Banach J. Math. Anal. (2023) 17:81
https://doi.org/10.1007/s43037-023-00309-7

Tusi Mathematical Research Group

ORIGINAL PAPER

Topological aspects of quasi *-algebras with sufficiently many *-representations

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Received: 8 May 2023 / Accepted: 14 September 2023

Abstract
Quasi *-algebras possessing a sufficient family $\mathcal{M}$ of invariant positive sesquilinear forms carry several topologies related to $\mathcal{M}$ which make every *-representation continuous. This leads to define the class of locally convex quasi GA*-algebras whose main feature consists in the fact that the family of their bounded elements, with respect to the family $\mathcal{M}$, is a dense C*-algebra.

Keywords Invariant positive sesquilinear form · *-Representation · Locally convex quasi *-algebra

Mathematics Subject Classification 46K05 · 46K10 · 47L60 · 47A07 · 08A55

1 Introduction

Locally convex quasi *-algebras $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ arise often when taking the completion $\mathfrak{A} := \widetilde{\mathfrak{A}_0}[\tau]$ of a locally convex *-algebra $\mathfrak{A}_0[\tau]$ with separately (but not jointly) continuous multiplication (this was, in fact, the case considered at an early stage of the theory, concerning applications in quantum physics). Concrete examples are provided by families of operators acting in rigged Hilbert spaces or by certain families of unbounded operators acting on a common domain $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. For a synthesis of the theory and of its applications, we refer to [6].

Communicated by Fedor Sukochev.

This work was completed with the support of our TeX-pert.

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Published online: 09 October 2023 Birkhäuser

The study of this structure and the analysis performed also in [1, 4, 5, 7, 9] made it clear that the most regular situation occurs when the locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ under consideration possesses a *sufficiently rich* family $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ of invariant positive sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$ (see below for definitions); they allow a GNS construction similar to that defined by a positive linear functional on a *-algebra $\mathfrak{A}_0$. The basic idea where this paper moves from is to consider a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ where one can introduce a locally convex topology by means of the set of sesquilinear forms $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ itself. In the best circumstances, we expect a behavior analogous to that of a *-algebra $\mathfrak{B}_0$ whose topology can be defined via families of C*-seminorms

$$p_M(x) = \sup_{\omega \in M; \omega(\mathsf{e})=1} \omega(x^* x)^{1/2},$$

where M is a convenient set of positive linear functionals on $\mathfrak{B}_0$ [10].

For this reason, we start from a pure algebraic setup, i.e., $(\mathfrak{A}, \mathfrak{A}_0)$ is a quasi *-algebra and we suppose that it has a sufficiently large $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ (in the sense that, for some convenient subset $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ and for every $a \in \mathfrak{A}$, $a \neq 0$, there exists $\varphi \in \mathcal{M}$, such that $\varphi(a, a) > 0$). Starting from this set $\mathcal{M}$, we undertake the construction of locally convex topologies on $\mathfrak{A}$ selecting in particular those under which each (sufficiently regular) *-representation is continuous. This analysis leads to the selection of a class of locally convex quasi *-algebras $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ (called locally convex quasi GA*-algebras) whose bounded elements constitute a C*-algebra. The paper is organized as follows. In Sect. 2, some preliminary notions on quasi *-algebras, their topologies, and their representations are summarized. In Sect. 3, we introduce the order defined by a family $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ whose related wedge becomes a cone when the family $\mathcal{M}$ is sufficiently rich. In Sect. 4, given $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$, we introduce two notions of bounded elements: those bounded with respect to a family $\mathcal{M}$ and those related to the order defined by $\mathcal{M}$. These two notions turn out to be equivalent and every *-representation produces a bounded operator when acting on a bounded element. In Sect. 5 the topologies generated by a family $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ are investigated, and in Sect. 6, we finally introduce locally convex quasi GA*-algebras and study some properties of them. Locally convex quasi GA*-algebras are characterized by the fact that their topology is equivalent to that generated by some $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ as in Sect. 5.

2 Basic definitions and facts

We begin with some preliminaries; we refer to [6] for details.

A *quasi *-algebra* $(\mathfrak{A}, \mathfrak{A}_0)$ is a pair consisting of a vector space $\mathfrak{A}$ and a *-algebra $\mathfrak{A}_0$ contained in $\mathfrak{A}$ as a subspace and such that

(i) $\mathfrak{A}$ carries an involution $a \mapsto a^*$ extending the involution of $\mathfrak{A}_0$;
(ii) $\mathfrak{A}$ is a bimodule over $\mathfrak{A}_0$ and the module multiplications extend the multiplication of $\mathfrak{A}_0$. In particular, the following associative laws hold:

$$(xa)y = x(ay); \quad a(xy) = (ax)y, \quad \forall\, a \in \mathfrak{A},\ x, y \in \mathfrak{A}_0;$$

(iii) $(ax)^* = x^* a^*$, for every $a \in \mathfrak{A}$ and $x \in \mathfrak{A}_0$.

The *identity* of $(\mathfrak{A}, \mathfrak{A}_0)$, if any, is a necessarily unique element $\mathsf{e} \in \mathfrak{A}_0$, such that $a\mathsf{e} = a = \mathsf{e}a$, for all $a \in \mathfrak{A}$.

We will always suppose that

$$\begin{aligned} ax = 0, \ \forall x \in \mathfrak{A}_0 \Rightarrow a = 0 \\ ax = 0, \ \forall a \in \mathfrak{A} \Rightarrow x = 0. \end{aligned}$$

These two conditions are clearly satisfied if $(\mathfrak{A}, \mathfrak{A}_0)$ has an identity e.

Definition 2.1 A quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be *locally convex* if $\mathfrak{A}$ is a locally convex vector space, with a topology τ enjoying the following properties:

(lc1) $x \mapsto x^*, \ x \in \mathfrak{A}_0$, is continuous;
(lc2) for every $a \in \mathfrak{A}$, the maps $x \mapsto ax$ and $x \mapsto xa$, from $\mathfrak{A}_0$ into $\mathfrak{A}$, $x \in \mathfrak{A}_0$, are continuous;
(lc3) $\overline{\mathfrak{A}_0}^{\tau} = \mathfrak{A}$; i.e., $\mathfrak{A}_0$ is dense in $\mathfrak{A}[\tau]$.

In particular, if τ is a norm topology, with norm $\| \cdot \|$, and

(bq*) $\|a^*\| = \|a\|, \ \forall a \in \mathfrak{A}$,

then $(\mathfrak{A}[\| \cdot \|], \mathfrak{A}_0)$ is called a *normed quasi *-algebra* and a *Banach quasi *-algebra* if the normed vector space $\mathfrak{A}[\| \cdot \|]$ is complete.

Let $\mathcal{D}$ be a dense vector subspace of a Hilbert space $\mathcal{H}$. Let us consider the following families of linear operators acting on $\mathcal{D}$:

$$\begin{aligned} \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) &= \{X \text{ closable}, D(X) = \mathcal{D}; \ D(X^*) \supset \mathcal{D}\} \\ \mathcal{L}^\dagger(\mathcal{D}) &= \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : X\mathcal{D} \subset \mathcal{D}; \ X^*\mathcal{D} \subset \mathcal{D}\} \\ \mathcal{L}^\dagger(\mathcal{D})_b &= \{Y \in \mathcal{L}^\dagger(\mathcal{D}); \ \overline{Y} \text{ bounded}\}, \end{aligned}$$

where $\overline{Y}$ denotes the closure of Y. The involution in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is defined by $X^\dagger := X^* \upharpoonright \mathcal{D}$, the restriction of X^*, the adjoint of X, to $\mathcal{D}$

The set $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra; more precisely, it is the maximal O*-algebra on $\mathcal{D}$, (for the theories of O*-algebras and *-representations, we refer to [8]).

Furthermore, $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is also a *partial *-algebra* [2] with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger := X^* \upharpoonright \mathcal{D}$, and the *(weak)* partial multiplication:

$$X_1 \square X_2 = {X_1}^{\dagger *} X_2, \tag{2.1}$$

defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$), that is, whenever $X_2\mathcal{D} \subset \mathcal{D}({X_1}^{\dagger *})$ and ${X_1}^*\mathcal{D} \subset \mathcal{D}({X_2}^*)$.

The following topologies on $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ will be used in this paper.

The *weak topology* t_w on $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is defined by the seminorms

$$r_{\xi,\eta}(X) = |\langle X\xi | \eta \rangle|, \quad X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \ \xi, \eta \in \mathcal{D}.$$

The *strong topology* t_s on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is defined by the seminorms

$$p_\xi(X) = \|X\xi\|, \quad X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),\ \xi \in \mathcal{D}.$$

The *strong* topology* t_{s^*} on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is usually defined by the seminorms

$$p^*_\xi(X) = \max\{\|X\xi\|, \|X^\dagger\xi\|\},\ \xi \in \mathcal{D}.$$

Then, $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathsf{t}_{s^*}], \mathcal{L}^\dagger(\mathcal{D})_b)$ is a complete locally convex quasi *-algebra [6, Section 6.1].

Let us denote by $t_\dagger$ the graph topology on $\mathcal{D}$ defined by the set of seminorms

$$\xi \in \mathcal{D} \to \|X\xi\|;\ X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}).$$

The family of all bounded subsets of $\mathcal{D}[t_\dagger]$ is denoted by $\mathfrak{B}$.

We will indicate by $\mathfrak{t}_u$, $\mathfrak{t}^u$ and $\mathfrak{t}^u_*$, respectively, the *uniform* topologies defined by the following families of seminorms:
for $\mathfrak{t}_u$: $p_{\mathcal{B}}(X) = \sup_{\xi,\eta\in\mathcal{B}} |\langle X\xi|\eta\rangle|, \quad \mathcal{B} \in \mathfrak{B}$;
for $\mathfrak{t}^u$: $p^{\mathcal{B}}(X) = \sup_{\xi\in\mathcal{B}} \|X\xi\|, \quad \mathcal{B} \in \mathfrak{B}$;
for $\mathfrak{t}^u_*$: $p^{\mathcal{B}}_*(X) = \max\{p^{\mathcal{B}}(X), p^{\mathcal{B}}(X^\dagger)\}, \quad \mathcal{B} \in \mathfrak{B}$.

It is easy to see that $\mathfrak{t}_u \preceq \mathfrak{t}^u \preceq \mathfrak{t}^u_*$

$$p_{\mathcal{B}}(X) \le \gamma_{\mathcal{B}}\, p^{\mathcal{B}}(X) \le \gamma_{\mathcal{B}} p^{\mathcal{B}}_*(X), \quad \forall X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H});$$

moreover,

$$p_{\mathcal{B}}(X^\dagger \square X) = p^{\mathcal{B}}(X)^2 \text{ whenever } X^\dagger \square X \text{ is well defined.}$$

As shown in [2, Proposition 4.2.3] $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathfrak{t}^u_*]$ is complete.

Definition 2.2 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and $\mathcal{D}_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$. A linear map π from $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ is called a **-representation* of $(\mathfrak{A}, \mathfrak{A}_0)$, if the following properties are fulfilled:

(i) $\pi(a^*) = \pi(a)^\dagger, \quad \forall\, a \in \mathfrak{A}$;
(ii) for $a \in \mathfrak{A}$ and $x \in \mathfrak{A}_0$, $\pi(a)\square\pi(x)$ is well defined and $\pi(a)\square\pi(x) = \pi(ax)$.

If $(\mathfrak{A}, \mathfrak{A}_0)$ has a unit $\mathsf{e} \in \mathfrak{A}_0$, we assume that for every *-representation π of $(\mathfrak{A}, \mathfrak{A}_0)$, $\pi(\mathsf{e}) = \mathbb{I}_{\mathcal{D}_\pi}$, the identity operator on the space $\mathcal{D}_\pi$.

If $\pi_o := \pi \restriction \mathfrak{A}_0$ is a *-representation of the *-algebra $\mathfrak{A}_0$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi)$, we say that π is a *qu*-representation* of $(\mathfrak{A}, \mathfrak{A}_0)$.

A *-representation π is called *bounded* if $\pi(a)$ is a bounded operator in $\mathcal{D}_\pi$, for every $a \in \mathfrak{A}$.

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. We denote by $\mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$ the set of all sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$, such that

(i) φ is positive, i.e., $\varphi(a, a) \ge 0, \quad \forall\, a \in \mathfrak{A}$;

(ii) $\varphi(ax, y) = \varphi(x, a^* y), \quad \forall\, a \in \mathfrak{A},\; x, y \in \mathfrak{A}_0$.

For every $\varphi \in \mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$, the set

$$N_\varphi := \left\{a \in \mathfrak{A} : \varphi(a, a) = 0\right\} = \left\{a \in \mathfrak{A} : \varphi(a, b) = 0,\; \forall\, b \in \mathfrak{A}\right\}$$

is a subspace of $\mathfrak{A}$.

Let $\lambda_\varphi : \mathfrak{A} \to \mathfrak{A}/N_\varphi$ be the usual quotient map, and for each $a \in \mathfrak{A}$, let $\lambda_\varphi(a)$ be the corresponding coset of $\mathfrak{A}/N_\varphi$, which contains a. An inner product $\langle\cdot|\cdot\rangle$ is then defined on $\lambda_\varphi(\mathfrak{A}) = \mathfrak{A}/N_\varphi$ by

$$\langle \lambda_\varphi(a) | \lambda_\varphi(b) \rangle := \varphi(a, b), \quad \forall\, a, b \in \mathfrak{A}.$$

Denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{A})$.

Definition 2.3 We denote by $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ the subset of forms $\varphi \in \mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$ for which $\lambda_\varphi(\mathfrak{A}_0)$ is dense in $\mathcal{H}_\varphi$. Elements of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ are also called *invariant positive sesquilinear forms* or briefly *ips-forms*.

Moreover, if $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is a locally convex quasi *-algebra, we denote by $\mathcal{P}^\tau_{\mathfrak{A}_0}(\mathfrak{A})$ the family of elements φ of $\mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$ that are jointly τ-continuous; i.e., there exists a continuous seminorm p_σ, such that

$$|\varphi(a, b)| \le p_\sigma(a) p_\sigma(b), \quad \forall a, b \in \mathfrak{A}.$$

The sesquilinear forms of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ allow building up a GNS representation [6]. Indeed,

Proposition 2.4 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a quasi *-algebra with unit* e *and* φ *a sesquilinear form on* $\mathfrak{A} \times \mathfrak{A}$*. The following statements are equivalent.*

(i) $\varphi \in \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$.
(ii) *There exist a Hilbert space* $\mathcal{H}_\varphi$*, a dense domain* $\mathcal{D}_\varphi$ *of the Hilbert space* $\mathcal{H}_\varphi$*, and a closed cyclic *-representation* π_φ *in* $\mathcal{L}^\dagger(\mathcal{D}_\varphi, \mathcal{H}_\varphi)$*, with cyclic vector* ξ_φ (in the sense that $\pi_\varphi(\mathfrak{A}_0)\xi_\varphi$ is dense in $\mathcal{H}_\varphi$)*, such that*

$$\varphi(a, b) = \langle \pi_\varphi(a)\xi_\varphi | \pi_\varphi(b)\xi_\varphi \rangle, \quad \forall\, a, b \in \mathfrak{A}.$$

Remark 2.5 The *-representation π_φ is in fact obtained by taking the closure of the *-representation π°_φ defined on $\lambda_\varphi(\mathfrak{A}_0)$ by

$$\pi^\circ_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax) \quad a \in \mathfrak{A}, x \in \mathfrak{A}_0.$$

If $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ is *rich enough* (in the sense that, for every $a \in \mathfrak{A}$, there exists $\varphi \in \mathcal{M}$, such that $\varphi(a, a) > 0$), then we can introduce a partial multiplication as in [6, Definition 3.1.30].

Indeed, in this case, we say that the *weak* multiplication, $a \diamond b$, $a, b \in \mathfrak{A}$, is well defined if there exists $c \in \mathfrak{A}$, such that

$$\varphi(bx, a^* y) = \varphi(cx, y), \quad \forall\, x, y \in \mathfrak{A}_0 \text{ and } \forall \varphi \in \mathcal{M}. \tag{2.2}$$

In this case, we put $a \diamond b := c$.

With these definitions, we conclude that [3, Proposition 4.4] $\mathfrak{A}$ is also a partial *-algebra with respect to the weak multiplication $\diamond$.

Remark 2.6 The uniqueness of $c = a \diamond b$ is guaranteed by Proposition 3.3 below. Clearly, this multiplication depends on the family $\mathcal{M}$.

3 Families of forms and order structure

As discussed extensively in [6], the notion of bounded element of a locally convex quasi *-algebra reveals to be important for undertaking a spectral analysis in this structure. We propose here two different approaches similar to those developed in [3] but without the continuity assumptions made therein.

Before going forth, we introduce some notions needed in what follows. In particular, in analogy to [10],

Definition 3.1 A subset $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ is said to be

balanced if $\varphi \in \mathcal{M}$ implies $\varphi^x \in \mathcal{M}$, for every $x \in \mathfrak{A}_0$, where $\varphi^x(a, b) := \varphi(ax, bx)$ for all $a, b \in \mathfrak{A}$.

sufficient if it is balanced and if, for every $a \in \mathfrak{A}\setminus\{0\}$, there exists $\varphi \in \mathcal{M}$, such that $\varphi(a, a) > 0$.

Remark 3.2 If $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is a locally convex quasi *-algebra, then

(a) $\mathcal{P}^{\tau}_{\mathfrak{A}_0}(\mathfrak{A}) \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$;
(b) $\mathcal{P}^{\tau}_{\mathfrak{A}_0}(\mathfrak{A})$ is balanced.

Both (a) and (b) follow directly from the joint continuity of elements of $\mathcal{P}^{\tau}_{\mathfrak{A}_0}(\mathfrak{A})$.

The following proposition allows us to deal with notion of sufficiency in other equivalent ways.

Proposition 3.3 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra, $\mathcal{M}$ a subset of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ and $a \in \mathfrak{A}$. Then, the following are equivalent:*

(i) $\varphi(ax, x) = 0$, *for all* $\varphi \in \mathcal{M}$, $x \in \mathfrak{A}_0$*;*
(ii) $\varphi(ax, y) = 0$, *for all* $\varphi \in \mathcal{M}$, $x, y \in \mathfrak{A}_0$*;*
(iii) $\varphi(ax, ax) = 0$, *for all* $\varphi \in \mathcal{M}$, $x \in \mathfrak{A}_0$.

If $(\mathfrak{A}, \mathfrak{A}_0)$ has unit e *and $\mathcal{M}$ is balanced, then the previous statements are equivalent to*

(iv) $\varphi(a, a) = 0$, *for every* $\varphi \in \mathcal{M}$.

In the case of a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$, positive elements have been defined in [7] as the members of the closure $\mathfrak{A}^+ := \overline{\mathfrak{A}_0^+}^{\tau}$, where

$$\mathfrak{A}_0^+ := \left\{ \sum_{k=1}^{n} x_k^* x_k,\ x_k \in \mathfrak{A}_0,\ n \in \mathbb{N} \right\}.$$

Here, as we have anticipated, we will start from a quasi *-algebra without a topology and we will introduce the notion of positive element via a family $\mathcal{M}$ of forms of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$.

Definition 3.4 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. We call $\mathcal{M}$-positive an element $a \in \mathfrak{A}$, such that

$$\varphi(ax, x) \geq 0, \quad \forall \varphi \in \mathcal{M}, \forall x \in \mathfrak{A}_0.$$

We put

$$\mathcal{K}_{\mathcal{M}} := \{a \in \mathfrak{A} :\ \varphi(ax, x) \geq 0,\ \forall \varphi \in \mathcal{M}, \forall x \in \mathfrak{A}_0\}.$$

If $\mathcal{M} = \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$, we denote the corresponding set by $\mathcal{K}_{\mathcal{I}}$.

Lemma 3.5 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with a sufficient $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$. If a is $\mathcal{M}$-positive, then $a = a^*$.*

Proof The conclusion is a consequence of Proposition 3.3. □

The set $\mathcal{K}_{\mathcal{M}}$ is a qm-admissible wedge, that is $a + b \in \mathcal{K}_{\mathcal{M}}$, $\lambda a \in \mathcal{K}_{\mathcal{M}}$ and $x^*ax \in \mathcal{K}_{\mathcal{M}}$ for all $a, b \in \mathcal{K}_{\mathcal{M}}$, $x \in \mathfrak{A}_0$ and $\lambda \geq 0$. If, moreover, $\mathfrak{A}$ has a unit e, then $\mathsf{e} \in \mathcal{K}_{\mathcal{M}}$.

As usual, one can define an order on the *real* vector space $\mathfrak{A}_h = \{a \in \mathfrak{A} :\ a = a^*\}$ by

$$a \leq_{\mathcal{M}} b \quad \Leftrightarrow \quad b - a \in \mathcal{K}_{\mathcal{M}}, \quad a, b \in \mathfrak{A}_h.$$

Proposition 3.6 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be quasi *-algebra with unit* e *and let $\mathcal{M}$ be a balanced subset of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$. Then, the following are equivalent:*

(i) *$\mathcal{M}$ is sufficient;*
(ii) $\mathcal{K}_{\mathcal{M}} \cap (-\mathcal{K}_{\mathcal{M}}) = \{0\}$.

Proof (i) $\Rightarrow$ (ii) Let $a \in \mathcal{K}_{\mathcal{M}} \cap (-\mathcal{K}_{\mathcal{M}})$. Then, $\varphi(ax, x) = 0$, for every $\varphi \in \mathcal{M}$ and every $x \in \mathfrak{A}_0$; hence, by Proposition 3.3, and by the sufficiency of $\mathcal{M}$, we get $a = 0$. (ii) $\Rightarrow$ (i) Let us suppose by absurd that there exists $a \in \mathfrak{A}$, $a \neq 0$, such that $\varphi(a, a) = 0$, for every $\varphi \in \mathcal{M}$. Then, again, by Proposition 3.3, it follows that $\varphi(ax, x) = 0$ for every $x \in \mathfrak{A}_0$ and every $\varphi \in \mathcal{M}$; this means that $a \in \mathcal{K}_{\mathcal{M}} \cap (-\mathcal{K}_{\mathcal{M}}) = \{0\}$ a contradiction. □

4 Bounded and order bounded elements

Definition 4.1 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ sufficient. We say that an element $a \in \mathfrak{A}$ is $\mathcal{M}$-bounded if there exists $\gamma_a = \gamma_{a,\mathcal{M}} > 0$, such that

$$|\varphi(ax, y)| \leq \gamma_a \varphi(x, x)^{1/2} \varphi(y, y)^{1/2}, \quad \forall \varphi \in \mathcal{M};\ \forall x, y \in \mathfrak{A}_0.$$

If a is $\mathcal{M}$-bounded, we put

$$\|a\|_{\mathrm{b}}^{\mathcal{M}} = \inf\{\gamma_a > 0 : |\varphi(ax, y)| \leq \gamma_a \varphi(x, x)^{1/2} \varphi(y, y)^{1/2}, \\ \varphi \in \mathcal{M},\ \ x, y \in \mathfrak{A}_0\}.$$

Remark 4.2 For future use, we notice that, in general and regardless to the $\mathcal{M}$-boundedness of $a \in \mathfrak{A}$, the following equalities hold:

$$\begin{aligned}\Lambda_a &:= \inf\{\gamma_a > 0 : |\varphi(ax, y)| \leq \gamma_a \varphi(x, x)^{1/2} \varphi(y, y)^{1/2},\ \varphi \in \mathcal{M},\ x, y \in \mathfrak{A}_0\} \\ &= \sup\{|\varphi(ax, y)|; \varphi \in \mathcal{M}, x, y \in \mathfrak{A}_0, \varphi(x, x) = \varphi(y, y) = 1\} \\ &= \sup\{\|\pi_\varphi(a)\|;\ \varphi \in \mathcal{M}\}.\end{aligned}$$

Moreover, if $a = a^*$

$$\Lambda_a = \sup\{|\varphi(az, z)|; \varphi \in \mathcal{M}, z \in \mathfrak{A}_0, \varphi(z, z) = 1\}.$$

The value of Λ_a is finite if and only if a is $\mathcal{M}$-bounded; by the definition itself, $\|a\|_{\mathrm{b}}^{\mathcal{M}} = \Lambda_a$.

Lemma 4.3 *Let $a, b \in \mathfrak{A}$ be $\mathcal{M}$-bounded. Then*

(i) *a^* is $\mathcal{M}$-bounded too, and $\|a^*\|_{\mathrm{b}}^{\mathcal{M}} = \|a\|_{\mathrm{b}}^{\mathcal{M}}$;*
(ii) *$a + b$ is $\mathcal{M}$-bounded and $\|a + b\|_{\mathrm{b}}^{\mathcal{M}} \leq \|a\|_{\mathrm{b}}^{\mathcal{M}} + \|b\|_{\mathrm{b}}^{\mathcal{M}}$;*
(iii) *αa is $\mathcal{M}$-bounded, $\forall \alpha \in \mathbb{C}$;*
(iv) *if $a \diamond b$ is well defined, the product $a \diamond b$ is $\mathcal{M}$-bounded and $\|a \diamond b\|_{\mathrm{b}}^{\mathcal{M}} \leq \|a\|_{\mathrm{b}}^{\mathcal{M}} \|b\|_{\mathrm{b}}^{\mathcal{M}}$.*

Proof We prove (iv). Suppose that $a \diamond b$ is well defined. Then, for every $\varphi \in \mathcal{M}$ and $x, y \in \mathfrak{A}_0$, with $\varphi(x, x) = \varphi(y, y) = 1$, we have

$$\begin{aligned}|\varphi((a \diamond b)x, y)| &= |\varphi(bx, a^* y)| \leq \varphi(bx, bx)^{1/2} \varphi(a^* y, a^* y)^{1/2} \\ &\leq \|a\|_{\mathrm{b}}^{\mathcal{M}} \|b\|_{\mathrm{b}}^{\mathcal{M}}.\end{aligned}$$

The statement then follows by taking the sup over $\varphi \in \mathcal{M}$. □

As in [3, Proposition 4.18], one can prove

Proposition 4.4 *Let a, b be $\mathcal{M}$-bounded elements of $\mathfrak{A}$ and let $\varphi \in \mathcal{M}$. Then, if $a \diamond b$ is well defined, $\pi_\varphi(a) \Box \pi_\varphi(b)$ is also well defined and $\pi_\varphi(a \diamond b) = \pi_\varphi(a) \Box \pi_\varphi(b)$.*

Remark 4.5 Remark 4.2 and Proposition 4.4 imply that if a is $\mathcal{M}$-bounded and $a^* \diamond a$ is well defined, then $\|a^* \diamond a\|_{\mathrm{b}}^{\mathcal{M}} = (\|a\|_{\mathrm{b}}^{\mathcal{M}})^2$.

The notion of $\mathcal{M}$-positive element can be used to give a formally different definition of bounded element. Let $a \in \mathfrak{A}$, put $\Re(a) = \frac{1}{2}(a + a^*)$, $\Im(a) = \frac{1}{2i}(a - a^*)$. Then, both $\Re(a), \Im(a) \in \mathfrak{A}_h$ and $a = \Re(a) + i\Im(a)$.

Definition 4.6 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. The element $a \in \mathfrak{A}$ is said $\mathcal{K}_{\mathcal{M}}$-*bounded* if there exists $\gamma \geq 0$, such that

$$\begin{cases} \pm\varphi(\Re(a)x, x) \leq \gamma\varphi(x, x) \\ \pm\varphi(\Im(a)x, x) \leq \gamma\varphi(x, x) \end{cases}, \quad \forall \varphi \in \mathcal{M}, x \in \mathfrak{A}_0. \tag{4.1}$$

If $(\mathfrak{A}, \mathfrak{A}_0)$ is unital, then we can rewrite (4.1), more syntetically, as

$$\pm\Re(a) \leq_{\mathcal{M}} \gamma \mathsf{e}, \qquad \pm\Im(a) \leq_{\mathcal{M}} \gamma \mathsf{e}.$$

We denote by $\mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}})$ the set of all $\mathcal{K}_{\mathcal{M}}$-bounded elements of $\mathfrak{A}$.

As in [7], the following result holds true:

Proposition 4.7 *The couple* $(\mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}), \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}) \bigcap \mathfrak{A}_0)$ *is a quasi *-algebra, and hence, in particular*

1. $\alpha a + \beta b \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}})$ *for any* $\alpha, \beta \in \mathbb{C}$ *and* $a, b \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}})$*;*
2. $a \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}) \Leftrightarrow a^* \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}})$*;*
3. $a \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}})$, $x \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}) \bigcap \mathfrak{A}_0 \Rightarrow xa \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}})$*;*
4. $x \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}) \bigcap \mathfrak{A}_0 \Leftrightarrow xx^* \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}) \bigcap \mathfrak{A}_0$.

In particular, if $(\mathfrak{A}, \mathfrak{A}_0)$ *has a unit, then also* $(\mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}), \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}}) \bigcap \mathfrak{A}_0)$ *has a unit.*

Theorem 4.8 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a quasi *-algebra and* $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$*. Then, the following are equivalent:*

(i) $a \in \mathfrak{A}_{\mathrm{b}}(\mathcal{K}_{\mathcal{M}})$*;*

(ii) *a is* $\mathcal{M}$*-bounded; i.e.,* (Definition 4.1)*, there exists* $\gamma_a > 0$*, such that*

$$|\varphi(ax, y)| \leq \gamma_a \varphi(x, x)^{1/2} \varphi(y, y)^{1/2}, \ \forall x, y \in \mathfrak{A}_0,$$

for every $\varphi \in \mathcal{M}$;

(iii) *there exists* $\gamma'_a > 0$*, such that*

$$\varphi(ax, ax) \leq \gamma'_a \varphi(x, x), \ \forall x \in \mathfrak{A}_0,$$

for every $\varphi \in \mathcal{M}$.

(iv) *there exists* $\gamma''_a > 0$*, such that*

$$|\varphi(ax, x)| \leq \gamma''_a \varphi(x, x), \ \forall x \in \mathfrak{A}_0,$$

for every $\varphi \in \mathcal{M}$.

Proof We prove it for symmetric elements.
(i) $\Rightarrow$ (ii) It is clear that if $a = a^* \in \mathfrak{A}_b(\mathcal{K}_\mathcal{M})$, then there exists $\gamma > 0$, such that

$$|\varphi(ax, x)| \le \gamma\varphi(x, x), \ \forall\varphi \in \mathcal{M}, \forall x \in \mathfrak{A}_0$$

hence

$$\sup\{|\varphi(ax, x)|; \ \varphi \in \mathcal{M}, x \in \mathfrak{A}_0, \varphi(x, x) = 1\} < \infty.$$

From Remark 4.2, it follows that $\|a\|_b^\mathcal{M} < \infty$, i.e., a is $\mathcal{M}$-bounded.
(ii) $\Rightarrow$ (iii) Assume that $a \in \mathfrak{A}$ is $\mathcal{M}$-bounded. If $\varphi \in \mathcal{M}$, denote by π_φ the corresponding GNS representation. Then

$$\begin{aligned}|\langle\pi_\varphi(a)\lambda_\varphi(x)|\lambda_\varphi(y)\rangle| &= |\varphi(ax, y)| \le \|a\|_b^\mathcal{M}\varphi(x, x)^{1/2}\varphi(y, y)^{1/2} \\ &= \|a\|_b^\mathcal{M} \|\lambda_\varphi(x)\| \, \|\lambda_\varphi(y)\|, \quad \forall x, y \in \mathfrak{A}_0.\end{aligned}$$

This implies that, for every $\varphi \in \mathcal{M}$, the operator $\pi_\varphi(a)$ is bounded and $\|\pi_\varphi(a)\| \le \|a\|_b^\mathcal{M}$. Hence

$$\begin{aligned}\varphi(ax, ax)^{1/2} &= \|\pi_\varphi(a)\lambda_\varphi(x)\| \\ &\le \|a\|_b^\mathcal{M}\|\lambda_\varphi(x)\| = \|a\|_b^\mathcal{M}\varphi(x, x)^{1/2}, \quad \forall x, y \in \mathfrak{A}_0. \end{aligned} \tag{4.2}$$

(iii) $\Rightarrow$ (iv) Suppose that a satisfies (iii). Let $\varphi \in \mathcal{M}$ and $x \in \mathfrak{A}_0$. Then

$$|\varphi(ax, x)| \le \varphi(ax, ax)^{1/2}\varphi(x, x)^{1/2} \le \gamma_a'^{1/2}\varphi(x, x).$$

(iv) $\Rightarrow$ (i) It is straightforward. □

Remark 4.9 By the previous theorem, we also deduce the following equalities for the norm of an $\mathcal{M}$-bounded element a (see also Remark 4.2):

$$\begin{aligned}\|a\|_b^\mathcal{M} &= \inf\{\gamma > 0 : \varphi(ax, ax) \le \gamma^2\varphi(x, x), \ \varphi \in \mathcal{M}, \ x \in \mathfrak{A}_0\} \\ &= \sup\{\varphi(ax, ax)^{1/2} : \varphi \in \mathcal{M}, \ x \in \mathfrak{A}_0, \ \varphi(x, x) = 1\}.\end{aligned}$$

In view of Theorem 4.8, we adopt the notation $\mathfrak{A}_b^\mathcal{M}$ for the set of either $\mathcal{M}$-bounded or $\mathcal{K}_\mathcal{M}$-bounded elements; i.e., we put $\mathfrak{A}_b^\mathcal{M} = \mathfrak{A}_b(\mathcal{K}_\mathcal{M})$.

Definition 4.10 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and let $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$. We say that a *-representation π of $\mathfrak{A}$ is $\mathcal{M}$-*regular* if, for every $\xi \in \mathcal{D}_\pi$, the vector form φ_ξ defined by

$$\varphi_\xi(a, b) := \langle\pi(a)\xi|\pi(b)\xi\rangle, \quad a, b \in \mathfrak{A} \tag{4.3}$$

is a form in $\mathcal{M}$.

In particular, if $\mathcal{M} = \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$, π is said to be *regular* [6]. We denote by $\mathsf{Rep}^{r,\mathcal{M}}(\mathfrak{A}, \mathfrak{A}_0)$ the set of $\mathcal{M}$-regular *-representations of $(\mathfrak{A}, \mathfrak{A}_0)$. If $\mathcal{M} = \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$, we denote it simply by $\mathsf{Rep}^{r}(\mathfrak{A}, \mathfrak{A}_0)$.

Remark 4.11 If $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ is balanced, then for every $\varphi \in \mathcal{M}$, the *-representation π°_φ is $\mathcal{M}$-regular; see [6, Proposition 2.4.16].

Proposition 4.12 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit* e *and let $a \in \mathfrak{A}$.*

*If $\pi(a) \geq 0$ for every *-representation π of $(\mathfrak{A}, \mathfrak{A}_0)$, then $a \in \mathcal{K}_{\mathcal{I}}$. Conversely, if $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ and $a \in \mathcal{K}_{\mathcal{M}}$, then $\pi(a) \geq 0$ for every $\mathcal{M}$-regular *-representation π of $(\mathfrak{A}, \mathfrak{A}_0)$.*

Proof If $\pi(a) \geq 0$ for every *-representation π of $(\mathfrak{A}, \mathfrak{A}_0)$, then for every $\varphi \in \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ and every $x \in \mathfrak{A}_0$,

$$\varphi(ax, x) = \langle \pi_\varphi(a)\lambda_\varphi(x) | \lambda_\varphi(x)\rangle \geq 0.$$

Hence, $a \in \mathcal{K}_{\mathcal{I}}$.
Conversely, let π be a $\mathcal{M}$-regular *-representation. Then, for every $\xi \in \mathcal{D}_\pi$, the vector form $\varphi_\xi(a, b) = \langle \pi(a)\xi | \pi(b)\xi\rangle$, with $a, b \in \mathfrak{A}$, belongs to $\mathcal{M}$. Thus, from $a \in \mathcal{K}_{\mathcal{M}}$, it follows that $\varphi_\xi(a, \mathsf{e}) = \langle \pi(a)\xi | \xi\rangle \geq 0$, for every $\xi \in \mathcal{D}_\pi$. Hence, $\pi(a) \geq 0$. □

Remark 4.13 The first implication is also true if we consider only the GNS representations constructed from the forms $\varphi \in \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$; if $\varphi \in \mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ and every $\pi_\varphi(a) \geq 0$, then $a \in \mathcal{K}_{\mathcal{M}}$.

Proposition 4.14 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit* e *and with sufficient $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$. Then*

(i) *if $a \in \mathfrak{A}^{\mathcal{M}}_{\mathrm{b}}$, then $\pi(a)$ is a bounded operator for every $\pi \in \mathsf{Rep}^{r,\mathcal{M}}(\mathfrak{A}, \mathfrak{A}_0)$ and $\|\pi(a)\| \leq \|a\|^{\mathcal{M}}_{\mathrm{b}}$;*
(ii) *if $\pi(a)$ is a bounded operator for every $\pi \in \mathsf{Rep}^{r,\mathcal{M}}(\mathfrak{A}, \mathfrak{A}_0)$ and $\sup\{\|\pi(a)\|; \pi \in \mathsf{Rep}^{r,\mathcal{M}}(\mathfrak{A}, \mathfrak{A}_0)\} < \infty$, then $a \in \mathfrak{A}^{\mathcal{M}}_{\mathrm{b}}$.*

Proof (i) By Theorem 4.8, $a \in \mathfrak{A}^{\mathcal{M}}_{\mathrm{b}}$ implies $\varphi(ax, ax)^{1/2} \leq \|a\|^{\mathcal{M}}_{\mathrm{b}}\varphi(x, x)^{1/2}$, $\forall\varphi \in \mathcal{M}$; $x \in \mathfrak{A}_0$. If π is $\mathcal{M}$-regular, for every $\xi \in \mathcal{D}_\pi$, $\varphi_\xi \in \mathcal{M}$ where $\varphi_\xi(a, b) = \langle \pi(a)\xi | \pi(b)\xi\rangle$. Then

$$\|\pi(ax)\xi\| = \varphi_\xi(ax, ax)^{1/2} \leq \|a\|^{\mathcal{M}}_{\mathrm{b}}\varphi_\xi(x, x)^{1/2} = \|a\|^{\mathcal{M}}_{\mathrm{b}}\|\pi(x)\xi\|.$$

The quasi *-algebra is supposed to be unital and also $\pi(\mathsf{e}) = I_{\mathcal{D}_\pi}$. Then

$$\|\pi(a)\xi\| \leq \|a\|^{\mathcal{M}}_{\mathrm{b}}\|\xi\|, \quad \forall\xi \in \mathcal{D}_\pi.$$

Hence, $\|\pi(a)\| \leq \|a\|^{\mathcal{M}}_{\mathrm{b}}$.
(ii) Put $\gamma := \sup\{\|\pi(a)\|; \pi \in \mathsf{Rep}^{r,\mathcal{M}}(\mathfrak{A}, \mathfrak{A}_0)\}$. By hypothesis

$$\|\pi^\circ_\varphi(a)\lambda_\varphi(x)\| \leq \gamma\|\lambda_\varphi(x)\|, \quad \forall\varphi \in \mathcal{M}, x \in \mathfrak{A}_0,$$

that is

$$\varphi(ax, ax) \le \gamma^2 \varphi(x, x), \quad \forall \varphi \in \mathcal{M}, x \in \mathfrak{A}_0,$$

and by Theorem 4.8, it is equivalent to say that $a \in \mathfrak{A}_b^{\mathcal{M}}$. □

Remark 4.15 Let $\varphi \in \mathcal{M}$ and denote, as before, by π_φ the corresponding GNS representation. If $a \in \mathfrak{A}_b^{\mathcal{M}}$, then $\pi_\varphi(a)$ is a bounded operator. Indeed, for every $x \in \mathfrak{A}_0$

$$\|\pi_\varphi(a)\lambda_\varphi(x)\|^2 = \varphi(ax, ax) \le (\|a\|_b^{\mathcal{M}})^2 \varphi(x, x) = (\|a\|_b^{\mathcal{M}})^2 \|\lambda_\varphi(x)\|^2,$$

regardless of whether π_φ is $\mathcal{M}$-regular or not.

The following additional condition will be used:

(C) if $a, b \in \mathfrak{A}_b^{\mathcal{M}}$, then there exists $c \in \mathfrak{A}$, such that $\pi_\varphi(a) \square \pi_\varphi(b) = \pi_\varphi(c)$, for every $\varphi \in \mathcal{M}$.

Remark 4.16 The uniqueness of c in condition (C) is guaranteed by the sufficiency of $\mathcal{M}$. Moreover, $c \in \mathfrak{A}_b^{\mathcal{M}}$, as we shall see in Theorem 4.17.

Theorem 4.17 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. Let $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ be sufficient and assume that condition (C) holds. Then, $\mathfrak{A}_b^{\mathcal{M}}$ is a normed *-algebra with the weak multiplication $\diamond$ and the norm $\|\cdot\|_b^{\mathcal{M}}$.*

Proof As we have seen until now, $\mathfrak{A}_b^{\mathcal{M}}$ is a normed space (the sufficiency of $\mathcal{M}$ guarantees that if $\|a\|_b^{\mathcal{M}} = 0$, then $a = 0$), such that if $a \in \mathfrak{A}_b^{\mathcal{M}}$, then a^* is $\mathcal{M}$-bounded and $\|a^*\|_b^{\mathcal{M}} = \|a\|_b^{\mathcal{M}}$ and, whenever $a \diamond b$ is well defined, the product $a \diamond b$ is $\mathcal{M}$-bounded and $\|a \diamond b\|_b^{\mathcal{M}} \le \|a\|_b^{\mathcal{M}} \|b\|_b^{\mathcal{M}}$. Now, if $\varphi \in \mathcal{M}$ and $a, b \in \mathfrak{A}_b^{\mathcal{M}}$, the operator $\pi_\varphi(a) \square \pi_\varphi(b)$ is well defined, since, by Remark 4.15, $\pi_\varphi(a)$ and $\pi_\varphi(b)$ are bounded operators; of course, $\pi_\varphi(a) \square \pi_\varphi(b)$ is also bounded. Thus, by (C), there exists a unique $c \in \mathfrak{A}_b^{\mathcal{M}}$, such that $\pi_\varphi(a) \square \pi_\varphi(b) = \pi_\varphi(c)$, for every $\varphi \in \mathcal{M}$. Hence, for all $\varphi \in \mathcal{M}$ and $x, y \in \mathfrak{A}_0$, we have

$$\begin{aligned}\varphi(bx, a^*y) &= \langle \pi_\varphi(b)\lambda_\varphi(x) | \pi_\varphi(a^*)\lambda_\varphi(y)\rangle = \langle \pi_\varphi(a) \square \pi_\varphi(b)\lambda_\varphi(x) | \lambda_\varphi(y)\rangle \\ &= \langle \pi_\varphi(c)\lambda_\varphi(x) | \lambda_\varphi(y)\rangle = \varphi(cx, y).\end{aligned}$$

Thus, $c = a \diamond b$ is well defined and is $\mathcal{M}$-bounded by Lemma 4.3. This completes the proof. □

Proposition 4.18 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit* e. *Let $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ be balanced and denote by $R^{\mathcal{M}}(\mathfrak{A})$ the intersection of the kernels of all $\mathcal{M}$-regular *-representations of $\mathfrak{A}$ on some Hilbert space. Then*

$$R^{\mathcal{M}}(\mathfrak{A}) = \{a \in \mathfrak{A} | \, \varphi(a, a) = 0, \; \forall \varphi \in \mathcal{M}\}.$$

Proof For every $\varphi \in \mathcal{M}$, the GNS representation π°_φ is $\mathcal{M}$-regular, then, if $a \in R^{\mathcal{M}}(\mathfrak{A})$, it is $\pi^\circ_\varphi(a) = 0$, and hence, $\varphi(a, a) = \|\pi^\circ_\varphi(a)\xi_\varphi\| = 0$.
Conversely, if $a \in \mathfrak{A}$ is such that $\varphi(a, a) = 0$, $\forall \varphi \in \mathcal{M}$, since $\mathcal{M}$ is balanced, it is $\varphi^x(a, a) = 0$ for all $x \in \mathfrak{A}_0$ and for all $\varphi \in \mathcal{M}$, and hence

$$\varphi^x(a, a) = \varphi(ax, ax) = \|\pi^\circ_\varphi(a)\lambda_\varphi(x)\|^2 = 0,$$

and by the density of $\lambda_\varphi(\mathfrak{A}_0)$ in $\mathcal{H}_\varphi$, this implies that $\pi^\circ_\varphi(a) = 0$. Now, let π be a $\mathcal{M}$-regular *-representation of $\mathfrak{A}$, then for every $\xi \in \mathcal{D}_\pi$, the form $\varphi_\xi \in \mathcal{M}$ and by what we have seen before it is $\pi^\circ_{\varphi_\xi}(a) = 0$. For every $\xi \in \mathcal{D}_\pi$, there exists a cyclic vector η for the GNS representation $\pi^\circ_{\varphi_\xi}$, such that

$$\|\pi(a)\xi\|^2 = \|\pi^\circ_{\varphi_\xi}(a)\eta\|^2 = 0;$$

this implies that $\pi(a) = 0$. By the arbitrariness of the $\mathcal{M}$-regular *-representation π of $\mathfrak{A}$, it follows that $a \in R^{\mathcal{M}}(\mathfrak{A})$. This concludes the proof. □

Remark 4.19 The set $R^{\mathcal{M}}(\mathfrak{A})$ is clearly a sort of *-radical; however, its nature is purely algebraic here.

5 Topologies defined by families of sesquilinear forms

The properties we have discussed in the previous section are all of pure algebraic nature; but families of sesquilinear forms of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ can be used to define, in rather natural way, topologies on $(\mathfrak{A}, \mathfrak{A}_0)$. Our next goal is in fact to define in $\mathfrak{A}$ topologies that mimick the uniform topologies of families of operators.

Throughout this section, we will suppose that $(\mathfrak{A}, \mathfrak{A}_0)$ is a quasi *-algebra with unit e and that $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ is a sufficient set of forms. Then, $\mathcal{M}$ defines the topologies $\tau^{\mathcal{M}}_w, \tau^{\mathcal{M}}_s, \tau^{\mathcal{M}}_{s^*}$ generated, respectively, by the following families of seminorms:

$\tau^{\mathcal{M}}_w$: $\quad a \mapsto |\varphi(ax, y)|, \quad a \in \mathfrak{A},\ \varphi \in \mathcal{M},\ x, y \in \mathfrak{A}_0;$
$\tau^{\mathcal{M}}_s$: $\quad a \mapsto \varphi(a, a)^{1/2}, \quad a \in \mathfrak{A},\ \varphi \in \mathcal{M};$
$\tau^{\mathcal{M}}_{s^*}$: $\quad a \mapsto \max\left\{\varphi(a, a)^{1/2}, \varphi(a^*, a^*)^{1/2}\right\}, \quad a \in \mathfrak{A},\ \varphi \in \mathcal{M}.$

Definition 5.1 Let $\mathcal{F}$ be a subset of $\mathcal{M}$. We say that $\mathcal{F}$ is *bounded* if

$$\sup_{\varphi \in \mathcal{F}} \varphi(a, a) < \infty, \quad \forall a \in \mathfrak{A}.$$

The family $\mathfrak{F}$ of bounded subsets of forms in $\mathcal{M}$ has the following properties:

(a) $\bigcap_{n \in \mathbb{N}} \mathcal{F}_n \in \mathfrak{F}, \quad \mathcal{F}_n \in \mathfrak{F};$
(b) $\mathcal{F} \cup \mathcal{G} \in \mathfrak{F}, \quad \mathcal{F}, \mathcal{G} \in \mathfrak{F}.$

If $\mathcal{F} \in \mathfrak{F}$, we put

$$p^{\mathcal{F}}(a) := \sup_{\varphi \in \mathcal{F}} \varphi(a, a)^{1/2}, \quad a \in \mathfrak{A}.$$

Lemma 5.2 *Let $\mathcal{F} \in \mathfrak{F}$. Then*

(a) $p^{\mathcal{F}}$ *is a seminorm on* $\mathfrak{A}$*;*
(b) *the set* $\mathcal{F}^x = \{\varphi^x, \varphi \in \mathcal{F}\}$, $x \in \mathfrak{A}_0$, *is bounded;*
(c) *for every* $x \in \mathfrak{A}_0$,

$$p^{\mathcal{F}}(ax) = p^{\mathcal{F}^x}(a), \quad \forall a \in \mathfrak{A}.$$

Proof As for (c), we have

$$p^{\mathcal{F}}(ax) = \sup_{\varphi \in \mathcal{F}} \varphi(ax, ax)^{1/2} = \sup_{\varphi \in \mathcal{F}} \varphi^x(a, a)^{1/2} = \sup_{\psi \in \mathcal{F}^x} \psi(a, a)^{1/2} = p^{\mathcal{F}^x}(a).$$

□

Since $\mathcal{M}$ is sufficient, then $\{p^{\mathcal{F}}; \mathcal{F} \in \mathfrak{F}\}$ is a separating family of seminorms; thus, it defines on $\mathfrak{A}$ a Hausdorff locally convex topology which we denote by $\tau^{\mathfrak{F}}$.

Let us assume that $(\mathfrak{A}, \mathfrak{A}_0)$ has a unit e. If $\mathcal{F} \in \mathfrak{F}$, we define

$$p_{\mathcal{F}}(a) := \sup_{\varphi \in \mathcal{F}} |\varphi(a, \mathsf{e})|, \quad a \in \mathfrak{A}.$$

Then

$$p_{\mathcal{F}}(a) \leq \gamma_{\mathcal{F}} p^{\mathcal{F}}(a), \quad \forall a \in \mathfrak{A},$$

with $\gamma_{\mathcal{F}} = \sup_{\varphi \in \mathcal{F}} \varphi(\mathsf{e}, \mathsf{e})^{1/2}$, and the following holds:

$$p_{\mathcal{F}}(a^*) = p_{\mathcal{F}}(a), \quad \forall a \in \mathfrak{A};$$
$$p_{\mathcal{F}}(ax) \leq p^{\mathcal{F}}(x) p^{\mathcal{F}}(a^*), \quad \forall a \in \mathfrak{A}, x \in \mathfrak{A}_0;$$
$$p_{\mathcal{F}}(a^* \diamond a) = p^{\mathcal{F}}(a)^2, \ \forall a \in \mathfrak{A}, \text{ such that } a^* \diamond a \text{ is well defined.}$$

By $\tau_{\mathfrak{F}}$, we will denote the locally convex topology on $\mathfrak{A}$ generated by the family of seminorms $\{p_{\mathcal{F}}; \mathcal{F} \in \mathfrak{F}\}$ (to simplify notations, we do not mention explicitly the dependence on $\mathcal{M}$). Note that $\tau_{\mathfrak{F}}$ need not be Hausdorff, in general.

Remark 5.3 We notice that if $a^* \diamond a$ is well defined and $a^* \diamond a = 0$, then $p^{\mathcal{F}}(a) = 0$ for every bounded set $\mathcal{F} \in \mathfrak{F}$ and, therefore, $a = 0$.

Proposition 5.4 *Let $\mathcal{M}$ be sufficient and suppose that $\tau_{\mathfrak{F}} = \tau^{\mathfrak{F}}$. Then, $\mathfrak{A}[\tau^{\mathfrak{F}}]$ is a locally convex space with the following properties:*

(i) *the involution $a \mapsto a^*$ is continuous;*
(ii) *for every bounded set $\mathcal{F} \in \mathfrak{F}$, there exists a bounded set $\mathcal{G} \in \mathfrak{F}$*

$$p_{\mathcal{F}}(ax) \leq p^{\mathcal{G}}(a) p^{\mathcal{G}}(x), \quad \forall a \in \mathfrak{A}, x \in \mathfrak{A}_0;$$

which implies that the left and right multiplications are jointly continuous.

In particular, if $\mathfrak{A}_0$ *is* $\tau^{\mathfrak{F}}$*-dense in* $\mathfrak{A}$*, then* $(\mathfrak{A}[\tau^{\mathfrak{F}}], \mathfrak{A}_0)$ *is a locally convex quasi *-algebra.*

In general, the involution $a \mapsto a^*$ in not continuous for $\tau^{\mathfrak{F}}$. To circumvent this problem, we define the topology $\tau_*^{\mathfrak{F}}$ generated by the family of seminorms

$$p_*^{\mathcal{F}}(a) = \max \left\{ p^{\mathcal{F}}(a), p^{\mathcal{F}}(a^*) \right\}, \quad a \in \mathfrak{A}, \mathcal{F} \in \mathfrak{F}.$$

Clearly, $\tau_{\mathfrak{F}} \preceq \tau^{\mathfrak{F}} \preceq \tau_*^{\mathfrak{F}}$, and if $\tau_{\mathfrak{F}} = \tau^{\mathfrak{F}}$, then $\tau_{\mathfrak{F}} = \tau^{\mathfrak{F}} = \tau_*^{\mathfrak{F}}$.

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and suppose that the set $\mathcal{M}$ is sufficient. It is clear that every $\varphi \in \mathcal{M}$ is automatically continuous for $\tau_s^{\mathcal{M}}$ and for any finer topology such as $\tau_{s^*}^{\mathcal{M}}$, $\tau_*^{\mathfrak{F}}$ or $\tau^{\mathfrak{F}}$.

Our next goal is to investigate the properties of $(\mathfrak{A}, \mathfrak{A}_0)$ when $\mathfrak{A}$ is endowed with one of the topologies defined by the family $\mathcal{M}$ defined above.

We could wonder whether $(\mathfrak{A}[\tau_*^{\mathfrak{F}}], \mathfrak{A}_0)$ is a locally convex quasi *-algebra. The left and right multiplications by an element $x \in \mathfrak{A}_0$ are continuous if we make an additional assumption: let us suppose that for every $x \in \mathfrak{A}_0$, there exists $\gamma_x > 0$, such that

$$\varphi(xa, xa) \leq \gamma_x \varphi(a, a), \quad \forall \varphi \in \mathcal{M}, \forall a \in \mathfrak{A}. \tag{5.1}$$

By (5.1), it follows that every $x \in \mathfrak{A}_0$ is $\mathcal{M}$-bounded and for every bounded subset $\mathcal{F} \subset \mathcal{M}$

$$p^{\mathcal{F}}(xa) \leq \gamma_x^{1/2} p^{\mathcal{F}}(a), \quad \forall a \in \mathfrak{A}.$$

This inequality, together with (c) of Lemma 5.2 and the continuity of the involution, implies that, for every $x \in \mathfrak{A}_0$, the maps $a \mapsto ax$, $a \mapsto xa$ are $\tau_*^{\mathfrak{F}}$-continuous. The *-algebra $\mathfrak{A}_0$ is not $\tau_*^{\mathfrak{F}}$-dense in $\mathfrak{A}$ in general, and hence, to get a locally convex quasi *-algebra with topology $\tau_*^{\mathfrak{F}}$, we could take as $\mathfrak{A}$ the completion $\widetilde{\mathfrak{A}_0}^{\tau_*^{\mathfrak{F}}}$. Now, we prove the following.

Lemma 5.5 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a quasi *-algebra. Assume that* $\mathcal{M}$ *is sufficient and directed upward w.r.to the order*

$$\varphi \leq \psi \;\Leftrightarrow\; \varphi(a, a) \leq \psi(a, a), \quad \forall a \in \mathfrak{A}.$$

Then, $\mathfrak{A}_0$ *is dense in* $\mathfrak{A}[\tau_s^{\mathcal{M}}]$.

Proof Let us begin with proving that given $a \in \mathfrak{A}$, we can find a net $\{x_\alpha\} \subset \mathfrak{A}_0$, such that $\varphi(x_\alpha - a, x_\alpha - a) \to 0$ for every $\varphi \in \mathcal{M}$. We put $\varphi[a] := \varphi(a, a)$, $\varphi \in \mathcal{M}$, $a \in \mathfrak{A}$.

Since $\varphi \in \mathcal{M}$, $\lambda_\varphi(\mathfrak{A}_0)$ is dense in $\mathcal{H}_\varphi$ (with $\mathcal{H}_\varphi$ defined as in Proposition 2.4). This implies that, for every $a \in \mathfrak{A}$, there exists a sequence $\{x_n^\varphi\}$, such that $\lambda_\varphi(x_n^\varphi - a) \to 0$

or, equivalently $\varphi[x_n^\varphi - a] \to 0$. Then

$$\forall n \in \mathbb{N}, \quad \exists n_\varphi \in \mathbb{N}: \ \varphi[x_{n_\varphi}^\varphi - a] < \frac{1}{n}.$$

If $\varphi, \psi \in \mathcal{M}$, we define $(\varphi, n_\varphi) \leq (\psi, n_\psi)$ if $\varphi \leq \psi$ and $n_\varphi \leq n_\psi$. Since $\mathcal{M}$ is directed, $\{(\varphi, n_\varphi)\}$ is directed and $\{x_{(\varphi,n_\varphi)}\}$ is a net, with $x_{(\varphi,n_\varphi)} := x_{n_\varphi}^\varphi$. We prove that, for every $\psi \in \mathcal{M}$ $\psi[x_{n_\varphi}^\varphi - a] \to 0$. Indeed, let $\epsilon > 0$ and $n \in \mathbb{N}$, such that $\frac{1}{n} < \epsilon$. Then, if $(\varphi, n_\varphi) \geq (\psi, n_\psi)$

$$\psi[x_{n_\varphi}^\varphi - a] \leq \varphi[x_{n_\varphi}^\varphi - a] < \frac{1}{n} < \epsilon.$$

This proves that $\mathfrak{A}_0$ is dense in $\mathfrak{A}[\tau_s^{\mathcal{M}}]$. □

The representation π_φ° is $(\tau^{\mathfrak{F}}, t_s)$-continuous. Indeed, if $\mathcal{F}$ is any bounded subset of $\mathcal{M}$ containing φ,

$$\|\pi_\varphi^\circ(a)\lambda_\varphi(x)\| = \varphi(ax, ax)^{1/2} \leq p^{\mathcal{F}}(ax) \leq p^{\mathcal{F}^x}(a), \quad \forall a \in \mathfrak{A};\ x \in \mathfrak{A}_0$$

as in Lemma 5.2.

Proposition 5.6 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with sufficient $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$. If $\mathfrak{A}$ is $\tau_{s^*}^{\mathcal{M}}$-complete, then $\mathfrak{A}$ is also $\tau_*^{\mathfrak{F}}$-complete.*

Proof Let $\{a_\alpha\}$ be a $\tau_*^{\mathfrak{F}}$-Cauchy net. Since $\tau_{s^*}^{\mathcal{M}} \preceq \tau_*^{\mathfrak{F}}$, there exists $a \in \mathfrak{A}$, such that $a = \tau_{s^*}^{\mathcal{M}} - \lim_\alpha a_\alpha$. From the Cauchy condition, for every $\epsilon > 0$ and every bounded set $\mathcal{F}$, there exists $\overline{\alpha}$, such that

$$\max\{\varphi(a_\alpha - a_{\alpha'}, a_\alpha - a_{\alpha'}), \varphi(a_\alpha^* - a_{\alpha'}^*, a_\alpha^* - a_{\alpha'}^*)\} < \epsilon,\ \forall \varphi \in \mathcal{F}, \alpha, \alpha' > \overline{\alpha}.$$

Then, taking limit over α'

$$\max\{\varphi(a_\alpha - a, a_\alpha - a), \varphi(a_\alpha^* - a^*, a_\alpha^* - a^*)\} \leq \epsilon,\ \forall \varphi \in \mathcal{F}, \alpha > \overline{\alpha}.$$

Therefore, $\mathfrak{A}$ is $\tau_*^{\mathfrak{F}}$-complete. □

Theorem 5.7 *Let $\mathcal{M}$ be sufficient and let property (C) hold too. If $\mathfrak{A}$ is $\tau_{s^*}^{\mathcal{M}}$-complete, then $\mathfrak{A}_{\mathrm{b}}^{\mathcal{M}}$ is a C*-algebra with the weak multiplication $\diamond$ and the norm $\|\cdot\|_{\mathrm{b}}^{\mathcal{M}}$.*

Proof By Theorem 4.17 and Remark 4.5, we only need to prove the completeness of $\mathfrak{A}_{\mathrm{b}}^{\mathcal{M}}$. Let $\{a_n\} \subset \mathfrak{A}_{\mathrm{b}}^{\mathcal{M}}$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathrm{b}}^{\mathcal{M}}$. Then, $\{a_n^*\}$ is $\|\cdot\|_{\mathrm{b}}^{\mathcal{M}}$-Cauchy too. By (4.2),

$$\varphi((a_n - a_m)x, (a_n - a_m)x) \leq (\|a_n - a_m\|_{\mathrm{b}}^{\mathcal{M}})^2 \varphi(x, x),\ \forall \varphi \in \mathcal{M}, \forall x \in \mathfrak{A}_0$$

and

$$\varphi((a_n^* - a_m^*)x, (a_n^* - a_m^*)x) \le (\|a_n^* - a_m^*\|_{\mathrm{b}}^{\mathcal{M}})^2 \varphi(x, x), \ \forall \varphi \in \mathcal{M}, \forall x \in \mathfrak{A}_0.$$

Therefore, both $\varphi((a_n - a_m)x, (a_n - a_m)x) \to 0$ and $\varphi((a_n^* - a_m^*)x, (a_n^* - a_m^*)x) \to 0$, as $n, m \to \infty$.

This is, in particular, true when $x = \mathsf{e}$; hence, $\{a_n\}$ is also Cauchy with respect to $\tau_{s^*}^{\mathcal{M}}$ and since $\mathfrak{A}$ is $\tau_{s^*}^{\mathcal{M}}$-complete, there exists $a \in \mathfrak{A}$, such that $a_n \xrightarrow{\tau_{s^*}^{\mathcal{M}}} a$. The limit $a \in \mathfrak{A}_{\mathrm{b}}^{\mathcal{M}}$; indeed, for every $\varphi \in \mathcal{M}$ and $x \in \mathfrak{A}_0$

$$\begin{aligned} |\varphi(ax, x)|^2 &\le \varphi(ax, ax)\varphi(x, x) = \varphi(x, x) \limsup_{n\to\infty} \varphi(a_n x, a_n x) \\ &\le \sup_{n\in\mathbb{N}} (\|a_n\|_{\mathrm{b}}^{\mathcal{M}})^2 \varphi(x, x)^2. \end{aligned}$$

Since $\{a_n\}$ is Cauchy with respect to the norm $\|\cdot\|_{\mathrm{b}}^{\mathcal{M}}$, for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$, such that $\|a_n - a_m\|_{\mathrm{b}}^{\mathcal{M}} < \epsilon^{1/2}$, for all $n, m > n_\epsilon$. This implies that $\varphi((a_n - a_m)x, (a_n - a_m)x) < \epsilon\varphi(x, x)$, $\forall \varphi \in \mathcal{M}, \forall x \in \mathfrak{A}_0$, forall $n, m > n_\epsilon$. Then, if we fix $n > n_\epsilon$ and let $m \to \infty$, we obtain $\varphi((a_n - a)x, (a_n - a)x) \le \epsilon\varphi(x, x)$, $\forall \varphi \in \mathcal{M}, \forall x \in \mathfrak{A}_0$. This implies that $\mathfrak{A}_{\mathrm{b}}^{\mathcal{M}}$ is complete with respect to the norm $\|\cdot\|_{\mathrm{b}}^{\mathcal{M}}$. □

To conclude, let us suppose that $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ is balanced. We pose the question: *under what conditions is $\mathcal{M}$ also sufficient?* Let us consider a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ and choose $\mathcal{M} = \mathcal{P}_{\mathfrak{A}_0}^{\tau}(\mathfrak{A})$. This set is certainly balanced, but it is not necessarily sufficient. This property can be characterized (by negation) by the following.

Proposition 5.8 *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi *-algebra with unit* e. *For an element $a \in \mathfrak{A}$, the following statements are equivalent:*

(i) $a \in \operatorname{Ker} \pi$ *for every strongly continuous (i.e., t_s-continuous) qu*-representation π of $(\mathfrak{A}[\tau], \mathfrak{A}_0)$;*
(ii) $\varphi(a, a) = 0$, *for every $\varphi \in \mathcal{P}_{\mathfrak{A}_0}^{\tau}(\mathfrak{A})$;*
(iii) $p^{\mathcal{F}}(a) = 0$, *for every bounded subset $\mathcal{F}$ of $\mathcal{P}_{\mathfrak{A}_0}^{\tau}(\mathfrak{A})$.*

6 Locally convex quasi GA*-algebras

The discussion of the previous sections suggests the following definition (which strengthen an analogous one for partial *-algebras [3, Definition 4.26]).

Definition 6.1 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. Let $\mathcal{M}$ be a family of forms of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$. We say that $\mathcal{M}$ is *strongly well behaved* if

(wb_1) $\mathcal{M}$ is sufficient;
(wb_2) every $x \in \mathfrak{A}_0$ is $\mathcal{M}$-bounded;
(wb_3) condition (C) holds;

(wb$_4$) $\mathfrak{A}$ is $\tau_*^{\mathfrak{F}}$-complete.

Definition 6.2 Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi *-algebra. We say that $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is a *locally convex quasi GA*-algebra* if there exists $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ which is strongly well behaved, and τ and $\tau_*^{\mathfrak{F}}$ are equivalent (in symbols $\tau \approx \tau_*^{\mathfrak{F}}$).

Example 6.3 Let us consider the quasi *-algebra $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \mathcal{L}^\dagger(\mathcal{D})_b)$ of Sect. 2. Assume that $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is endowed with the topology $\mathfrak{t}_*^u$ and denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_u$ the $\mathfrak{t}_*^u$-closure of $\mathcal{L}^\dagger(\mathcal{D})_b$ in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, then $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_u[\mathfrak{t}_*^u], \mathcal{L}^\dagger(\mathcal{D})_b)$ is a locally convex quasi *-algebra. Let us take as $\mathcal{M}$ the space consisting of the restrictions to $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_u$ of the $\mathfrak{t}_*^u$-continuous ips-forms on $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \mathcal{L}^\dagger(\mathcal{D})_b)$. We will see that $\mathcal{M}$ is strongly well behaved and $\mathfrak{t}_*^u \approx \tau_*^{\mathfrak{F}}$: this makes of $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_u[\mathfrak{t}_*^u], \mathcal{L}^\dagger(\mathcal{D})_b)$ a locally convex quasi GA*-algebra. Due to the $\mathfrak{t}_*^u$-density of $\mathcal{L}^\dagger(\mathcal{D})_b$ in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_u$, we can identify $\mathcal{M}$ with the space of all $\mathfrak{t}_*^u$-continuous ips-forms on $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_u, \mathcal{L}^\dagger(\mathcal{D})_b)$. This implies [3, Theorem 3.10] that every $\psi \in \mathcal{M}$ can be written as follows: $\psi(A, B) = \sum_{i=1}^n \langle A\xi_i | B\xi_i \rangle$ $A, B \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_u$, for some vectors $\xi_1, ..., \xi_n \in \mathcal{D}$. Hence, the set of $\mathcal{M}$-bounded elements coincides with the set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_b$ of all bounded operators of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, which can be identified with the C*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$.

These facts allow us to conclude easily that $\mathcal{M}$ is strongly well behaved. In particular, we notice that (wb$_3$) holds, since if $A, B \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})_b$, then the multiplication $\diamond$ (see (2.2)) is well defined and coincides with the weak multiplication $\Box$ of operators (see (2.1)): $A \diamond B = A \Box B$ is certainly well defined; then if $\varphi \in \mathcal{M}$, we have $\pi_\varphi(A) \Box \pi_\varphi(B) = \pi_\varphi(A \Box B) = \pi_\varphi(A \diamond B)$, by definition of *-representation.

Example 6.4 Let K denote a compact subset of the real line with $m(K) > 0$, where m denotes the Lebesgue measure. Then, the pair $(L^p(K, m), C(K))$, where $C(K)$ denotes the C*-algebra of continuous functions on K, is a Banach quasi *-algebra. Let $\mathcal{M}$ be the space of all jointly continuous ips-forms on $(L^p(K, m), C(K))$. Then, as shown in [6, Example 3.1.44], if $p \geq 2$, $\mathcal{M}$ can be completely described by functions of $L^s(K, m)$, where $s = \frac{p}{p-2}$ $(\frac{1}{0} = \infty)$, in the following sense:

$$\varphi \in \mathcal{M} \Leftrightarrow \exists w \in L^s(K, m),\ w \geq 0 :\ \varphi(f, g) = \int_K f \overline{g} w \mathrm{d}m, \quad \forall f, g \in L^p(K, m).$$

For this reason, we identify $\mathcal{M}$ with $L^s(K, m)$. With this in mind

(a) a subset $\mathcal{F}$ of $\mathcal{M}$ is bounded, if and only if it is contained in a ball centered at 0 in $L^s(K, m)$;

(b) the topology $\tau^{\mathfrak{F}}$ (which equals $\tau_*^{\mathfrak{F}}$, in this case) is normed and the norm coincides with $\| \cdot \|_p$, since

$$\sup_{\|w\|_s = 1} \int_K |f|^2 w \mathrm{d}m = \| |f|^2 \|_{p/2} = \|f\|_p^2;$$

(c) the topology $\tau_{\mathfrak{F}}$ is also a norm topology and the norm coincides with $\| \cdot \|_{p/2}$;

(d) the set of $\mathcal{M}$-bounded elements is the C*-algebra $L^\infty(K, m)$.

In conclusion, $(L^p(K, m), L^\infty(K, m))$ is a Banach quasi GA*-algebra.

Example 6.5 The space $L^p_{\text{loc}}(\mathbb{R}, m)$ of all (classes of) measurable functions on $\mathbb{R}$, such that the restriction $f_{\upharpoonright K}$ of f to K is in $L^p(K, m)$, for every compact subset $K \subset \mathbb{R}$, behaves similarly to the case discussed in Example 6.4. The main difference consists, of course, in the fact that we will not deal with norm topologies. More precisely, let us consider the pair $(L^p_{\text{loc}}(\mathbb{R}, m), C_b(\mathbb{R}))$ (where $C_b(\mathbb{R})$ denotes the continuous bounded functions on $\mathbb{R}$), which is, as it is easy to check, a quasi *-algebra. The natural topology τ_p of $L^p_{\text{loc}}(\mathbb{R}, m)$ is then defined as the inductive limit of the norm topologies of the spaces $L^p(K)$, when K runs in the family of compact subsets of $\mathbb{R}$.

Let $\mathcal{M}$ denote the space of all ips-forms on $(L^p_{\text{loc}}(\mathbb{R}, m), C_b(\mathbb{R}))$ whose restriction to $L^p(K, m)$ is continuous for every compact subset $K \subset \mathbb{R}$. Then, if $p \geq 2$, one can easily prove that $\mathcal{M}$ can be described by functions of $L^s_{\text{loc}}(\mathbb{R}, m)$ where, as before, $s = \frac{p}{p-2}$ (again, $\frac{1}{0} = \infty$). It is easily seen that $\mathcal{M}$ is strongly well behaved. In this case, the set of $\mathcal{M}$-bounded elements is the C*-algebra $L^\infty(\mathbb{R}, m)$. The pair $(L^p_{\text{loc}}(\mathbb{R}, m), L^\infty(\mathbb{R}, m))$ is a locally convex quasi GA*-algebra.

The following theorem motivates in our opinion the attention devoted to locally convex quasi GA*-algebras.

Theorem 6.6 *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi GA*-algebras with unit and a well behaved $\mathcal{M} \subset \mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$. Then:*

(a) every $\varphi \in \mathcal{M}$ is jointly τ-continuous;
*(b) every $\mathcal{M}$-regular *-representation of $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is (τ, t_{s^*})-continuous;*
(c) the set $\mathfrak{A}^{\mathcal{M}}_{\mathrm{b}}$ of bounded elements is a C-algebra with respect to the norm $\|\cdot\|^{\mathcal{M}}_{\mathrm{b}}$.*

Proof (a): Each φ is $\tau^{\mathfrak{F}}_*$-continuous by the construction itself of $\tau^{\mathfrak{F}}_*$; the statement then follows from the assumption $\tau \approx \tau^{\mathfrak{F}}_*$.
(b): This follows from (a). Indeed, if π is $\mathcal{M}$-regular, then for every $\xi \in \mathcal{D}_\pi$, the sesquilinear form φ_ξ (see (4.3)) is in $\mathcal{M}$; then, it is $\tau^{\mathfrak{F}}_*$-continuous. Then, there exists $\mathcal{F} \in \mathfrak{F}$, such that

$$|\langle \pi(a)\xi | \pi(b)\xi \rangle| \leq p^{\mathcal{F}}_*(a) p^{\mathcal{F}}_*(b);$$

hence

$$\|\pi(a)\xi\| \leq p^{\mathcal{F}}_*(a), \quad \text{and} \quad \|\pi(a^*)\xi\| \leq p^{\mathcal{F}}_*(a^*) = p^{\mathcal{F}}_*(a), \quad \forall a \in \mathfrak{A},$$

then for every $\xi \in \mathcal{D}_\pi$, there exists $\mathcal{F} \in \mathfrak{F}$, such that

$$p^*_\xi(\pi(a)) = \max\{\|\pi(a)\xi\|, \|\pi(a)^\dagger \xi\|\} \leq p^{\mathcal{F}}_*(a).$$

(c): We have just to prove the completeness of the set $\mathfrak{A}^{\mathcal{M}}_{\mathrm{b}}$ with respect to the norm $\|\cdot\|^{\mathcal{M}}_{\mathrm{b}}$. Let $\{a_n\} \subset \mathfrak{A}^{\mathcal{M}}_{\mathrm{b}}$ be a $\|\cdot\|^{\mathcal{M}}_{\mathrm{b}}$-Cauchy sequence, then for every $\epsilon > 0$, there exists

$n_\epsilon \in \mathbb{N}$, such that for all $n, m \geq n_\epsilon$, it is both $\|a_n - a_m\|_b^{\mathcal{M}} < \epsilon$ and $\|a_n^* - a_m^*\|_b^{\mathcal{M}} < \epsilon$. Since $\{a_n\} \subset \mathfrak{A}_b^{\mathcal{M}}$, for every $\varphi \in \mathcal{M}$ and every $x_0 \in \mathfrak{A}_0$, it is

$$\varphi((a_n - a_m)x, (a_n - a_m)x) \leq (\|a_n - a_m\|_b^{\mathcal{M}})^2 \varphi(x, x), \ \forall n, m \in \mathbb{N}$$

and

$$\varphi((a_n^* - a_m^*)x, (a_n^* - a_m^*)x) \leq (\|a_n^* - a_m^*\|_b^{\mathcal{M}})^2 \varphi(x, x), \ \forall n, m \in \mathbb{N};$$

hence, if $\mathcal{F} \in \mathfrak{F}$:

$$\sup_{\varphi \in \mathcal{F}} \varphi((a_n - a_m)x, (a_n - a_m)x)^{1/2} \leq \|a_n - a_m\|_b^{\mathcal{M}} \sup_{\varphi \in \mathcal{F}} \varphi(x, x)^{1/2}, \ \forall n, m \in \mathbb{N}$$

and

$$\sup_{\varphi \in \mathcal{F}} \varphi((a_n^* - a_m^*)x, (a_n^* - a_m^*)x)^{1/2} \leq \|a_n^* - a_m^*\|_b^{\mathcal{M}} \sup_{\varphi \in \mathcal{F}} \varphi(x, x)^{1/2}, \ \forall n, m \in \mathbb{N}$$

by the previous inequalities, for every $\mathcal{F} \in \mathfrak{F}$, we get

$$p_*^{\mathcal{F}}(a_n - a_m) = \max \left\{ p^{\mathcal{F}}(a_n - a_m), \, p^{\mathcal{F}}((a_n - a_m)^*) \right\} < \epsilon p^{\mathcal{F}}(\mathsf{e}), \ \forall n, m \geq n_\epsilon.$$

Then, $\{a_n\}$ is a $\tau_*^{\mathfrak{F}}$-Cauchy sequence. Since $\mathfrak{A}$ is $\tau_*^{\mathfrak{F}}$-complete, there exists $a \in \mathfrak{A}$, such that $a_n \xrightarrow{\tau_*^{\mathfrak{F}}} a$.

The limit a is $\mathcal{M}$-bounded; indeed, if $\varphi \in \mathcal{M}$ and $x \in \mathfrak{A}_0$, we have

$$\varphi(ax, ax) = \lim_{n \to \infty} \varphi(a_n x, a_n x) \leq \limsup_{n \to \infty} (\|a_n\|_b^{\mathcal{M}})^2 \varphi(x, x).$$

The sequence $\{\|a_n\|_b^{\mathcal{M}}\}$ is Cauchy too and bounded; therefore, a is $\mathcal{M}$-bounded. To prove that $\|a_n - a\|_b^{\mathcal{M}} \to 0$ as $n \to \infty$, it suffices to use the same arguments as in Theorem 5.7. □

7 Conclusion

In this paper we have constructed some topologies on a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ starting from a sufficiently rich family of sesquilinear forms that behave regularly. This study led us to introduce a new class of locally convex quasi *-algebras, that we have named GA*, since their definition closely recalls that one of A*-algebras. Several questions remain, however, still open. We mention some of them.

(a) When does a (locally convex) quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ possess a sufficient family $\mathcal{M}$ of sesquilinear forms of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$?

(b) Under what conditions is a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ a locally convex quasi GA*-algebra? We already know that there exist Banach quasi *-algebras $(\mathfrak{A}[\|\cdot\|], \mathfrak{A}_0)$ for which the set of continuous elements of $\mathcal{I}_{\mathfrak{A}_0}(\mathfrak{A})$ reduces to $\{0\}$ [6, Example 3.1.29] and the sesquilinear forms of a well-behaved family $\mathcal{M}$ of ips-forms are automatically continuous in a locally convex quasi GA*-algebra. Hence, in general, the two notions do not coincide.

(c) Under which conditions is it possible to lighten the definition of *well-behaved* family of ips-forms (Definition 6.1) by removing (wb_3) and/or (wb_4)?

We hope to discuss these problems in a future paper.

Acknowledgements This work has been done within the activities of Gruppo UMI Teoria della Approssimazione e Applicazioni and of GNAMPA of the INdAM.

Funding Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement.

Data Availability Not applicable.

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