend to increase the variance in performance. Furthermore, the performance of the ILS-EEs algorithm based on 1,000 a posteriori simulations is relatively weak overall. Its embedded 2.5-opt local search algorithm might not be sufficiently explorative to capture the interactions between more distant customers. Moreover, the algorithm seems insufficiently competitive in terms of CPU runtime to repeatedly assess every potentially improving candidate move for a representative, yet manageable, sample of correlated a posteriori outcomes.

9.3. Illustration of a Real-World Application

Failed deliveries due to absent customers remain a costly burden to many parcel-delivery companies. Parcel-

Table 1. Differences in Customer Sequence

Parameters			Tour difference vs. BSF ($\Delta\tau^*$) (%)				Tour difference vs. PTSP ($\Delta\tau_{PTSP}$) (%)			
Prob.	Dep.	Copula	ILSM	cpACS(8,3)	cpACS(4,2)	ILS-EEs	ILSM	cpACS(8,3)	cpACS(4,2)	ILS-EEs
Low	High	Gaussian	0.44	9.25	9.34	17.35	11.41	10.95	11.95	19.49
		MTCJ	2.04	8.61	7.85	17.85	11.57	11.13	9.80	20.90
		Gumbel	2.22	9.11	9.47	19.19	12.14	12.19	10.46	19.90
Low	Low	Gaussian	5.14	4.65	5.59	17.45	9.97	7.77	9.37	17.53
		MTCJ	6.58	3.28	7.57	19.66	8.92	6.45	7.80	17.14
		Gumbel	3.39	5.43	7.57	18.59	10.84	8.53	8.73	17.56
Low	Ind.	Ind.	8.69	4.99	8.34	17.45				
High	High	Gaussian	3.04	7.59	7.72	16.44	9.97	9.07	9.48	18.04
		MTCJ	2.02	8.72	7.29	16.11	7.61	8.40	8.55	17.10
		Gumbel	1.72	7.87	7.32	17.11	7.09	7.96	8.75	15.37
High	Low	Gaussian	3.89	5.62	5.56	18.83	5.13	5.06	6.93	14.01
		MTCJ	1.88	7.30	6.01	18.37	4.84	6.00	9.35	15.53
		Gumbel	2.80	6.81	8.93	16.51	11.19	9.71	11.46	16.62
High	Ind.	Ind.	4.07	6.66	10.01	17.52				

delivery companies therefore increasingly urge customers to cancel upcoming delivery attempts in case of anticipated absence through track-and-trace systems. A track-and-trace system typically presents a customer the opportunity to deliver the parcel to a nearby collection-and-delivery point instead or to postpone the delivery to a later date. If a customer cancels the delivery while the delivery driver is already en route, the cancellation is communicated in real time to the delivery driver. The driver is then instructed to treat customers who cancel in the same way as the (C)PTSP treats its absent customers—namely, by simply skipping them on the delivery route.

As the previous results illustrate, anticipating potential customer absences and their dependencies in the determination of an a priori delivery tour can lead to different route cost estimates and route decisions. Assuming that a route is not dynamically reoptimized when a driver is notified of a canceled home delivery en route, the CPTSP allows us to estimate the errors and opportunity costs of disregarding potential cancellations and their dependencies in the design of an a priori tour.

For this purpose, real-world data on parcel-delivery attempts for a region in the southwest of Amsterdam are sourced from a parcel-delivery company. The data contain delivery information about 34,516 delivery attempts across 134 neighborhoods, spanning an area of 31.9 km^2, for the period February 1, 2019, to April 30, 2019. Each delivery attempt is associated with one of the 134 included neighborhoods (five-digit postcodes). The outcomes of the delivery attempts in each neighborhood are expressed in terms of delivery efficiency—that is, the proportion of successful deliveries out of the total number of deliveries. The center of each neighborhood is taken to be the target delivery location. Seven different instances are extracted from the data set, based on the actual delivery routes described in the data. The name of each instance reads Ams-n, where n denotes the number of neighborhoods included in the instance. An artificially compiled eighth instance, Ams-134, contains all 134 neighborhoods.

The delivery efficiency in each neighborhood varies throughout the day and exhibits potential positive or negative relationships with the efficiency in other neighborhoods (see Figure 5). Therefore, a D-Vine PCC is used to capture the relationships between the delivery successes across neighborhoods. The selection algorithm outlined in Panagiotelis et al. (2017) is used to select the appropriate D-Vine structure and fit the copulas that serve as its building blocks. In line with the previous experiments, the copula selection for the algorithm includes Gaussian, MTCJ/Clayton, and Gumbel copulas. Fitting the D-Vine PCC to the data yields a copula mix consisting of 20% Gaussian copulas, 57% MTCJ/Clayton copulas, and 23% Gumbel copulas. The D-Vine is truncated at $\lambda = 1$, as the conditional dependencies estimated beyond this tree do not significantly differ from independence. The resulting D-Vine PCC for each instance is embedded in a CPTSP framework and solved with the ILSM, cpACS(8,3), cpACS(4,2), and ILS-EEs. These algorithms maintain the same implementations as in the previous subsection. The experiments are repeated with PTSP and TSP length implementations for the same instances.

Figure 5. Illustration of the Average Delivery Efficiency Evolving over Delivery Hours Across Three Selected Neighborhoods

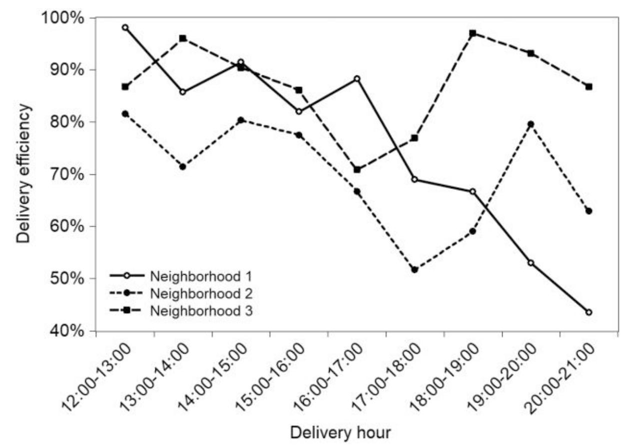


Table 2 reports the expected length of the BSF among the applied algorithms ($\mathbb{E}[L(\tau^*)]$) with the fitted D-Vine PCC, along with the computation time that the superior algorithm took to obtain the BSF in CPU seconds (CPU). The delivery driver's a priori tour is determined thrice for different levels of customer cancellations ("Canc." column), expressed in terms of the average number of total failed deliveries in each instance. For example, a cancellation level of 50% indicates that half of the deliveries that typically fail are also communicated to the driver while en route, whereas the remaining 50% are not communicated and, thus, still result in failed deliveries. If none of the customers notify the driver of their absence (cancellation level 0%), the problem reduces to a TSP. The $\mathbb{E}[L(\tau^*_{\text{PTSP}})]$ and its adjacent CPU columns report the results generated under the assumption of independent delivery efficiencies across neighborhoods. The impact of stochastics is reported in the $\Delta\mathbb{E}[L(\tau^*_{\text{TSP}})]$ and $\Delta\tau^*_{\text{TSP}}$ columns, which compares the results of the CPTSP row-wise with the 0% cancellation level for each instance. Similarly, the impact of dependencies is reported in the $\Delta\mathbb{E}[L(\tau^*_{\text{PTSP}})]$ and $\Delta\tau^*_{\text{PTSP}}$ columns, which compare the results of the CPTSP column-wise with the results of the PTSP.

In spite of the relatively high average delivery efficiencies ($\overline{p}$), the results in Table 2 indicate that disregarding the potential of delivery cancellations during operations altogether results in estimation errors of the expected length ($\Delta\mathbb{E}[L(\tau^*_{\text{PTSP}})]$) to be traveled ranging between [0.41%, 5.49%]. The difference between both estimation

Table 2. CPTSP, PTSP, and TSP Results for Different Neighborhood Sets and Levels of Customer Cancellations

Instance	Canc. (%)	$\overline{p}$	$\overline{K}$	$\mathbb{E}[L(\tau^*)]$	CPU	$\Delta\mathbb{E}[L(\tau^*_{\text{TSP}})]$ (%)	$(\Delta\tau^*_{\text{TSP}})$ (%)	$\mathbb{E}[L(\tau^*_{\text{PTSP}})]$	CPU	$\Delta\mathbb{E}[L(\tau^*_{\text{PTSP}})]$ (%)	$(\Delta\tau^*_{\text{PTSP}})$ (%)
Ams-14	0	1.000		9.912	<1						
	50	0.980	0.046	9.836	1	−0.77	(0.00)	9.848	1	−0.12	(0.00)
	100	0.961	0.046	9.738	6	−1.76	(0.00)	9.766	5	−0.29	(0.00)
Ams-18	0	1.000		9.682	<1						
	50	0.980	0.029	9.447	27	−2.43	(0.00)	9.447	7	−0.01	(0.00)
	100	0.961	0.029	9.208	600	−4.89	(0.00)	9.210	600	−0.02	(0.00)
Ams-20	0	1.000		6.867	<1						
	50	0.926	0.045	6.692	46	−2.55	(0.00)	6.697	11	−0.07	(0.00)
	100	0.851	0.045	6.490	16	−5.49	(0.00)	6.503	2	−0.20	(0.00)
Ams-26	0	1.000		7.114	<1						
	50	0.914	0.039	6.955	45	−2.23	(0.00)	6.957	2	−0.03	(0.00)
	100	0.828	0.039	6.790	44	−4.55	(0.00)	6.796	3	−0.08	(0.00)
Ams-47	0	1.000		19.360	1						
	50	0.933	0.033	18.953	600	−2.11	(0.00)	18.956	600	−0.02	(0.00)
	100	0.865	0.033	18.531	600	−4.28	(0.00)	18.541	600	−0.05	(0.00)
Ams-50	0	1.000		11.947	2						
	50	0.945	0.039	11.668	600	−2.34	(16.00)	11.670	600	−0.01	(0.00)
	100	0.890	0.039	11.379	600	−4.76	(24.00)	11.383	600	−0.04	(0.00)
Ams-57	0	1.000		21.063	1						
	50	0.932	0.031	20.674	600	−1.85	(0.00)	20.676	600	−0.01	(0.00)
	100	0.864	0.031	20.269	600	−3.77	(0.00)	20.676	600	−1.97	(0.00)
Ams-134	0	1.000		37.502	25						
	50	0.939	0.034	37.350	600	−0.41	(11.19)	37.357	600	−0.02	(0.00)
	100	0.878	0.034	36.368	600	−3.02	(6.72)	36.385	600	−0.05	(0.00)

errors is significant at the 5% level, implying that the accuracy of the length estimates benefits from taking the uncertainty surrounding canceling customers into account. That is, the delivery driver's route cost estimates are prone to significant estimation errors if uncertainty among customer presences is ignored. Furthermore, four out of the 16 investigated cases suggest also that an alternative route should be taken, with customer sequences differing ($\Delta\tau^*_{\text{TSP}}$) between 6.72% and 24.00%. Disregarding dependencies show more modest estimation errors of the expected length ($\Delta\mathbb{E}[L(\tau^*_{\text{PTSP}})]$) ranging between $[0.01\%, 1.97\%]$. The increase in accuracy of the expected length estimates comes at slightly bigger computational cost for the instances that can be solved within the computational time limit. The relatively weak values for the average Kendall's tau correlation in each instance ($\overline{K}$) suggest that the overall impact of dependencies on the route are, in this particular case study, limited to route estimates, rather than route decisions ($\Delta\tau^*_{\text{PTSP}}$). Nevertheless, the improved route cost estimates and the modest increase in computation time suggest that incorporating stochastic dependencies bears sufficient potential to increase the accuracy of parcel-delivery operations with customer cancellations.

10. Conclusions

This paper introduces the CPTSP, a new stochastic combinatorial optimization problem that is concerned with finding an a priori tour of minimal expected length along a set of customers whose presences are both stochastic and correlated. The CPTSP generalizes the PTSP proposed by Jaillet (1988) by relaxing the assumption that the random variables associated with the stochastic customer presences are independently distributed. D-Vine PCCs (Panagiotelis, Czado, and Joe 2012) are used to model dependencies among the stochastic customer presences in heterogeneous CPTSP applications. The proposed formulation of the CPTSP exhibits a number of desirable properties from both a modeler's and a practitioner's perspective. For example, the CPTSP objective function only requires the specification of marginal probabilities, pertaining to the stochastic presence of customers, and pairwise correlations, specifying the degree of dependence between customers. Moreover, the objective function of the CPTSP reduces to the objective function of the PTSP in the absence of correlation and to the objective function of the TSP in the absence of uncertainty.

This paper contributes a novel approach to the literature that enables one to model dependencies between stochastic customer presences in routing problems. To the best of my knowledge, such an approach has not been proposed before. Dependencies allow practitioners to reflect relationships and interactions between customers—for example, presence behavior that results from (dis)similarities within and between different customer segments. Although the TSP is selected as the underlying problem to illustrate the concept, parts of the methodology can be readily extended to related routing problems, most notably the VRP with stochastic customers (Bertsimas 1988).

Generally speaking, representations of routing problems that incorporate dependencies between stochastic customer presences warrant more accurate estimations of the routing costs that can be observed in reality (Gendreau, Jabali, and Rei 2016). An application to the last-mile delivery problem with customer cancellations supports this premise: The results suggest that implementations based on the CPTSP produce up to 5.49% more accurate length estimates than implementations that do not take into account the expected costs of customer cancellations during operations and up to 1.97% more accurate length estimates than implementations that do not exploit correlations between customer presences. Furthermore, I illustrate with several computational experiments that, even for low levels of dependence, neither a good tour for the TSP nor a good tour for the PTSP needs to coincide with a good tour for the CPTSP, regardless of the dependence structure. I propose two heuristic approaches based on a priori and a posteriori approximations that support the notion that good tours under stochastic and dependent conditions can be obtained with limited computational effort. I also propose an exact algorithm based on the integer L-shaped method. A comparative study suggests that further, seemingly small, improvements in route costs can lead to relatively big differences in route decisions. Practitioners are therefore recommended to explicitly take into account dependencies between customer presences that can influence route decisions.

The CPTSP creates ample opportunity for future theoretical and practical experimentation, supported by the results in this paper. Firstly, improved bounds and other theoretical properties (cf. Bertsimas and Howell 1993) for CPTSP implementations equipped with specific copula families may be derived. These results can help to further reduce the computational burden of the objective function. Secondly, the setup of the CPTSP can be extended to increase its practical value for industry applications—for example, by extending the setup of the case study with time windows, multiple vehicles, and dynamic features. Finally, the empirical results suggest that it can be promising to develop novel simulation-based solution procedures that are capable of accounting for dependencies between the presences of more distant stochastic customers.

Acknowledgments

The author is greatly indebted to Patrick Jaillet (MIT) for his valuable suggestions and feedback. He also thanks the two anonymous referees and the associate editor for their constructive comments. Furthermore, he is thankful to the parcel-delivery company (who wishes to remain anonymous) for providing delivery data. Finally, he thanks Betty Johanna Garzon Rozo (University of Edinburgh) for her input on discrete copulas and Wout Dullaert (Vrije Universiteit Amsterdam), Bart Keijsers (University of Amsterdam), and Jamal Ouenniche (University of Edinburgh) for their editorial suggestions.

References

Agresti A (2013) *Categorical Data Analysis*, 3rd ed. (John Wiley & Sons, New York).

Applegate D (2006) *The Traveling Salesman Problem: A Computational Study* (Princeton University Press, Princeton, NJ).

Bahadur R (1961) A representation of the joint distribution of responses to n dichotomous items. *Stud. Item Anal. Prediction* 6:158–168.

Bakach I, Campbell A, Ehmke J, Urban T (2021) Solving vehicle routing problems with stochastic and correlated travel times and makespan objectives. *EURO J. Transportation Logist.* 10:100029.

Balaprakash P, Birattari M, Stützle T, Dorigo M (2010) Estimation-based metaheuristics for the probabilistic traveling salesman problem. *Comput. Oper. Res.* 37(11):1939–1951.

Bent R, Van Hentenryck P (2004) Scenario-based planning for partially dynamic vehicle routing with stochastic customers. *Oper. Res.* 52(6):977–987.

Beraldi P, Ghiani G, Laporte G, Musmanno R (2005) Efficient neighborhood search for the probabilistic pickup and delivery travelling salesman problem. *Networks* 45(4):195–198.

Berhan E, Beshah B, Kitaw D, Abraham A (2014) Stochastic vehicle routing problem: A literature survey. *J. Inform. Knowledge Management* 13(3):1450022.

Bertsimas D (1988) The probabilistic vehicle routing problem. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA.

Bertsimas D (1992) A vehicle routing problem with stochastic demand. *Oper. Res.* 40(3):574–585.

Bertsimas D, Howell L (1993) Further results on the probabilistic traveling salesman problem. *Eur. J. Oper. Res.* 65(1):68–95.

Bianchi L, Gambardella L, Dorigo M (2002) An ant colony optimization approach to the probabilistic traveling salesman problem. Guervós J, Adamidis P, Beyer H, Schwefel H, Fernández-Villacañas J, eds. *Proc. 7th Internat. Conf. Parallel Problem Solving Nature PPSN 2002, Lecture Notes in Computer Science*, vol. 2439 (Springer-Verlag, Berlin), 883–892.

Birattari M, Balaprakash P, Stützle T, Dorigo M (2008) Estimation-based local search for stochastic combinatorial optimization using delta evaluations: A case study on the probabilistic traveling salesman problem. *INFORMS J. Comput.* 20(4):644–658.

Boyer K, Prud'homme A, Chung W (2009) The last mile challenge: Evaluating the effects of customer density and delivery window patterns. *J. Bus. Logist.* 30(1):185–201.

Brechmann E, Czado C, Aas K (2012) Truncated regular vines in high dimensions with application to financial data. *Canad. J. Statist.* 40(1):68–85.

Campbell A, Thomas B (2008) Probabilistic traveling salesman problem with deadlines. *Transportation Sci.* 42(1):1–21.

Campbell AM, Thomas BW (2009) Runtime reduction techniques for the probabilistic traveling salesman problem with deadlines. *Comput. Oper. Res.* 36(4):1231–1248.

Carlsson JG, Behroozi M, Mihic K (2018) Wasserstein distance and the distributionally robust TSP. *Oper. Res.* 66(6):1603–1624.

Cherubini U, Luciano E, Vecchiato W (2004) *Copula Methods in Finance* (John Wiley & Sons, Chichester, UK).

Dantzig G, Ramser J (1959) The truck dispatching problem. *Management Sci.* 6(1):80–91.

Dantzig G, Fulkerson R, Johnson S (1954) Solution of a large-scale traveling-salesman problem. *J. Oper. Res. Soc. Amer.* 2(4):393–410.

Declerck L, Aerts M, Molenberghs G (1998) Behaviour of the likelihood ratio test statistic under a Bahadur model for exchangeable binary data. *J. Stat. Comput. Simulation* 61(1–2):15–38.

Delage E, Ye Y (2010) Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Oper. Res.* 58(3):595–612.

Dell'Amico M, Iori M, Novellani S, Subramanian A (2018) The bike sharing rebalancing problem with stochastic demands. *Transportation Res. Part B Methodological* 118:362–380.

Denuit M, Lambert P (2005) Constraints on concordance measures in bivariate discrete data. *J. Multivariate Anal.* 93(1):40–57.

Dinh T, Fukasawa R, Luedtke J (2018) Exact algorithms for the chance-constrained vehicle routing problem. *Math. Program.* 172(1–2):105–138.

Dorigo M, Gambardella L (1997) Ant colony system: A cooperative learning approach to the traveling salesman problem. *IEEE Trans. Evolutionary Comput.* 1(1):53–66.

European Commission (2021) EU Open Data Portal: Domestic postal traffic, letter mail and parcel services. Accessed July 29, 2022, https://data.europa.eu/euodp/en/data/dataset/6rFQHnqYW7HDFi1hIuDHg.

Fitzmaurice G, Laird N, Rotnitzky A (1993) Regression models for discrete longitudinal responses. *Statist. Sci.* 8(3):284–299.

Frey R, McNeil A (2003) Dependent defaults in models of portfolio credit risk. *J. Risk* 6:59–92.

Gambardella L, Montemanni R, Weyland D (2012) Coupling ant colony systems with strong local searches. *Eur. J. Oper. Res.* 220(3):831–843.

Gendreau M, Jabali O, Rei W (2014) Stochastic vehicle routing problems. Toth P, Vigo D, eds. *Vehicle Routing: Problems, Methods, and Applications*, 2nd ed. (MOS-SIAM, Philadelphia), 213–239.

Gendreau M, Jabali O, Rei W (2016) 50th anniversary invited article—Future research directions in stochastic vehicle routing. *Transportation Sci.* 50(4):1163–1173.

Gendreau M, Laporte G, Séguin R (1996) Stochastic vehicle routing. *Eur. J. Oper. Res.* 88(1):3–12.

Genest C, Neálehová J (2007) A primer on copulas for count data. *Astin Bull.* 37(2):475–515.

Ghosal S, Wiesemann W (2020) The distributionally robust chance-constrained vehicle routing problem. *Oper. Res.* 68(3):716–732.

Golden B, Yee Y (1979) A framework for probabilistic vehicle routing. *AIIE Trans.* 11(2):109–112.

Gordy M (2000) A comparative anatomy of credit risk models. *J. Banking Finance* 24(1–2):119–149.

Gounaris C, Repoussis P, Tarantilis C, Wiesemann W, Floudas C (2016) An adaptive memory programming framework for the robust capacitated vehicle routing problem. *Transportation Sci.* 50(4):1239–1260.

IMRG (2018) IMRG valuing home delivery review 2018. Technical report, Interactive Media in Retail Group, London.

Jaillet P (1985) Probabilistic traveling salesman problems. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA.

Jaillet P (1988) A priori solution of a traveling salesman problem in which a random subset of the customers are visited. *Oper. Res.* 36(6):929–936.

Joe H (1997) *Multivariate Models and Multivariate Dependence Concepts* (Chapman & Hall/CRC, London).

Kao E (1978) A preference order dynamic program for a stochastic traveling salesman problem. *Oper. Res.* 26(6):1033–1045.

Kleywegt A, Shapiro A, Homem-de Mello T (2002) The sample average approximation method for stochastic discrete optimization. *SIAM J. Optim.* 12(2):479–502.

Köster F, Ulmer M, Mattfeld D, Hasle G (2018) Anticipating emission-sensitive traffic management strategies for dynamic delivery routing. *Transportation Res. Part D Transportation Environ.* 62:345–361.

Kruskal J (1983) An overview of sequence comparison: Time warps, string edits, and macromolecules. *SIAM Rev.* 25(2):201–237.

Kübler J, Reiffer A, Briem L, Vortisch P (2022) Integrating neighbours into an agent-based travel demand model to analyse success rates of parcel deliveries. *Procedia Comput. Sci.* 201:181–188.

Laporte G, Louveaux F (1993) The integer L-shaped method for stochastic integer programs with complete recourse. *Oper. Res. Lett.* 13(3):133–142.

Laporte G, Louveaux F, Mercure H (1994) A priori optimization of the probabilistic traveling salesman problem. *Oper. Res.* 42(3):543–549.

Laurent J, Gregory J (2005) Basket default swaps, CDOs and factor copulas. *J. Risk* 7(4):1–20.

Lei H, Laporte G, Guo B (2012) Districting for routing with stochastic customers. *EURO J. Transportation Logist.* 1(1–2):67–85.

Leipälä T (1978) On the solutions of stochastic traveling salesman problems. *Eur. J. Oper. Res.* 2(4):291–297.

Letchford A, Nasiri S (2015) The Steiner travelling salesman problem with correlated costs. *Eur. J. Oper. Res.* 245(1):62–69.

Li W (2017) A simulation-based algorithm for the probabilistic traveling salesman problem. Emmerich M, Deutz A, Schütze O, Legrand P, Tantar E, Tantar A-A, eds. *EVOLVE—A Bridge Between Probability*, Set Oriented Numerics and Evolutionary Computation VII (Springer, Cham, Switzerland), 157–183.

Liu C, Wang Q, Susilo Y (2019) Assessing the impacts of collection-delivery points to individual's activity-travel patterns: A greener last mile alternative? *Transportation Res. Part E Logist. Transportation Rev.* 121:84–99.

Liu YH (2008) A memetic algorithm for the probabilistic traveling salesman problem. *Proc. IEEE Congress Evolutionary Comput. CEC 2008* (IEEE, Piscataway, NJ), 146–152.

Macioszek E (2017) First and last mile delivery—Problems and issues. Sierpinski G, ed. *Advanced Solutions of Transport Systems for Growing Mobility: 14th Scientific and Technical Conference "Transport Systems. Theory & Practice 2017" Selected Papers, Advances in Intelligent Systems and Computing*, vol. 631 (Springer, Cham, Switzerland), 147–154.

Mangiaracina R, Perego A, Seghezzi A, Tumino A (2019) Innovative solutions to increase last-mile delivery efficiency in B2C e-commerce: A literature review. *Internat. J. Physical Distribution Logist. Management* 49(9):901–920.

Marinakis Y, Marinaki M (2010) A hybrid multi-swarm particle swarm optimization algorithm for the probabilistic traveling salesman problem. *Comput. Oper. Res.* 37(3):432–442.

Marinakis Y, Marinaki M, Migdalas A (2015) Adaptive tunning of all parameters in a multi-swarm particle swarm optimization algorithm: An application to the probabilistic traveling salesman problem. Migdalas A, Karakitsiou A, eds. *Optimization, Control, and Applications in the Information Age* (Springer, Cham, Switzerland), 187–207.

Molenberghs G, Verbeke G (2005) *Models for Discrete Longitudinal Data* (Springer-Verlag, New York).

Nelsen RB (2006) *An Introduction to Copulas*, 2nd ed. (Springer-Verlag, New York).

Nešlehová J (2007) On rank correlation measures for non-continuous random variables. *J. Multivariate Anal.* 98(3):544–567.

Nikoloulopoulos A (2013) Copula-based models for multivariate discrete response data. Jaworski P, Durante F, Härdle W, eds. *Copulae in Mathematical and Quantitative Finance, Lecture Notes in Statistics*, vol. 213 (Springer, Berlin), 231–249.

Oyola J, Arntzen H, Woodruff D (2017) The stochastic vehicle routing problem, a literature review, part II: Solution methods. *EURO J. Transportation Logist.* 6(4):349–388.

Oyola J, Arntzen H, Woodruff D (2018) The stochastic vehicle routing problem, a literature review, part I: Models. *EURO J. Transportation Logist.* 7(3):193–221.

Panagiotelis A, Czado C, Joe H (2012) Pair copula constructions for multivariate discrete data. *J. Amer. Statist. Assoc.* 107(499):1063–1072.

Panagiotelis A, Czado C, Joe H, Stöber J (2017) Model selection for discrete regular vine copulas. *Comput. Statist. Data Anal.* 106:138–152.

Plackett R (1954) A reduction formula for normal multivariate integrals. *Biometrika* 41(3/4):351–360.

Rahimian H, Mehrotra S (2019) Distributionally robust optimization: A review. Preprint, submitted August 13, https://arxiv.org/abs/1908.05659.

Rai H, Verlinde S, Macharis C (2021) Unlocking the failed delivery problem? Opportunities and challenges for smart locks from a consumer perspective. *Res. Transportation Econom.* 87:100753.

Rajabi-Bahaabadi M, Shariat-Mohaymany A, Babaei M, Vigo D (2021) Reliable vehicle routing problem in stochastic networks with correlated travel times. *Oper. Res.* 21:299–330.

Rostami B, Desaulniers G, Errico F, Lodi A (2021) Branch-price-and-cut algorithms for the vehicle routing problem with stochastic and correlated travel times. *Oper. Res.* 69(2):436–455.

Schaefer J, Figliozzi M (2021) Spatial accessibility and equity analysis of Amazon parcel lockers facilities. *J. Transportation Geography* 97:103212.

Sklar M (1959) Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris* 8:229–231.

Smith M, Khaled M (2012) Estimation of copula models with discrete margins via Bayesian data augmentation. *J. Amer. Statist. Assoc.* 107(497):290–303.

Song P (2007) *Correlated Data Analysis: Modeling, Analytics, and Applications* (Springer-Verlag, New York).

Stewart W, Golden B (1983) Stochastic vehicle routing: A comprehensive approach. *Eur. J. Oper. Res.* 14(4):371–385.

Sungur I, Ren Y, Ordóñez F, Dessouky M, Zhong H (2010) A model and algorithm for the courier delivery problem with uncertainty. *Transportation Sci.* 44(2):193–205.

Tang H, Miller-Hooks E (2004) Approximate procedures for probabilistic traveling salesperson problem. *Transportation Res. Rec.* 1882(1):27–36.

Tang H, Miller-Hooks E (2007) Solving a generalized traveling salesperson problem with stochastic customers. *Comput. Oper. Res.* 34(7):1963–1987.

Tillman F (1969) The multiple terminal delivery problem with probabilistic demands. *Transportation Sci.* 3(3):192–204.

Toriello A, Haskell W, Poremba M (2014) A dynamic traveling salesman problem with stochastic arc costs. *Oper. Res.* 62(5):1107–1125.

Van Duin J, De Goffau W, Wiegmans B, Tavasszy L, Saes M (2016) Improving home delivery efficiency by using principles of address intelligence for B2C deliveries. *Transportation Res. Procedia* 12:14–25.

Van Loon P, Deketele L, Dewaele J, McKinnon A, Rutherford C (2015) A comparative analysis of carbon emissions from online retailing of fast moving consumer goods. *J. Cleaner Production* 106:478–486.

Voccia S, Campbell A, Thomas B (2013) The probabilistic traveling salesman problem with time windows. *EURO J. Transportation Logist.* 2(1–2):89–107.

Xiao Q, Zhou S (2019) Matching a correlation coefficient by a Gaussian copula. *Commun. Statist. Theory Methods* 48(7):1728–1747.

Zhang M, Qin J, Yugang Y, Liang L (2018) Traveling salesman problems with profits and stochastic customers. *Internat. Trans. Oper. Res.* 25(4):1297–1313.