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Implicit Variable-radius Arc Canal Surfaces for Solid Modeling

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ABSTRACT

In this paper we consider the problem of obtaining an implicit form for the canal surface whose spine is the arc and the radius changes linearly in respect to the angle. We present a number of different solutions to the problem including exact and approximated ones and discuss the scenarios where each of the solutions is appropriate to use in solid modeling with real functions.

Keywords: implicit surfaces, canal surfaces, implicit form, sphere sweeping.

1 INTRODUCTION

Canal surfaces, which are surfaces obtained by sweeping a sphere over a given curve, are one of the important primitives in solid modeling. Being a special case of sweep surfaces, they are generally easier to define and have applications in different areas of computer-aided design such as industrial design or organic modeling. Canal surfaces represented in the parametric form (boundary representation) were widely studied [2]. On the other hand, when the object is treated as a solid with the surface defined in the implicit form, not much research can be found. In this paper, we consider canal surfaces from the point of view of a solid/volume modeling system dealing with the objects defined in the implicit form. A solid object in such a system is represented by a real function defined in the entire domain and whose value at the given point specifies whether the point is located inside, outside or on the boundary of the solid. Examples of such systems are HyperFun [15] and BlobTree [14]. Unlike boundary representation methods, the research on canal surfaces in implicit modeling is very limited to the cases with simple spines and constant shape of the moving object.

In this paper, we are discussing an important case of canal surfaces whose spine is an arc and the radius of the canal surface changes linearly between the ends of the arc in respect to the angle of the arc (see Fig. 1). For simplicity, we are considering the case where the arc lies in XY-plane with the center at the origin and the start point lies on the x-axis. The radius of the sweeping sphere linearly changes from the start point of the arc to the end point of the arc. We are not considering cases where the radius is constant, as the surface becomes a part of a torus, and the case where the arc is degenerate, because the sweeping object becomes a sphere. All the other cases of arc canal surfaces can be obtained from this simple case by using affine transformations.

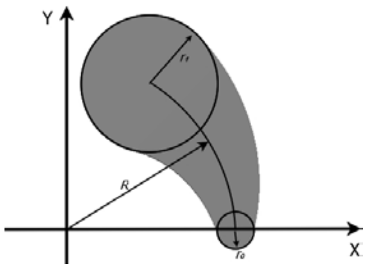

Fig. 1: The form of arc canal surface with start radius $r_0 = 0.1$, end radius $r_1 = 0.2$ and angle of the arc $\alpha = \frac{\pi}{3}$

We aim for representation of this kind of object by using a real function with positive values for all the points inside the canal surface, zero values for all the points belonging to the surface itself and negative values for all the points outside the canal surface. For the defining function we are setting the following requirements:

- The function should be defined in $\Re^3$ and be at least C^0-continuous;
- The function cannot be equal zero at any points outside the boundary of the specified object;
- It should be efficient and easy to evaluate.

In this work, we show few possible ways to define such a function with their pros and cons and discuss how these functions can be used in the modeling system dealing with surfaces defined in the implicit form. The contributions of this paper are:

- We present several ways to define arc canal surface in an implicit form including exact and approximate and with different complexity;
- We present several applications in shape modelling and outline which representation is better to be used in similar applications;
- We discuss possible ways to efficiently use such an object from the user's perspective.

1.1 Related Works

Canal surfaces swept by a sphere moving along a curve were studied since 19 century. A recent detailed survey discussing sweeping methods shows a number of various methods [2] applied for parametric representation as well as for implicit representation, yet the main focus is done on the surfaces rather than on the point sets bounded by these surfaces (i.e. volumes). A numerical solution discussing the sweeping operation is also well known in solid modeling with real functions [3].

To define a model obtained with sweeping, two general approaches can be used: theory of envelopes [5] and Minkowski sums [7]. In practice, however, theory of envelopes is applicable for the limited set of sweeping primitives and Minkowski sums are computationally very expensive. This results in the methods dealing with numerical and approximation methods to obtain the defining function for sweeping. Thus, in [10] a numerical procedure is described for a sweep by an arbitrary solid involving global maximum search. A numerical and broadly applicable analytic formulation based on the Jacobian rank deficiency was proposed in [1].

2 DEFINITION AND EXACT IMPLICIT EQUATION

The arc in the case we are discussing is easily defined in the parametric form as following:

$$\begin{cases} x(t) = R\cos(t) \\ y(t) = R\sin(t) \\ z(t) = 0 \end{cases} \tag{2.1}$$

Here R denotes the radius of the arc, for $t = 0$ we have a start point of the arc and for $t = \alpha$ we have end point of the arc defined by the angle of the arc, $0 < \alpha < 2\pi$. The linearly changing radius in respect to an angle means that $r(t) = r_0 + \frac{t}{\alpha}(r_1 - r_0)$.

2.1 Obtaining implicit equation with elimination

The first and the common technique allowing getting an implicit form for sweep surfaces is the standard elimination technique. Let us apply the theory of envelopes [5] to the given problem. According to the theory of envelopes, we have to eliminate *t* from the following system:

$$\begin{cases} F(x,y,z,t) = (r_0 + t(r_1 - r_0))^2 - (x - R\cos(t))^2 - (y - R\sin(t))^2 - z^2 = 0 \\ \dfrac{\partial F(x,y,z,t)}{\partial t} = 0 \end{cases} \tag{2.2}$$

It can be simplified by using the following substitutions: $d = r_0^2 - x^2 - y^2 - z^2 - R^2$, $a = 2r_0(r_1 - r_0)$, $b = (r_1 - r_0)^2$, $u = 2Rx$, $v = 2Ry$. The system becomes the following:

$$\begin{cases} F(x,y,z,t) = d + at + bt^2 + u\cos(t) + v\sin(t) = 0 \\ \dfrac{\partial F(x,y,z,t)}{\partial t} = a + 2bt - u\sin(t) + v\cos(t) = 0 \end{cases} \tag{2.2}$$

As *t* is present in both polynomial and trigonometric terms, it is not possible to express *t* in terms of *x*, *y* and *z*. However elimination of *t* is still possible using the method similar to one presented in [6]. Let $s = \sin(t)$ and $c = \cos(t)$ be two parameters we want to add to the system in order to make it linear. From the equation 2.2 and trigonometric identities we have the following system:

$$\begin{cases} d + at + bt^2 + uc + vs = 0 \\ a + 2bt - us + vc = 0 \\ s^2 + c^2 = 1 \end{cases} \tag{2.3}$$

It can be seen that the resulting system is polynomial. By eliminating *s* and *c* from the system we obtain the following quartic equation for *t*:

$$b^2t^4 + 2abt^3 + (a^2 + b^2 + 2bd)t^2 + 2a(d + 2b)t + (a^2 + d^2 - u^2 - v^2) = 0 \tag{2.4}$$

Also, by eliminating *c* and *t* from the same system we have

$$\begin{gathered} (u^2 + v^2)^2 s^4 - 8bv(u^2 + v^2)s^3 - 2\big((a^2 - 4bd)(u^2 - v^2) + (v^2 - 8b^2)(u^2 + v^2)\big)s^2 - \\ 8bv(a^2 - 4bd - 2u^2 - v^2)s + (a^2 - 4b(d + u) - v^2)(a^2 - 4b(d - u) - v^2) = 0 \end{gathered} \tag{2.5}$$

The point lies on the desired arc canal surface if one of the solutions of the equation 2.4 corresponds to the sine of the one of the solutions of the equation 2.3. Formally it can be presented as follows:

$$F(x,y,z) = \arg\min_{s,t} \left| s - \sin(t) \right| = 0 \tag{2.6}$$

Here s is a real root of the equation 2.5 and t is a real root of the equation 2.4.

While the equation 2.6 gives us the predicate to distinguish points that lie on the arc canal surface and those that lie outside, this result can hardly be used in the context of solid modeling. The main reason is that the equations 2.4 and 2.5 may result in zero real roots for some points in space. Because of that the function is not defined at all the points of the domain, and in areas where it is defined, the function is not C^0 - continuous. It can be seen that the elimination technique is not suitable for practical purposes and hence some approximations should be used in order to obtain a function with the desired properties.

3 APPROXIMATE EQUATIONS FOR ARC CANAL SURFACE

In this section, we discuss an approximate solution in the sense that the obtained arc canal surface differs from the exact shape by some parameters and the resulting error or displacement can be evaluated. We can vaguely distinguish the methods of approximation of the arc canal surface into the following categories:

- Spine approximation
- Approximation of the radius of the rolling sphere
- Approximation of the shape

In the first category, we approximate the arc by another curve or the number of curves.
In the second category, we approximate the function that defines the change of the radius depending on the parameter on the arc. The third category includes the combination of the arc and radius approximation.

3.1 Approximation of the radius of the rolling sphere

Approximation by polynomial can be achieved through the approximation of the radius change. We discussed above that the elimination of the t parameter from the system 1.1 cannot be done without expanding the system because of the polynomial and trigonometric terms in the equation. In case we subdivide the arc into a number of a small segments each corresponding to the angle β or smaller, $\beta < \alpha$, bearing in mind that $\sin(t) \approx t$ for t being close to zero, the following formulation can be obtained with the approximation of the radius change: $r(t) = r_0 + \beta \dfrac{\sin(t)}{\sin(\beta)} d_r$. The system 1.1 becomes the following:

$$\begin{cases} \left(r_0 + \dfrac{\sin(t)}{\sin(\beta)} d_r \right)^2 - x - R\cos^2(t) - y - R\sin^2(t) = p_0 + p_1 \sin(t) + p_2 \cos(t) + p_3 \sin^2(t) = 0 \\ p_1 \cos(t) - p_2 \sin(t) + 2 p_3 \sin(t)\cos(t) = 0 \end{cases}$$

Where $p_0 = r_0^2 - x^2 - y^2 - z^2 - R^2$, $p_2 = 2xR$, $p_1 = \dfrac{2r_0 d_r}{\sin(\beta)} + 2yR$ and $p_3 = \dfrac{d_r^2}{\sin^2(t)}$. By substituting $\sin(t) = u$ and $\cos(t) = \sqrt{1-u^2}$, we have the following system:

$$\begin{cases} p_0 + p_1 u + p_2\sqrt{1-u^2} + p_3 u^2 = 0 \\ p_1\sqrt{1-u^2} - p_2 u + 2p_3 u\sqrt{1-u^2} = 0 \end{cases}$$

As each equation in this system can be traced to the polynomial of degree 4, the parameter *u* can be eliminated by common techniques. The result is the following:

$$P(x,y,z) = (p_1^2+p_2^2)^2(-p_0^2+p_1^2+p_2^2) + 8p_0(-4p_0^2+p_1^2+4p_2^2)p_3^3 + 16(-p_0^2+p_2^2)p_3^4 - \\ p_3\Big(2p_0\big(5p_1^4+p_1^2p_2^2-4p_2^4-4p_0^2(p_1-p_2)(p_1+p_2)\big) + \big(16p_0^4+p_1^4+20p_1^2p_2^2-8p_2^4-8p_0^2(4p_1^2+p_2^2)\big)p_3\Big) \quad (2.7)$$

So the resulting shape can be expressed as an analytical geometric object. We should note that the resulting shape approximates the arc canal surface only between $t \in [0,\beta]$ and additional set-theoretic operations are needed to cut the extra parts of the surface (see Fig. 2). Therefore to get the desired arc canal surface, two half-spheres should be added with set-theoretic union to the set-theoretic intersection of the resulting object with two half-spaces:

$$F(x,y,z) = \big(P(x,y,z) \wedge \big(y \wedge (x\tan(\beta) - y)\big)\big) \vee \big(r_0^2 - (x-R)^2 - y^2 - z^2\big) \vee \\ \vee \big(r_\beta^2 - (x - R\cos(\beta))^2 - (y - R\sin(\beta))^2 - z^2\big)$$

where $\vee$ denotes an appropriate R-function for the set-theoretic union and $\wedge$ denotes an appropriate R-function for the set theoretic intersection, both allowing for C^n-continuity [9].

Fig. 2: Approximation of a radius function for the arc canal surface results in a different shape that encloses the needed shape within some epsilon

The maximum approximation error in this case is $\beta\dfrac{\sin(u^*)}{\sin(\beta)} - u^*$, where $u^* = \arccos\dfrac{\sin(\beta)}{\beta}$.

It can be noted that the resulting algebraic surface belongs to the family of cyclides. To approximate the desired shape with other types of cyclides, for example, Dupin cyclides of degree 4 [11] and of degree 3 [12] is possible. However, such as approximation involves both spine and radius approximation and cannot be used in the general case.

3.2 Approximation of the spine

The spine of the canal surface can be approximated by another curve in case the quality of the approximation can be estimated or defined. Unfortunately approximation often involves different parameterization along the curve which is not suitable for our purposes. For example, the arc can be parameterized with rational quadratic functions $x(t)=\frac{1-t^2}{1+t^2}$, $y(t)=\frac{2t}{1+t^2}$, however in this case no linear correspondence between the angle and the parameter t is available, and because of that the requirement of the radius changing linearly cannot be supported. Similarly the arc can be approximated by the quadratic curve such as $x=t$, $y=1-\frac{t^2}{2}$ in the area of $t=0$. The system of equations 2.1 in this case becomes polynomial where the parameter t can be relatively easily eliminated. However this parameterization does not only approximate the spine, but also changes the variation of the radius from linear to the quadratic, hence doing it approximately as well.

Still the solution with approximation of the desired shape is possible as we can benefit from the set-theoretic approach, i.e., the spine can be subdivided into the number of approximating curves and the result can be obtained by applying set-theoretic union to the resulting models.

The easiest approximation we can apply is an approximation of the arc using line segments. From [8] the closed-form solution for the canal surface with the linearly changing radius can be obtained as:

$$F(x,y,z)=\left(R_1+t(R_2-R_1)\right)^2-\left(x-x_1-t(x_2-x_1)\right)^2-\left(y-y_1-t(y_2-y_1)\right)^2-\left(z-z_1-t(z_2-z_1)\right)^2$$

where $t=\frac{\mathbf{l}\bullet(\mathbf{x}-\mathbf{p}_1)+R_1(R_2-R_1)}{\mathbf{l}\bullet\mathbf{l}-(R_2-R_1)^2}$, the start point of the segment $\mathbf{p}_1=(x_1,y_1,z_1)$ the endpoint of the segment is $\mathbf{p}_2=(x_2,y_2,z_2)$ and $\mathbf{l}=\mathbf{p}_2-\mathbf{p}_1$ is the direction vector of the segment.

For the line segment approximating a part of the arc between the angle α_1 and α_2 the start point is $\mathbf{p}_1=(\cos(\alpha_1),\sin(\alpha_1),0)$, the end point is $\mathbf{p}_2=(\cos(\alpha_2),\sin(\alpha_2),0)$ and the radius value at the ends of the segment can be obtained using a simple linear interpolation such as $R_1=r_0+\frac{\alpha_1}{\alpha}(r_1-r_0)$ and $R_2=r_0+\frac{\alpha_2}{\alpha}(r_1-r_0)$.

The equation for the approximated arc canal surface can be obtained by successive application of the R-functions corresponding to the set-theoretic union operation to the segments. In case we approximate the arc by n segments, $n-1$ set-theoretic operations are needed. The continuity of the resulting function depends on the choice of R-functions (see [9]).

3.3 Combination of a rolling circle and rolling sphere

The shape of the arc canal surface created as a sweep of the sphere over the arc is different from the shape of the sweep by a disk (see Fig. 3). On the other hand, the formulation for the sweep by a disk with the linearly changing radius and the spine as a circle can be easily obtained. The model obtained with sweeping by a disk is also called normal ringed surface [13]. We could not find the defining function formulation for it in the literature; however it can be derived from [4] as follows:

$$t(x,y)=\min(\alpha,\arctan\frac{y}{x})$$

$$F(x,y,z)=\left(r_0+t(x,y)d_r\right)^2-\left(x-R\cos(t(x,y))\right)^2-\left(y-R\sin(t(x,y))\right)^2-z^2$$

As it can be seen, this formulation allows for the one-to-one correspondence between the angle and the radius of the sweeping disk. The dependency between the radius and the angle can be obtained also in the case of the canal surface created as the sweep of a sphere. In the case of a sweeping sphere for the given point the radius becomes a non-linear function as we discuss below.

Fig. 3: Difference between the solid object obtained by sweeping with a disk (green) and a sphere (red).

Consider the case where we aim to calculate the radius for the given point in the two-dimensional space (see Fig. 4). For the parameter value t the radius of the sweeping disk is $r(t) = r_0 + \frac{t}{\alpha}(r_1 - r_0)$. The actual value of the radius function is represented by the segment AT because of the influence of the rolling sphere and it might be bigger than the radius of a sweeping disk. The points T and U correspond to the parameters t and u, and the point A is the point of intersection of the sphere with the centre in U and radius $r(u) = r_0 + \frac{u}{\alpha}(r_1 - r_0)$ with the line passing through the points P and T. Therefore, the distance between U and A is $r(u)$.

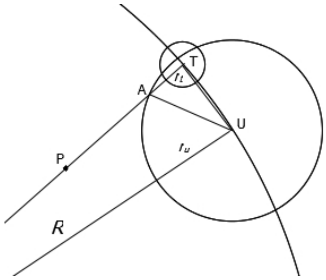

Fig. 4: Influence of the sweeping sphere on the radius function for the given point.

The angle UTA depends on the coordinates of the point for which we seek the value of the radius function, we denote it as $\gamma(x, y, z, u)$. For the given point P whose radius function we are seeking knowing that P lies on the line OA:

$$\cos(\gamma) = \cos(\gamma(x, y, z, u)) = \frac{TP \cdot UT}{|TP||UT|} = \frac{(x - t_x)(t_x - R\cos(u)) + (y - t_y)(t_y - R\sin(u))}{\sqrt{(x - t_x)^2 + (y - t_y)^2 + z^2}\sqrt{(t_x - R\cos(u))^2 + (t_y - R\sin(u))^2}}$$

Where $t_x = t_x(x,y) = R\frac{x}{\sqrt{x^2+y^2}}$ and $t_x = t_x(x,y) = R\frac{y}{\sqrt{x^2+y^2}}$ are points on the arc corresponding to the point P.

Given that the distance between T and U is the length of the chord corresponding to the angle $|t-u|$, from the triangle ATU we have:

$$AU^2 = AT^2 + TU^2 - 2AT \cdot TU\cos(\gamma)$$

$$AT^2 = TU\cos(\gamma) + \sqrt{AU^2 + TU^2\cos^2(\gamma) - TU^2} = 2R\sin\left(\frac{t-u}{2}\right)\cos(\gamma) + \sqrt{r(u)^2 - \left(2R\sin\left(\frac{t-u}{2}\right)\right)^2\left(\cos^2(\gamma)-1\right)}$$

The actual radius of the canal surface obtained by a sweeping of a sphere is a maximum for all the possible values u for the given parameter t:

$$r_s(x,y,z) = \arg\max_u\left(2R\sin\left(\frac{t-u}{2}\right)\cos(\gamma) + \sqrt{r(u)^2 - \left(2R\sin\left(\frac{t-u}{2}\right)\right)^2\left(\cos^2(\gamma)-1\right)}\right) \quad (2.8)$$

where we are seeking for a maximum on the interval

$$u \in \begin{cases} [0,t), & r_0 < r_1 \\ (t,\alpha], & r_0 > r_1 \\ t & r_0 = r_1 \end{cases}$$

It can be seen that the maximum for the equation 2.8 cannot be found analytically for u and therefore numerical methods have to be used. Overall, an exact and continuous solution with the finite machine precision can be obtained, however the numerical search for the global maximum means that the point query is computationally expensive. For the numerical computations we can use the value of the function $r(t)$ as the initial approximation.

4 APPLICATIONS AND DISCUSSION

Our main motivation for this research was to introduce the geometric primitive, namely the arc canal surface, to the modeling system dealing with the geometry defined in the implicit form. However as the exact solution could not be obtained and the approximate ones should be used instead, we have to test different approaches in real-life scenarios. Our testing models include the artistic design model, the “Swan” (see Fig. 5), and the architecture-inspired model “Dome” (see Fig. 6). All the timings mentioned below were done on the PC with 6-core Intel Xeon W3680 processor and 8GB of memory.

4.1 Case study 1: the swan model

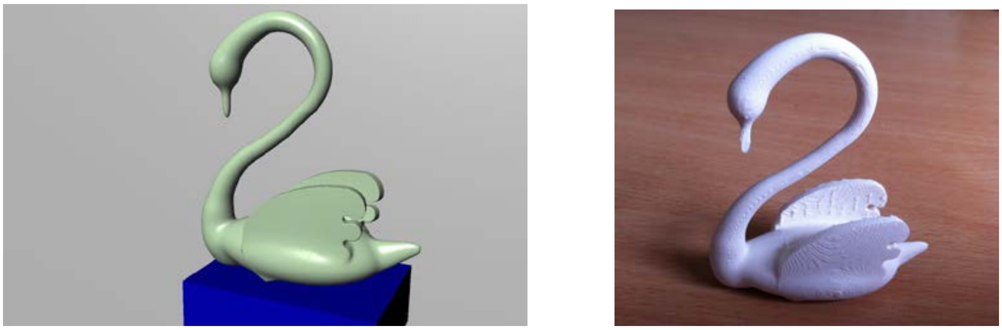

Fig. 5: Swan model with 6 arc canal surfaces containing the neck of the model: (a) rendered version, (b) physical version produced with 3D printer

In modeling with arc canal surfaces one of the case studies is a situation where we have a curve approximated by arcs and the radius of the rolling sphere approximated by linear function. In this case each sweeping segment for the arc is approximated by an arc canal surface. The swan model is an example of this approach: in this model 6 arcs were connected together for the spine and the radius changes linearly for each arc (see Fig. 5a). In this model we have 6 connected arc canal surfaces, where set-theoretic union was used to connect these canal surfaces together.

For this model from designer's point of view the spine shape and the smooth transitions between adjacent arcs are more important than the nature of the changes in the radius. That is why the obvious strategy here was to use the lightweight polynomial solution presented in the section 3.1. This allows us to obtain the desired shape and at the same time keep the defining function relatively easy to evaluate. While sampling the function to polygonise the object, each sample for the whole swan model took approximately 1.83e-6 seconds where the calculation of canal surfaces took approximately 3.5e-7 seconds, hence being approximately 20% of the calculations for the defining function of the swan model. Because of the approximation, the maximum difference between the expected radius of the arc canal surface and the resulting radius in the shape is approximately 19% for all the segments in the model. This difference, however, is not visible and the resulting shape is in line with designer's needs.

Because of the lightweight defining function, it was also possible to quickly convert the model from the function representation to the slice format which modern 3D printers understand and hence fabricate the model to get a physical copy (see Fig. 5b). The mesh generation by using adaptive resolution 550*190*700 took approximately 135 seconds. The model was 3D printed using an Ultimaker 2 machine in around 2 hours.

4.2 Case study 2: the dome model

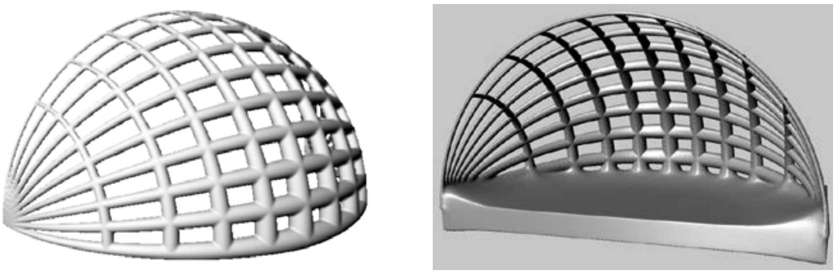

Fig. 6: Architectural-like model "half-dome": (a) Model containing 29 arc canal surfaces unioned together, (b) Further blending operation was performed to complete the model.

Another case study deals with a type of shapes where different sweeping objects intersect, and it is assumed that in the place of intersection the radius of sweeping sphere is the same for different objects. A dome model is such an example with a number of arc canal surfaces that are not only connected by endpoints of the arc spines, but also have intersections. For the model like that, it is important that the radius of the rolling sphere is changed exactly linearly; otherwise the condition of having the same radius of the sweeping sphere at the intersection points is violated. Looking at the possible ways to represent the arc canal surface, it can be seen that no approximation of the radius is possible, and approximation of the spine or approximation of the moving object should be used instead. In our experiments with a dome model, the approximation of the spine by a large set of line segments (see section 3.2) and the approximation with rolling circle and radius correction (see section 3.3) produces visually equal results, however the timings are significantly different. Sampling the function for the dome model, where each arc canal surface is approximated by 50 line segments took approximately 8.5e-6 seconds per sample. Sampling of the function where radius correction is used took several milliseconds, mainly because of the operation of maximization. In general, the nature of the numerical correction of the radius when we approximate the arc canal surface by the rolling circle results in larger timings for computation of the defining function, and hence it becomes slower to visualize and process the model. Another important property of the approximation of the spine with linear segments is C^1-continuity which is useful for further operations applied to the arc canal surface. In the case of the dome model these operations include blending union operation which required the defining function of the model to be smooth and continuous.

4.3 Recommendations on approximations of variable-radius arc canal surfaces

After preparing these case studies and further tests with applications of the arc canal surface for the purposes of solid modeling with real functions, we can identify the following strategies which allow for efficiently creating the desired shape with most efficient defining function:

1. In the case of the small radius of the spine it is more convenient to approximate the radius change function rather than the spine. Also this approximation is more convenient to use in the applications, which do not require exactly linear change of the radius (see discussion in 4.1 above).
2. In the case of the significant radius change and the large angle of the arc, it is more convenient to use the spine approximation, because the radius approximation might result in large difference between the desired shape and the approximated one. Also, the spine approximation is better to use when the linear change of radius is essential (see discussion in 4.2 above).
3. The approximation of the spine of the canal surface is the easiest to implement and evaluate, however it depends on the choice of the function for the set-theoretic union. In the case of R-functions providing C^1 continuity and a large number of segments, the resulting timings might be slow.
4. If the speed is not important, but the shape should be precisely described, for example, for robotics applications, the combination of disk sweeping with the radius from sphere sweeping should be used. However, by researching arc canal surface, we were unable to identify such applications, mainly because current robotics applications use sweeping with constant radius and linearly changing radius is a rather unusual application. Still we do not want to leave this option behind, yet stating it as a rare one.

The resulting canal surface can be used in a solid modeling system as a base primitive allowing additional operations and deformations on top of it.

5 CONCLUSIONS

In this paper we presented different methods allowing for solving the problem of obtaining an implicit form of the canal surface whose spine is an arc and the radius changes linearly in respect to the angle of the spine. It can be seen that various methods to define the same object can be used. Let us briefly summarize all the methods presented so far:

- Using the elimination we can have an exact and easy to evaluate query to check if the point is on the desired surface or not; however it is not C^0- continuous (simply not defined) at all the points outside the surface;
- The spine can be approximated with line segments resulting in a very simple definition of the desired canal surface; however to calculate with finite machine precision we should use large number of segments that slows down the function evaluation significantly and, in addition, depends on the choice of the set-theoretic operation implementation;
- The change of the radius can be approximated with a trigonometric function; the result is a polynomial equation and additional set-theoretic operations are needed to get the desired shape;
- The combination of the disk sweeping formulation with the radius of the shape obtained from sphere sweep provides an exact solution; however it involves the global maximum search and therefore can be computationally expensive and be considered as an approximation with the finite precision.

Unfortunately, there is no method which would be a perfect fit to the criteria we set at the beginning of the paper. At the same time, we can see that in real life applications all the computations are being done with the finite machine precision and therefore we can consider the approximation methods as well as the exact ones as long as we can set the finite machine precision as the approximation tolerance. We should bear in mind that all the approximation methods presented above rely on the fact that we can subdivide the spine onto smaller arcs and then get the resulting shape by using set-theoretic union of the canal surfaces build for each arc.

The problem of the arc canal surface seems to be very important for solid modeling systems with implicit forms of the geometric objects because of the lack of canal surfaces that can be used in such systems with the spine as a curve. As we have not found the perfect solution, we feel that more methods to solve this problem should be found. During our investigations, we have found that the desired shape can be approximated by Dupin cyclides by applying formulation from [11] and finding parameters of the cyclide such that start radius and end radius coincide with those from arc canal surface. However we have also found out that such a geometric object is an approximation in both radius and the spine, so more research is needed to find an appropriate balance. To address the problem of the arbitrary canal surface representation in the implicit form, we are going to approximate the given spine curve with an arc spline [16], assign radius values to the endpoints of each arc, and then to apply set-theoretic union to the obtained implicit arc canals.

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