Elementenmethode I

Uitwerkingen van vraagstukken

Studiemateriaal bij de P.a.t.o.-cursus

Methode der eindige elementen - fundamenten

Uitwerking Opgaven 3,4 en 9.

Voorjaar 1979

Vakgroep Technische Mechanica

Afdeling der Werktuigbouwkunde

Technische Hogeschool Eindhoven

Code: V.4 aanhangsel.

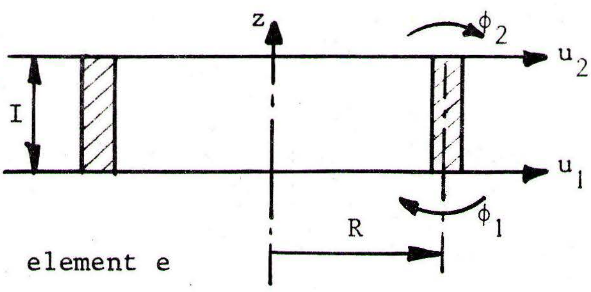
Uitwerking opgave 3

a. De bepaling van de stijfheidsmatrix

Berekening van de stijfheidsmatrix Q_e voor het pijpelement e

De stijfheidsmatrix is:

$$Q_e = \overset{\blacktriangledown}{A}{}_e^{-1} \, . \, K_e \, . \, A_e^{-1} \qquad (1)$$



We hebben dus de matrices A_e^{-1} en K_e nodig.
Voor het verplaatsingsveld $\hat{u}_e$ nemen we een derdegraads polynoom.

$$\text{Stel } \hat{u}_e = \alpha_0 + \alpha_1.z + \alpha_2.z^2 + \alpha_3.z^3 = \overset{\blacktriangledown}{\alpha}_e.\hat{\Phi}_e = \overset{\blacktriangledown}{\hat{\Phi}}_e.\alpha_e \qquad (2)$$

Hierin zijn α_e en $\hat{\Phi}_e$ de kolomvectoren gedefinieerd door:

$$\overset{\blacktriangledown}{\alpha}_e = (\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3) \; ; \; \overset{\blacktriangledown}{\hat{\Phi}}_e = (1 \quad z \quad z^2 \quad z^3) \qquad (3)$$

De onbekende grootheden α_0, α_1, α_2 en α_3 kunnen worden uitgedrukt in de verplaatsingen en verdraaiingen $u_e^{(1)}$, $\phi_e^{(1)}$, $u_e^{(2)}$ en $\phi_e^{(2)}$ aan de randen van het element e.

$$\text{Hierbij geldt: } \hat{\phi}_e = \frac{d\hat{u}_e}{dz} = \alpha_1 + z.\alpha_2.z + 3\alpha_3.z^2 \qquad (4)$$

De uit (3), (4) en (5) volgende vier vergelijkingen in matrixvorm geschreven zijn:

$$\begin{bmatrix} u_e^{(1)} \\ \phi_e^{(1)} \\ u_e^{(2)} \\ \phi_e^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & z_1 & z_1^2 & z_1^3 \\ 0 & 1 & 2z_1 & 3z_1^2 \\ 1 & z_2 & z_2^2 & z_2^3 \\ 0 & 1 & 2z_2 & 3z_2^2 \end{bmatrix} . \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \qquad (5)$$

$$u_e \quad = \quad A_e \quad \alpha_e$$

Omdat de matrix A_e regulier is kan α_e worden opgelost volgens

$$\alpha_e = A^{-1} \cdot u_e \qquad (6)$$

Omdat de stijfheidsmatrix voor ieder element identiek is mogen we voor ieder element $z_1 = 0$ en $z_2 = H = \frac{\ell}{ne}$ nemen. Hierin is H de hoogte van elk element bij ne elementen.

De matrix A_e wordt dan:

$$A_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & H & H^2 & H^3 \\ 0 & 1 & 2H & 3H^2 \end{bmatrix} \qquad (7)$$

Inverse van A_e

$$A_e^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3H^{-2} & -2H^{-1} & 3H^{-2} & -H^{-1} \\ 2H^{-3} & H^{-2} & -2H^{-3} & H^{-2} \end{bmatrix} \qquad (8)$$

Ter controle: $A_e \cdot A_e^{-1} = I$

De elastische energie voor element e kan geschreven worden als

$$U_e = \tfrac{1}{2} \overset{\blacktriangledown}{\alpha}_e K_e \alpha_e \qquad (9)$$

Met behulp van U_e kan matrix K_e bepaald worden

$$U_e = \frac{\pi RE}{(1-\nu^2)} \int_{z_1=0}^{z_2=H} \{\frac{1}{12} t^3 (u'')^2 + t(1-\nu^2)\left(\frac{u}{R}\right)^2\} dz \qquad (10)$$

Met $\hat{u}_e = \overset{\blacktriangledown}{\alpha}_e \hat{\Phi}_e$

$$K_e = \frac{2\pi RE}{1-\nu^2} \int_0^H \{\frac{1}{12} t^3 \frac{d^2\Phi_e}{dz_e^2} \frac{d^2\overset{\blacktriangledown}{\Phi}_e}{dz_e^2} + \frac{t}{R^2}(1-\nu^2)\Phi_e \cdot \overset{\blacktriangledown}{\Phi}_e\} dz_e \qquad (11)$$

Met $\overset{\blacktriangledown}{\Phi}_e = [1 \quad z \quad z^2 \quad z^3]$ en $\dfrac{d^2\overset{\blacktriangledown}{\Phi}}{dz^2_e} = [0 \quad 0 \quad 2 \quad 6z]$ en

substitutie van $C = \dfrac{t^2R^2}{(1-\nu^2)}$ geeft dit na integratie van (11):

$$K_e = \frac{2\pi Et}{R}\begin{bmatrix} H & \frac{1}{2}H^2 & \frac{1}{3}H^3 & \frac{1}{4}H^4 \\ \frac{1}{2}H^2 & \frac{1}{3}H^3 & \frac{1}{4}H^4 & \frac{1}{5}H^5 \\ \frac{1}{3}H^3 & \frac{1}{4}H^4 & \frac{1}{5}H^5+\frac{1}{3}CH & \frac{1}{6}H^6+\frac{1}{2}CH^2 \\ \frac{1}{4}H^4 & \frac{1}{5}H^5 & \frac{1}{6}H^6+\frac{1}{2}CH^2 & \frac{1}{7}H^7+CH^3 \end{bmatrix}$$

$$Q_e = \overset{\blacktriangledown}{A}{}_e^{-1} \cdot K_e \cdot A_e^{-1}$$

Uitgewerkt:

$$Q_e = \frac{2\pi Et}{R}\begin{bmatrix} \frac{13}{35}H + CH^{-3} & \frac{11}{120}H^2 + \frac{1}{2}CH^{-2} & \frac{9}{70}H - CH^{-3} & \frac{-13}{420}H^2 + \frac{1}{2}CH^{-2} \\ \frac{11}{210}H^2 + \frac{1}{2}CH^{-2} & \frac{1}{105}H^3 + \frac{1}{3}CH^{-1} & \frac{13}{420}H^2 - \frac{1}{2}CH^{-2} & \frac{-1}{140}H^3 + \frac{1}{6}CH^{-1} \\ \frac{9}{70}H - CH^{-3} & \frac{13}{420}H^2 - \frac{1}{2}CH^{-2} & \frac{13}{35}H + CH^{-3} & \frac{-11}{210}H^2 - \frac{1}{2}CH^{-2} \\ \frac{-13}{420}H^2 + \frac{1}{2}CH^2 & \frac{-1}{140}H^3 + \frac{1}{6}CH^{-1} & \frac{-11}{210}H^2 - \frac{1}{2}CH^{-2} & \frac{1}{105}H^3 + \frac{1}{3}CH^{-1} \end{bmatrix}$$

b. $$\Omega_{ce} = -\overset{\blacktriangledown}{u}_e\ f_{ce} = -\int_G \overset{\blacktriangledown}{\hat{u}}_e\ \hat{q}_e^0\ dG$$

en met $\hat{u}_e = \overset{\blacktriangledown}{\Phi}_e\ \alpha_e$ en $u_e = A_e\ \alpha_e$

$$\hat{u}_e = \overset{\blacktriangledown}{\hat{\Phi}}_e\ A_e^{-1}\ u_e \quad \text{dus}$$

$$\Omega_{ce} = -\int_G \overset{\blacktriangledown}{u}_e\ \overset{\blacktriangledown}{A}_e^{-1}\ \hat{\Phi}_e\ \hat{q}_e^0\ dG$$

$$f_{ce} = \overset{\blacktriangledown}{A}_e^{-1} \int_G \hat{\Phi}_e\ \hat{q}_e^0\ dG = \overset{\blacktriangledown}{A}_e^{-1} . 2\pi R \int_{z=0}^{H} \hat{\Phi}_e\ \hat{q}_e^0\ dz$$

$$\hat{\Phi}_e = \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{bmatrix} \qquad \hat{q}_e^0 = q_1 + \frac{z}{H}(q_2 - q_1)$$

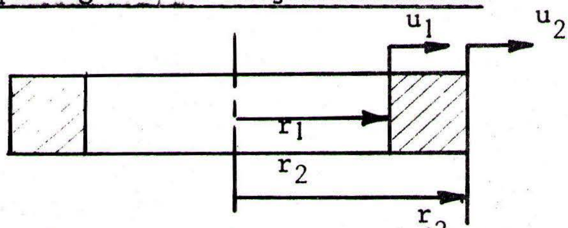
$$f_{ce} = 2\pi R \overset{\blacktriangledown}{A}_e^{-1} \int_{z=0}^{H} \{q_1 + \frac{z}{H}(q_2-q_1)\} \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{bmatrix} dz$$

$$= 2\pi R \overset{\blacktriangledown}{A}_e^{-1} \begin{bmatrix} q_1 H + \frac{H}{2}(q_2-q_1) \\ \frac{1}{2} q_1 H^2 + \frac{H^2}{3}(q_2-q_1) \\ \frac{1}{3} q_1 H^3 + \frac{H^3}{4}(q_2-q_1) \\ \frac{1}{4} q_1 H^4 + \frac{H^4}{5}(q_2-q_1) \end{bmatrix}$$

$$\mathbf{f_{ce}} = 2\pi R \begin{bmatrix} \frac{1}{2} q_1 H + \frac{3}{20} H (q_2-q_1) \\ \frac{1}{12} q_1 H^2 + \frac{1}{30} H^2 (q_2-q_1) \\ \frac{1}{2} q_1 H + \frac{7}{20} H (q_2-q_1) \\ -\frac{1}{12} q_1 H - \frac{1}{20} H^2 (q_2-q_1) \end{bmatrix}$$

Uitwerking opgave 4

b. Bepaling van de stijfheidsmatrix



$$u = C_1 + C_2 r$$

Elastische energie: $U_e = \frac{1}{2} h \int_{r_1}^{r_2} (\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta) 2\pi r \, dr$

$$\sigma_r = \frac{E}{1-\nu^2}\{\varepsilon_r + \nu\varepsilon_\theta\} \qquad \sigma_\theta = \frac{E}{1-\nu^2}\{\varepsilon_\theta + \nu\theta_r\}$$

$\rightarrow$

$$U_e = \frac{1}{2}\int_{r_1}^{r_2} \frac{Eh}{1-\nu^2}\{\varepsilon_r^2 + \varepsilon_\theta^2 + 2\nu\varepsilon_r\varepsilon_\theta\} 2\pi r \, dr$$

$$\varepsilon_r = u' \qquad \varepsilon_\theta = \frac{u}{r}$$

$\rightarrow$

$$U_e = \frac{1}{2}\frac{Eh}{1-\nu^2}\int_{r_1}^{r_2}\{u'^2 + \left(\frac{u}{r}\right)^2 + 2\nu u' \frac{u}{r}\} 2\pi r \, dr$$

$$u = C_1 + C_2 r = \overset{\blacktriangledown}{\Phi}_e \alpha_e \qquad \Phi_e = \begin{bmatrix} 1 \\ r \end{bmatrix} \qquad \alpha_e = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$\rightarrow$

$$U_e = \frac{1}{2}\frac{Eh}{1-\nu^2}\int_{r_1}^{r_2}\{(\overset{\blacktriangledown}{\hat{\Phi}}{}'_e \alpha_e)^2 + (\frac{1}{r}\overset{\blacktriangledown}{\hat{\Phi}}_e \alpha_e)^2 + 2\nu(\overset{\blacktriangledown}{\hat{\Phi}}_e \alpha_e)'(\frac{1}{r}\overset{\blacktriangledown}{\hat{\Phi}}_e \alpha_e)\} 2\pi r \, dr$$

$$= \frac{1}{2}\frac{Eh}{1-\nu^2}\int_{r_1}^{r_2}\{\overset{\blacktriangledown}{\alpha}_e \hat{\Phi}'_e \overset{\blacktriangledown}{\Phi}{}'_e \alpha_e + \overset{\blacktriangledown}{\alpha}_e \frac{1}{r^2}\hat{\Phi}_e \overset{\blacktriangledown}{\Phi}_e \alpha_e + \overset{\blacktriangledown}{\alpha}_e \frac{\nu}{r}\overset{\blacktriangledown}{\Phi}{}'_e \overset{\blacktriangledown}{\Phi}_e \alpha_e +$$

$$+ \overset{\blacktriangledown}{\alpha}_e \frac{\nu}{r}\hat{\Phi}_e \overset{\blacktriangledown}{\hat{\Phi}}{}'_e \alpha_e\} 2\pi r \, dr$$

$$= \tfrac{1}{2}\, \overset{\blacktriangledown}{\alpha}_e \left(\frac{Eh}{1-\nu^2} \int_{r_1}^{r_2} \{ \hat{\Phi}'_e\, \overset{\blacktriangledown}{\Phi}'_e + \frac{1}{r^2}\, \hat{\Phi}_e\, \overset{\blacktriangledown}{\Phi}_e + \frac{\nu}{r}\, \hat{\Phi}'_e\, \overset{\blacktriangledown}{\Phi}_e + \frac{\nu}{r}\, \hat{\Phi}_e\, \overset{\blacktriangledown}{\Phi}'_e \} 2\pi r\, dr \right) \alpha_e$$

$$= \tfrac{1}{2}\, \overset{\blacktriangledown}{\alpha}_e\, K_e \alpha_e$$

$$K_e = \frac{Eh}{1-\nu^2} \int_{r_1}^{r_2} \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} [0 \;\; 1] + \frac{1}{r^2} \begin{pmatrix} 1 \\ r \end{pmatrix} [1 \;\; r] + \frac{\nu}{r} \begin{pmatrix} 0 \\ 1 \end{pmatrix} [1 \;\; r] + \frac{\nu}{r} \begin{pmatrix} 1 \\ r \end{pmatrix} [0 \;\; 1] \} 2\pi r\, dr$$

$$= \frac{Eh}{1-\nu^2} \int_{r_1}^{r_2} \{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{r^2} \begin{bmatrix} 1 & r \\ r & r^2 \end{bmatrix} + \frac{\nu}{r} \begin{pmatrix} 0 & 0 \\ 1 & r \end{pmatrix} + \frac{\nu}{r} \begin{pmatrix} 0 & 1 \\ 0 & r \end{pmatrix} \} 2\pi r\, dr$$

$$= \frac{Eh}{1-\nu^2} \int_{r_1}^{r_2} \begin{bmatrix} \frac{1}{r^2} & \frac{1+\nu}{r} \\ \frac{1+\nu}{r} & 2+2\nu \end{bmatrix} 2\pi r\, dr = \frac{E 2\pi h}{1-\nu^2} \int_{r_1}^{r_2} \begin{bmatrix} \frac{1}{r} & 1+\nu \\ 1+\nu & (2+2\nu) r \end{bmatrix} dr$$

$$= 2\pi \frac{Eh}{1-\nu^2} \begin{bmatrix} \ln \frac{r_2}{r_1} & (1+\nu)(r_2-r_1) \\ (1+\nu)(r_2-r_1) & (1+\nu)(r_2^2-r_1^2) \end{bmatrix}$$

$$u_e = A_e\, \alpha_e \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ 1 & r_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longrightarrow \begin{array}{l} u_1 = c_1 + c_2\, r_1 \longrightarrow c_1 = u_1 - c_2\, r_1 \\ u_2 = c_1 + c_2\, r_2 \end{array}$$

$$u_2 = u_1 - c_2\, r_1 + c_2\, r_2 \longrightarrow u_2 - u_1 = c_2 (r_2 - r_1) \qquad c_2 = \frac{u_2 - u_1}{r_2 - r_1}$$

$$c_1 = u_1 - c_2\, r_1 = u_1 - r_1 \frac{u_2}{r_2 - r_1} + r_1 \frac{u_1}{r_2 - r_1} = u_1 \{ 1 + \frac{r_1}{r_2 - r_1} \} +$$

$$+ u_2 \{ - \frac{r_1}{r_2 - r_1} \}$$

$$= u_1 \{ \frac{r_2}{r_2 - r_1} \} + u_2 \{ - \frac{r_1}{r_2 - r_1} \}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{r_2}{r_2-r_1} & -\frac{r_1}{r_2-r_1} \\ \frac{-1}{r_2-r_1} & \frac{1}{r_2-r_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \alpha_e = A_e^{-1} \cdot K_e$$

$$A_e^{-1} = \begin{bmatrix} \frac{r_2}{r_2-r_1} & -\frac{r_1}{r_2-r_1} \\ -\frac{1}{r_2-r_1} & \frac{1}{r_2-r_1} \end{bmatrix} \qquad \overset{\blacktriangledown}{A}{}_e^{-1} = \begin{bmatrix} \frac{r_2}{r_2-r_1} & -\frac{1}{r_2-r_1} \\ -\frac{r_1}{r_2-r_1} & \frac{1}{r_2-r_1} \end{bmatrix}$$

$$U_e = \tfrac{1}{2}\, \overset{\blacktriangledown}{\alpha}_e K_e \alpha_e = \tfrac{1}{2}\, \overset{\blacktriangledown}{u}_e Q_e u_e \qquad Q_e = \overset{\blacktriangledown}{A}{}_e^{-1} K_e A_e^{-1}$$

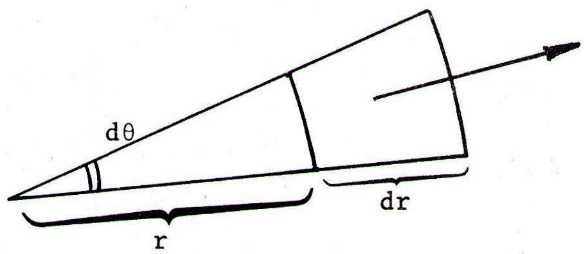
$$\text{met } \alpha_e = A_e^{-1} u_e$$

$$Q_e = 2\pi \frac{Eh}{1-\nu^2} \begin{bmatrix} \frac{r_2}{r_2-r_1} & -\frac{1}{r_2-r_1} \\ -\frac{r_1}{r_2-r_1} & \frac{1}{r_2-r_1} \end{bmatrix} \begin{bmatrix} \ln \frac{r_2}{r_1} & (1+\nu)(r_2-r_1) \\ (1+\nu)(r_2-r_1) & (1+\nu)(r_2^2-r_1^2) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \frac{r_2}{r_2-r_1} & -\frac{r_1}{r_2-r_1} \\ -\frac{1}{r_2-r_1} & \frac{1}{r_2-r_1} \end{bmatrix}$$

$$= 2\pi \frac{Eh}{1-\nu^2} \frac{1}{(r_2-r_1)^2} \begin{bmatrix} r_2^2 \ln \frac{r_2}{r_1} - (1+\nu)(r_2-r_1)^2 & -r_1 r_2 \ln \frac{r_2}{r_1} \\ -r_1 r_2 \ln \frac{r_2}{r_1} & r_1^2 \ln \frac{r_2}{r_1} + (1+\nu)(r_2-r_1)^2 \end{bmatrix}$$

<u>Bepaling van de kinematisch consistente belastingvector f_{ce}</u>



$$\rho\omega^2 \, h \, r^2 \, dr \, d\theta = \hat{q}_e^0$$

potentiaal van de uitw. belasting: $\Omega_{ce} = - \int_{r=r_1}^{r_2} \int_{\theta=0}^{2\pi} \overset{\blacktriangledown}{\hat{u}}_e \, \hat{\mathbf{q}}_e^0 =$

$$= - \overset{\blacktriangledown}{\alpha}_e \int_{r=r_1}^{r_2} \int_{\theta=0}^{2\pi} \hat{\Phi}_e \, \hat{q}_e^0$$

$$= - \overset{\blacktriangledown}{\alpha}_e \int_{r_1}^{r_2} \int_{0}^{2\pi} \begin{pmatrix} 1 \\ r \end{pmatrix} \rho\omega^2 \, h \, r^2 \, dr \, d\theta = - \overset{\blacktriangledown}{u}_e \overset{\blacktriangledown}{A}{}_e^{-1} \{ 2\pi\rho\omega^2 \, h \int_{r_1}^{r_2} \begin{bmatrix} r^2 \\ r^3 \end{bmatrix} dr \}$$

$$= - \overset{\blacktriangledown}{u}_e \overset{\blacktriangledown}{A}{}_e^{-1} \, 2\pi\rho\omega^2 \, h \begin{bmatrix} \frac{1}{3} r^3 \\ \frac{1}{4} r^4 \end{bmatrix} \Bigg|_{r_1}^{r_2} = - \overset{\blacktriangledown}{u}_e \overset{\blacktriangledown}{A}{}_e^{-1} \, 2\pi\rho\omega^2 \, h \begin{bmatrix} \frac{1}{3}(r_2^3 - r_1^3) \\ \frac{1}{4}(r_2^4 - r_1^4) \end{bmatrix} =$$

$$= - \overset{\blacktriangledown}{u}_e \, f_{ce}$$

$$f_{ce} = 2\pi\rho\omega^2 \, h \, \overset{\blacktriangledown}{A}{}_e^{-1} \begin{bmatrix} \frac{1}{3}(r_2^3 - r_1^3) \\ \frac{1}{4}(r_2^4 - r_1^4) \end{bmatrix} = 2\pi\rho\omega^2 \, h \begin{bmatrix} \frac{r_2}{r_2 - r_1} & -\frac{1}{r_2 - r_1} \\ -\frac{r_1}{r_2 - r_1} & \frac{1}{r_2 - r_1} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3}(r_2^3 - r_1^3) \\ \frac{1}{4}(r_2^4 - r_1^4) \end{bmatrix}$$

$$f_{ce} = \frac{2\pi\rho\omega^2 \, h}{r_2 - r_1} \begin{bmatrix} \frac{1}{3} r_2 (r_2^3 - r_1^3) - \frac{1}{4}(r_2^4 - r_1^4) \\ -\frac{1}{3} r_1 (r_2^3 - r_1^3) + \frac{1}{4}(r_2^4 - r_1^4) \end{bmatrix}$$

Uitwerking opgave 9

Evenwichtrelaties: $\Sigma_H = 0 \longrightarrow H_2 = -(H_1 + H_3)$ (1)

$\Sigma M_{t.o.v.2} = 0 \longrightarrow V_1 = -\frac{1}{2} H_3$ (2)

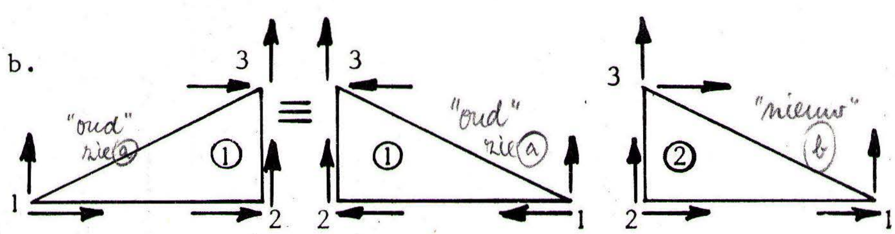
$\Sigma V = 0 \longrightarrow V_2 = -(V_1 + V_3)$ (3)

Dan:

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 8 & -8 & 0 & 0 & 4 & -4 \\ -8 & 20 & -12 & 6 & -10 & 4 \\ 0 & -12 & 12 & -6 & 6 & 0 \\ 0 & 6 & -6 & 3 & -3 & 0 \\ 4 & -10 & 6 & -3 & 35 & -32 \\ -4 & 4 & 0 & 0 & -32 & 32 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$f_e = Q_e \, u_e$$

b.



$\longrightarrow Q_2$ ontstaat uit Q_1 door zowel de rijen als de kolommen 1, 2 en 3 met -1 te vermenigvuldigen.

$$\text{Dus } Q_2 = \begin{bmatrix} 8 & -8 & 0 & 0 & -4 & 4 \\ -8 & 20 & -12 & -6 & 10 & -4 \\ 0 & -12 & 12 & 6 & -6 & 0 \\ 0 & -6 & 6 & 3 & -3 & 0 \\ -4 & 10 & -6 & -3 & 35 & -32 \\ 4 & -4 & 0 & 0 & -32 & 32 \end{bmatrix}$$

Hier is lokale nummering gebruikt.

c.

$$\underset{\sim}{u}_t = \overset{\;1 \quad 2 \qquad 3 \qquad 4 \quad 5 \quad 6 \quad 7 \quad 8}{\left[\underbrace{u_2 \; u_4}_{u_\ell} \; \underbrace{v_4}_{u_p} \; \underbrace{u_1 \; u_3 \; v_1 \; v_2 \; v_3}_{u_s} \right]}$$

$$\underset{\sim}{f}_t = \left[\underbrace{H_2 \quad H_4}_{f_\ell} \quad \underbrace{V_4}_{f_p} \quad \underbrace{H_1 \quad H_3 \quad V_1 \quad V_2 \quad V_3}_{f_s} \right] \; ; \; \underset{\sim}{f}_e = [0 \quad 0]$$

d. $$\begin{bmatrix} f_\ell \\ f_p \\ f_s \end{bmatrix} = \begin{bmatrix} Q_{\ell\ell} & Q_{\ell p} & Q_{\ell s} \\ \overset{\blacktriangledown}{Q}_{\ell p} & Q_{pp} & Q_{ps} \\ \overset{\blacktriangledown}{Q}_{\ell s} & \overset{\blacktriangledown}{Q}_{ps} & Q_{ss} \end{bmatrix} \begin{bmatrix} u_\ell \\ u_p \\ u_s \end{bmatrix}$$

e. LOK

	1	2	3	4	5	6
	u_1	u_2	u_3	v_1	v_2	v_3
el. 1	4	1	5	6	7	8
el. 2	2	1	5	3	7	8

elementvrijheidsgraden

vrijheidsgraden van de constructie

f. $Q_{\ell\ell} = \begin{bmatrix} 40 & -8 \\ -8 & 8 \end{bmatrix}$; $Q_{\ell p} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$

g. $$\begin{bmatrix} H_2 \\ H_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 & -8 \\ -8 & 8 \end{bmatrix} \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \end{bmatrix} \cdot v_4$$

$$40\,u_2 - 8\,u_4 = 6\,v_4$$
$$-8\,u_2 + 8\,u_4 = 0 \quad \longrightarrow \quad u_4 = u_2$$
$$\overline{32\,u_2 \qquad = 6\,v_4}\;+$$

Dus $u_2 = u_4 = \frac{6}{32}\, v_4 = \frac{3}{16} V$

h. 1. $$f_p = \overset{\blacktriangledown}{Q}_{\ell p} \cdot u_\ell + Q_{pp} \cdot u_p + Q_{ps} \cdot \overset{0}{u_s}$$
$$f_s = \overset{\blacktriangledown}{Q}_{\ell s} \cdot u_\ell + \overset{\blacktriangledown}{Q}_{ps} \cdot u_p + Q_{ss} \cdot \underset{0}{u_s}$$

Dus eerst u_ℓ oplossen en overige relevante matrices (Q_{pp}, $Q_{\ell s}$ en Q_{ps}) bepalen; daarna f_p en f_s direct uit te schrijven.

2. Bepaal u_ℓ en daaruit u_e voor elk element.
Bepaal voor elk element $f_e = Q_e \cdot u_e$.
Assembleer f_e, e = 1,2 tot f_t. (bevat alle krachtgrootheden in de constructie knooppunten).