Twisted Covariant Form Hierarchies: From Hidden Symmetries in M-theory to Anomalies in Heterotic Backgrounds.

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Twisted Covariant Form Hierarchies: From Hidden Symmetries in M-theory to Anomalies in Heterotic Backgrounds.

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Abstract

In this thesis, we compute the twisted covariant form hierarchies (TCFHs) of minimal supergravity theories in four and five dimensions, as well as, eleven-dimensional supergravity and the internal space of its warped AdS backgrounds. As a consequence, the form bilinears satisfy a generalised conformal Killing-Yano equation with respect to the TCFH connection. Then, we find the (hidden) symmetries generated by the form bilinears in spinning particle actions propagating in certain supersymmetric backgrounds of $D = 4$, $N = 2$ and $D = 5$, $N = 1$ minimal supergravities, M-brane backgrounds which include the M2-brane, M5-brane, pp-wave and KK-monopole, maximal supersymmetric AdS backgrounds and some AdS backgrounds that arise as near horizon geometries of intersecting M-branes. In addition, we explore whether the form bilinears are sufficient to prove the integrability of particle probe dynamics on M-brane backgrounds. Moreover, we show that the covariantly constant forms of heterotic backgrounds with $SU(2)$ and $SU(3)$ holonomy generate a W-symmetry algebra in two-dimensional non-linear supersymmetric sigma models with the previous backgrounds as target spaces and analyse the consistency conditions of the chiral anomalies arising from all symmetry generators required for the closure of the algebra.

Statement of originality

This thesis is based on four collaborative publications.

1. G. Papadopoulos and E. Pérez-Bolaños, "Symmetries, spinning particles and the TCFH of $D = 4, 5$ minimal supergravities," Phys. Lett. B **819**, 136441 (2021) doi:10.1016/j.physletb.2021.136441 [arXiv:2101.10709 [hep-th]].

2. G. Papadopoulos and E. Pérez-Bolaños, "TCFHs, hidden symmetries and M-theory backgrounds," Class. Quant. Grav. **39**, no.24, 245015 (2022) doi:10.1088/1361-6382/aca1a2 [arXiv:2201.11563 [hep-th]].

3. G. Papadopoulos and E. Pérez-Bolaños, "The TCFHs of D = 11 AdS backgrounds and hidden symmetries," JHEP **06**, 015 (2023) doi:10.1007/JHEP06(2023)015 [arXiv:2206.04369 [hep-th]].

4. L. Grimanellis, G. Papadopoulos and E. Pérez-Bolanos, "W-symmetries, anomalies and heterotic backgrounds with SU holonomy," [arXiv:2305.19793 [hep-th]].

My contribution on the last paper involving the symmetries of heterotic sigma models was centred on developing sections 2 and 3. I have also been involved in the checking of the paper throughout.

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Contents

Chapter 1

Introduction

The notion of symmetry has evolved as our comprehension of nature improves, from being closely related to harmony, unity and beauty in Plato's theories to our current most common definition as "invariance under a group of transformations." Symmetries play a fundamental role in our understanding of Physics and often they help to describe a system. These are extremely useful since they can be applied in diverse contexts, from relativistic to non-relativistic theories, as well as, classical and quantum regimes.

Symmetry is a powerful tool to study physical systems since it imposes constraints that limit the types of quantities that may appear in a theory or the form of its equations of motion or Lagrangian. It is often related to conserved quantities and allows us to get insights of certain properties of physical theories. It is instructive to explain the work done in this thesis to distinguish between two types of symmetries in physical systems, Noether-type symmetries and gauge symmetries. The first ones are continuous global symmetries that leave the action invariant and by Noether's theorem give conserved quantities. These are physical symmetries which sometimes are also referred to as rigid symmetries. While gauge symmetry is a local symmetry that strictly speaking is a redundancy in the way we describe a system rather than a symmetry that transforms a physical state into a different physical state of the system. One identifies all the states related by a gauge transformation with the same physical state. To see this consider Maxwell's equations of motion which do not uniquely define the evolution of the gauge potential A_μ but rather its equivalence class subject to a gauge transformation. Many properties desired in physical theories such as Lorentz invariance, locality and unitarity can be described manifestly in the dynamics using the redundancy of gauge fields. For instance, in electromagnetism gauge symmetry allows us to describe a photon with a four component field that transforms under the Lorentz group and remove two of its degrees of freedom to leave only two polarisation states.

In this thesis, for the most part we consider Noether-type symmetries which are not apparent in the spacetime but arise by analysing the dynamics of test particles propagating in different gravitational backgrounds. In this context these symmetries are often called "hidden symmetries." The ones analysed here are associated either with Killing-Stäckel or Killing-Yano tensors found in relativistic and spinning particles propagating on supersymmetric backgrounds. The generalised Killing-Yano forms investigated here emerge from a geometric structure called twisted covariant form hierarchy (TCFH) and the probe actions studied are expressed as one-dimensional non-linear supersymmetric sigma models.

Killing-Stäckel and (conformal) Killing-Yano forms have a long and distinguished history in general relativity as they have been used to investigate the integrability and separability properties of many classical equations, like the geodesic, Hamilton-Jacobi, Klein-Gordon, Dirac and Maxwell equations, on black hole spacetimes, see selected references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and reviews [11, 12]. In particular, Killing-Stäckel tensors generate (hidden) symmetries for relativistic particle probes propagating on gravitational backgrounds and so symmetries of the geodesic flow. While Killing-Yano forms, which can be thought of as the "square root" of Killing Stäckel tensors, generate (hidden) symmetries for spinning particle probes [13, 14, 15] propagating on gravitational backgrounds [16]. For some other applications, see also [17, 18, 19, 20, 21].

More recently, it has been demonstrated in [22] that the conditions imposed by the Killing spinor equations on the (Killing spinor) form bilinears of any supergravity theory, which may include higher order curvature corrections, can be arranged as a twisted covariant form hierarchy (TCFH) [23]. This means that these conditions can be written as

$$\mathcal{D}^{\mathcal{F}}_X\Omega = i_X\mathcal{P} + X \wedge \mathcal{Q} \ , \tag{1.1}$$

for every spacetime vector field X, where Ω is a multiform with components the form bilinears, $\mathcal{P}$ and $\mathcal{Q}$ are appropriate multi-forms which depend on the bilinears and the fields of the theory. Note that X also denotes the associated 1-form constructed from the vector field X after using the spacetime metric g, $X(Y) = g(X, Y)$. Furthermore $\mathcal{D}^{\mathcal{F}}$ is a connection on the space of forms which depends on the fluxes $\mathcal{F}$ of the supergravity theory that it is not necessarily form degree preserving. A consequence of the TCFH is that the form bilinears Ω satisfy a generalisation of the conformal Killing-Yano equation with respect the $\mathcal{D}^{\mathcal{F}}$ connection

$$(\mathcal{D}^{\mathcal{F}}_X\Omega)_k = i_X((d^{\mathcal{D}^{\mathcal{F}}}\Omega)_{k+1}) - \frac{1}{n-k+1}X \wedge (\delta^{\mathcal{D}^{\mathcal{F}}}\Omega)_{k-1} \ , \tag{1.2}$$

as one can easily verify by skew-symmetrising and taking the contraction with respect to the spacetime metric of (1.1)[1]. This raises the question of whether the form bilinears generate symmetries in appropriate probes propagating on supersymmetric backgrounds. Chapters 3-5 investigate the conditions under which such symmetries[2] occur in 4- and 5-dimensional minimal supergravities [27], as well as 11-dimensional supergravity [28] and the internal space of its warped AdS backgrounds [29].

Symmetries generated by Killing-Yano forms in sigma models were studied before the realisation of TCFHs, see [30, 31, 32, 33]. In particular, it has been known for a while that covariantly constant forms[3] with respect to a connection with torsion, $\hat{\nabla} = \nabla + \frac{1}{2}H$, generate symmetries in two-dimensional non-linear supersymmetric sigma models. Such connection corresponds to the TCFH for heterotic supergravity. Therefore, as an additional application of the TCFHs, chapter 6 focuses on the W-symmetry algebra generated by $\hat{\nabla}$-parallel forms and their chiral anomalies in (1,0) sigma models with heterotic backgrounds with SU(2) and SU(3) holonomy as target spaces [34].

This thesis is organized as follows. In chapter 2, we provide a review of the basics about Killing-Stäckel tensors, Killing-Yano forms, 11d supergravity, the chiral anomaly,

[1] An alternative way to state the above condition is that the highest weight representation in the decomposition of the tensor $\mathcal{D}^{\mathcal{F}}\Omega$ in orthogonal irreducible representations vanishes.

[2] This has also been explored in Type II supergravity theories [24, 25, 26]

[3] Covariantly constant forms can be seen as a special case of the Killing-Yano equation (2.11) with vanising right-hand side.

W-symmetries and the main developments utilized for the analysis done in the following chapters are introduced, such as the definition of TCFHs, the construction of spinning particles as one-dimensional non-linear supersymmetric sigma models and Wess-Zumino consistency conditions. Next, in chapter 3, we present the TCFHs of $D = 4$, $\mathcal{N} = 2$ and $D = 5$, $\mathcal{N} = 1$ minimal supergravities and the spinning particle actions invariant under symmetries generated by the TCFHs of these theories. In chapter 4, we present the TCFH of $D = 11$ supergravity and compute the Killing-Stäckel, Killing-Yano and closed conformal Killing-Yano tensors of all spherical symmetric M-branes which include M2-brane, M5-brane, KK-monopole and pp-wave, and demonstrate that their geodesic flows are completely integrable by giving all independent conserved charges in involution. Then we turn back our attention to the TCFH and analyse the symmetries generated by the Killing spinor bilinears in each M-brane. In chapter 5, we present the TCFH of the internal space of all warped AdS backgrounds of 11-dimensional supergravity and explore the hidden symmetries of spinning particle probes propagating on these backgrounds. We give examples of hidden symmetries for probes on some AdS backgrounds arising as near horizon geometries of intersecting M-branes as well as the maximal supersymmetric AdS backgrounds. Afterwards, in chapter 6, we show that the sigma models with supersymmetric heterotic backgrounds with SU(2) and SU(3) holonomy as target spaces are invariant under a W-symmetry algebra generated by the covariantly constant forms of these backgrounds. We prove that the chiral anomalies of all these symmetries are consistent at one-loop in perturbation theory and explore the conditions required to cancel them. Finally, in chapter 7 we explore the question of whether the form bilinears are sufficient to prove the integrability of particle probe dynamics on supersymmetric backgrounds and provide the conclusions of this thesis and some future directions.

Chapter 2

Review of background material

2.1 Killing-Stäckel and Killing-Yano tensors

Spacetime symmetries are transformations which preserve the spacetime geometry. Killing vectors (isometries) are responsible for continuous symmetry transformations called explicit symmetries. However, there are other types of symmetries that do not manifest explicitly in spacetime, i.e. do not generate spacetime diffeomorphisms, but can be discovered by studying the dynamics of the system. These symmetries are described with tensors of rank 2 or higher called Killing-Stäckel (KS) tensors and we refer to them as hidden symmetries. For a classical system, these are transformations that left invariant the dynamics; whereas for a quantum system, they are related to a set of phase space operators that commutes with the evolution operator and maps solutions into solutions.

Historically, Carter's work in 1968 had an influence on the study of hidden symmetries in the following years. He derived an additional integral of motion quadratic in momentum, nowadays known as Carter's constant, by showing the separability of the Hamilton-Jacobi and Klein-Gordon equations in the Kerr geometry [1, 2]. Followed in 1970 by Walker and Penrose' demonstration that the Kerr metric admits a rank 2 KS tensor responsible for the extra integral of motion [35]. Then Penrose [3] and Floyd [4] in 1973 showed that the KS tensor can be obtained from "squaring" a Killing-Yano (KY) tensor. Several discoveries regarding the separability and integrability of dynamical systems, Killing-Stäckel and Killing-Yano tensors have taken place since Carter's work. In the context of higher-dimensional black holes and the desire for a better understanding of the nature of gravitational theory, this topic has gained interest due to the discovery of non-trivial hidden symmetries in higher-dimensional rotating black holes which are related to the existence of KS and KY tensors. Moreover, the most general Kerr-NUT-(A)dS spacetimes admit a special geometric object called principal tensor which uniquely determines the geometry. In the sense that from it, one can construct a set of hidden and explicit symmetries. This principal tensor can be identified as a rank 2 closed conformal Killing-Yano (CCKY) tensor. For a detailed review of hidden symmetries in this context check [11, 12]

Phase space formalism is a natural choice to describe dynamical symmetries. It requires a symplectic manifold equipped with a symplectic form and a Hamiltonian function. In general, without further structures, all dynamical symmetries are described infinitesimally by flows in phase space that leave the symplectic structure and the Hamiltonian invariant. However, if one can identify the phase space as the cotangent bundle, T^*M, of a well-defined configuration space, M, then a distinction between explicit and hidden symmetries arises. In natural Hamiltonian systems,[1] explicit symmetries are those that admit conserved quantities linear in momenta, while conserved quantities of higher order in momenta are defined as hidden symmetries.

We called conserved quantities or constants/integrals of motion to those observables K which remains constant along the dynamical trajectories (geodesics). These quantities must (Poisson) commute with the Hamiltonian.

Killing vectors:

Continuous transformations of the spacetime into itself preserving the metric g are called isometries. These are generated by base manifold (configuration space) vectors $K = K^\mu(x)\frac{\partial}{\partial x^\mu}$ satisfying

$$\nabla_{(\mu}K_{\nu)} = 0 \iff \nabla_\mu K_\nu = \nabla_{[\mu}K_{\nu]} \ . \tag{2.1}$$

After projecting the Hamiltonian vector field of explicit symmetries into the configuration space, it reduces to a quantity that depends only on the spacetime variables, i.e. independent of momenta. Thus, we say that these symmetries are well-behaved transformations of the phase space into the base manifold. Let us give an example, consider a free relativistic particle with Hamiltonian $H = \frac{1}{2m}g^{ab}p_\mu p_\nu$ and the existence of Killing vectors, such that the constants of motion can be expressed as $C = K^\mu p_\mu$, $\{C, H\} = 0$. The corresponding Hamiltonian vector field reads $X_C = K^\mu\partial_{x^\mu} - \frac{\partial K^\nu}{\partial x^\mu}p_\nu\partial_{p_\mu}$. It is common to find in the literature that upon a canonical projection to the spacetime manifold, $\pi : T^*M \to M$, $\pi_* X_C$ reduces to the Killing vector, $K = K^\mu\partial_{x^\mu}$. However, it is more precise to say that $\pi_* X_C$ is the vector field that is π-related[2] to X_C as the push-forward map π, in general, does not map vector fields to vector fields.[3]

One can also see that a transformation is well defined in spacetime when the variation δx^μ is not proportional to p and $\delta H = 0$. In our previous example, this is the case since we get

$$\delta x^\mu = \epsilon\xi^\mu \ , \quad \delta p_\mu = -\epsilon\frac{\partial\xi^\nu}{\partial x^\mu}p_\nu, \ , \quad \delta H = 0 \ , \tag{2.2}$$

where ϵ is an infinitesimal parameter.

Killing tensors:

Conserved quantities of higher order in momenta lead to hidden symmetries described by Killing tensors. For every integral of motion that is monomial in momenta, there exists a Killing tensor. Suppose there is an integral of motion $C = K^{\mu_1\cdots\mu_s}(x)p_{\mu_1}\cdots p_{\mu_s}$, then its Poisson bracket with the Hamiltonian for the free relativistic particle gives

$$\{C, H\} = \nabla^{\mu_0}K^{\mu_1\cdots\mu_s}p_{\mu_0}p_{\mu_1}\cdots p_{\mu_s} \ . \tag{2.3}$$

For C to be a constant of motion, it must Poisson commute with the Hamiltonian. Since this must hold for an arbitrary number of p_a's, the tensor K must satisfy

$$\nabla^{(\mu_0}K^{\mu_1\cdots\mu_s)} = 0 \ . \tag{2.4}$$

[1]Physical systems with a metric defined on configuration space and Hamiltonian function quadratic in momenta. From now on we will consider only natural Hamiltonians and drop the word natural.

[2]A vector field X on M and a vector field Y on N are φ-related if for all $f \in C^\infty(N)$, $X(\pi^* f) = \varphi^*(Y(f))$ or equivalently $\varphi(X_m) = Y_{\varphi(m)}$ for all points $m \in M$.

[3]There are two problems: if π is not surjective there is no candidate to define the push-forward for the section $M \to T^*M$ away from the image of π. If π is not injective there may be more than one candidate over some points in the image of π.

This equation defines a symmetric tensor of rank s called Killing-Stäckel tensor. For $s = 1$ we recover Killing vectors, hence KS tensors are symmetric generalizations of the latter ones. The phase space symmetry associated is generated by the Hamiltonian vector field

$$X_C = sK^{\mu\nu_2\cdots\nu_s} p_{\nu_2} \cdots p_{\nu_s} \partial_{x^\mu} - \frac{\partial K^{\nu_1\cdots\nu_s}}{\partial x^\mu} p_{c_1} \cdots p_{\nu_s} \partial_{p_\mu} \ . \tag{2.5}$$

After projecting into the base manifold gives

$$\pi_* X_C = sK^{\mu\nu_2\cdots\nu_s} p_{\nu_2} \cdots p_{\nu_s} \partial_{x^\mu} \ . \tag{2.6}$$

Here again, one should say that $\pi_* X_C$ is the vector field that is π-related to X_C. For $s \geq 2$, $\pi_* X_C$ cannot be seen as a pure spacetime quantity since it still depends on momenta. Hidden symmetries do not have a simple description in spacetime.

As before, the same conclusion can be derived from the perspective of the canonical transformations generated by the conserved quantity C

$$\delta x^\mu = \epsilon s K^{\mu\nu_2\cdots\nu_s} p_{c_2} \cdots p_{\nu_s} \ , \quad \delta p_\mu = -\epsilon \frac{\partial K^{\nu_1\cdots\nu_s}}{\partial x^\mu} p_{\nu_1} \cdots p_{\nu_s} \ . \tag{2.7}$$

Now both variations are proportional to p, and hence its transformation to the spacetime is not well defined.

A symmetrized product of two KS tensors gives again a KS tensor of rank $s = s_1 + s_2$. If a KS tensor can be decomposed in terms of symmetrized products other KS tensors and Killing vectors it is called reducible.

Conformal Killing vectors and Killing-Stäckel tensors:
These are generalizations for the propagation of light of the objects previously discussed given by

$$\nabla^{(\mu_0} K^{\mu_1\cdots\mu_s)} = g^{(\mu_0\mu_1} \alpha^{\mu_2\cdots\mu_s)}, \tag{2.8}$$

where α is a symmetric tensor of rank $s-1$. It is related to the divergence of K and derivatives of its traces. For $s = 1$ we get conformal Killing vectors whereas for $s > 2$ we obtain conformal Killing-Stäckel (CKS) tensors. They provide conserved quantities along null geodesics [35, 36]. Similarly, one can get another CKS tensor by the symmetrized product of CKS tensors and for $s \geq 2$ they give rise to hidden symmetries.

Conformal Killing-Yano forms:
One way to define the Killing-Yano tensor is by studying the decomposition of the covariant derivative of an antisymmetric form ω into its irreducible parts [37]. The Killing-Yano family is given by the decomposition depending only on the exterior derivative and the divergence (co-derivative) parts. The most general case is called conformal Killing-Yano (CKY) p-form given by

$$\nabla_\mu \omega_{\mu_1\cdots\mu_p} = \nabla_{[\mu} \omega_{\mu_1\cdots\mu_p]} + \frac{p}{D-p+1} g_{\mu[\mu_1} \nabla^\nu \omega_{|\nu|\mu_2\cdots\mu_p]} \tag{2.9}$$

or in differential notation

$$\nabla_X \omega = \frac{1}{p+1} i_X d\omega - \frac{1}{D-p+1} \alpha_X \wedge \delta\omega \ . \tag{2.10}$$

Killing-Yano forms are those with vanishing co-derivative

$$\nabla_\mu \omega_{\mu_1 \cdots \mu_p} = \nabla_{[\mu} \omega_{\mu_1 \cdots \mu_p]} \iff \nabla_X \omega = \frac{1}{p+1} i_X d\omega \ . \tag{2.11}$$

Note that this equation is the antisymmetric generalisation of the Killing vector equation [38]. On the other hand, if the covariant derivative depends only on the divergence part we get closed conformal Killing-Yano (CCKY) forms [39, 40]

$$\nabla_\mu \omega_{\mu_1 \cdots \mu_p} = \frac{p}{D-p+1} g_{\mu[\mu_1} \nabla^\nu \omega_{|\nu|\mu_2 \cdots \mu_p]} \iff \nabla_X \omega = -\frac{1}{D-p+1} \alpha_X \wedge \delta\omega \ . \tag{2.12}$$

Properties:

- The Hodge dual of a CKY form is again a CKY form.
- The Hodge dual of a CCKY form is a KY form.
- (C)KY "square" to (C)KS, i.e. the symmetrized product of two (C)KY p-forms, ω_1 and ω_2, is a rank 2 (C)KS tensor.

$$K^{\mu\nu} = \omega^{(\mu}_{(1)\rho_2 \cdots \rho_p} \omega^{\nu)\rho_2 \cdots \rho_p}_{(2)} \ . \tag{2.13}$$

- Having two CCKY p-form and q-form. Their exterior (wedge) product is a CCKY $(p+q)$ form

Principal tensor:
A principal h is a non-degenerate[4] CCKY 2-form satisfying

$$\nabla_\rho h_{\mu\nu} = g_{\rho\mu} \xi_\nu - g_{\rho\nu} \xi_\mu \ , \quad \xi_\mu = \frac{1}{D-1} \nabla^\nu h_{\nu\mu} \ , \tag{2.14}$$

where the indices are raised with g and it can be shown that ξ is a Killing vector called the primary Killing vector. In order to check this one needs to plug the expression above in the Killing vector equation (2.1) and use the Einstein space condition, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, where Λ is the cosmological constant.

From a principal tensor h one can construct a sequence of various symmetry objects called the Killing tower [8, 41, 42].

Here we will show a sketch of the objects we can build:

- CCKY $h^{(j)}$ of rank $2j$ by taking its wedge powers

$$h^{(j)} = \frac{1}{j!} h^{\wedge j} \ . \tag{2.15}$$

- KY forms $\omega^{(j)}$ of rank $(D-2j)$ by taking the Hodge duals

$$\omega^{(j)} = *h^{(j)} \ . \tag{2.16}$$

[4]The condition of non-degeneracy implies that h has the maximal (matrix) rank and the maximal number of functionally independent eigenvalues.

- Rank 2-KS tensors $K_{(j)}$ by the symmetrized product of two KY

$$K^{\mu\nu}_{(j)} = \frac{1}{(D-2j-1)!}\omega^{(j)(\mu}{}_{\rho_2\cdots\rho_{D-2j-1}}\omega^{(j)\nu)\rho_2\cdots\rho_{D-2j-1}} \ . \tag{2.17}$$

- Rank 2-CKS tensors $Q_{(j)}$ by the symmetrized product of two CCKY

$$Q^{\mu\nu}_{(j)} = \frac{1}{(2j-1)!}h^{(j)(\mu}{}_{\rho_2\cdots\rho_{D-2j-1}}h^{(j)\nu)\rho_2\cdots\rho_{D-2j-1}} \ . \tag{2.18}$$

- Killing vectors $f_{(j)}$

$$f_{(j)} = K_{(j)} \cdot \xi \ . \tag{2.19}$$

where " $\cdot$ " means the contraction of two tensors in adjacent indices.

All the description discussed until now is for vacuum solutions, in the presence of fluxes, at the classical level is not always possible to build symmetries to generalised the ones presented without flux and at the quantum level there might be anomalies. There are different generalisations of KS and KY tensors, in supergravity theories, one is that of a connection twisted by a 3-form torsion [43, 44, 45, 46] which can be identified naturally with one of the fluxes in the theory. For instance in 5D minimal supergravity the torsion can be identified with $\frac{1}{\sqrt{3}} * F$. However, the generalisation used in this thesis arises from the twisted covariant form hierarchies which will be introduced later.

2.2 Separability and Integrability

Integrable systems are nonlinear differential equations which in principle can be solved analytically. This means that the solution can be reduced to a finite number of algebraic operations and integrations. A very informal and naive approach to integrability can be stated in the following ways:

- Classical level: An integrable system is a dynamical system in which all its classical solutions can be expressed in a closed form in terms of simple functions.

- Quantum level: An integrable system requires that the eigenvalues and eigenfunctions of a set of characteristic observables can be evaluated in a closed form.

Integrals of motion facilitate the study of dynamical systems. A dynamical system with N degrees of freedom is called completely integrable if the number of independent integrals of motion that are in involution (Poisson commuting) is equal to the number of degrees of freedom, as a consequence, its solution can be written in terms of integrals. This result is known as the Liouville theorem.

The existence of a principal tensor and therefore a Killing tower implies complete integrability of geodesic motion in all dimensions.

Another method to study the geodesic motion in any dimension is using the Hamilton-Jacobi equation. This a partial differential equation for Hamilton's principal function $S(q,t)$

$$\frac{\partial S}{\partial t} + H(x^\mu, \frac{\partial S}{\partial x^\nu}, t) = 0 \ . \tag{2.20}$$

A complete integral is defined as a function $S = \bar{S}(t, q^\mu, P_\nu) + C$, where P_ν and C are constants. The complete integral can be interpreted as the generating function of a canonical transformation between the original Hamiltonian system and a new one with variables Q^μ, P_ν and H'. The relations between the variables are given by

$$\begin{aligned} p_\nu(t, q, P) &= \frac{\partial S}{\partial q^\nu} \ , \\ Q^\mu(t, q, P) &= \frac{\partial S}{\partial P_\mu} \ , \\ H' &= H + \frac{\partial S}{\partial t} \ . \end{aligned} \tag{2.21}$$

The function S generates a canonical transformation that trivialises the system since eq. (2.20) implies $H' = 0$. Then Q and P are constants of motion after assuming that the second equation in (2.21) is invertible one gets $q = q(Q, P, t)$ and $p = p(Q, P, t)$

For a time-independent Hamiltonian, the principal function can be solved with the ansatz $S(q, P, t) = W(q, P) - Et$. The Hamilton-Jacobi equation reads

$$H(q^\mu, \frac{\partial S}{\partial q^\nu}) = E \ , \tag{2.22}$$

where the function S is called Hamilton's characteristic function and the constant E is the energy.

For a natural Hamiltonian with a positive definite, time-independent Riemannian metric and orthogonal coordinates such that $g^{\mu\nu} = 0$ for $\mu \neq \nu$, the coordinate system is separable if the Hamiltonian-Jacobi equation

$$\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\nu} = E \ , \tag{2.23}$$

admits a complete solution of the form

$$S(q, P) = \sum_{\mu=1}^{n} S_\mu(q^\mu, P) \ , \tag{2.24}$$

with $\det \left[\frac{\partial^2 S}{\partial P \partial q} \right] \neq 0$.

The Hamiltonian-Jacobi equation is separable on a Riemannian manifold if and only if it admits r Killing vectors $\kappa_{(i)}$ $(i = 0, \cdots r-1)$ and $D - r$ rank 2 Killing tensors $K_{(\alpha)}$ $(\alpha = 1, \cdots D - r)$, all of them independent and satisfy the following two properties

- All mutually (Nijenhuis-Schouten) commute

$$\left[K_{(\alpha)}, K_{(\beta)}\right]_{NS} = 0 \ , \quad \left[\kappa_{(i)}, K_{(\beta)}\right]_{NS} = 0 \ , \quad \left[\kappa_{(i)}, \kappa_{(j)}\right]_{NS} = 0 \ . \tag{2.25}$$

- Killing tensors K_α have in common $D - r$ eigenvectors m_α, such that

$$\left[K_{(\alpha)}, K_{(\beta)}\right] = 0 \ , \quad \left[K_{(\alpha)}, \kappa_{(i)}\right] = 0 \ , \quad g(m_{(\alpha)}, \kappa_{(i)}) = 0 \ . \tag{2.26}$$

2.3 Kerr geometry

The Kerr geometry was the first example where it was realized that the existence of Killing-Stäckel and Killing-Yano tensors play a big role in the integrability of the geodesic flow. This discovery motivated several discoveries about the separability and integrability of dynamical systems and put Killing-Yano tensors on the map for many physicists. For these reasons, it is instructive to discuss the Kerr geometry from a perspective of hidden symmetries. The details of the Kerr metric can be found in standard textbooks [47, 48, 49, 5]

The Kerr metric is the most general stationary vacuum solution of Einstein's equation in an asymptotically flat spacetime describing a rotating black hole with a regular event horizon. In Boyer-Lindquist coordinates it is given by

$$g = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt d\phi + \frac{A\sin^2\theta}{\Sigma} d\phi^2 + \frac{\Sigma}{\Delta_r} dr^2 + \Sigma d\theta^2 \ , \tag{2.27}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta_r = r^2 - 2Mr + a^2$, $A = (r^2+a^2)^2 - \Delta_r a^2 \sin^2\theta$. Note that it has two commuting Killing vectors, $\xi_{(t)} = \partial_t$ and $\xi_{(\phi)} = \partial_\phi$, since it is independent of t and ϕ. The first one generates translations in time while the second one is the generator of axial rotations. The conditions to specify these Killing vectors uniquely[5] are the following: $\xi_{(t)}$ is timelike at infinity, meaning the metric is stationary, and the integral lines of $\xi_{(\phi)}$ are closed. In the exterior of the black hole, the points where $\xi_{(\phi)} = 0$, i.e. when $\theta = 0, \pi$, form a regular two-dimensional geodesic submanifold, called the axis of symmetry; the metric is axisymmetric. The induced metric in the axis is

$$\gamma = -F dt^2 + F^{-1} dr^2 \ , \quad F = \frac{\Delta_r}{r^2 + a^2} \ . \tag{2.28}$$

The Kerr metric is characterized by two parameters, the mass, M, and rotation parameter, $a \leq M$, associated with the angular momentum of the black hole, $J = aM$. Taking the limit $r \to \infty$, the metric simplifies to

$$g \approx -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4Ma\sin^2\theta}{r} dt d\phi + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ . \tag{2.29}$$

The Boyer-Lindquist form gives the Kerr metric in a convenient coordinate system that generalises the Schwarzschild coordinates. However, one can perform a coordinate transformations in which the hidden symmetry is more evident.

$$y = a\cos\theta \ , \quad \psi = \frac{\phi}{a} \ , \quad \tau = t - a\phi \ . \tag{2.30}$$

Then the Kerr metric reads

$$g = \frac{1}{\Sigma}\left[-\Delta_r \left(d\tau + y^2 d\psi\right)^2 + \Delta_y \left(d\tau - r^2 d\psi\right)^2\right] + \Sigma\left[\frac{dr^2}{\Delta_r} + \frac{dy^2}{\Delta_y}\right] \ , \tag{2.31}$$

where $\Sigma = r^2 + y^2$, $\Delta_r = r^2 - 2Mr + a^2$ and $\Delta_y = a^2 - y^2$. These coordinates (τ, r, y, ψ) are sometimes referred as canonical coordinates. It is common not to specify the functions

[5] Any linear combination of the Killing vectors with constant coefficients is again a Killing vector.

Δ_r and Δ_y and work with arbitrary functions instead. The downside is that in this form the metric is not a solution to Einstein's equations, and it is called off-shell metric.

The metric (2.31) admits a principal tensor [11]

$$h = ydy \wedge \left(d\tau - r^2 d\psi\right) - rdr \wedge \left(d\tau + y^2 d\psi\right) \ , \tag{2.32}$$

which can be generated from a potential b, $h = db$, given by

$$b = -\frac{1}{2}\left[\left(r^2 - y^2\right) d\tau + r^2 y^2 d\psi\right] \ . \tag{2.33}$$

It can be verified by a straightforward computation that h satisfies the CCKY equation. In addition, one can construct the following Killing tower from it.

- Killing-Yano tensor $f = \star h$

$$f = rdy \wedge \left(d\tau - r^2 d\psi\right) + ydr \wedge \left(d\tau + y^2 d\psi\right) \ . \tag{2.34}$$

- Conformal Killing tensor $Q_{\mu\nu} = h_{\mu\rho} h_\nu{}^\rho$

$$Q = \frac{1}{\Sigma}\left[r^2 \Delta_r \left(d\tau + y^2 d\psi\right)^2 + y^2 \Delta_y \left(d\tau - r^2 d\psi\right)^2\right] + \Sigma\left[\frac{y^2 dy^2}{\Delta_y} - \frac{r^2 dr^2}{\Delta_r}\right] \ . \tag{2.35}$$

- Killing tensor $K_{\mu\nu} = f_{\mu\rho} f_\nu{}^\rho$

$$K = \frac{1}{\Sigma}\left[y^2 \Delta_r \left(d\tau + y^2 d\psi\right)^2 + r^2 \Delta_y \left(d\tau - r^2 d\psi\right)^2\right] + \Sigma\left[\frac{r^2 dy^2}{\Delta_y} - \frac{y^2 dr^2}{\Delta_r}\right] \ . \tag{2.36}$$

- A primary Killing vector $\xi^\mu_{(\tau)} = \frac{1}{3}\nabla_\nu h^{\nu\mu} = \partial^\mu_\tau$.
- Secondary Killing vectors $\xi^\mu_{(\psi)} = -K^\mu{}_\nu \xi^\nu_{(\tau)} = \partial^\mu_\psi$.

The primary Killing vector is timelike at infinity, as expected since the metric is stationary. Furthermore, one can construct a Killing vector as a linear combination, $\xi_{(\phi)} = a^{-1}\xi_\psi - a\xi_{(\tau)} = \partial_\phi$ whose fixed points form the axis of symmetry and its integral lines are closed cycles, indicating the metric is axisymmetric.

The Killing vectors, $\xi_{(\tau)}$ and $\xi_{(\psi)}$, the Killing tensor, K, and the metric, g, are all independent and mutually (Nijenhuis-Schouten) commute. As a consequence, the following integrals of motion associated with these objects are independent and in involution, hence, the geodesic motion is completely integrable [1, 2, 35, 3, 4].

$$\begin{aligned} &g^{\mu\nu} p_\mu p_\nu = -m^2 \ , \quad K^{\mu\nu} p_\mu p_\nu = \mathcal{K} \ , \quad p_\tau \equiv \xi^\mu_{(\tau)} p_\mu = -E \ , \\ &p_\psi \equiv \xi^\mu_{(\psi)} p_\mu = L_\psi = aL_\phi - a^2 E \ , \end{aligned} \tag{2.37}$$

where p_μ is the four-momentum of a free relativistic particle with mass m, and E and L_ϕ are its energy and angular momentum, respectively. The conserved quantity $\mathcal{K}$ is the analogue of the Carter constant for the off-shell metric.

2.4 Non-linear sigma models

Non-linear sigma models have been studied for more than 60 years, going back to the work of Gell-Mann and Lévy [50]. Since then they have found widespread applications. In this brief review, we have skipped some details and encourage the reader to check [51, 52, 53, 54] and references within.

A non-linear sigma model is a scalar field theory whose fields take values in a Riemannian manifold M, called the target space. The action of a theory of n free scalar fields $X^i(x)$, $i = 1, \dots n$, in a d-dimensional spacetime N with coordinates x^μ, $\mu, \nu, \dots = 0, 1, \dots d-1$, and metric $\gamma_{\mu\nu}$ is given by

$$S = \frac{1}{2}\mu^{2-d} \int d^d x \sqrt{\gamma}\gamma^{\mu\nu} g_{ij}(X(x)) \partial_\mu X^i \partial_\nu X^j \ . \tag{2.38}$$

The fields X^i can be thought of as coordinates of the curved manifold M with metric g_{ij}. The X^i are dimensionless and the mass scale μ is required to make the action dimensionless. Note that this coupling makes the theory non-renormalizable for $d \geq 3$ whereas in $d = 2$ is absent and hence renormalizable. It is common to choose units where $\mu = 1$.

The action is independent of the choice of coordinates on M and N since it is invariant under the general coordinate transformation

$$X^i \to X'^i, \quad g_{ij}(X) \to g'_{ij}(X') = g_{k\ell} \frac{\partial X^k}{\partial X'^i} \frac{\partial X^\ell}{\partial X'^j} \ , \tag{2.39}$$

or, for some transformation with respect to an infinitesimal vector field v^i

$$X'^i = X^i + v^i(X), \quad g'_{ij}(X') = g_{ij} - 2\nabla_{(i} v_{j)}(X) \ . \tag{2.40}$$

Two sigma models with metrics g_{ij} and g'_{ij}, which are related by coordinate transformations as in (2.39) or (2.40) are equivalent; hence, they describe the same physics. Consider the action (2.38) with metric as in (2.40), then it satisfies

$$S[\phi, g'_{ij}(X)] = S[\tilde{X}, g_{ij}(\tilde{X})] \ , \tag{2.41}$$

where $\tilde{X}^i = X^i - v^i$. This means that $S[X, g'(X)]$ is related to (2.38) by redefining $X^i \to \tilde{X}^i$. Thus the sigma model is determined by an equivalence class of metrics related by diffeomorphisms.

In addition, one can introduce a spacetime reparameterization invariant term, called the Wess-Zumino term, constructed from the alternating tensor on N $\epsilon_{\mu\nu\cdots\rho}$, providing that $n \leq d$.

$$S_{WZ} = \frac{q}{n!} \int d^n x \sqrt{\gamma} \epsilon^{\mu_1 \cdots \mu_n} b_{i_1 \cdots i_n}(X) \partial_{\mu_1} X^{i_1} \cdots \partial_{\mu_n} X^{i_n} \ , \tag{2.42}$$

where q is a constant. This action is M-coordinate independent if b is a n-form on M, $b = \frac{1}{n!} b_{i_1 \cdots i_n} d\phi^{i_1} \wedge \cdots \wedge d\phi^{i_n}$. The variation of this term under shifts $b \to b + d\lambda$ for any $(n-1)$ form λ is only a surface term

$$\delta b_{i_1 \cdots i_n} = \partial_{[i_1 \lambda_{i_2} \cdots i_n]} \ . \tag{2.43}$$

The field equation for X^i obtained from $\delta(S + S_{WZ})$ is given by

$$-g_{ij}\left(\frac{1}{\sqrt{\gamma}}\partial_\mu\gamma^{\mu\nu}\sqrt{\gamma}\partial_\nu X^j + \Gamma^j_{k\ell}\partial_\mu X^k\partial^\mu X^\ell\right)$$
$$+q\frac{(n+1)}{n!}H_{i_1\cdots i_n}\epsilon^{\mu_1\cdots\mu_n}\partial_{\mu_1}X^{i_1}\cdots\partial_{\mu_n}X^{i_n} = 0 \ , \qquad (2.44)$$

where $H = db$ is the field strength. It is well known that if H is closed but not exact, the charge q must be quantized to have a consistent quantum theory. If b is globally defined then there is no quantization condition. The field equation (2.44) is well-defined since it does not depend explicitly on b. However, the action (2.42) is not well defined since it involves b, so one could find potential obstacles in quantization.

One can couple spacetime fermions $\lambda^A(x)$ labelled by an internal index $A = 1, \ldots, r$ to the sigma model through

$$S_F = i\int d^dx\sqrt{\gamma}G_{AB}(\phi(x))\bar{\lambda}^A\rho^\mu D_\mu\lambda^B \ , \qquad (2.45)$$

where ρ^μ denotes the spacetime gamma matrices[6] satisfying $\rho^\mu\rho^\nu + \rho^\nu\rho^\mu = 2\gamma^{\mu\nu}$ and the covariant derivative is $D_\mu\lambda^A = \nabla_\mu\lambda^A + \mathsf{A}_i{}^A{}_B(X(x))\partial_\mu X^i\lambda^B$. Here ∇_μ is a derivative covariant with respect to spacetime Lorentz transformations[7].

2.4.1 1D-Supersymmetric non-linear sigma models

In this thesis, we write the actions for supersymmetric sigma models using superfields, which has the advantage that supersymmetry is manifest and writing down the actions is straightforward. First, let us review some basic aspects about supersymmetry and Minkowski superspace [55, 56, 57, 58, 59, 60].

At the algebra level, supersymmetry is an extension of the d-dimensional Poincaré algebra which incorporates anticommuting charges Q. The graded Poincaré algebra is generated by the Lorentz generators, M, the translation generators, P, and the supersymmetry generators, Q. It might include an internal symmetry group G and central charge generators, Y and Z, which commute with all other generators. The central charges exist only in extended supersymmetry, $N > 1$.

Supersymmetry is inherently manifest in superspace by construction. Besides the even coordinates x^μ from Minkowski space, superspace introduces odd Grassmann coordinates carrying a spinor index, θ^α. Minkowski superspace can be defined as the coset space (super Poincaré group)/(Lorentz group).

A superfield is a function encoding the spacetime coordinates and the Grassmann coordinates, $X = X(x, \theta)$. Superfields can be expanded in a terminating Taylor series in θ.[8] In particular for 1-dimension and $N = 1$, we have

$$X^\mu(t,\theta) = X^\mu| + \theta DX^\mu| =: x^\mu(t) + \theta\lambda(t) \ , \qquad (2.46)$$

where $|$ indicates the part independent of θ, i.e. $|_\theta = 0$. The number of supersymmetries N is given by the α-index in θ^α and is the same as that on Q^α. The supercharges Q are defined to anti-commute with the supersymmetry covariant derivatives, $\{D, Q\} = 0$.

[6] Here following standard notation $\gamma^{\mu\nu}$ corresponds to the spacetime metric

[7] This means that ∇_μ is built as an Ehresmann connection on a bundle of orthonormal frames.

[8] This is a consequence of Grassmann coordinates being nilpotent.

These objects can be written in terms of differential operators. For 1-dimension and N=1, the supersymmetry charge and the covariant derivative D are given by

$$Q = i\frac{\partial}{\partial_\theta} + \theta\frac{\partial}{\partial_t} \ , \quad D = \frac{\partial}{\partial_\theta} + i\theta\frac{\partial}{\partial_t} \ , \tag{2.47}$$

which satisfy the supersymmetry algebra $\{Q,Q\} = 2i\partial_t = 2P$ and $D^2 = i\partial_t$. The components of the superfield in (2.46) are derived by projecting out the θ-independent part. This means that the leading term $X| = x$ is a bosonic scalar field and the components $DX| = \lambda$ are fermionic (Grasmmann odd) since the operators D are odd. The physical fields with propagating degrees of freedom are x and λ.

The action of the 1d, $N = 1$ supersymmetric sigma model reads

$$S = -\frac{i}{2}\int dtd\theta g_{\mu\nu}DX^\mu\partial_t X^\nu \ , \tag{2.48}$$

where $\partial_t = \frac{\partial}{\partial_t}$. Note that this action contains the bosonic action after integrating over the odd coordinate, θ, using the properties of the Berezin integral and expanding in the components of the superfield X

$$\begin{aligned} S &= -\frac{i}{2}\int dtD\left(g_{\mu\nu}DX^\mu\partial_t X^\nu\right)| \\ &= \int dt\left(\underbrace{\frac{1}{2}g_{\mu\nu}\partial_t x^\mu\partial_t x^\nu}_{\text{Bosonic action}} + \frac{i}{2}g_{\mu\nu}\lambda^\mu\partial_t\lambda^\nu - \frac{i}{2}\partial_\rho g_{\mu\nu}\lambda^\rho\lambda^\mu\partial_t x^\nu\right) \\ &= \int dt\left(\underbrace{\frac{1}{2}g_{\mu\nu}\partial_t x^\mu\partial_t x^\nu + \frac{i}{2}g_{\mu\nu}\lambda^\mu\nabla_t\lambda^\nu}_{\text{Spinning particle action}}\right) \ , \end{aligned} \tag{2.49}$$

where $\nabla_t\lambda^\nu = \partial_t\lambda^\nu + \Gamma^\nu_{\rho\sigma}\partial_t x^\rho\lambda^\sigma$. Therefore, the non-linear supersymmetric sigma model in 1 dimension describes spinning particle actions.

The most general $N = 1$ sigma model in 1d [61] with dimensionless couplings and attaching mass zero to X and introducing a fermionic superfield ψ with mass dimension $[\frac{1}{2}]$ is given by

$$\begin{aligned} S = -i\int dtd\theta\Big(&\frac{1}{2}g_{\mu\nu}DX^\mu\partial_t X^\nu + \frac{1}{3!}c_{\mu\nu\rho}DX^\mu DX^\nu DX^\rho - \frac{i}{2}h_{ab}\psi^a\nabla\psi^b + \frac{1}{3!}l_{abc}\psi^a\psi^b\psi^c \\ &+ f_{\mu a}\partial_t X^\mu\psi^a + \frac{i}{2!}m_{\mu ab}\psi^a\psi^b DX^\mu + \frac{1}{2!}n_{\mu\nu a}DX^\mu DX^\nu\psi^a\Big) \ , \end{aligned} \tag{2.50}$$

where $\nabla\psi^a = D\psi^a + DX^\mu A_\mu{}^a{}_b\psi^b$, with A being a connection.

2.5 Symmetries in probe actions

We have introduced Killing-Stäckel and the Killing-Yano family from the Hamiltonian formalism. However, in this thesis we look at the symmetries generated by Killing-Yano forms in spinning particle probe actions propagating in supersymmetric backgrounds. We shall summarise the applications of KS, KY and CKY tensors in generating symmetries for particle actions. Consider the action of a relativistic particle probe propagating on a spacetime M with metric g

$$S = \frac{1}{2} \int dt \, g_{\mu\nu} \, \dot{x}^\mu \, \dot{x}^\nu \ , \tag{2.51}$$

where $\dot{x}$ denotes the derivative of the coordinate x with respect to t. The equations of motion are those of the geodesic flow on M with affine parameter t. Given a rank k KS tensor on M, i.e. a symmetric $(0, k)$ tensor d on M which satisfies

$$\nabla_{(\mu} d_{\nu_1 \nu_2 \cdots \nu_k)} = 0 \ , \tag{2.52}$$

where ∇ is the Levi-Civita connection of g, the action (2.51) is invariant under the infinitesimal transformations

$$\delta x^\mu = \epsilon d^\mu{}_{\nu_1 \cdots \nu_{k-1}} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_{k-1}} \ , \tag{2.53}$$

with parameter ϵ. The associated conserved charge is

$$Q(d) = d_{\nu_1 \nu_2 \cdots \nu_k} \, \dot{x}^{\nu_1} \, \dot{x}^{\nu_2} \cdots \dot{x}^{\nu_k} \ . \tag{2.54}$$

For $k = 1$, d is a Killing vector field. The symmetrised tensor product of two KS tensors is also a KS tensor. Hidden symmetries are those generated by rank $k \geq 2$ KS tensors d with $d \neq g$.

Recall that a spinning particle probe propagating on a spacetime M with metric g is described by the action

$$S = -\frac{i}{2} \int dt \, d\theta \ g_{\mu\nu} \, DX^\mu \, \dot{X}^\nu \ , \tag{2.55}$$

where t and θ are the even and odd coordinates of the worldline superspace, respectively, X are worldline superfields $X = X(t, \theta)$ and $D^2 = i\partial_t$. Spinning particles are supersymmetric extensions of relativistic particles.

Given a KY form, α, on M, the infinitesimal transformation

$$\delta X^\mu = \epsilon \, \alpha^\mu{}_{\nu_1 \cdots \nu_{k-1}} DX^{\nu_1} \cdots DX^{\nu_{k-1}} \ , \tag{2.56}$$

with parameter ϵ leaves the spinning particle action (2.55) invariant[9].

$$\begin{aligned}
\delta S \quad &= -\frac{i}{2} \int dt \, d\theta \ (-2\delta X^\mu g_{\mu\nu} \, \nabla_t DX^\nu) \\
&= i \int dt \, d\theta g_{\mu\nu} (\epsilon \alpha^\mu{}_{\nu_1 \cdots \nu_{k-1}} DX^{\nu_1} \cdots DX^{\nu_{k-1}}) \nabla_t DX^\nu \\
&= i \int dt \, d\theta \frac{\epsilon}{k} \alpha_{\nu\nu_1 \cdots \nu_{k-1}} \nabla_t (DX^{\nu_1} \cdots DX^{\nu_{k-1}} DX^\nu) \\
&= -i \int dt \, d\theta \frac{\epsilon}{k} \nabla_\mu \alpha_{\nu\nu_1 \cdots \nu_{k-1}} \partial_t x^\mu DX^{\nu_1} \cdots DX^{\nu_{k-1}} DX^\nu \\
&= - \int dt \, d\theta \frac{\epsilon}{k(k+1)} \nabla_\mu \alpha_{\nu\nu_1 \cdots \nu_{k-1}} D(DX^\mu DX^{\nu_1} \cdots DX^{\nu_{k-1}} DX^\nu)
\end{aligned}$$

[9]It is clear that this result can be extended to sigma models in two dimensions with adequate modifications to DX and $\partial_t X$ such that the corresponding superspace differential operators anti-commute with the supercharges Q.

$$= 0 \iff \nabla_\mu \alpha_{\nu\nu_1\cdots\nu_{k-1}} = \nabla_{[\mu}\alpha_{\nu\nu_1\cdots\nu_{k-1}]} \ , \tag{2.57}$$

where $\nabla_t DX^\nu = \partial_t DX^\nu + \Gamma^\nu{}_{\rho\sigma}\partial_t X^\rho DX^\sigma$. The associated conserved charge is

$$Q(\alpha) = (k+1)\alpha_{\nu_1\nu_2\cdots\nu_k}\partial_t X^{\nu_1} DX^{\nu_2}\cdots DX^{\nu_k} - \frac{i}{k+1}(d\alpha)_{\nu_1\cdots\nu_{k+1}} DX^{\nu_1}\cdots DX^{\nu_{k+1}} \tag{2.58}$$

Observe that $Q(\alpha)$ is preserved, $DQ(\alpha) = 0$, subject to the equations of motion of (2.55). Note that if $d\alpha = 0$ and so α is covariantly constant (or equivalently parallel) with respect to the Levi-Civita connection, then

$$\tilde{Q}(\alpha) = \alpha_{\nu_1\nu_2\cdots\nu_k} DX^{\nu_1} DX^{\nu_2}\cdots DX^{\nu_k} \ , \tag{2.59}$$

is also conserved subject to the field equations of (2.55), $\partial_t \tilde{Q}(\alpha) = 0$. There are several generalisations of the KS and CKY tensors, see e.g. [62, 43, 44, 46, 18, 63, 30, 31].

The commutator algebra of transformations (2.56) generated by spacetime forms has been examined in detail in [32, 33]. Given two symmetries (2.56) generated by the k-form α and ℓ-form β, the commutator contains two types of terms. One term depends on the Nijenhuis tensor of α and β and the other term is the transformation

$$\delta X^\mu = \epsilon(\alpha \cdot_s \beta)^\mu{}_{\nu\lambda_1\ldots\lambda_{k+\ell-4}}\partial_t X^\nu DX^{\lambda_1}\ldots DX^{\lambda^{k+\ell-4}} \ , \tag{2.60}$$

generated by the tensor

$$(\alpha \cdot_s \beta)_{\mu\nu\lambda_1\ldots\lambda_{k+\ell-4}} = \alpha_{\mu\kappa[\lambda_1\ldots\lambda_{k-2}}\beta^\kappa{}_{|\nu|\lambda_{k-1}\ldots\lambda_{k+\ell-4}]} + \alpha_{\nu\kappa[\lambda_1\ldots\lambda_{k-2}}\beta^\kappa{}_{|\mu|\lambda_{k-1}\ldots\lambda_{k+\ell-4}]} \ , \tag{2.61}$$

where ϵ is an infinitesimal parameter. Clearly if α and β are rank 2 KY tensors, then $\alpha \cdot_s \beta$ is a KS tensor. In the case that both α and β are covariantly constant with respect to the Levi-Civita connection, the Nijenhuis tensor vanishes and so the transformation (2.60) is a symmetry of the spinning particle action (2.55). This will be the case for all symmetries generated by the form bilinears of pp-wave and KK-monopole solutions covered in Chapter 4.

Consider a dynamical system with $2n$-dimensional phase space P. This is completely integrable, according to Liouville, provided that P admits n independent functions (observables) Q_r, $r = 1, \ldots, n$, including the Hamiltonian, in involution. Q_r are independent provided that the map $Q : P \to \mathbb{R}^n$, where $Q = (Q_1, \ldots, Q_n)$, has rank n. Moreover Q_r are in involution, iff $\{Q_r, Q_s\}_{\mathrm{PB}} = 0$, i.e. Poisson bracket of any two Q_rs' vanishes[10].

Returning to the relativistic particle, the conserved charges (2.54) can be written in phase space variables as

$$Q(d) = d^{\nu_1\cdots\nu_k} p_{\nu_1}\cdots p_{\nu_k} \ , \tag{2.62}$$

where p_μ is the conjugate momentum of x^μ and we have raised the indices of d with the spacetime metric g. These clearly commute with the Hamiltonian $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$ as they are constants of motion. Furthermore the Poisson bracket algebra of two constants of motion $Q(d_1)$ and $Q(d_2)$ is $\{Q(d_1), Q(d_2)\}_{\mathrm{PB}} = Q([d_1, d_2]_{\mathrm{NS}})$, where

$$([d_1, d_2]_{\mathrm{NS}})^{\nu_1\cdots\nu_{k+\ell-1}} = k d_1^{\mu(\nu_1\cdots\nu_{k-1}}\partial_\mu d_2^{\nu_k\cdots\nu_{k+\ell-1})} - \ell d_2^{\mu(\nu_1\cdots\nu_{\ell-1}}\partial_\mu d_1^{\nu_k\cdots\nu_{k+\ell-1})} \ , \tag{2.63}$$

is the Nijenhuis-Schouten bracket of the KS tensors d_1 and d_2. Observe that if d_1 is a vector, then $[d_1, d_2]_{\mathrm{NS}} = \mathcal{L}_{d_1} d_2$, i.e. the Nijenhuis-Schouten bracket is the Lie derivative of d_2 with respect to the vector field d_1. Therefore two charges are in involution provided that the Nijenhuis-Schouten bracket of the associated KS tensors vanishes.

[10] Complete integrability is related to the separability of the equations of motion of a dynamical system. In the phase space coordinates $(Q_1, \ldots, Q_n, \psi_1, \ldots, \psi_n)$ defined by the charges $Q_1, \ldots, Q_n$ and the action-angle coordinates $(\psi_1, \ldots, \psi_n)$ adapted to the Hamiltonian vector fields, $X_{Q_i} = \partial_{\psi_i}$, the time evolution of the system is at most linear.

2.6 Supersymmetric backgrounds

A supergravity background is an n-tuple $(M,\ g\ ,\ \mathcal{F},\ \phi)$ satisfying the classical equations of motion for the bosonic fields, i.e. the metric g, scalar fields ϕ and fluxes $\mathcal{F}$ in the theory. If these solutions also preserve some residual supersymmetry of the original supergravity theory, they are called supersymmetric backgrounds. The SUSY variations leave the fields invariant and transform bosons into fermions and vice versa, however classically the fermonic fields must vanish, which in turn implies that the supersymmetry variations of the bosons are identically zero. On the other hand, the SUSY variations of the gravitino and the remaining fermions λ will give rise to a parallel transport equation for the supercovariant connection, $\mathcal{D}$, and some algebraic equations depending on the fields, respectively. These vanishing conditions of the SUSY variations of the fields are called the Killing spinor equations (KSEs).

$$\delta\psi_M|_{\psi,\lambda=0} = \mathcal{D}_\mu\epsilon = 0\ , \quad \delta\lambda_\mu|_{\psi,\lambda=0} = \mathcal{A}_\mu\epsilon = 0\ , \tag{2.64}$$

where $\mathcal{A}$ is a Clifford algebra element that depends on the fields and the supercovariant connection is

$$\mathcal{D}_\mu := \nabla_\mu + \sigma_\mu(e, F), \tag{2.65}$$

where ∇ is the spin connection of the spacetime acting on spinors

$$\nabla_\mu := \partial_\mu + \frac{1}{4}\Omega_{\mu,\alpha\beta}\Gamma^{\alpha\beta}\ , \tag{2.66}$$

and $\sigma(e, F)$ is a Clifford algebra element which depends on the coframe e and the fluxes F.

In other words, the classical solution $(M,\ g\ ,\ \mathcal{F},\ \phi)$ is supersymmetric if there exists a nonzero spinor ϵ which is parallel with respect to the supercovariant connection and satisfies the constraints imposed by the algebraic KSEs. The amount of linearly independent solutions ϵ admitted by the KSEs is called the number, N, of supersymmetries preserved. Thus, we call Killing spinors those nonzero spinors ϵ which are parallel (or covariantly constant) with respect to the supercovariant connection, $\mathcal{D}$, and satisfy the algebraic KSEs.

Before proceeding let us clarify that the usual geometrical notion of Killing spinor is given by

$$\nabla_X\epsilon = \eta X\cdot\epsilon \tag{2.67}$$

for a vector field X, where $\eta\in\mathbb{C}$ and $\cdot$ is the Clifford product. The word Killing is standard in the supergravity literature and alludes to the property that these spinors are "square roots" of Killing vectors, i.e. the one form, K_ϵ, defined for all $X\in TM$ by $K_\epsilon = \langle\epsilon, X\cdot\epsilon\rangle$[11] is dual to a Killing vector field.

The linearity of KSEs and especially the gravitino KSE which is a parallel transport equation uniquely defines a Killing spinor on all points of spacetime from its value at any given point. As a consequence, the dimension of the vector space formed by the Killing

[11]The notation $\langle\cdot,\cdot\rangle$ indicates Dirac inner product, i.e. $\langle\epsilon,\epsilon\rangle := \langle\Gamma_0\epsilon,\epsilon\rangle$

spinors is at most the rank of the spinor bundle S of which ϵ is a section. The amount of supersymmetries preserved can be defined by the fraction ν as the ratio

$$\nu = \frac{\text{dim(Killing spinors)}}{\text{rank } \mathsf{S}} \ . \tag{2.68}$$

The maximal amount of sypersymmetries preserved by a supergravity background is given by the number of supersymmetry charges of the theory, i.e. the number of components of the spinor representation in given dimensions times the number of gravitinos. In the absence of other fermionic fields beside the gravitino, as it happens in 11 dimensional supergravity, the kernel of the connection $\mathcal{D}$ corresponds to the parallel sections of the spinor bundle, i.e. the Killing spinors, and maximally supersymmetric solutions are those which saturate the kernel. However, generally the kernel of $\mathcal{D}$ cannot be identified with the Killing spinors for all supergravity theories since one must take into consideration the remaining algebraic KSEs. In general, maximal supersymmetry implies the flatness of the connection D and the determination of such backgrounds requires the analysis of the integrability conditions. In addition, one can find generic supergravity backgrounds that do not preserve any amount of sypersymmetry.

The integrability conditions of the KSEs are useful to study the holonomy and the field equations of supersymmetric backgrounds. Schematically these conditions can be expressed as

$$\mathcal{R}_{\mu\nu}\epsilon = [\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0 \ , \quad [\mathcal{D}_\mu, \mathcal{A}] = 0 \ , \quad [\mathcal{A}, \mathcal{A}] = 0 \ . \tag{2.69}$$

First, let us consider the (reduced) holonomy group, hol(∇), of the spin connection ∇ for a d-dimensional spacetime

$$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{4} R_{\mu\nu}{}^{\alpha\beta}\Gamma_{\alpha\beta} \ . \tag{2.70}$$

This corresponds to vacuum solutions, i.e. the fluxes vanish, $\mathcal{F} = 0$, and $\Gamma_{\alpha\beta}$ are the generators of the $Spin(d-1,1)$ algebra. As a consequence, hol(∇)$\subseteq Spin(d-1,1)$. However, for generic supergravity backgrounds, one has to consider the presence of fluxes, $\mathcal{F} \neq 0$. In this case, the integrability condition for the gravitino KSE includes more Cifford algebra element terms due to σ in (2.65) and the holonomy, hol($\mathcal{D}$), generically is contained in a GL group instead of a $Spin$ group. The computation of the Lie algebra of hol($\mathcal{D}$) is determined by the span of the values of the supercovariant curvature $\mathcal{R}$ and its covariant derivatives $\mathcal{D}^k\mathcal{R}$ evaluated on spacetime vector fields[12], in general, these expressions are given by all possible skew-symmetric products of gamma matrices. As a consequence of Clifford algebra representation theory, the hol($\mathcal{D}$) is the Lie algebra generated by all the anti-symmetric product of gamma matrices $\{1, \Gamma_\alpha, \Gamma_{\alpha\beta}, ...\}$ which is identified as a $\mathfrak{gl}(n, \mathbb{F})$ algebra and leads to the claim that the holonomy is contained in a GL$(n, \mathbb{F})$ group, where n is associated with the dimension of the gamma matrices. As an example the holonomy of the supercovariant derivative for $11d$ [64, 65, 66] and type II [67] supergravities is $SL(32, \mathbb{R})$ and a list of lower dimensional supergravities is given in [68].

Another key feature employed in the analysis of the supersymmetric solutions is the gauge symmetry in the KSEs. These gauge transformations transform a spacetime coframe e, fluxes $\mathcal{F}$ and spinor ϵ but leave the KSE covariant

[12]This is a consequence of the Ambrose-Singer theorem.

$$\mathcal{D}_\mu(e,\mathcal{F}) \to U^{-1}\mathcal{D}_\mu(e,\mathcal{F})U = \mathcal{D}_\mu(e',\mathcal{F}') \ , \quad \mathcal{A}_\mu \to U^{-1}\mathcal{A}_\mu(e,\mathcal{F})U = \mathcal{A}_\mu(e',\mathcal{F}') \ . \tag{2.71}$$

The holonomy group of $\mathcal{D}$ in most supergravity theories is bigger than the gauge group, which always contains $Spin(d-1,1)$ as a subgroup.

2.7 Twisted Covariant Form Hierarchies

It has been known for sometime that Killing spinor bilinears constructed from Killing spinors ϵ

$$\tau^k = \frac{1}{k!}\langle\epsilon, \Gamma_{N_1\cdots N_k}\epsilon\rangle dx^{N_1}\wedge\cdots\wedge dx^{N_k} \ , \tag{2.72}$$

satisfy the CKY equation. This can be seen as follows if ϵ satisfies the geometric Killing condition, see (2.67)

$$\nabla_M\epsilon + \eta\Gamma_M\epsilon = 0 \ . \tag{2.73}$$

Then, using the Dirac inner product $\langle\cdot,\cdot\rangle$, a direct computation

$$\nabla_M\tau_{N_1\ldots N_k} = \langle\nabla_M\epsilon, \Gamma_{N_1\ldots N_k}\epsilon\rangle + \langle\epsilon, \Gamma_{N_1\ldots N_k}\nabla_M\epsilon\rangle \ , \tag{2.74}$$

reveals that for Lorentzian signature manifolds

$$\nabla_X\tau^k = \left(\bar{\eta} - (-1)^k\eta\right) i_X\tau^{k+1} + \left(\bar{\eta} + (-1)^k\eta\right)\alpha_X\wedge\tau^{k-1} \ , \tag{2.75}$$

whereas for Euclidean signature manifolds

$$\nabla_X\tau^k = -\left(\bar{\eta} + (-1)^k\eta\right) i_X\tau^{k+1} - \left(\bar{\eta} - (-1)^k\eta\right)\alpha_X\wedge\tau^{k-1} \ , \tag{2.76}$$

where ∇ is the Levi-Civita connection, i_X is the inner derivation on the space of forms with the vector field X and $\alpha_X(Y) = g(X,Y)$.

Comparing the equations above with (2.10) one concludes that the bilinear τ^k satisfies the CKY equation. However, the KSEs of generic supergravity theories impose more complicated conditions on the Killing spinor bilinears and a suitable generalisation of the CKY is required.

Let M be a n-dimensional manifold with metric g and signature (r,s), $\Lambda^*_c(M)$ be the complexified bundle of all forms on M and $\mathcal{F}$ a multi-form, i.e. a collection of (complex) forms of non-necessarily different degrees. A twisted covariant form hierarchy (TCFH) with $\mathcal{F}$ is a collection of forms χ^p [23], with possibly different degrees p which satisfy

$$\nabla^{\mathcal{F}}_X(\{\chi^p\}) = i_X\mathcal{P}(\mathcal{F},\{\chi^p\}) + \alpha_X\wedge\mathcal{Q}(\mathcal{F},\{\chi^p\}) \ , \tag{2.77}$$

where $\mathcal{P},\mathcal{Q} : \Gamma(\Lambda^*_c(M)) \to \Gamma(\Lambda^*_c(M))$ and $\nabla^{\mathcal{F}}$, the covariant hierarchy connection, is a connection acting on $\Gamma(\oplus^m\Lambda^*_c(M))$ constructed from the Levi-Civita connection and $\mathcal{F}$. Such connection is not necessarily degree preserving.

In supergravity theories, $\mathcal{F}$ are the form field strengths and $\{\chi^p\}$ are the Killing spinor bilinears. Comparing the left and right-hand sides of (2.77) one finds [22]

$$\left(\nabla^{\mathcal{F}}_X\{\chi_q\}\right)|_p = \frac{1}{p+1}\left(i_Xd^{\mathcal{F}}(\{\chi_q\})\right)|_p - \frac{1}{n-p+1}\alpha_X\wedge\left(\delta^{\mathcal{F}}(\{\chi_q\})\right)|_{p-1} \ , \tag{2.78}$$

where $(\cdots)|_p$ denotes a restriction of the expression to p-forms, $d^{\mathcal{F}}$ and $\delta^{\mathcal{F}}$ are the exterior derivative and the adjoint constructed using $\nabla^{\mathcal{F}}_X$. This equation can be seen as a generalisation of the CKY equation which differs with respect to other generalisations since it relates a collection of forms with possibly different degrees and the connection $\nabla^{\mathcal{F}}_X$ may not be degree preserving in contrast to the conditions on a form with a definite degree imposed by the standard CKY and other generalisations.

It should be noted that equation (2.78) is implied by (2.77) however one must impose additional conditions to guarantee the converse is true

$$\frac{1}{p+1}\left(i_X d^{\mathcal{F}}(\{\chi_q\})\right)|_p = (i_X\mathcal{P})|_p \ , \qquad -\frac{1}{n-p+1}\left(\delta^{\mathcal{F}}(\{\chi_q\})\right)|_{p-1} = \mathcal{Q}|_{p-1} \ . \tag{2.79}$$

Thus there might be solutions of (2.78) which are not solutions of (2.77).

Let us give a sketch of how to prove that KSEs of a supergravity theory give rise to a TCFH, one starts with the structure of a supercovariant connection of such theories, see [22] for all the details of this proof.

$$\mathcal{D}_X = \nabla_X + c\,(i_X\mathcal{H}) + c\,(\alpha_X \wedge \mathcal{G}) \ , \tag{2.80}$$

where c denotes the Clifford algebra element associated with the multiforms $i_X\mathcal{H} = \sum_p i_X H^p$, $\mathcal{G} = \sum_p G^p$ made with the p-form field strengths H^p and G^p. ∇_X usually is the Levi-Civita connection but it can also be twisted by a gauge connection.

Then, considering the Killing spinor form bilinears $\{\chi^p\}$ with respect to Dirac inner product

$$\begin{aligned}\nabla_X\chi^p = &-\frac{1}{p!}\left(\langle c(i_XH)\epsilon, \Gamma_{A_1\dots A_p}\epsilon\rangle + \langle\epsilon, \Gamma_{A_1\dots A_p}c(i_XH)\epsilon\rangle\right)e^{A_1}\wedge\dots\wedge e^{A_p}\\ &-\frac{1}{p!}\left(\langle c(\alpha_X\wedge G)\epsilon, \Gamma_{A_1\dots A_p}\epsilon\rangle + \langle\epsilon, \Gamma_{A_1\dots A_p}c(\alpha_X\wedge G)\epsilon\rangle\right)e^{A_1}\wedge\dots\wedge e^{A_p} \ ,\end{aligned} \tag{2.81}$$

where $\{e^A\}$ is a (pseudo)-orthonormal frame adapted to the spacetime metric. Here note that since the action of multiforms $\mathcal{H}$ and $\mathcal{G}$ is linear on the right-hand side, it is sufficient to take them as single forms $H = H^\ell$ and $G = G^\ell$ of degree ℓ.

After using the Hermiticity properties of the inner product, the definition of the form bilinears and extensive Clifford algebra manipulation one derives the expression

$$\begin{aligned}\nabla_X\chi_p \quad &+ \left(\sum_q\left((c^1_q i_XH + c^2_q i_XG)\cdot\chi^q + (\tilde{c}^1_q i_X\bar{H} + \tilde{c}^2_q i_X\bar{G})\cdot\chi^q\right)\right)|_p = \\ &-i_X\left(\sum_q(c^3_q G\cdot\chi^q + \tilde{c}^3_q\bar{G}\cdot\chi^q)\right)|_p\\ &-\alpha_X\wedge\left(\sum_q(c^4_q G\cdot\chi^q + \tilde{c}^4_q\bar{G}\cdot\chi^q)\right)|_{p-1} \ ,\end{aligned} \tag{2.82}$$

where $c^1_q, \tilde{c}^1_q, c^2_q, \tilde{c}^2_q, c^3_q, \tilde{c}^3_q, c^4_q, \tilde{c}^4_q$ are combinatorial coefficients which depend on p, ℓ and the Dirac inner product $\langle\cdot,\cdot\rangle$ and whose values can be omitted for the proof, $\bar{H}$ and $\bar{G}$ are the comples conjugates of H and G, respectively.

Finally, the expression above defines a TCFH (2.78) with the following identifications

$$\nabla^{\mathcal{F}}_X\chi^p \quad \equiv \nabla_X\chi_p + \left(\sum_q\left((c^1_q i_XH + c^2_q i_XG)\cdot\chi^q + (\tilde{c}^1_q i_X\bar{H} + \tilde{c}^2_q i_X\bar{G})\cdot\chi^q\right)\right)|_p \ ,$$

$$\begin{aligned}(i_X \mathcal{P})\,|_p \quad &\equiv -i_X \left(\sum_q (c^3_q G \cdot \chi^q + \tilde{c}^3_q \bar{G} \cdot \chi^q) \right) |_p \ , \\ \mathcal{Q}|_{p-1} \quad &\equiv -\alpha_X \wedge \left(\sum_q (c^4_q G \cdot \chi^q + \tilde{c}^4_q \bar{G} \cdot \chi^q) \right) |_{p-1} \ , \end{aligned} \tag{2.83}$$

where $\mathcal{F} = \{\mathcal{H}, \mathcal{G}\}_{\rm ind}$ are the linearly independent field strengths.

Let us give some remarks about this proof. Note that no restriction on the signature of the spacetime were imposed, the proof works for all (r,s)-signature manifolds. Furthermore, all the arguments used can be generalized to Killing spinor bilinears constructed from different Killing spinors η, ϵ with respect to any Spin invariant inner product, i.e. $\chi^p = \frac{1}{p!}\langle \eta, \Gamma_{A_1 \dots A_p} \epsilon \rangle_s e^{A_1} \wedge \dots \wedge e^{A_p}$. One can also generalise it to include supergravity with higher-order corrections since the supercovariant connections are expected to have the same general structure as in (2.80). The main goal of this connection is to give a geometrical interpretation to the form bilinears as generalised CKY forms with respect to the TCFH connection $\nabla^{\mathcal{F}}$. Taking this into consideration, there might be terms $\mathcal{F} \cdot \chi_p$ which generate an ambiguity in the definition of $\nabla^{\mathcal{F}}$, note that

$$i_X \left(\mathcal{F} \cdot \chi_p \right) = i_X \mathcal{F} \cdot \chi_p \ , \tag{2.84}$$

e.g. terms for which all the indices of χ_p are contracted to indices of $\mathcal{F}$ since these terms can be placed on the connection side or on the right-hand side and in both cases the bilinears satisfy a generalised CKY equation. If all such terms contribute to $\nabla^{\mathcal{F}}$ the TCFH is defined with respect to a maximal connection whereas if all those terms contribute to $\mathcal{P}$ the TCFH is defined with respect to a minimal connection.

Another feature of the TCFHs arising from supergravity theories is that depending on the set of bilinears $\{\chi^p\}$ chosen, one can have the Hodge duality operation on $\{\chi^p\}$ as an automorphism if such set consists of all the bilinears and their Hodge duals, in this case the hierarchy is twisted by $\mathcal{F}$. On the other hand, if one chooses the set $\{\chi^p\}$ up to a Hodge duality operation, it might not be an automorphism of the TCFH which now will be twisted by $\mathcal{F}$ and its dual ${}^*\mathcal{F}$. Furthermore, the fluxes of the theory can be chosen such that the supercovariant derivative $\mathcal{D}_X$ depends only on $i_X \mathcal{F}$, as a consequence there is a choice $\{\chi^p\}$for which the associated TCFH is a parallel transport equation with respect to $\nabla^{\mathcal{F}}$ connection. This can be done with an appropriate basis for the fluxes and bilinears, some examples include the TCFH of heterotic supergravity whose $\nabla^{\mathcal{F}}$ connection is identified with $\nabla^H = \nabla + \frac{1}{2} H$, where H is the three-form flux which is understood as torsion. Another example can be found in the appendix A.2 where we construct such TCFH for minimal 4d supergravity.

As an example, consider the TCFH associated with the one- and two-form Killing spinor bilinears of 11-dimensional supergravity[13]. First, the conditions imposed on the two-form by the supercovariant derivative read

$$\begin{aligned}\nabla_\mu \omega_{\nu_1 \nu_2} &= \frac{1}{3 \cdot 3!} F_{\mu \rho_1 \rho_2 \rho_3} \tau^{\rho_1 \rho_2 \rho_3}{}_{\nu_1 \nu_2} - \frac{1}{3} F_{\mu \nu_1 \nu_2 \rho} k^\rho \\ &- \frac{1}{3 \cdot 3!} \tau_{\mu [\nu_1}{}^{\rho_1 \rho_2 \rho_3} F_{\nu_2] \rho_1 \rho_2 \rho_3} + \frac{1}{3 \cdot 4!} g_{\mu [\nu_1} \tau_{\nu_2]}{}^{\rho_1 \rho_2 \rho_3 \rho_4} F_{\rho_1 \rho_2 \rho_3 \rho_4} \ , \end{aligned} \tag{2.85}$$

Here, k, is a one-form killing spinor bilinear, ω is a two-form and τ is a five-form. Then

[13]The full TCFH of 11d supergravity will be presented in Chapter 4

after some manipulation

$$\mathcal{D}^{\mathcal{F}}_{\mu}\omega_{\nu_1\nu_2} \equiv \nabla_\mu\omega_{\nu_1\nu_2} \underline{- \frac{1}{2\cdot 3!}F_{\mu\rho_1\rho_2\rho_3}\tau^{\rho_1\rho_2\rho_3}{}_{\nu_1\nu_2}} = -\frac{1}{3}F_{\mu\nu_1\nu_2\rho}k^\rho$$
$$-\frac{1}{2\cdot 3!}\tau_{[\mu\nu_1}{}^{\rho_1\rho_2\rho_3}F_{\nu_2]\rho_1\rho_2\rho_3} + \frac{1}{3\cdot 4!}g_{\mu[\nu_1}\tau_{\nu_2]}{}^{\rho_1\rho_2\rho_3\rho_4}F_{\rho_1\rho_2\rho_3\rho_4} \ , \tag{2.86}$$

This expression is not degree preserving. Now look at the connection side of the TCFH, the underlined term twists the Levi-Civita connection and is responsible for this expression to satisfy a generalised CKY equation since it arises by reorganizing the terms of this type such that the antisymmetrization bracket includes the index of the supercovariant derivative. Similarly, the TCFH of the one form bilinear is given by

$$\mathcal{D}^{\mathcal{F}}_{\mu}k_\nu \equiv \nabla_\mu k_\nu = \frac{1}{6}F_{\mu\nu\alpha\beta}\omega^{\alpha\beta} - \frac{1}{6!}{}^*F_{\mu\nu\rho_1\rho_2\rho_3\rho_4\rho_5}\tau^{\rho_1\rho_2\rho_3\rho_4\rho_5} \ , \tag{2.87}$$

In this case, the TCFH connection is the standard Levi-Civita connection. We have included this equation to remark on the fact that the connection side is different for every bilinear in the theory. Nevertheless, these forms satisfy a generalised CKY equation

$$\mathcal{D}^{\mathcal{F}}_{\mu}k_\nu = \mathcal{D}^{\mathcal{F}}_{[\mu}k_{\nu]}, \qquad \mathcal{D}^{\mathcal{F}}_{\mu}\omega_{\nu_1\nu_2} = \mathcal{D}^{\mathcal{F}}_{[\mu}\omega_{\nu_1\nu_2]} - \frac{1}{5}g_{\mu[\nu_1}\mathcal{D}^{\mathcal{F}\rho}\omega_{\nu_2]\rho} \ . \tag{2.88}$$

Note there are terms having all the indices in the form bilinears contracted, these terms can contribute to the connection or the right-hand side. The maximal TCFH connection arises when all such terms are included in the connection side.

$$\nabla^{\mathcal{F}}_{\mu}k_\nu \equiv \nabla_\mu k_\nu - \frac{1}{6}F_{\mu\nu\alpha\beta}\omega^{\alpha\beta} + \frac{1}{6!}{}^*F_{\mu\nu\rho_1\rho_2\rho_3\rho_4\rho_5}\tau^{\rho_1\rho_2\rho_3\rho_4\rho_5} = 0 \ ,$$
$$\nabla^{\mathcal{F}}_{\mu}\omega_{\nu_1\nu_2} \equiv \nabla_\mu\omega_{\nu_1\nu_2} \underline{- \frac{1}{2\cdot 3!}F_{\mu\rho_1\rho_2\rho_3}\tau^{\rho_1\rho_2\rho_3}{}_{\nu_1\nu_2}} + \frac{1}{3}F_{\mu\nu_1\nu_2\rho}k^\rho$$
$$= -\frac{1}{2\cdot 3!}\tau_{[\mu\nu_1}{}^{\rho_1\rho_2\rho_3}F_{\nu_2]\rho_1\rho_2\rho_3} + \frac{1}{3\cdot 4!}g_{\mu[\nu_1}\tau_{\nu_2]}{}^{\rho_1\rho_2\rho_3\rho_4}F_{\rho_1\rho_2\rho_3\rho_4} \ , \tag{2.89}$$

satisfying the following CKY equations

$$\nabla^{\mathcal{F}}_{\mu}k_\nu = 0, \qquad \nabla^{\mathcal{F}}_{\mu}\omega_{\nu_1\nu_2} = \nabla^{\mathcal{F}}_{[\mu}\omega_{\nu_1\nu_2]} - \frac{1}{5}g_{\mu[\nu_1}\nabla^{\mathcal{F}\rho}\omega_{\nu_2]\rho} \ . \tag{2.90}$$

Finally, let us give some comments about the curvature associated to the TCFH connection and explain how to compute it for certain cases. In general, the curvature characterises a connection and is related to parallel transport around a loop. In order to calculate it, we usually take the commutator of two covariant derivatives which measures the difference between parallel transporting a tensor in two opposite orderings. The TCFH connection is not degree preserving which makes unclear what connection must be used to derive the curvature and how such TCFH connection acts on the fluxes and the metric. To illustrate this consider again the TCFH of the two-form in 11d supergravity (2.86). Then if one takes the commutator

$$[\mathcal{D}^{\mathcal{F}}_{\mu'}, \mathcal{D}^{\mathcal{F}}_{\mu}]\omega_{\nu_1\nu_2} = \mathcal{D}^{\mathcal{F}}_{\mu'}\mathcal{D}^{\mathcal{F}}_{\mu}\omega_{\nu_1\nu_2} - \mathcal{D}^{\mathcal{F}}_{\mu}\mathcal{D}^{\mathcal{F}}_{\mu'}\omega_{\nu_1\nu_2}$$
$$= \mathcal{D}^{\mathcal{F}}_{\mu'}\Big(-\frac{1}{3}F_{\mu\nu_1\nu_2\rho}k^\rho - \frac{1}{2\cdot 3!}\tau_{[\mu\nu_1}{}^{\rho_1\rho_2\rho_3}F_{\nu_2]\rho_1\rho_2\rho_3}$$

$$+\frac{1}{3\cdot 4!}g_{\mu[\nu_1}\tau_{\nu_2]}{}^{\rho_1\rho_2\rho_3\rho_4}F_{\rho_1\rho_2\rho_3\rho_4}\Big) - (\mu \leftrightarrow \mu') \ . \tag{2.91}$$

Immediately, we can see that one must act with a different TCFH connection for each term in the second equality. Therefore, in general, the standard procedure to compute the curvature associated with the TCFH seems to be not applicable. Now let us assume there is a theory where the TCFH reads

$$\nabla_\mu k_\nu - \frac{1}{2}F_{\mu\rho}\omega^\rho{}_\nu = 0 \ , \quad \nabla_\mu\omega_{\nu_1\nu_2} - F_{\mu[\nu_1}k_{\nu_2]} = 0 \ . \tag{2.92}$$

The curvature expression of the two-form reads

$$\begin{aligned}
[\nabla_{\mu'},\nabla_\mu]\omega_{\nu_1\nu_2} \quad &= \nabla_{\mu'}\nabla_\mu\omega_{\nu_1\nu_2} - \nabla_\mu\nabla_{\mu'}\omega_{\nu_1\nu_2} \\
&= \nabla_{\mu'}(F_{\mu[\nu_1}k_{\nu_2]}) - (\mu \leftrightarrow \mu') \\
&= \nabla_{\mu'}F_{\mu[\nu_1}k_{\nu_2]} + \frac{1}{2}F_{\mu[\nu_1|}F_{\mu'\rho|}\omega^\rho{}_{\nu_2]} - (\mu \leftrightarrow \mu') \\
2R_{\mu'\mu}{}^\rho{}_{[\nu_1}\omega_{|\rho|\nu_2]} \quad &= \nabla_{\mu'}F_{\mu[\nu_1}k_{\nu_2]} + \frac{1}{2}F_{\mu[\nu_1|}F_{\mu'\rho|}\omega^\rho{}_{\nu_2]} - (\mu \leftrightarrow \mu') \ ,
\end{aligned} \tag{2.93}$$

where we used the fact that the bilinears are parallel and replace $\mathcal{D}^{\mathcal{F}}$ with the Levi-Civita connection ∇. This approach can be extended to TCFHs for which the Killing spinor bilinears are not $\mathcal{D}^{\mathcal{F}}$-covariantly constant.

2.8 11D Supergravity

The eleven-dimensional construction of supergravity [69] can be seen as the effective theory of M-theory obtained from the strong coupling limit of Type IIA string theory [70, 71]. Consider 11d supergravity compactified on a circle with radius R_{11}, then the D0-brane is identified with the first Kaluza Klein excitation of the supergraviton given by $M_N^2 = (\frac{N}{R_{11}})^2$ and $R_{11} = \ell_s g_s$. Taking the limit $R_{11} \to \infty$ implies the decompactification of the eleventh circular dimension, leading to an eleven-dimensional theory called M-theory whose low energy limit is 11d supergravity. The M2- and M5-branes [72, 73], as well as their superpositions and intersections [74, 75, 76] are some examples of supersymmetric solutions of 11d supergravity. These have a wide range of applications regarding the existence of M-theory and the web of string dualities and more recently in the AdS/CFT correspondence. There are several detailed review articles and books that discuss M-theory and the web of dualities [77, 78, 79, 80, 81, 82, 83] and describe the sypersymmetric solutions of 11d supergravity [84, 85].

Let us review the main aspects of 11d supergravity. The bosonic flied content includes the metric g and a 4-form field strength F.The action for these fields [69, 86] is

$$\int_M \left(\frac{1}{2}R\mathrm{dvol} - \frac{1}{4}F\wedge\star F + \frac{1}{12}F\wedge F\wedge A\right) \ , \tag{2.94}$$

where $F = dA$, A is the 3-form gauge potential, R is the scalar curvature of the metric and dvol is the spacetime volume form.

The field equations read

$$d\star F = -\frac{1}{2}F\wedge F \ ,$$

$$R_{\mu\nu} = \frac{1}{12} F_{\mu}{}^{\rho_1\rho_2\rho_3} F_{\nu}{}^{\rho_1\rho_2\rho_3} - \frac{1}{144} g_{\mu\nu} F_{\rho_1\rho_2\rho_3\rho_4} F^{\rho_1\rho_2\rho_3\rho_4} \ . \tag{2.95}$$

We will work throughout this thesis with the Clifford algebra

$$\Gamma_\alpha \Gamma_\beta + \Gamma_\beta \Gamma_\alpha = +2\eta_{\alpha\beta} \mathbb{1} \ , \tag{2.96}$$

with the mostly plus signature of the metric η. The Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{10,1})$ is isomorphic (as an algebra) to $\mathrm{Mat}(32,\mathbb{R}) \oplus \mathrm{Mat}(32,\mathbb{R})$ which has two irreducible (pinor) representations $S^{\pm}$ corresponding to each factor in the decomposition above and given by the standard action of $\mathrm{Mat}(32,\mathbb{R})$ on $\mathbb{R}^{32}$. These are distinguished by the action of volume element $\Gamma^{12} = \pm 1$. There is a unique real-spinor S (Majorana) representation of $\mathrm{Spin}(10,1) \subset \mathrm{Cliff}^{\mathrm{even}}(\mathbb{R}^{10,1}) = \mathrm{Mat}(32,\mathbb{R})$.

The supercovariant connection is

$$\mathcal{D}_\mu = \nabla_\mu + \frac{1}{288} \left(\Gamma_{\mu}{}^{\nu_1\nu_2\nu_3\nu_4} F_{\nu_1\nu_2\nu_3\nu_4} - 8 F_{\mu\nu_1\nu_2\nu_3} \Gamma^{\nu_1\nu_2\nu_3} \right) \tag{2.97}$$

The supercovariant curvature reads [87]

$$\begin{aligned}
\mathcal{R}_{\mu\nu} = \quad & \frac{1}{4} R_{\mu\nu,\alpha\beta} \Gamma^{\alpha\beta} + \frac{2}{(288)^2} F_{\alpha_1\alpha_2\alpha_3\alpha_4} F_{\beta_1\beta_2\beta_3\beta_4} \epsilon_{\mu\nu}{}^{\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4}{}_{\delta} \Gamma^{\delta} \\
& + \frac{48}{(288)^2} \Big(4 F_{\mu\alpha_1\alpha_2\alpha_3} F^{\alpha_1\alpha_2\alpha_3}{}_{\beta} \Gamma^{\beta}{}_{\nu} - 4 F_{\nu\alpha_1\alpha_2\alpha_3} F^{\alpha_1\alpha_2\alpha_3}{}_{\beta} \Gamma^{\beta}{}_{\mu} \\
& - 36 F_{\alpha\beta\mu\gamma} F^{\alpha\beta}{}_{\nu\delta} \Gamma^{\gamma\delta} + F_{\alpha_1\alpha_2\alpha_3\alpha_4} F^{\alpha_1\alpha_2\alpha_3\alpha_4} \Gamma_{\mu\nu} \Big) \\
& + \frac{1}{36} \Big(\nabla_\mu F_{\nu\alpha_1\alpha_2\alpha_3} - \nabla_\nu F_{\mu\alpha_1\alpha_2\alpha_3} \Big) \Gamma^{\alpha_1\alpha_2\alpha_3} \\
& - \frac{8}{3(288)^2} \Big(F_{\beta_1\beta_2\beta_3\beta_4} F_{\gamma_1\gamma_2\gamma_3\nu} \epsilon_{\mu}{}^{\beta_1\beta_2\beta_3\beta_4\gamma_1\gamma_2\gamma_3}{}_{\alpha_1\alpha_2\alpha_3} - (\nu \leftrightarrow \mu) \Big) \Gamma^{\alpha_1\alpha_2\alpha_3} \\
& - \frac{1}{432} \Big(4 F_{\beta\alpha_1\alpha_2\alpha_3} F^{\beta}{}_{\mu\nu\alpha_4} \Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4} \\
& + 3 F_{\beta\gamma\alpha_1\alpha_2} F^{\beta\gamma\alpha_3}{}_{\nu} \Gamma^{\alpha_1\alpha_2}{}_{\mu\alpha_3} - 3 F_{\beta\gamma\alpha_1\alpha_2} F^{\beta\gamma\alpha_3}{}_{\mu} \Gamma^{\alpha_1\alpha_2}{}_{\nu\alpha_3} \Big) \\
& - \frac{1}{288} \Big(\nabla_\mu F_{\alpha_1\alpha_2\alpha_3\alpha_4} \Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4}{}_{\nu} - (\nu \leftrightarrow \mu) \Big) \\
& - \frac{1}{5!(72)^2} \Big(-6 F_{\mu\beta_1\beta_2\beta_3} F_{\nu\gamma_1\gamma_2\gamma_3} \epsilon^{\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3}{}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \\
& - 6 F_{\mu\delta\beta_1\beta_2} F^{\delta}{}_{\gamma_1\gamma_2\gamma_3} \epsilon_{\nu}{}^{\beta_1\beta_2\gamma_1\gamma_2\gamma_3}{}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \\
& - 6 F_{\nu\delta\beta_1\beta_2} F^{\delta}{}_{\gamma_1\gamma_2\gamma_3} \epsilon_{\mu}{}^{\beta_1\beta_2\gamma_1\gamma_2\gamma_3}{}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \\
& + 9 F_{\delta\kappa\beta_1\beta_2} F^{\delta\kappa}{}_{\gamma_1\gamma_2} \epsilon_{\mu\nu}{}^{\beta_1\beta_2\gamma_1\gamma_2}{}_{\alpha_1\alpha_2\alpha_3\alpha_3\alpha_4\alpha_5} \Big) \Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \ .
\end{aligned} \tag{2.98}$$

Contracting with Γ^ν and using the Bianchi identity of the curvature tensor $R_{\mu[\nu\rho\sigma]}$ one gets an expression in terms of the field equations and Bianchi identity for the field strength [88]

$$\begin{aligned}
\Gamma^\nu \mathcal{R}_{\mu\nu} = \quad & E_{\mu\nu} \Gamma^\nu - \frac{1}{36} L F_{\alpha_1\alpha_2\alpha_3} \left(\Gamma_{\mu}{}^{\alpha_1\alpha_2\alpha_3} = 6 \delta_{\mu}^{\alpha_1} \Gamma^{\alpha_2\alpha_3} \right) + \\
& \frac{1}{6!} B F_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \left(\Gamma_{\mu}{}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} - 10 \delta_{\mu}^{\alpha} \Gamma^{\alpha_2\alpha_3\alpha_4\alpha_5} \right) \ ,
\end{aligned} \tag{2.99}$$

where

$$
\begin{aligned}
E_{\mu\nu} =: R_{\mu\nu} &= \frac{1}{12} F_{\mu}{}^{\rho_1\rho_2\rho_3} F_{\nu\rho_1\rho_2\rho_3} - \frac{1}{144} g_{\mu\nu} F_{\rho_1\rho_2\rho_3\rho_4} F^{\rho_1\rho_2\rho_3\rho_4} \ , \\
LF_{\mu\nu\rho} &= \star(d\star F := -\frac{1}{2} F\wedge F)_{\mu\nu\rho} \ , \\
BF_{\mu_1\mu_2\mu_3\mu_4\mu_5} &:= (dF)_{\mu_1\mu_2\mu_3\mu_4\mu_5} \ .
\end{aligned} \tag{2.100}
$$

The expression above vanishes provided the background satisfies the equations of motion and F is closed as in 11d supergravity.

The Lie algebra of the holonomy group of the supercovariant connection, $\mathrm{hol}(\mathcal{D})$, is given by all possible products of gamma matrices of first order and above in (2.98), i.e. the expression does not contain terms proportional to $\mathbb{1}_{32}$. Thus the trace

$$
\mathrm{tr}(\mathcal{R}(X,Y)) = 0 \ , \tag{2.101}
$$

on the spinor indices vanishes. Hence the values of $\mathcal{R}(X,Y)$ are spanned by the subset $\mathrm{Mat}^0(32,\mathbb{R}) \subset \mathrm{Mat}(32,\mathbb{R})$ of 32×32 matrices with vanishing trace. One can identify the Lie algebra $\mathfrak{sl}(32,\mathbb{R}) = \mathrm{Mat}^0(32,\mathbb{R})$ of $\mathrm{SL}(32,\mathbb{R})$. Therefore the reduced holonomy group of $\mathcal{D}$ is contained in $SL(32,\mathbb{R})$, $\mathrm{hol}(\mathcal{D}) \subseteq \mathrm{SL}(32,\mathbb{R})$ [64, 65, 66].

2.9 Holography M-branes

The AdS/CFT correspondence [89, 90] associates the vacuum state of a conformal field theory (CFT) to a gravitational solution that contains an anti-de-Sitter (AdS) subspace. The latter is known as the gravitational dual of the correspondence and its fluctuations are related to the gauge invariant operators of the CFT. This correspondence allows to model strong coupling effects in CFT's in terms of gravitational theory considerations which are out of reach of perturbation theory.

The AdS/CFT correspondence in its simplest form proposed that in the large N limit, the non-Abelian gauge theory on a stack of N branes in string/M-theory is equivalent to the string/M-theory living on the near horizon geometry of the branes, see Table 2.1[14].

Eleven-dimensional supergravity has four types of elementary solutions preserving half supersymmetry: M2-brane, M5-brane, pp-wave and Kaluza-Klein monopole. For now, we will focus on the M2-brane and M5-brane near horizon geometries are supersymmetric solutions of the type $AdS_{p+2}\times S^{d-n}$, where p indicates the brane, and their worldvolume theory is a superconformal field theory. In such cases, there is a holographic duality between M-theory in AdS space and the superconformal field theory (SCFT) at the boundary. The AdS supersymmetry in $p+2$ dimensions can be matched with the superconformal symmetry in $p+1$ dimensions.

Case	Near Horizon Geometry	Gauge theory
N D3-branes	Type IIB on $\mathrm{AdS}_5\times S^5$	4d $\mathcal{N}$=4 Super-Yang-Mills
N M2-branes	M-theory on $\mathrm{AdS}_4\times S^7$	ABJM model
N M5-branes	M-theory on $\mathrm{AdS}_7\times S^4$	6d $\mathcal{N}$=(2,0) theory

Table 2.1: Basic examples of the AdS/CFT.

[14] A large number of extensions and generalisations have been done since the AdS/CFT inception.

M2-brane

A system of N parallel M2-branes is described by

$$g = h^{-\frac{2}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + h^{\frac{1}{3}}(dr^2 + r^2 d\Omega_7^2) \ , \quad F = \pm dx^0 \wedge dx^1 \wedge dx^2 \wedge dh^{-1} \ , \quad (2.102)$$

where $\mu, \nu = 0, 1, 2$. Taking the harmonic function to depend only on the transverse radial coordinate r

$$h = 1 + \frac{a^6}{r^6} \ , \quad a^6 \equiv 2^5\pi^2 N\ell_p^6 \ , \quad (2.103)$$

where ℓ_p is the Planck-length.

In the near horizon limit

$$\lim_{r\to 0} h(r) \sim \frac{a^6}{r^6} \ , \quad (2.104)$$

the metric becomes

$$g = a^{-4}r^4 g_{2+1} + a^2 r^{-2} dr^2 + a^2 g_{S^7} \ , \quad (2.105)$$

where $g_{2+1} = \eta_{\mu\nu}dx^\mu dx^\nu$. Note that the last term is the metric on S^7 with radius a, whereas the first two terms under a change of coordinates, $u = \frac{r^2}{4R_{AdS}}$ with $R_{AdS} = \frac{a}{2}$, give the metric on 4d AdS spacetime with radius R_{AdS}.

$$g_{AdS} = R_{AdS}^2 \left[\frac{du^2}{u^2} + \left(\frac{u}{R_{AdS}}\right)^2 \frac{g_{2+1}}{R_{AdS}^2}\right] \ . \quad (2.106)$$

Therefore, the near horizon geometry is of the form $\mathrm{AdS}_4 \times S^7$ with the radii of curvature $2R_{Ads} = R_{S^7} = \ell_p(32\pi N)^{\frac{1}{6}}$. The bosonic symmetry is $SO(3,2) \times SO(8)$. In addition, this is a maximally supersymmetric solution preserving 32 supercharges which implies just by symmetry arguments that the field theory describing a stack of M2-branes must be some $\mathcal{N}$=8 3d SCFT. The $SO(3,2)$ part is identified with the conformal group of the 3d SCFT and $SO(8)$ with the R-symmetry.

The relation between the parameters of the field theory and supergravity indicates that one might trust the supergravity description, as long as the radius of the AdS space is much larger than the Planck scale, which happens in the large N limit.

The M2-brane is a solution that preserves 16 supersymmetries and it interpolates between two maximally supersymmetric solutions, $\mathrm{AdS}_4 \times S^7$ in the near horizon limit, and flat Minkowski space far away from the brane, $\lim_{r\to 0} h(r) \sim 1$.

M5-brane

A stack of N parallel M5-branes is described by

$$g = h^{-\frac{1}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + h^{\frac{2}{3}}(dr^2 + r^2 d\Omega_4^2) \ , \quad F = \pm \star_5 dh \ , \quad (2.107)$$

where $\mu, \nu = 0, 1, 2, 3, 4, 5$. We can choose the harmonic function to depend only on the transverse radial coordinate r

$$h = 1 + \frac{a^3}{r^3} \ , \quad a^3 \equiv \pi^N \ell_p^3 \ , \quad (2.108)$$

where ℓ_p is the Planck-length.

Taking the near horizon limit

$$\lim_{r\to 0} h(r) \sim \frac{a^3}{r^3} \ , \tag{2.109}$$

the metric becomes

$$g = a^{-1} r g_{5+1} + a^2 r^{-2} dr^2 + a^2 g_{S^4} \ , \tag{2.110}$$

where $g_{5+1} = \eta_{\mu\nu} dx^\mu dx^\nu$. Here the last term is the metric on S^4 with radius a, whereas the first two terms under a change of coordinates, $u^2 = 2R_{AdS} r$ with $R_{AdS} = 2a$, produce the metric on 7d AdS spacetime with radius R_{AdS}.

$$g_{AdS} = R_{AdS}^2 \left[\frac{du^2}{u^2} + \left(\frac{u}{R_{AdS}} \right)^2 \frac{g_{5+1}}{R_{AdS}^2} \right] \ . \tag{2.111}$$

Hence, the near horizon geometry is of the form $\text{AdS}_7 \times S^4$ with the radii of curvature $R_{Ads} = 2R_{S^4} = 2\ell_p(\pi^2 N)^{\frac{1}{3}}$. The bosonic symmetry is $SO(6,2) \times SO(5)$. As in the previous case, this is a maximally supersymmetric solution preserving 32 supercharges. Symmetry arguments tell us that the field theory describing a stack of M5-branes must be some $\mathcal{N}$=(2,0) 6d SCFT. The $SO(6,2)$ part is identified with the conformal group of the 6d SCFT and $SO(5)$ with the R-symmetry.

Similar to the M2-brane case, one might trust the supergravity description of the M2-brane in the large N limit when the radius of the AdS space is much larger than the Planck scale. The M5-brane also interpolates between two maximally supersymmetric solutions, $\text{AdS}_7 \times S^4$ in the near horizon limit, and flat Minkowski space far away from the brane.

2.10 AdS Backgrounds of 11D Supergravity

The most general expression of the metric and the 4-form flux of all the warped AdS backgrounds in 11-dimensional supergravity can be expressed as near horizon geometries using Gaussian null coordinates [91, 92] as

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + ds^2(\mathcal{S}) \ , \\ F &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + r\mathbf{e}^+ \wedge d_h Y + X \ , \end{aligned} \tag{2.112}$$

where we have introduced the frame

$$\mathbf{e}^+ = du \ , \quad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du \ , \quad \mathbf{e}^i = e^i{}_J dy^J \ ; \quad g_{IJ} = \delta_{ij} e^i{}_I e^j{}_J \tag{2.113}$$

and

$$ds^2(\mathcal{S}) = g_{ij}\mathbf{e}^i\mathbf{e}^j \ . \tag{2.114}$$

is a metric on the near horizon section $\mathcal{S}$ given by $r = u = 0$. The dependence on the coordinates r, u is given explicitly, $h = h_i \mathbf{e}^i$ and $d_h Y = dY - h \wedge Y$. In addition, Δ, h, Y and X are a 0-form, 1-form, 2-form and 4-form respectively that depend only on the coordinates y of $\mathcal{S}$.

An example of a near horizon geometry is the metric

$$ds^2 = A^2 ds^2(AdS_n) + ds^2(M^{11-n}) \tag{2.115}$$

on a warped product of AdS_n with an internal space M^{11-n}, where A is the warp factor which depends only on the coordinates of M^{11-n}. The $AdS_n \times_w M^{11-n}$ backgrounds are invariant under the isometry group $SO(n-1, 2)$ of AdS_n. After imposing this symmetry, the expression (2.112) for AdS_2 now reads

$$\begin{aligned} ds^2 &= 2du(dr + rh - \frac{1}{2}r^2\Delta du) + ds^2(M^9) \ , \\ F &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + X \ , \end{aligned} \tag{2.116}$$

with $h = -d\log A^2 = d\log\Delta$, $d_h Y = 0$ and where A, Y and X are a 0-form, 2-form and 4-form on M^9 respectively.

The expression (2.112) for the rest of $\mathrm{AdS}_n, n > 2$ reads

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2(dz^2 + e^{\frac{2z}{\ell}} \sum_{a=1}^{n-3}(dx^a)^2) + ds^2(M^{11-n}) \ , \\ F &= dvol(AdS_n) \wedge W + X = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + X \ , \end{aligned} \tag{2.117}$$

with $\Delta = 0, h = -(\frac{2}{\ell}dz + d\log A^2), d_h Y = 0$ and where ℓ is the radius of AdS, W, X and Y are a $(4-n)$-form, 4-form and 2-form respectively.

$$\begin{aligned} &Y = dz \wedge W \ , \quad (n = 3) \ ; \quad Y = W(A^2 e^{\frac{z}{\ell}} dz \wedge dx^1) \ , \quad (n = 4) \ ; \\ &Y = 0 \ , \quad (n > 4) \ . \end{aligned} \tag{2.118}$$

Later W will be identified with the 1-form Q or the 0-form S when we work in AdS_3 or AdS_4 respectively.

Now let us turn to the KSEs of $D = 11$ supergravity given by[15]

$$\nabla_\mu \epsilon - \left(\frac{1}{288} \Gamma\!\!\!/ F_\mu - \frac{1}{36} F\!\!\!/_\mu \right) \epsilon = 0 \ , \tag{2.119}$$

where ∇ is the spin connection and F is the 4-form field strength of the theory and ϵ is a Majorana $Spin(10,1)$ spinor[16]. The KSEs evaluated on the ansatz (2.112) are integrable along the lightcone directions to yield

$$\begin{aligned} &\epsilon = \epsilon_+ + \epsilon_- \ , \quad \Gamma_\pm \epsilon_\pm = 0 \ , \\ &\epsilon_+ = \phi_+ + u\Gamma_+\Theta_-\phi_- \ , \quad \epsilon_- = \phi_- + r\Gamma_-\Theta_+\epsilon_+ \ , \end{aligned} \tag{2.120}$$

[15]See Appendix A.1 for the forms notation.

[16]Here we used the standard terminology in supergravity. More precisely, these should be pinors as one requires the odd part of the Clifford algebra.

where $\Gamma_\pm$ are lightcone projections, $\phi_\pm$ depend only on the coordinates y of $\mathcal{S}$ and

$$\Theta_\pm = \frac{1}{4}\not{h} + \frac{1}{288}\not{X} \pm \frac{1}{12}\not{Y} \ . \tag{2.121}$$

Furthermore, one can show after plugging (2.120) in (2.119) and using the Bianchi identities and field equations, see [93] for a more detailed description of the analysis, the remaining independent KSEs are

$$\mathcal{D}_i^\pm \phi_\pm \equiv \tilde{\nabla}_i \phi_\pm + \Psi_i^\pm \phi_\pm = 0 \ , \tag{2.122}$$

where

$$\Psi_i^\pm = \mp\frac{1}{4}h_i - \frac{1}{288}\Gamma\!\!\not{X}_i + \frac{1}{36}\not{X}_i \pm \frac{1}{24}\Gamma\!\!\not{Y}_i \mp \frac{1}{6}\not{Y}_i \ . \tag{2.123}$$

In addition, one derives two useful integrability conditions of the solution of the KSEs along the remaining AdS directions.

$$\begin{aligned}
&\left(\frac{1}{2}\Delta + 2(\frac{1}{4}\not{h} - \frac{1}{288}\not{X} + \frac{1}{12}\not{Y})\Theta_+\right)\phi_+ = 0 \ , \\
&\left(-\frac{1}{2}\Delta + 2(-\frac{1}{4}\not{h} + \frac{1}{288}\not{X} + \frac{1}{12}\not{Y})\Theta_-\right)\phi_- = 0
\end{aligned} \tag{2.124}$$

Then one must reduce the KSEs from $\mathcal{S}$ to the internal manifold, evaluate (2.122) along the z-direction to find

$$\partial_z \phi_\pm = \Xi^\pm \phi_\pm \ , \tag{2.125}$$

where

$$\Xi^\pm = -\frac{1}{2}\Gamma_z \not{\partial} A \mp \frac{1}{2\ell} + \frac{1}{288}A\Gamma_z\not{X} \pm \frac{1}{6}\not{Z} \ , \tag{2.126}$$

where $\not{Z}$ is taken according to the corresponding value of Y in (2.118). Taking the second derivative with respect to z and using (2.124) gives

$$\partial_z^2 \phi_\pm \pm \frac{1}{\ell}\partial_z \phi_\pm = 0 \ . \tag{2.127}$$

This can be solved to yield

$$\phi_\pm = \sigma_\pm + e^{\mp\frac{z}{\ell}}\tau_\pm \ , \tag{2.128}$$

where

$$\begin{aligned}
&\partial_z \sigma_\pm = \partial_z \tau_\pm = 0 \ , \\
&\Xi^\pm \sigma_\pm = 0 \ , \quad \Xi^\pm \tau_\pm = \mp\frac{1}{\ell}\tau_\pm \ .
\end{aligned} \tag{2.129}$$

The remaining x^a coordinates of AdS do not generate additional integrability conditions and after integrating them one gets expressions similar to the second line in (2.129). For more details see [94] and references within. Hence spinors for $\text{AdS}_{n\geq 4}$ can be written

$$\phi_+ = \sigma_+ - \frac{1}{\ell}x^a\Gamma_{az}\tau_+ + e^{\frac{-z}{\ell}}\tau_+ \ , \quad \phi_- = \sigma_- + e^{\frac{z}{\ell}}(-\frac{1}{\ell}x^a\Gamma_{az}\sigma_- + \tau_-) \ , \tag{2.130}$$

and the independent KSEs are given by

$$\mathcal{D}_i^\pm \sigma_\pm = 0 \ , \quad \mathcal{D}_i^\pm \tau_\pm = 0 \ , \tag{2.131}$$

and the algebraic ones are

$$\Xi^\pm \sigma_\pm = 0 \ , \quad \Xi^\pm \pm \frac{1}{\ell}\tau_\pm = 0 \ . \tag{2.132}$$

2.11 Anomalies

The basic idea of an anomaly can be explained as follows. Consider a quantum theory whose classical action is invariant under a symmetry group G, $\delta S_{cl} = 0$. Thus, an anomaly is the failure of a symmetry of the classical theory to be preserved in the quantum effective action. If the symmetry G is global, then anomalies in G imply that classical rules are not obeyed in the quantum theory and previously forbidden processes may occur. However, it does not indicate an inconsistency of the theory. On the other hand, if G is an anomalous gauge symmetry, this would indicate an inconsistency in the theory since gauge symmetries are redundancies of our theory that help us to remove negative norm states and to prove unitarity and renormalizability. Therefore, all gauge anomalies must vanish if one tries to build a consistent theory. The best way to ensure gauge symmetries are non-anomalous is to work with Dirac fermions and gauge fields which couple to left-handed and right-handed fermions in the same manner. For a review of anomalies, see [95, 96]

Below, we will explain the chiral anomaly as it provides the introductory background required to understand the anomalies analysed in chapter 6, which occur as a consequence of the presence of worldsheet chiral fermions in the sigma model action.

2.11.1 Chiral anomaly

Let us introduce the Chiral anomaly following Fujikawa's method [97, 98, 99, 100] and for simplicity let us carry out the computation in 4 dimensions. Consider

$$e^{-W[A]} = \int D\Psi D\bar{\Psi} e^{-\int d^4x \bar{\Psi} i \nabla\!\!\!/ \Psi} \ , \tag{2.133}$$

where $\nabla\!\!\!/ = \gamma^\mu(\partial_\mu + \omega_\mu + A_\mu)$. One can always compactify the space in such a way that the geometry (spin connection) plays no role.

The classical action $\int d^4x \bar{\Psi} i \nabla\!\!\!/ \Psi$ is invariant under

$$\Psi \to e^{i\gamma^5\alpha}\Psi \ , \quad \bar{\Psi} \to \bar{\Psi} e^{i\gamma^5\alpha} \ , \tag{2.134}$$

where γ^5 is chirality gamma matrix operator. We can expand the spinors in terms of Dirac eigenspinors

$$\Psi = \sum_i a_i \psi_i \ , \quad \bar{\Psi} = \sum_i \bar{b}_i \bar{\psi} \ , \tag{2.135}$$

where a_i and $\bar{b}_i$ are Grassmann-valued numbers. The operator $\nabla\!\!\!/$ has eigenspinors satisfying

$$i\nabla\!\!\!/ \psi_i = \lambda_i \psi_i \ , \tag{2.136}$$

since $i\nabla\!\!\!/$ is Hermitian, λ_i is real. If we consider a compact manifold, ψ_i can be normalized

$$\langle \psi_i | \psi_j \rangle = \int d^4x \psi_i^\dagger(x) \psi_j(x) \delta_{ij} = \delta_{ij} \ . \tag{2.137}$$

Consider the transformations

$$\Psi(x) \to \Psi(x) + i\alpha(x)\gamma^5\Psi(x) \ , \quad \bar{\Psi}(x) \to \bar{\Psi}(x) + i\bar{\Psi}(x)\alpha(x)\gamma^5 \ . \tag{2.138}$$

Then

$$
\begin{aligned}
\int d^4x \bar{\Psi} i\not{\nabla}\Psi \quad &\to \int d^4x \bar{\Psi} i\not{\nabla}\Psi + i\int d^4x \left(\alpha\bar{\Psi}\gamma^5 i\not{\nabla}\Psi + \bar{\Psi} i\not{\nabla}(\alpha\gamma^5\Psi)\right) \\
&= \int d^4x \bar{\Psi} i\not{\nabla}\Psi + \int d^4x \alpha(x)\partial_\mu j_5^\mu(x) \ , \qquad (2.139)
\end{aligned}
$$

where $\partial_\mu j_5^\mu(x) = \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x)$.

In quantum theory, there is a change in the path integral measure

$$
\int D\Psi D\bar{\Psi} = \int \prod_i da_i d\bar{b}_i \to \int \prod_i da_i' d\bar{b}_i' \ . \qquad (2.140)
$$

From the orthonormality of ψ_i, we find that

$$
\begin{aligned}
a_i' = \langle \psi_i | \Psi' \rangle \quad &= \langle \psi_i | (1 + i\alpha\gamma^5)\Psi \rangle \\
&= \sum_j \langle \psi_i | (1 + i\alpha\gamma^5)\psi_j \rangle a_j \equiv \sum_j c_{ij} a_j \ , \qquad (2.141)
\end{aligned}
$$

where $c_{ij} = \delta_{ij} + i\alpha\langle \psi_i | \gamma^5 \psi_j \rangle$. Then

$$
\begin{aligned}
\prod da_j' \quad &= (\det c_{ij})^{-1} \prod da_i = e^{-\mathrm{tr}\ln c_{ij}} \prod da_i \\
&\approx e^{-i\mathrm{tr}\alpha\langle \psi_i | \gamma^5 \psi_j \rangle} \prod da_i \\
&= e^{-i\alpha \sum_i \langle \psi_i | \gamma^5 \psi_i \rangle} \prod da_i \ . \qquad (2.142)
\end{aligned}
$$

Similar Jacobian computation can be done the change $\bar{b}_i \to \bar{b}_i'$. Thus,

$$
\prod_i da_i d\bar{b}_i \to \prod_i da_i' d\bar{b}_i' e^{-2i\int d^4x \alpha(x) \sum \psi_n^\dagger(x)\gamma^5\psi_n(x)} \ . \qquad (2.143)
$$

Now, the effective action reads

$$
\begin{aligned}
e^{-W[A]} \quad &= \int \prod_i da_i d\bar{b}_i e^{-\int d^4x \bar{\Psi} i\not{\nabla}\Psi} \\
&= \int \prod_i da_i' d\bar{b}_i' e^{-\int d^4x \bar{\Psi} i\not{\nabla}\Psi - \int d^4x \alpha(x)\partial_\mu j_5^\mu(x) - 2i\int d^4x \alpha(x)\mathcal{A}(x)} \ , \qquad (2.144)
\end{aligned}
$$

where $\mathcal{A} \equiv \sum_i \psi_i^\dagger(x)\gamma^5\psi_i(x)$. Since $\alpha(x)$ is arbitrary, the axial current is not conserved in quantum theory. This is called the chiral anomaly.

$$
\partial_\mu j_5^\mu(x) = -2i\mathcal{A}(x) \ . \qquad (2.145)
$$

Note that the integral (2.143) is not well defined and must be regularized[17]. After introducing a Gaussian cut-off

$$
\int d^4x \mathcal{A} = \sum \langle \psi_i | \gamma^5 e^{-(\frac{i\not{\nabla}}{\Lambda})^2} \psi_i \rangle |_{\Lambda\to\infty} \ , \qquad (2.146)
$$

[17]We will skip the details of this computation.

where Λ is a regularisation scale. Then, one can prove that the contribution to the right hand side of (2.146) comes only from the zero-energy modes. After applying the fact that $(i\not{\nabla})^2 = -\nabla_\mu \nabla^\mu - \frac{1}{4}[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu}$, where $\mathcal{F}_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$, one gets

$$\mathcal{A}(x) = \sum_i \langle \psi_i | x \rangle \langle x | \gamma^5 e^{\frac{\nabla^2 + \frac{1}{4}[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu}}{\Lambda^2}} | \psi_i \rangle |_{\Lambda \to \infty} \ . \tag{2.147}$$

After some computation, it can be shown that one gets the well-known result

$$\mathcal{A}(x) = -\frac{1}{32\pi^2} \mathrm{tr} \epsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma} \ . \tag{2.148}$$

Dealing with anomalies [101, 102] implies the computation of $\mathrm{tr}\mathcal{F}^n$, and finding ways to cancel such terms. It is instructive to explain the relation of this expression with Chern-Simons forms as it is a crucial part of the anomaly cancellation method.

We define[18]

$$P_{2n} = \mathrm{tr}\mathcal{F}^n \ , \quad \mathcal{F} = dA + A \wedge A \ , \quad \mathcal{D} = d + [A, \cdot] \ , \tag{2.149}$$

where A is a Lie-algebra valued one form connection and the graded commutator satisfies $[\alpha, \beta] = \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha$ for a $\mathfrak{g}$-valued p-form, α, and a $\mathfrak{g}$-valued q-form, β.

Note that

$$dP_{2n} = d\mathrm{tr}\mathcal{F}^n = n\mathrm{tr}(d\mathcal{F}\mathcal{F}^{n-1}) = n\mathrm{tr}(\mathcal{D}\mathcal{F}\mathcal{F}^{n-1}) = 0 \ , \tag{2.150}$$

where we have used the Bianchi identity, $\mathcal{D}\mathcal{F} = 0$, in the last step.

Let A_1, A_2 be two connections on a principal bundle P over a manifold, M, and let $A(t), 0 \leq t \leq 1$, be an interpolation between A_1, A_2, $A(t=0) = A_1$, $A(t=1) = A_2$. We can define $P_{2n}(t)$ for each value of t. Then

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_t &= d\left(\frac{dA(t)}{dt}\right) + \frac{dA(t)}{dt}A(t) + A(t)\frac{dA(t)}{dt} \\ &= \mathcal{D}_t\left(\frac{dA(t)}{dt}\right) \ . \end{aligned} \tag{2.151}$$

Thus

$$\begin{aligned} \frac{d}{dt}P_{2n}(t) &= \frac{d}{dt}\mathrm{tr}\mathcal{F}^n \\ &= n\mathrm{tr}\left(\frac{d}{dt}\mathcal{F}(t)\mathcal{F}^{n-1}(t)\right) \\ &= n\mathrm{tr}\left(\mathcal{D}_t\left(\frac{dA(t)}{dt}\right)\mathcal{F}^{n-1}(t)\right) \\ &= nd\mathrm{tr}\left(\frac{dA(t)}{dt}\mathcal{F}^{n-1}(t)\right) \ , \end{aligned} \tag{2.152}$$

where we have used in the last step the Bianchi identity and the fact that $\mathcal{D}_t$ commutes with the tr operator. Then, integrating from $t = 0$ to $t = 1$

$$\begin{aligned} P_{2n}(A_2) - P_{2n}(A_1) &= dn\int_0^1 dt\mathrm{tr}\left(\frac{dA(t)}{dt}\mathcal{F}^{n-1}(t)\right) \\ &= dQ^0_{2n-1}(A_t, \mathcal{F}_t) \ , \end{aligned} \tag{2.153}$$

where Q^0_{2n-1} is the Chern-Simons form.

[18] There are different ways to define P, here we follow the notation where the subscript indicates the degree of the form.

2.12 Wess-Zumino consistency conditions and Descent equations

The anomaly equation under infinitesimal gauge transformations of the fermionic effective action implicitly is given by [101]

$$\delta_v \Gamma[A] = \int \mathrm{tr} v \alpha(A) = W[v, A] \ , \tag{2.154}$$

where $\alpha(A)$ is a local functional which corresponds to the anomaly. The commutator of two infinitesimal gauge transformations $[\delta_u, \delta_v]$ gives another infinitesimal gauge transformation

$$[\delta_u, \delta_v] = \delta_{[u,v]} \ . \tag{2.155}$$

As a consequence, we have

$$\delta_u W[v, A] - \delta_v W[u, A] = W([u, v], A) \ . \tag{2.156}$$

This statement is known as the Wess-Zumino consistency conditions. These can be written in terms of differential forms (Chern-Simons forms) in what is called the descent equations[19].

$$\begin{aligned}
&\delta Q^0_{2n-1}(v, A) + dQ^1_{2n-2}(v, A) = 0 \ , \\
&\delta Q^1_{2n-2}(v, A) + dQ^2_{2n-3}(v, A) = 0 \ , \\
&\cdots \ , \\
&\delta Q^{2n-2}_1(v, A) + dQ^{2n-1}_0(v, A) = 0 \ , \\
&\delta Q^{2n-1}_0(v, A) = 0 \ ,
\end{aligned} \tag{2.157}$$

where the upperscript index indicates up to which power of v we can include in the terms (ghost number[20]), whereas the subscript indicates the degree of the form.

2.13 W-symmetries

W-algebras first appeared in 2-dimensional conformal field theory. Usually, they are introduced with the standard example of W_3-algebra [103, 104]. However, here we will explain them with 2 examples applied to non-linear sigma modes.

Consider a non-linear two-dimensional sigma model defined on some D-dimensional target space M with metric $g_{\mu\nu}$[105].

$$S_0 = \frac{1}{2} \int d^2 x g_{\mu\nu} \partial_{\ddagger} X^\mu \partial_= X^\nu \ , \tag{2.158}$$

where the two-dimensional space-time has null coordinates $x^\mu = (x^{\ddagger}, x^=)$. If $d_{\mu\nu\rho}$ is a covariantly constant symmetric tensor on M, $\nabla_\mu d_{\nu\rho\sigma}$, then the sigma model is invariant under the semi-local transformations

$$\delta X^\mu = k_= \partial_{\ddagger} X^\mu + \lambda_{==} d^\mu{}_{\nu\rho} \partial_{\ddagger} X^\nu \partial_{\ddagger} \phi^\rho \ , \tag{2.159}$$

[19] The details of this computation can be found in [101, 102, 100]

[20] In the language of Faddeev-Popov and BRST formulation.

where the parameters satisfy

$$\partial_{=}k_{=} = 0 \ , \quad \partial_{=}\lambda_{==} = 0 \ . \tag{2.160}$$

The conserved currents are the energy-momentum tensor and one constructed from the tensor d given respectively by

$$T_{\ddagger\ddagger} = \frac{1}{2} g_{\mu\nu}\partial_{\ddagger}X^{\mu}\partial_{\ddagger}X^{\nu} \ , \quad W_{\ddagger\ddagger\ddagger} = \frac{1}{3} d_{\mu\nu\rho}\partial_{\ddagger}X^{\mu}\partial_{\ddagger}X^{\nu}\partial_{\ddagger}X^{\rho} \ . \tag{2.161}$$

In general, the closure of two symmetries as in (2.159) will lead to an infinite sequence of currents. However, if the tensor d satisfies

$$d_{(\mu\nu}{}^{\tau} d_{\sigma)\rho\tau} = \kappa g_{\mu\nu} g_{\sigma\rho} \ , \tag{2.162}$$

then the algebra closes to give

$$[\delta_{k_1} + \delta_{\lambda_1}, \delta_{k_2} + \delta_{\lambda_2}] = \delta_{k_3} + \delta_{\lambda_3} \ , \tag{2.163}$$

where

$$\begin{aligned} k_3 &= k_2\partial_{\ddagger}k_1 + 4\kappa(\lambda_2\partial_{\ddagger}\lambda_1)T_{\ddagger\ddagger} - (1 \leftrightarrow 2) \ , \\ \lambda_3 &= 2\lambda_2\partial_{\ddagger}k_1 + k_2\partial_{\ddagger}\lambda_1 - (1 \leftrightarrow 2) \ . \end{aligned} \tag{2.164}$$

This type of algebra, where the parameters depend on the currents, is often called a W-algebra[21] of the sigma model.

Now, let us consider the $N = 1$ supersymmetric sigma model defined on some D-dimensional target space M

$$S_1 = \int d^2\sigma d^2\theta g_{\mu\nu} D_+X^{\mu} D_-X^{\nu} \ , \tag{2.165}$$

where $(\sigma^{=}, \sigma^{\ddagger}, \theta^{+}, \theta^{-})$, $D_+^2 = i\partial_{\ddagger}$, $D_-^2 = i\partial_{=}$, $\{D_+, D_-\} = 0$. The sigma model is invariant under the following transformations

$$\begin{aligned} \delta_L X^{\mu} &= a_L L^{\mu}{}_{\lambda_1\ldots\lambda_\ell} D_+X^{\lambda_1} \ldots D_+X^{\lambda_\ell} \ , \\ \delta_T X^{\mu} &= 2i\alpha_T\partial_{\ddagger}X^{\mu} + D_+\alpha_T D_+X^{\mu} \ , \end{aligned} \tag{2.166}$$

provided that L is covariantly constant, where a_L is the parameter chosen such that $\delta_L X^{\mu}$ is even under Grassmannian parity and a_T is a parameter with even Grassmannian parity.

The corresponding charges associated to these transformations are given by

$$J_L = L_{\mu_1\ldots\mu_{\ell+1}} D_+X^{\mu_1\ldots\mu_{\ell+1}} \ , \quad T = g_{\mu\nu}D_+X^{\mu}\partial_{\ddagger}X^{\nu} \ , \tag{2.167}$$

which are conserved, i.e $D_-J_L = 0$, $D_-T = 0$. For a Calabi-Yau target space where the holonomy group is $SU(3)$, we will give only the commutator between two δ_L symmetries to illustrate the additional symmetries, admitted in certain supersymmetric non-linear sigma models, which are of W-type. For more details check [106].

For a_L with even Grassmannian parity we have

$$[\delta_L, \delta_L] = \delta_I \ , \tag{2.168}$$

with $a_I = -\frac{\ell\dot{\ell}!}{2^{\ell-1}}(a'_L D_+a_L - a_L D_+a'_L)J_I^{\ell-1}$, where δ_I is the transformation generated by the complex structure which is associated with the Kähler form and J_I is its corresponding charge. While for odd a_L

$$[\delta_L, \delta_L] = \delta_T + \delta_I \ , \tag{2.169}$$

with $a_T = -\frac{\ell\dot{\ell}!}{2^{\ell}}a'_L a_L J_I^{\ell-1}$ and $a_I = -\frac{\ell(\ell-1)\ell!}{2^{\ell-2}}a'_L a_L T J_I^{\ell-2}$.

[21] The W-algebras found in sigma models are the classical Poisson versions of those in Conformal Field Theory.

Chapter 3

Symmetries, spinning particles and the TCFH of D=4,5 Minimal supergravities

3.1 Introduction

The presence of the TCFH, and consequently, the existence of generalized CKY for all supersymmetric solutions raises the question on whether the form bilinears can be used to investigate the separability properties[1] of many classical equations, like the Hamilton-Jacobi, Klein-Gordon, Dirac and Maxwell equations, of these backgrounds, and on whether they generate symmetries in spinning particles propagating on such backgrounds. In this chapter, we shall demonstrate that the form bilinears of a large class of supersymmetric $D = 4$, $N = 2$ and $D = 5$, $N = 1$ minimal supergravity backgrounds generate symmetries in spinning particle actions with appropriate couplings. The key observation is that some of the conditions for invariance of the particle actions of [61] under some fermionic transformations can also be expressed as TCFHs. In this case, the associated TCFH connection depends on the couplings of the particle action and acts on forms that determine the infinitesimal fermionic symmetries of the system. Thus the task is to match the TCFHs of supersymmetric backgrounds with those of the spinning particle symmetries after an appropriate identification of the supergravity fields with the couplings of the particle system and of the form bilinears with the forms that generate the fermionic symmetries, respectively. We shall demonstrate that this can be achieved in a variety of cases. We shall also comment on the use of the form bilinears to investigate the separability properties of supersymmetric backgrounds.

3.2 Minimal D=4, N=2 supergravity

The supercovariant connection of minimal $D = 4$, $N = 2$ supergravity is

$$\mathcal{D}_\mu \equiv \nabla_\mu + \frac{i}{4} F_{ab}\Gamma^{ab}\Gamma_\mu \ , \tag{3.1}$$

[1] The separability properties of differential operators are described by separability structures. Separability structures are classes of separable charts for which the differential operators allow a separation of variables. For each separability structure there exists a family of separable coordinates that admits a maximal number of, r, ignorable coordinates.

where F is a 2-form field strength, $dF = 0$. The field equations also imply that F is co-closed, $d^*F = 0$. If ϵ is a Killing spinor, $\mathcal{D}_\mu \epsilon = 0$, the form bilinears of the theory up to a Hodge duality are

$$
\begin{aligned}
& f = \langle \epsilon, \epsilon \rangle \ , \quad h = \langle \epsilon, \Gamma_5 \epsilon \rangle \ , \quad k = \langle \epsilon, \Gamma_a \epsilon \rangle \, e^a \ , \\
& Y^1 = \langle \epsilon, \Gamma_a \Gamma_5 \epsilon \rangle \, e^a \ , \quad Y^3 + iY^2 = \langle \tilde{\epsilon}, \Gamma_a \Gamma_5 \epsilon \rangle \, e^a \ , \\
& \omega^1 = \frac{1}{2} \langle \epsilon, \Gamma_{ab} \epsilon \rangle \, e^a \wedge e^b \ , \quad \omega^3 + i\omega^2 = \frac{1}{2} \langle \tilde{\epsilon}, \Gamma_{ab} \epsilon \rangle \, e^a \wedge e^b \ ,
\end{aligned}
\tag{3.2}
$$

where the spacetime metric $g = \eta_{ab} e^a e^b$ with $e^a = e^a_\mu dx^\mu$ a local co-frame, $\langle \cdot, \cdot \rangle_D$ is the Dirac inner product, C is a charge conjugation matrix such that $C * \Gamma_a = -\Gamma_a C*$ and $C * C* = -1$, and $\tilde{\epsilon} = C * \epsilon$. $C = \Gamma_3$ in the conventions of [107], where $*$ indicates complex conjugation of the following object. Observe that if ϵ is a Killing spinor so is $\tilde{\epsilon}$. The TCFH of the theory [22] reads

$$
\begin{aligned}
& \nabla_\mu f = i F_{\mu\nu} k^\nu \ , \quad \nabla_\mu h = {}^*F_{\mu\nu} k^\nu \ , \quad \nabla_\mu k_\nu = i f F_{\mu\nu} - h \, {}^*F_{\mu\nu} \ , \\
& \nabla_\mu Y^r_\nu + 2 {}^*F_{\mu\rho} \omega^{r\rho}{}_\nu = 2 {}^*F_{[\mu|\rho|} \omega^{r\rho}{}_{\nu]} - \frac{1}{2} g_{\mu\nu} {}^*F_{\rho\lambda} \omega^{r\rho\lambda} \ , \quad r = 1,2,3 \ , \\
& \nabla_\mu \omega^r_{\nu\rho} - 4 \, {}^*F_{\mu[\nu} Y^r_{\rho]} = -3 \, {}^*F_{[\mu\nu} Y^r_{\rho]} - 2 \, g_{\mu[\nu} {}^*F_{\rho]\lambda} Y^{r\lambda} \ , \quad r = 1,2,3 \ .
\end{aligned}
\tag{3.3}
$$

In what follows we shall also consider the TCFH associated with the dual 2-forms χ^r of ω^r which can be defined as

$$
\chi^1 = -\frac{i}{2} \langle \epsilon, \Gamma_{ab} \Gamma_5 \epsilon \rangle \, e^a \wedge e^b \ , \quad \chi^3 + i\chi^2 = -\frac{i}{2} \langle \tilde{\epsilon}, \Gamma_{ab} \Gamma_5 \epsilon \rangle \, e^a \wedge e^b \ .
\tag{3.4}
$$

One can show that the Killing spinor equations imply the TCFH

$$
\begin{aligned}
& \nabla_\mu Y^r_\nu + 2 F_{\mu\rho} \chi^{r\rho}{}_\nu = 2 F_{[\mu|\rho|} \chi^{r\rho}{}_{\nu]} - \frac{1}{2} g_{\mu\nu} F_{\rho\lambda} \chi^{r\rho\lambda} \ , \quad r = 1,2,3 \\
& \nabla_\mu \chi^r_{\nu\rho} - 4 \, F_{\mu[\nu} Y^r_{\rho]} = -3 \, F_{[\mu\nu} Y^r_{\rho]} - 2 \, g_{\mu[\nu} F_{\rho]\lambda} Y^{r\lambda} \ , \quad r = 1,2,3 \ .
\end{aligned}
\tag{3.5}
$$

It is clear from (3.3) that k is a Killing vector which also leaves F invariant, $\mathcal{L}_K F = 0$.

To determine whether the above TCFH generates symmetries in a particle system propagating in the supersymmetric backgrounds of $D = 4$, $N = 2$ supergravity consider the worldline action

$$
S = \int dt d\theta \big(-\frac{i}{2} g_{\mu\nu} DX^\mu \partial_t X^\nu + i q_{\mu\nu} DX^\mu DX^\nu \psi + \frac{1}{2} \psi D \psi \big) \ ,
\tag{3.6}
$$

where X is a bosonic and ψ is a fermionic superfield that depend on the worldline time t and the odd coordinate θ and $D = \partial_\theta + i\theta \partial_t$ with $D^2 = i\partial_t$. The fields have components $X = X|, \lambda = DX|$, $\psi = \psi|$ and $A = D\psi|$, where the restriction means evaluation at $\theta = 0$. X and A are worldline bosons while the rest of the components are worldline fermions. The couplings of the theory are the spacetime metric g and the 2-form q which depend on X. Later q will be identified with either F or its dual ${}^*F$. This action is manifestly invariant under one worldline supersymmetry. To write the action above we adopted the reality conditions[2]

$$
(i\partial_t)^* = i\partial_t \ , \quad \theta^* = \theta \ , \quad X^* = X \ , \quad \psi^* = -\psi \ , \quad (\chi\lambda)^* = \chi^* \lambda^* \ ,
\tag{3.7}
$$

[2] The reality conditions are anti-linear maps that square to -1.

for every two worldline fermions χ and λ. With these reality conditions, the coupling of the theory g and q are real. Such a choice of reality conditions is not unique. For example, one could have chosen $\psi^* = \psi$ at the cost of removing the imaginary unit i from the coupling term $q(DX)^2\psi$ in the action. However, such a choice is not suitable for the application we are investigating. The action (3.6) is a special case of a general class of actions for spinning particles presented in [61].

Identifying q with either F or ${}^*F$, the Killing vector k of the TCFH (3.3) generates the infinitesimal transformation

$$\delta X^\mu = a k^\mu \ , \quad \delta\psi = 0 \ , \tag{3.8}$$

which is a symmetry of the action. Thus the isometries of the supersymmetric backgrounds of $D = 4$, $N = 2$ supergravity generate a symmetry in the particle system action (3.6).

It remains to see whether the remaining conditions of the TCFH (3.3) are associated with symmetries. For this consider the fermionic transformations

$$\delta X^\mu = i\alpha I^\mu{}_\nu DX^\nu + \alpha L^\mu \psi \ , \quad \delta\psi = i\alpha M_\mu \partial_t X^\mu \ , \tag{3.9}$$

where I, L and M depend on X and α is an anti-commuting infinitesimal parameter. The reality condition for α is chosen as $\alpha^* = -\alpha$ which has as a consequence the presence of an imaginary unit in the IDX term of the infinitesimal transformation of X. Again this is essential for the application we shall present below. With this choice of reality condition the tensors I, L and M are real. After some simplification, the conditions for the invariance of the action (3.6) under the infinitesimal transformations (3.9) can be expressed[3] as

$$\begin{aligned}
&\nabla_\mu I_{\nu\rho} - 4q_{\mu[\nu} M_{\rho]} = -6q_{[\mu\nu} M_{\rho]} \ , \quad I_{[\nu\rho]} = I_{\nu\rho} \ , \\
&L_\mu = -M_\mu \ , \quad \nabla_\mu M_\nu + 2q_{\mu\rho} I^\rho{}_\nu = 0 \ , \quad dq_{\lambda[\mu\nu} I^\lambda{}_{\rho]} = 0, \quad L^\mu q_{\mu\nu} = 0 \ .
\end{aligned} \tag{3.10}$$

Note that if instead we had chosen as reality conditions $\psi^* = \psi$ and $\alpha^* = \alpha$ with the rest remaining the same, the sign of the term qI in the conditions above would have been different. The differential conditions as stated in (3.10) on the tensors associated to the infinitesimal transformations (3.9) are in a TCFH form with connection which depends on the coupling q of the theory.

The associated conserved charge, Q, of the symmetry generated by (3.9) is

$$Q = I_{\mu\nu} DX^\mu \partial_t X^\nu + \frac{i}{6}(dI)_{\mu\nu\rho} DX^\mu DX^\nu DX^\rho - iM_\mu \partial_t X^\mu \,\psi - \frac{1}{2}(dM)_{\mu\nu} DX^\mu DX^\nu \,\psi \tag{3.11}$$

It can be verified using the equations of motion of (3.6) and (3.10) that $DQ = 0$.

To compare (3.3) with (3.10), one has to consider three copies of the transformation (3.9) generated by the tensors I^r and M^r, $r = 1, 2, 3$ and set

$$I^r = \omega^r \ , \quad M^r = Y^r \ , \quad q = {}^*F \ . \tag{3.12}$$

With these identifications the connection part of the TCFHs in (3.3) and (3.10) match. However, consistency requires that the right-hand-side of the last two equations in (3.3) must vanish. As a result, ω^r, Y^r are parallel with respect to the TCFH connection. Note

[3]The are many inequivalent ways to write the conditions for the invariance of the action (3.6) under the transformations (3.9). However, the form given below is suitable for the investigation of this example.

also that $d^*F = 0$ as a consequence of the field equations and so the last condition in (3.10) is satisfied.

The commutators of the symmetries (3.9) can be easily computed [61]. After the identification (3.12) and for the backgrounds investigated below, it can be easily seen that they do not close to the standard supersymmetry algebra $\{Q^r, Q^s\} = \delta^{rs} H$ in one dimension. This is in agreement with the commutators of the fermionic symmetries generated by KY forms in [16].

The supersymmetric solutions of minimal $D = 4$, $N = 2$ supergravity have been classified in [108]. A class of backgrounds for which the right-hand side of the last two equations in (3.3) vanishes are those that admit a null Killing spinor, i.e. a spinor for which the bilinear K is null. For all such backgrounds, one can demonstrate as a consequence of the Killing spinor equations that the non-vanishing components of the fluxes and form bilinears are

$$\begin{aligned}
&k = k_- e^- \ , \quad Y^r = Y^r_- e^- \ , \quad \omega^r = \omega^r_{-i}\, e^- \wedge e^i \ , \\
&F = F_{-i}\, e^- \wedge e^i \ , \quad {}^*F = {}^*F_{-i}\, e^- \wedge e^i \ ,
\end{aligned} \tag{3.13}$$

see [107] for more details, where (e^+, e^-, e^i) is a co-frame such that the metric $g = 2e^+e^- + \delta_{ij} e^i e^j$, $i, j = 1, 2$, i.e. the form bilinears and the flux F are null forms. Using this, one can easily verify that the right-hand-side of the last two equations in (3.3) vanishes. Therefore particles systems described by (3.6) propagating on backgrounds with a null Killing spinor and couplings the spacetime metric g and $q = {}^*F$ admit symmetries (3.9) generated by the associated form bilinears. Such solutions include for example pp-wave type of backgrounds.

One can also consider the symmetries generated by the TCFH (3.5). The investigation for this is similar to the one we have presented above for the TCFH (3.3). The only difference is that in this case $I^r = \chi^r$ and $q = F$. Thus again the spinning particles described by the action (3.6) with couplings the spacetime metric g and $q = F$ admit symmetries (3.9) generated by the form bilinears Y^r and χ^r.

A similar analysis can be performed for supersymmetric backgrounds with a time-like Killing spinor, i.e. k is a time-like vector. However in this case one can show that either the condition ${}^*F_{\mu\nu} Y^{r\nu} = 0$ which arises from the comparison of (3.3) with (3.10) or $F_{\mu\nu} Y^{r\nu} = 0$ which arises from the comparison of (3.5) with (3.10), for all $r = 1, 2, 3$, require that $F = 0$. This is because the 1-forms Y^r are spacelike and span the three spatial directions of the spacetime, see [107]. The only solutions with $F = 0$ are locally isometric to Minkowski spacetime.

3.3 Minimal D=5, N=1 supergravity

Next let us turn to investigate the TCFH of $D = 5$, $N = 1$ minimal supergravity. The supercovariant connection of the theory is

$$\mathcal{D}_\mu \equiv \nabla_\mu - \frac{i}{4\sqrt{3}} \left(\Gamma_\mu{}^{\nu\rho} F_{\nu\rho} - 4 F_{\mu\nu} \Gamma^\nu \right) \ . \tag{3.14}$$

If ϵ is a Killing spinor, $\mathcal{D}_\mu \epsilon = 0$ the independent (Killing spinor) form bi-linears up to a Hodge duality operation are

$$f = \langle \epsilon, \epsilon \rangle \ , \quad k = \langle \epsilon, \Gamma_a \epsilon \rangle\, e^a \ , \quad \omega^1 = \frac{1}{2} \langle \epsilon, \Gamma_{ab} \epsilon \rangle\, e^a \wedge e^b \ ,$$

$$\omega^2 + i\omega^3 = \frac{1}{2}\langle\epsilon, \Gamma_{ab}\tilde{\epsilon}\rangle\, e^a \wedge e^b \ , \tag{3.15}$$

where e^a, $a = 0, 1, 2, 3, 4$ is a co-frame such that the metric is $g = \eta_{ab} e^a e^b$, $\tilde{\epsilon} = \Gamma_{12} * \epsilon$ in the conventions of [107]. The spinor representation is quaternionic and $\mathcal{D}$ acts quaternionic-linearly, so its kernel is a quaternionic vector space. The supersymmetric backgrounds have been classified in [109].

The conditions imposed by the Killing spinor equation on the form bilinears have been derived in [109]. Writing them in a TCFH form, one finds [22] that

$$\begin{aligned} &\nabla_\mu f = -\frac{2i}{\sqrt{3}} F_{\mu\nu} k^\nu \ , \quad \nabla_\mu k_\nu = \frac{1}{\sqrt{3}} {}^*F_{\mu\nu\rho} k^\rho - \frac{2i}{\sqrt{3}} F_{\mu\nu} f \ , \\ &\nabla_\mu \omega^r_{\nu\rho} - 2\sqrt{3}\, {}^*F_{\lambda\mu[\nu} \omega^{r\lambda}{}_{\rho]} = -2\sqrt{3}\, {}^*F_{\lambda[\nu\rho} \omega^{r\lambda}{}_{\mu]} + \frac{2}{\sqrt{3}} g_{\mu[\nu}\, {}^*F_{\rho]\alpha\beta} \omega^{r\alpha\beta} \ , \end{aligned} \tag{3.16}$$

where $\mu, \nu, \rho = 0, 1, 2, 3, 4$ are spacetime indices and $r = 1, 2, 3$. In what follows, it is also useful to state the TCFH for the form bilinears

$$\lambda^1 = \frac{1}{3!}\langle\epsilon, \Gamma_{abc}\epsilon\rangle\, e^a \wedge \cdots \wedge e^c \ , \quad \lambda^2 + i\lambda^3 = \frac{1}{3!}\langle\epsilon, \Gamma_{abc}\tilde{\epsilon}\rangle\, e^a \wedge \cdots \wedge e^c \ , \tag{3.17}$$

which are Hodge duals to ω^r. This reads

$$\nabla_\mu \lambda^r_{\nu_1\nu_2\nu_3} - 3\sqrt{3}\, {}^*F_{\alpha\mu[\nu_1} \lambda^{r\alpha}{}_{\nu_2\nu_3]} = -4\sqrt{3}\, {}^*F_{\alpha[\mu\nu_1} \lambda^{r\alpha}{}_{\nu_2\nu_3]} + 2\sqrt{3} g_{\mu[\nu_1}\, {}^*F_{\nu_2|\alpha\beta|} \lambda^{r\alpha\beta}{}_{\nu_3]} \ . \tag{3.18}$$

To find whether the above TCFHs are associated with symmetries of a particle system propagating on the spacetime consider the action

$$S = -\frac{1}{2}\int dt d\theta \big(i\, g_{\mu\nu} DX^\mu \partial_t X^\nu + \frac{1}{6} c_{\mu\nu\rho} DX^\mu DX^\nu DX^\rho \big) \ , \tag{3.19}$$

where the superfields X are as in (3.6) and c is a spacetime 3-form which depends on X. Actions with such couplings have been considered before in [61]. This action is manifestly invariant under one supersymmetry.

Next consider the fermionic symmetry

$$\delta X^\mu = \alpha I^\mu{}_\nu DX^\nu \ , \tag{3.20}$$

where the infinitesimal parameter α satisfies the reality condition $\alpha^* = \alpha$. Invariance of the action under this fermionic symmetry implies [62] that

$$\begin{aligned} &\hat{\nabla}_\mu I_{\nu\rho} = \hat{\nabla}_{[\mu} I_{\nu\rho]} \ , \quad I_{\mu\nu} = I_{[\mu\nu]} \ , \\ &d i_I c - 3 i_I dc = 0 \ , \end{aligned} \tag{3.21}$$

where

$$\hat{\nabla}_\mu X^\nu = \nabla_\mu X^\nu + \frac{1}{2} c^\nu{}_{\mu\rho} X^\rho \ , \tag{3.22}$$

is a connection with skew-symmetric torsion c and i_I denotes the inner derivation[4] with respect to I. The associated conserved charge, Q, is

$$Q = I_{\mu\nu} DX^\mu \partial_t X^\nu + \frac{i}{9}(dI)_{\mu\nu\rho} DX^\mu DX^\nu DX^\rho + \frac{i}{6} I^\lambda{}_\mu c_{\lambda\nu\rho} DX^\mu DX^\nu DX^\rho \ , \tag{3.23}$$

[4] The inner derivation of a n-form χ with respect to the vector $(k-1)$-form L is $i_L\chi = \frac{1}{(k-1)!(n-1)!} L^\nu{}_{\mu_1\dots\mu_{k-1}} \chi_{\nu\mu_k\dots\mu_{n+k-2}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{k+n-2}}$.

where $DQ = 0$ subject to the equations of motion and the conditions in (3.21).

One can also consider the invariance of the action (3.19) under the infinitesimal (bosonic) transformations

$$\delta X^\mu = aL^\mu{}_{\nu\rho} DX^\nu DX^\rho \ . \tag{3.24}$$

These transformations leave the action invariant provided [30] that

$$L_{\mu\nu\rho} = L_{[\mu\nu\rho]} \ , \quad \hat\nabla_\mu L_{\nu_1\nu_2\nu_3} = \hat\nabla_{[\mu} L_{\nu_1\nu_2\nu_3]} \ , \quad di_L c + 4 i_L dc = 0 \ . \tag{3.25}$$

We shall explore these in relation with the TCFH in (3.18). The conserved charge is

$$\begin{aligned} Q \ &= \ L_{\mu_1\mu_2\mu_3} DX^{\mu_1} DX^{\mu_2} \partial_t X^{\mu_3} - \frac{i}{16} (dL)_{\mu_1\dots\mu_4} DX^{\mu_1} \dots DX^{\mu_4} \\ &\quad - \frac{i}{8} c^\lambda{}_{\mu_1\mu_2} L_{\lambda\mu_3\mu_4} DX^{\mu_1} DX^{\mu_2} DX^{\mu_3} DX^{\mu_4} \ . \end{aligned} \tag{3.26}$$

To identify the symmetries of a particle system with action (3.19) propagating in the supersymmetric $D = 5$, $\mathcal{N} = 1$ supergravity background, one has to match the conditions of the TCFH (3.16) with those of the invariance (3.21) of the particle system (3.19). For this let us consider three independent transformations (3.20) generated by the tensors I^r, $r = 1, 2, 3$ and identify I^r with the 2-form bilinears ω^r of TCFH, i.e. $I^r = \omega^r$. Comparing the TCFH connection on ω^r with that on I^r in (3.21), one concludes that the coupling c of the particle system should be chosen as

$$c = 2\sqrt{3}\, {}^*F \ . \tag{3.27}$$

Then consistency of (3.16) with (3.21) after this identification requires that

$${}^*F_{\rho\alpha\beta}\, \omega^{r\alpha\beta} = 0 \ , \quad d i_{\omega^r} {}^*F - 3 i_{\omega^r} d^*F = 0 \ . \tag{3.28}$$

These two conditions impose strong restrictions on the possible backgrounds for which the particle system (3.19) admits (3.20) as a symmetry.

Before we turn to investigate (3.28) for various backgrounds, observe that k is a Killing vector that leaves F invariant. As a result $\delta X^\mu = a k^\mu$ is a symmetry of (3.19).

To find the backgrounds that satisfy (3.28), let us begin with the supersymmetric backgrounds of $D = 5$, $\mathcal{N} = 1$ supergravity that admit a time-like Killing spinor, i.e. a Killing spinor such that vector bilinear k is time-like [109]. For such backgrounds, without loss of generality the Killing spinor can be chosen as $\epsilon = 1V$[5] in the conventions of [107], where V is a spacetime function and the metric and 2-form flux are given as

$$\begin{aligned} ds^2 &= -V^4 (dt + \beta)^2 + V^{-2} d\mathring{s}^2 \ , \\ F &= \frac{\sqrt{3}}{2} de^0 - \frac{1}{3} (d\beta)_{\text{asd}} \ , \end{aligned} \tag{3.29}$$

with $k = \partial_t$ and $e^0 = V^2(dt + \beta)$, where $(d\beta)_{\text{asd}}$ is the anti-self dual component of $d\beta$ and $d\mathring{s}^2$ is a 4-dimensional hyper-Kähler metric. In our conventions ω^r are self-dual and in addition $d\omega^r = 0$, $i_k \omega^r = 0$ and

$$\omega^r_{\rho\mu} \omega^{s\rho}{}_\nu = \delta^{rs} (V^4 g_{\mu\nu} k_\mu k_\nu) + \epsilon^{rs}{}_q V^2 \omega^q_{\mu\nu} \ . \tag{3.30}$$

[5]The Killing spinor can be expanded in the space of forms as $\epsilon = a1 + a^i e_i + \frac{1}{2} a^{ij} e_i \wedge e_j$, where e_i denotes an orthonormal coframe and a, a^i, a^{ij} are complex valued functions. Then, one can prove that the representative of the gauge group of the theory on the space of spinors can be chosen as $\epsilon = 1V$.

Also one finds that $\mathcal{L}_k \omega^r = 0$.

The first condition in (3.28) implies that

$$V = \text{const} \ , \quad (d\beta)_{ij}\omega^{rij} = 0 \ . \tag{3.31}$$

As V is constant set for convenience $V = 1$. Furthermore, the equations of motion imply that $(d\beta)_{\text{asd}} = 0$. As $d\omega^r = 0$, set

$$d\beta = u_r \omega^r \ , \tag{3.32}$$

where u is a constant vector. Without loss of generality pick $(u_r) = (1, 0, 0)$. Implementing all the restrictions mentioned above, the resulting solution is expressed as

$$ds^2 = -(dt + \beta)^2 + d\mathring{s}^2 \ , \quad F = \frac{\sqrt{3}}{2}\omega^1 \ . \tag{3.33}$$

The solution can be viewed locally as a circle fibration over a 4-dimensional hyper-Kähler manifold whose fibre $U(1)$ curvature is given by ω^1. It turns out that the last condition in (3.28) is also satisfied for the transformations (3.20) generated by ω^2 and ω^3. Thus the action (3.19) with couplings given in (3.33) is invariant under the transformations (3.20) generated by ω^2 and ω^3.

Note that this is unlike what has been encountered before in the context of supersymmetric sigma models where two supersymmetries of the type (3.20) generated by I^2 and I^3, respectively, always imply the existence of a third supersymmetry generated by $I^2 I^3$. However to derive this, there have been some assumptions. In particular, it had been taken that I^2 and I^3 are invertible and the sigma model manifold is (almost) hyper-complex. However here ω^2 and ω^3 are not invertible as spacetime tensors and ω^1 is singled out as the curvature of the $U(1)$ bundle over the hyper-Kähler manifold.

It remains to solve (3.28) for $D = 5$, $\mathcal{N} = 1$ supergravity backgrounds that admit a null Killing spinor. In such a case the Killing vector bilinear $k = \partial_u$ is null and one can show that there is a co-frame

$$e^+ = du + V dv + n_I dx^I \ , \quad e^- = h^{-1} dv \ , \quad e^i = h\delta^i_I dx^I \ , \quad i = 1, 2, 3 \ , \tag{3.34}$$

where (u, v, x^I), $I = 1, 2, 3$ are the spacetime coordinates and V, h, n_I depend only on x^I and v. Moreover one has that

$$\omega^r = e^- \wedge e^r \ , \tag{3.35}$$

and

$$\begin{aligned} ds^2 &= 2e^- e^+ + \delta_{ij} e^i e^j \ , \\ F &= -\frac{1}{4\sqrt{3}} \mathring{\epsilon}_I{}^{JK} h^{-2} (dn)_{JK} dv \wedge dx^I - \frac{\sqrt{3}}{4} \mathring{\epsilon}_{IJ}{}^K \partial_K h dx^I \wedge dx^J \ , \end{aligned} \tag{3.36}$$

where $\mathring{\epsilon}$ is the Levi-Civita tensor of the flat metric.

The first condition in (3.28) for all ω^r implies that h must depend only on v, $h = h(v)$. It turns out that this condition is also sufficient for the second condition in (3.28) to be satisfied. There are many solutions with $h = h(v)$. For example one can take $n = 0$, $h = 1$ in which case the field equations imply that V is a harmonic function on $\mathbb{R}^3$ with delta function sources and the solution a multi pp-wave. Another solution is to take $h = 1$, $n = n(x^I)$. Then the field equations imply, see e.g. [107], that

$$\partial^I dn_{IJ} = 0 \ , \quad \partial^I \partial_J V = \frac{1}{6} dn^{IJ} dn_{IJ} \ , \tag{3.37}$$

i.e. dn satisfies the Maxwell equation and V the Poisson equation on $\mathbb{R}^3$. For a solution set $n_I = \lambda_{IJ} x^J$, λ constant 2-form and $V = (1/3)\delta^{IJ}\lambda_{IK}\lambda_{JL}x^K x^L + V_0$, where V_0 is a harmonic function on $\mathbb{R}^3$ with delta-function sources.

The commutators of two (3.20) transformations can easily be computed and involve the Nijenhuis tensor of the I's that generate the transformations. In the examples explored above, these do not satisfy the standard supersymmetry algebra in one dimension. This can be easily seen as ω^r does not satisfy the algebra of imaginary unit quaternions, see (3.30). Nevertheless the commutator is a symmetry of the action (3.19).

Next, let us consider whether the 3-form bilinears (3.17) generate symmetries for the action (3.19). For this consider three transformations as in (3.24) generated by the tensors L^r and identify $L^r = \lambda^r$. Then consistency of (3.18) with (3.25) requires that

$$c = 2\sqrt{3}{}^*F \ , \quad di_{\lambda^r}{}^*F + 4 i_{\lambda^r} d^*F = 0 \ , \quad F_{\gamma\mu}\omega^{r\gamma}{}_\nu - F_{\gamma\nu}\omega^{r\gamma}{}_\mu = 0 \ . \tag{3.38}$$

The third condition above arises from the requirement that the last term in the TCFH (3.18) must vanish.

There are solutions to the conditions (3.38) for supersymmetric backgrounds with both a timelike and null Killing spinors. In the former case, the last condition in (3.38) together with the field equations imply that $V = 1$ and $(d\beta)_{\text{ads}} = 0$. Without loss of generality one again can choose $d\beta = \omega^1$. The spinning particles described by the action (3.19) on such such backgrounds are invariant under the transformation generated by λ^1 but they are not invariant under the transformations generated by λ^2 and λ^3. For backgrounds with a null Killing spinor, one again finds as a consequence of the last equation in (3.38) that $h = h(v)$. Then the analysis proceeds as for the symmetries generated by ω^r giving the same backgrounds as solutions.

A dynamical system with a $2n$-dimensional phase space is Liouville integrable or equivalently completely integrable provided it admits n independent conserved charges, including the Hamiltonian, which are in involution, i.e. all n conserved charges Poisson commute. This definition can be extended to the spinning particle systems described by the actions (3.6) and (3.19). The phase space of these systems is a supermanifold with dimension $(2D|\kappa)$, where D is the spacetime dimension and κ is the number of worldline fermions. A generalisation of the Liouville's definition is to declare that such spinning particle systems are completely integrable provided that they admit $(D|[\frac{\kappa+1}{2}])$ independent conserved charges in involution including the Hamiltonian. This is a rather strong concept of complete integrability as there is no dynamics in the worldvolume fermion directions because the Lagrangians of spinning particle systems are at most linear in the worldvolume fermion velocities. Alternatively one can declare that the spinning particle systems described by the actions (3.6) and (3.19) are completely integrable provided that they admit D independent (even) conserved charges in involution including the Hamiltonian, where the Poisson bracket used is that of the phase space supermanifold of these systems. This concept of integrability is weaker than that of the direct generalisation of the Liouville's definition. One can easily construct first order systems coupled to second order ones where the Hamilton-Jacobi equation can be separated under the weaker definition of integrability. The complete integrability of the spinning particle systems, under either of the two definitions, implies that the underlying bosonic system which can be obtained by setting all worldline fermions to zero must also be completely integrable. We shall use this below to determine whether the TCFHs imply the complete integrability of the spinning particle systems associated with supergravity backgrounds.

The 2-form bilinears of both $D = 4$ and $D = 5$ supergravities that generate symmetries in the spinning particle actions we have investigated are not principal. This means that they do not have 2 independent eigenvalues. The existence of a principal CCKY form on a background implies the separability of the geodesic equations and some of the classical field equations, see e.g. [12, 11]. Therefore one should not expect that the backgrounds we have investigated exhibit similar separability properties unless they admit additional symmetries, e.g. additional rotation or axial symmetries. To give an explicit example consider the solution of $D = 5$ supergravity which is locally a circle bundle over a 4-dimensional hyper-Kähler manifold. We have found that a particle system in such a background admits additional fermionic symmetries. However if one chooses as a hyper-Kähler manifold one without additional isometries, e.g K_3, one should not expect that the geodesic equations of the 5-dimensional solution to be separable. In turn the associated spinning particle system described by the action (3.19) is not completely integrable.

The separability properties of $D = 5$ $N = 1$ supergravity backgrounds have been investigated before in [43, 44, 46]. These authors explored the properties of the generalized CKY equation which is the CKY equation with respect to a connection with skew-symmetric torsion, like $\hat{\nabla}$ in (3.22). In particular, they considered generalized closed CKY 2-forms, i.e. 2-forms which are closed with respect to $\hat{d}$ the exterior covariant derivative associated to $\hat{\nabla}$. The 2-form bilinears ω^r we have considered here do not satisfy the same conditions as the generalized closed CKY forms. In particular, ω^r satisfy the generalized CKY equation as a consequence of (3.16) with skew-symmetric torsion c given in (3.27). However for general supersymmetric solutions ω^r do not satisfy the closure (or indeed the co-closure) condition with respect to $\hat{\nabla}$, i.e. $\hat{d}\omega^r \neq 0$. Of course as a consequence of the TCFH in (3.16) ω^r are closed, $d\omega^r = 0$, in the standard sense. Therefore the gravitational backgrounds investigated in [43, 44] and in this chapter are different.

Chapter 4

TCFHs, hidden symmetries and M-theory backgrounds

4.1 Introduction

One purpose of this chapter is to present the full TCFH of 11-dimensional supergravity. We shall find that the reduced holonomy of the minimal[1] TCFH connection is included in $SO(10,1) \times GL(517) \times GL(495)$ while the reduced holonomy of the maximal TCFH connection is included in $GL(528) \times GL(496)$. The latter holonomy is the same as that of the maximal connection of IIA and IIB TCFH [24]. Then we shall explore the question on whether the TCFH conditions can be identified with the invariance conditions of a probe action under transformations generated by the form bilinears. As the supersymmetric backgrounds of 11-dimensional supergravity have not been classified, we shall focus our investigation on the M-brane solutions[2] which include the M2- and M5-branes as well as the pp-wave and KK-monopole.

Before we proceed with the investigation of the TCFH for M-branes, we shall give the KS tensors and KY forms associated with the complete integrability of the geodesic flow of spherically symmetric M-brane solutions, i.e. those that depend on a harmonic function with one centre. The geodesic equations of these backgrounds are separable in angular variables. Here we shall present all independent conserved charges which are in involution. Moreover, we shall demonstrate that a relativistic particle probe propagating on spherically symmetric M-branes admits an infinite number of hidden symmetries generated by KS tensors. In addition, we shall find that the spinning particle probe action admits 2^8, 2^7 and 2^4 symmetries generated by KY forms on the pp-wave, M2-brane and M5-brane backgrounds, respectively. Spinning particle probes exhibit enhanced worldline supersymmetry propagating on the KK-monopole.

After this, we shall return to investigate under which conditions the form bilinears of M-brane backgrounds, which may now depend of a general harmonic function and so they are not necessarily spherically symmetric, generate symmetries for spinning particle type of probes. For this, we match the conditions required for a transformation generated by the form bilinears to leave a spinning particle probe action invariant with the TCFH conditions on the form bilinears. We shall find that all form bilinears of pp-wave and KK-monopole backgrounds generate symmetries for the spinning particle probes. This is because as a consequence of the TCFH and the vanishing of the 4-form field strength for

[1]See [22] for the definition of these connections.

[2]These have been instrumental in the understanding of string dualities [110, 111].

these solutions, the form bilinears are covariantly constant with respect to the Levi-Civita connection. Furthermore, we demonstrate that there are Killing spinors such that the 1-form, 2-form and 3-form bilinears of the M2-brane are KY forms and so generate symmetries for spinning particle probes propagating on this background. A similar analysis for the M5-brane reveals that only the 1-form bilinear generates symmetries for spinning particle probes. To demonstrate these results, we have computed all the form bilinears of M-brane backgrounds using spinorial geometry [112].

This chapter is organised as follows. In section 2, we present the TCFH of 11-dimensional supergravity and give the reduced holonomy of TCFH connections. In section 3, we give the KS and KY tensors of spherically symmetric M-brane backgrounds and prove the complete integrability of their geodesic flows. In section 4, we identify the form bilinears of M-branes that generate symmetries for probe actions, and in section 5 we give our conclusions. In appendix A, we give the form bilinears of the M5-brane. In appendix B, we explore the symmetries of spinning particle probes with 4-form couplings.

4.2 The TCFH of D=11 supergravity

The supercovariant connection of 11−dimensional supergravity [69] is

$$D_\mu = \nabla_\mu + \frac{1}{288}(\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4}F_{\nu_1\nu_2\nu_3\nu_4} - 8F_{\mu\nu_1\nu_2\nu_3}\Gamma^{\nu_1\nu_2\nu_3}) , \tag{4.1}$$

where ∇ is the spin connection of the spacetime metric and F is the 4-form field strength of the theory. The reduced holonomy of supercovariant connection on generic backgrounds is included in $SL(32,\mathbb{R})$ [65, 66].

Supersymmetric backgrounds with N-dimensional space of Killing spinors, ϵ^r, $r = 1,\dots,N$, are those that admit N linearly independent solutions to the KSE, $D_\mu\epsilon^r = 0$, where ϵ is a Majorana $\mathfrak{spin}(10,1)$ spinor[3]. Given N Killing spinors, one can construct the form bilinears

$$\begin{aligned}
&f^{rs} = \langle\epsilon^r,\epsilon^s\rangle\ , \quad k^{rs}_\mu = \langle\epsilon^r,\Gamma_\mu\epsilon^s\rangle\ , \quad \omega^{rs}_{\mu\nu} = \langle\epsilon^r,\Gamma_{\mu\nu}\epsilon^s\rangle\ , \quad \varphi^{rs}_{\mu_1\mu_2\mu_3} = \langle\epsilon^r,\Gamma_{\mu_1\mu_2\mu_3}\epsilon^s\rangle\ ,\\
&\theta^{rs}_{\mu_1\mu_2\mu_3\mu_4} = \langle\epsilon^r,\Gamma_{\mu_1\mu_2\mu_3\mu_4}\epsilon^s\rangle\ , \quad \tau^{rs}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \langle\epsilon^r,\Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}\epsilon^s\rangle\ .
\end{aligned}\tag{4.2}$$

Note that the form bilinears k, ω and τ are symmetric in the exchange of ϵ^r and ϵ^s while the rest are skew-symmetric. There is no classification of supersymmetric solutions of 11-dimensional supergravity. However, there are many partial results. For example, the maximally supersymmetric solutions have been classified in [87] and the KSE has been solved for one Killing spinor in [88, 113, 112], see review [107] for the current state of the art.

The TCFH of 11−dimensional supergravity for the form bilinears which are symmetric in the exchange of the two Killing spinors has been given in [22]. Here we shall present the TCFH for all form bilinears. The TCFH of 11−dimensional supergravity expressed in terms of the minimal connection $\mathcal{D}^{\mathcal{F}}_\mu$ reads

$$\begin{aligned}
&\mathcal{D}^{\mathcal{F}}_\mu k_\nu := \nabla_\mu k_\nu = \frac{1}{6}F_{\mu\nu\alpha\beta}\omega^{\alpha\beta} - \frac{1}{6!}{}^*F_{\mu\nu\rho_1\rho_2\rho_3\rho_4\rho_5}\tau^{\rho_1\rho_2\rho_3\rho_4\rho_5}\ ,\\
&\mathcal{D}^{\mathcal{F}}_\mu \omega_{\nu_1\nu_2} := \nabla_\mu\omega_{\nu_1\nu_2} - \frac{1}{2\cdot 3!}F_{\mu\rho_1\rho_2\rho_3}\tau^{\rho_1\rho_2\rho_3}{}_{\nu_1\nu_2} = -\frac{1}{3}F_{\mu\nu_1\nu_2\rho}k^\rho
\end{aligned}$$

[3] Here we follow standard terminology. We can be more precise and call these pinors as we need to know how they act in the whole Clifford algebra.

$$
\begin{aligned}
&-\frac{1}{2\cdot 3!}\tau_{[\mu\nu_1}{}^{\rho_1\rho_2\rho_3}F_{\nu_2]\rho_1\rho_2\rho_3}+\frac{1}{3\cdot 4!}g_{\mu[\nu_1}\tau_{\nu_2]}{}^{\rho_1\rho_2\rho_3\rho_4}F_{\rho_1\rho_2\rho_3\rho_4}\ ,\\
\mathcal{D}^{\mathcal{F}}_\mu\tau_{\nu_1\nu_2\nu_3\nu_4\nu_5} &:= \nabla_\mu\tau_{\nu_1\nu_2\nu_3\nu_4\nu_5}+5F_{\mu[\nu_1\nu_2\nu_3}\omega_{\nu_4\nu_5]}-\frac{5}{6}{}^*F_{\mu[\nu_1\nu_2\nu_3|\rho_1\rho_2\rho_3|}\tau_{\nu_4\nu_5]}{}^{\rho_1\rho_2\rho_3}=\\
&-\frac{1}{6}{}^*F_{\mu\nu_1\nu_2\nu_3\nu_4\nu_5\rho}k^\rho+\frac{5}{2}F_{[\mu\nu_1\nu_2\nu_3}\omega_{\nu_4\nu_5]}-\frac{5}{6}\tau_{[\mu\nu_1}{}^{\rho_1\rho_2\rho_3*}F_{\nu_2\nu_3\nu_4\nu_5]\rho_1\rho_2\rho_3}\\
&-\frac{10}{3}g_{\mu[\nu_1}\omega^\rho{}_{\nu_2}F_{\nu_3\nu_4\nu_5]\rho}-\frac{5}{18}g_{\mu[\nu_1}\tau_{\nu_2}{}^{\rho_1\rho_2\rho_3\rho_4*}F_{\nu_3\nu_4\nu_5]\rho_1\rho_2\rho_3\rho_4}\ ,\\
\mathcal{D}^{\mathcal{F}}_\mu f &:= \nabla_\mu f=\frac{1}{18}F_{\mu\nu_1\nu_2\nu_3}\varphi^{\nu_1\nu_2\nu_3}\ ,\\
\mathcal{D}^{\mathcal{F}}_\mu\varphi_{\nu_1\nu_2\nu_3} &:= \nabla_\mu\varphi_{\nu_1\nu_2\nu_3}-\frac{3}{4}F_{\mu[\nu_1|\rho_1\rho_2|}\theta^{\rho_1\rho_2}{}_{\nu_2\nu_3]}=\frac{1}{6}g_{\mu[\nu_1}F_{\nu_2|\rho_1\rho_2\rho_3|}\theta^{\rho_1\rho_2\rho_3}{}_{\nu_3]}\\
&-\frac{1}{36}{}^*F_{\mu\nu_1\nu_2\nu_3\rho_1\rho_2\rho_3}\varphi^{\rho_1\rho_2\rho_3}-\frac{1}{2}F_{[\mu\nu_1|\rho_1\rho_2|}\theta^{\rho_1\rho_2}{}_{\nu_2\nu_3]}-\frac{1}{3}F_{\mu\nu_1\nu_2\nu_3}f\ ,\\
\mathcal{D}^{\mathcal{F}}_\mu\theta_{\nu_1\nu_2\nu_3\nu_4} &:= \nabla_\mu\theta_{\nu_1\nu_2\nu_3\nu_4}-\frac{1}{3}{}^*F_{\mu[\nu_1\nu_2\nu_3|\rho_1\rho_2\rho_3|}\theta^{\rho_1\rho_2\rho_3}{}_{\nu_4]}+3F_{\mu[\nu_1\nu_2|\rho|}\varphi^\rho{}_{\nu_3\nu_4]}=\\
&\frac{1}{18}g_{\mu[\nu_1}{}^*F_{\nu_2\nu_3\nu_4]\rho_1\rho_2\rho_3\rho_4}\theta^{\rho_1\rho_2\rho_3\rho_4}-\frac{5}{18}{}^*F_{[\mu\nu_1\nu_2\nu_3|\rho_1\rho_2\rho_3|}\theta^{\rho_1\rho_2\rho_3}{}_{\nu_4]}\\
&-g_{\mu[\nu_1}F_{\nu_2\nu_3|\rho_1\rho_2|}\varphi^{\rho_1\rho_2}{}_{\nu_4]}+\frac{5}{3}F_{[\mu\nu_1\nu_2|\rho|}\varphi^\rho{}_{\nu_3\nu_4]}\ ,
\end{aligned}
\tag{4.3}
$$

where for simplicity we have suppressed the indices r and s on the form bilinears with label the independent Killing spinors. In our conventions $\epsilon_{0123456789\natural}=-1$, ${}^*F_{\mu_1\cdots\mu_7}=\frac{1}{4!}\epsilon_{\mu_1\cdots\mu_7}{}^{\nu_1\cdots\nu_4}F_{\nu_1\cdots\nu_4}$ and $\Gamma_\natural:=\Gamma_{0\ldots 9}$, where $\natural$ denotes the 11th direction. Clearly, the equations above are of the form stated in (1.1), where Ω is the multiform spanned by the form bilinears (4.2), $\mathcal{Q}$ can be read from the terms in the right-hand side of (4.3) that explicitly contain the spacetime metric g and $\mathcal{P}$ is spanned by the remaining terms on the right-hand side of (4.3). Clearly (4.3) provides a geometric interpretation of the conditions induced by the KSE on the form bilinears as it relates them to a generalisation of the CKY equations.

Viewing $\mathcal{D}^{\mathcal{F}}_\mu$ as degree non-preserving connection on k-forms, $k=0,1,2,3,4,5$, the reduced holonomy of $\mathcal{D}^{\mathcal{F}}_\mu$ factorises as the connection preserves the subspaces of k-degree forms for $k=1,2,5$ and for $k=0,3,4$, i.e. it preserves the subspaces of the form bilinears which are symmetric and skew-symmetric under the exchange of the two Killing spinors. This is also the case for the maximal connection defined in [22] which we do not consider here in detail. In addition, $\mathcal{D}^{\mathcal{F}}_\mu$ preserves the subspace of 1-forms, and the subspace of 2- and 5-forms, and acts trivially on 0-forms. As a result the reduced holonomy of $\mathcal{D}^{\mathcal{F}}_\mu$ is included in $SO(10,1)\times GL(517)\times GL(495)$ group. Note that the reduced holonomy of the maximal connection is included in $GL(528)\times GL(496)$ as it does not preserve the subspace of 1-forms but instead, it mixes them with the subspace of 2- and 5-forms and it acts non-trivially on 0-forms. The reduced holonomy of the maximal connection is the same as that of the maximal TCFH connections of type IIA and type IIB supergravities [24]. Of course, for special backgrounds, the holonomy of $\mathcal{D}^{\mathcal{F}}$ reduces further.

4.3 Symmetries of probes on M-brane backgrounds

4.3.1 Symmetries and integrability

In the examples that follow below, the complete integrability of the geodesic flow of the spacetimes considered is due to the large number of isometries that these spacetimes admit[4]. As the Lie algebra of these isometries is not abelian, the associated conserved charges are not in involution. Nevertheless, it is possible to use these charges to construct new ones associated with KS tensors which are in involution, see the example below.

4.3.2 Complete integrability of black hole geodesic flow

Before we proceed to investigate the symmetries of probes on M-theory backgrounds, let us present some examples. The standard example is the integrability of the geodesic flow of the Kerr black hole. However more suitable for the results that follow are the examples of Schwarzschild and Reissner-Nordström black holes in four and higher dimensions. The metric of both these solutions in four dimensions can be written as

$$g = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{4.4}$$

The associated geodesic equations of the metric above can be explicitly separated in the stated coordinates. However, it is instructive to provide a symmetry argument for the complete integrability of the geodesic equations.

The isometry group of the above backgrounds is $\mathbb{R} \times SO(3)$. There are two commuting isometries given by $k_0 = \partial_t$ and $k_1 = \partial_\phi$ which give rise to the conserved charges $K_0 = p_t$ and $K_1 = p_\phi$. These together with the Hamiltonian $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$ give three conserved charges in involution. Note that $[K_r, H]_{\rm NS} = \mathcal{L}_{k_r} g^{\mu\nu}p_\mu p_\nu = 0$, $r = 0, 1$, as k_r are isometries. It remains to find a fourth conserved charge in involution for the complete integrability of the geodesic system. For this consider the Killing vector fields

$$k_1 = \partial_\phi \ , \quad k_2 = -\sin\phi\cot\theta\partial_\phi + \cos\phi\partial_\theta \ , \quad k_3 = \cos\phi\cot\theta\partial_\phi + \sin\phi\partial_\theta \ , \tag{4.5}$$

which generate the $SO(3)$ isometry group and notice that $[k_a, k_b] = -\epsilon_{ab}{}^c k_c$. Then another conserved charge can be constructed utilising the (quadratic) Casimir operator of the Lie algebra of $SO(3)$ which can be used to construct a symmetric tensor that commutes with all the isometries of the background. As the quadratic Casimir is proportional to the identity matrix in the basis chosen for the Lie algebra, the associated symmetric tensor is

$$d = \delta^{ab}k_a \otimes k_b = \frac{1}{\sin^2\theta}(\partial_\phi)^2 + (\partial_\theta)^2 \ . \tag{4.6}$$

This is a KS tensor because k_a are Killing vectors. Thus

$$Q(d) = \frac{1}{\sin^2\theta}p_\phi^2 + p_\theta^2 \ , \tag{4.7}$$

is a conserved charge of the geodesic flow of the metric (4.4). It turns out that K_r, H and $Q(d)$ are independent and in involution implying that the geodesic equations are completely integrable for any function $A = A(r)$ in (4.4).

The metric (4.4) also admits a CCKY 2-form [11]. This is given by

$$\beta = r dt \wedge dr \ , \tag{4.8}$$

which can be verified after a computation. The dual

$$\alpha = {}^*\beta = r^3 \sin\theta d\theta \wedge d\phi \ , \tag{4.9}$$

[4]See 2.5 for a review of the main concepts used in this chapter.

is a KY 2-form. As a result, it generates a symmetry for the spinning particle action (2.55) given by the infinitesimal variation in (2.56). There are four additional KY 1-forms constructed from the Killing vector fields k_0, k_1, k_2, k_3 using the metric. All of which generate symmetries for the action (2.55). One can also square the KY tensor (4.9) to construct a KS tensor. It turns out that this is not independent from (4.6).

The analysis we have done can be extended to black holes in higher than four dimensions. Indeed consider the metric

$$g = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2 g(S^n) \ , \tag{4.10}$$

where $g(S^n)$ is the round metric on S^n with $n \geq 2$. Again the geodesic equation can be separated in angular coordinates and the geodesic flow is completely integrable. The above metric admits a $\mathbb{R} \times SO(n+1)$ group of isometries. Viewing S^n embedded as the hypersurface, $\sum_i (x^i)^2 = 1$, in $\mathbb{R}^{n+1}$, the Killing vectors of the spacetime metric g can be written as

$$k_0 = \partial_t \ , \quad k_{ij} = x_i \partial_j - x_j \partial_i \ , \quad i < j \ , \tag{4.11}$$

where $i, j = 1, \dots, n+1$ and $x_i = x^i$. Note that k_{ij} are tangent to S^n as $(d(x^2-1))(k_{ij}) = 2x_k dx^k(k_{ij}) = 0$. The associated conserved charges are $Q_0 = p_t$ and $Q_{ij} = x_i p_j - x_j p_i$, where p_i is the momentum on S^n and so $x^i p_i = 0$. These conserved charges are not in involution. However

$$Q_0 \ , \quad D_m = \frac{1}{4} \sum_{i,j \geq n+2-m} (Q_{ij})^2 \ , \quad m = 2, \dots, n+1 \ , \tag{4.12}$$

are independent conserved charges of the geodesic flow and in involution which together with the Hamiltonian, H, of the geodesic motion imply the complete integrability of the geodesic flow of the metric (4.10).

To explain the choice of D_m charges in (4.12), note that D_{n+1} is the Hamiltonian of the geodesic flow on S^n and it is constructed using the quadratic Casimir operator of $\mathfrak{so}(n+1)$. The $\mathfrak{so}(n+1)$ algebra admits a decomposition

$$\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(n) \subset \mathfrak{so}(n+1) \ . \tag{4.13}$$

The D_m conserved charge is constructed using the quadratic Casimir operator of the $\mathfrak{so}(m)$ subalgebra of $\mathfrak{so}(n+1)$. At each stage as the quadratic Casimir operator of $\mathfrak{so}(m)$ is invariant under $\mathfrak{so}(m)$, it is also invariant under the $\mathfrak{so}(m-1)$ subalgebra of $\mathfrak{so}(m)$. Therefore the quadratic Casimir operator of $\mathfrak{so}(m-1)$ commutes with that of $\mathfrak{so}(m)$. As a consequence, D_{m-1} is in involution with D_m. This method of constructing observables in involution has been generalised and used in [114] to investigate the integrability of geodesic flows on homogeneous manifolds.

Moreover, a direct computation reveals that β in (4.8) is a CCKY form for the metric (4.10) and therefore its dual α is a KY n-form. It turns out that β in (4.8) is a CCKY with respect to a metric as in (4.10) with $g(S^n)$ now replaced with the metric, $g(N)$, of any n-dimensional manifold N provided it is independent from the coordinates r and t.

4.3.3 Hidden symmetries and spherically symmetric M-branes

Next, let us turn to investigate the symmetries of relativistic and spinning particle probes described by the actions (2.51) and (2.55), respectively, propagating on M-branes. The focus will be on those KS and KY tensors which give rise to conserved charges related to the integrability of the geodesic flow on some of these backgrounds.

M-theory pp-waves

The M-theory pp-wave solution is

$$g = 2du(dv + \frac{1}{2}h(y,v)du) + \delta_{ij}dy^i dy^j \ , \tag{4.14}$$

with $F = 0$, where (u, v, y) are the coordinates of 11-dimensional spacetime and h is a Harmonic function on $\mathbb{R}^9$, $\partial_y^2 h = 0$. As the $\partial_y^2 h = 0$ condition appears in other M-brane backgrounds below, the solutions of this equation that we shall be considering on $\mathbb{R}^n$, $n > 2$, are

$$h = q_0 + \sum_{m=1}^{\ell} \frac{q_m}{|y - y_m|^{n-2}} \ , \quad q_0 = 0, 1 \ , \tag{4.15}$$

where q_m are constants, $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$ and y_m are the centres or positions of the harmonic function h.

Here we shall investigate the symmetries of probes propagating on a spherically symmetric pp-wave, i.e. a pp-wave that depends on a harmonic function with one centre. After a coordinate transformation to put the centre at 0, $h = \frac{q}{|y|^7}$, where q is constant denoting the momentum of the pp-wave. This solution has an $\mathbb{R}^2 \times SO(9)$ symmetry generated by the Killing vector fields $k_+ = \partial_u$, $k_- = \partial_v$ and

$$k_{ij} = y_i \partial_j - y_j \partial_i \ , \quad i < j \ , \tag{4.16}$$

where $y_i = y^i$. The latter vector fields are generated by the action of $SO(9)$ on the y coordinates.

Clearly all the above vector fields generate symmetries for the probe action (2.51) with conserved charges

$$Q_\pm = Q(k_\pm) = p_\pm \ , \quad Q_{ij} = Q(k_{ij}) = y_i p_j - y_j p_i \ . \tag{4.17}$$

In addition, one can demonstrate with a direct calculation that

$$d_{i_1 \cdots k} = y^{j_1} \dots y^{j_q} a_{j_1 \dots j_q, i_1 \dots i_k} \ , \tag{4.18}$$

are KS tensors of the pp-wave spacetime provided that the constant tensor a satisfies the condition[5]

$$a_{(j_1 \dots j_q, i_1) \dots i_k} = a_{j_1 \dots (j_q, i_1 \dots i_k)} = 0 \ . \tag{4.19}$$

These in turn generate transformations as those in (2.53) which leave the action (2.51) invariant. The associated conserved charges are given in (2.54) or equivalently in (2.62). It is evident from the above analysis that a probe described by the action (2.51) and propagating on this pp-wave spacetime, and so the geodesic flow, admits infinite number of hidden symmetries. Note that KS and KY tensors on 4-dimensional pp-wave spacetimes have been investigated before, see e.g. [115, 116].

Although the probe (2.51) admits an infinite number of symmetries propagating on a pp-wave background, it does not immediately imply that the dynamics is completely

[5]For $q = k$, the $(0, 2q)$ tensors that lie in the irreducible representation of $GL(9)$ associated with the 2 rows and q columns Young tableau solve the condition on a. A similar statement is true for the KS tensors of the M2- and M5-branes below.

integrable. Clearly the conserved charges $Q_{ij} = Q(k_{ij})$ and $Q_\pm = Q(k_\pm)$ generated by the vector fields k_{ij} and $k_\pm$ are not in involution-the Poisson bracket algebra of Q_{ij} is $\mathfrak{so}(9)$. However, Q_{ij} can be used to construct conserved charges which are in involution. In particular, one can show that the 10 conserved charges

$$Q_\pm \ , \quad D_m = \frac{1}{4} \sum_{i,j\geq 10-m} (Q_{ij})^2 \ , \quad m = 2, \dots, 9 \ , \tag{4.20}$$

are in involution. These together with the Hamiltonian of the geodesic system $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ give 11 independent conserved charges in involution leading to the complete integrability of the geodesic flow. As in the black hole analysis, D_9 is the Hamiltonian of the geodesic flow on S^8 which is constructed from the quadratic Casimir operator of $\mathfrak{so}(9)$.

Turning to the investigation of the symmetries of the probe (2.55) propagating on a pp-wave, one has to determine the KY tensors of the background. One can verify after some calculation that

$$\beta(\varphi) = y_i dy^i \wedge \varphi \wedge du \wedge dv \ , \tag{4.21}$$

are CCKY forms for any constant k-form φ on $\mathbb{R}^9$, where $y_i = y^i$. As a result $\alpha(\varphi) = {}^*\beta(\varphi)$ are KY forms. These generate the transformations (2.56) which leave the spinning particle action (2.55) invariant with associated conserved charges given in (2.58). Therefore the probe (2.55) propagating on a pp-wave background admits 2^8 linearly independent conserved charges[6] generated by the KY forms $\alpha(\varphi)$.

M2-branes

The M2-brane solution [72] can be expressed as

$$g = h^{-\frac{2}{3}} \eta_{ab} d\sigma^a d\sigma^b + h^{\frac{1}{3}} \delta_{ij} dy^i dy^j \ , \quad F = \pm d\sigma^0 \wedge d\sigma^1 \wedge d\sigma^2 \wedge dh^{-1} \ , \tag{4.22}$$

where σ^a, $a = 0, 1, 2$, are the worldvolume coordinates of the brane, y^i, $i = 1, \dots, 8$, are the transverse coordinates and h is a harmonic function $\partial_y^2 h = 0$ on the transverse space $\mathbb{R}^8$. An explicit expression for h is as in (4.15) with $q_0 = 1$ and $n = 8$.

For the spherically symmetric M2-brane solution that we shall consider in this section $h = 1 + \frac{q}{|y|^6}$. This solution is invariant under the action of the $SO(1,2) \ltimes \mathbb{R}^3 \times SO(8)$ group, where the Poincaré group acts on the worldvolume coordinates of the M2-brane while $SO(8)$ acts on the transverse coordinates with standard rotations. The Killing vector fields are $k_a = \partial_a$, $k_{ab} = \sigma_a \partial_b - \sigma_b \partial_a$ and $k_{ij} = y_i \partial_j - y_j \partial_i$, where $\sigma_a = \eta_{ab} \sigma^b$ and $y^i = y_i$. It is clear that the probe (2.51) propagating on this background admits symmetries generated by these vector fields and the associated conserved charges are

$$Q_a = Q(k_a) = p_a \ , \quad Q_{ab} = \sigma_a p_b - \sigma_b p_a \ , \quad Q_{ij} = Q(k_{ij}) = y_i p_j - y_j p_i \ . \tag{4.23}$$

As for the pp-wave, the probe (2.51) admits additional symmetries generated by KS tensors. To find these tensors we use an ansatz which preserves the worldvolume Poincaré symmetry of the solution. Then after some computation, one can verify that

$$d_{a_1 \dots a_{2m} i_1 \dots i_k} = h^{\frac{1}{3}(k-2m)} y^{j_1} \dots y^{j_q} a_{j_1 \dots j_q, i_1 \dots i_k} \eta_{(a_1 a_2} \dots \eta_{a_{2m-1} a_{2m})} \ , \tag{4.24}$$

[6] The maximal number of independent KY k-forms [17] on a n-dimensional spacetime is $(n+1)!/((k+1)!(n-k)!)$.

are KS tensors provided that the constant tensors a satisfy

$$a_{(j_1\dots j_q,i_1)\dots i_k} = a_{j_1\dots(j_q,i_1\dots i_k)} = 0 \ . \tag{4.25}$$

These in turn give additional conserved charges (2.54) for the relativistic particle probe (2.51). Therefore the probe (2.51), and so the geodesic flow on this M2-brane, admits an infinite number of hidden conserved charges.

The dynamics of the relativistic particle (2.51) propagating on this M2-brane background, and so the geodesic flow, is completely integrable. Indeed one can verify after some calculations that the conserved charges

$$Q_a \ , \quad D_m = \frac{1}{4} \sum_{i,j\geq 9-m} (Q_{ij})^2 \ , \quad m = 2, \dots, 8 \ , \tag{4.26}$$

are in involution. These together with the Hamiltonian of the relativistic particle (2.51) yield 11 independent conserved charges in involution.

Next, let us turn to investigate the symmetries of the spinning particle probe (2.55) propagating on the spherically symmetric M2-brane. Clearly the Killing vector fields of the M2-brane generate symmetries for the probe (2.55). Additional symmetries are generated by the KY forms of this M2-brane. To find these, we adapt an ansatz which is invariant under the worldvolume Poincaré group of the M2-brane. Then after some computation, one finds that

$$\beta(\varphi) = h^{\frac{1}{6}(k-4)} y_i dy^i \wedge \varphi \wedge d\mathrm{vol}(\mathbb{R}^{2,1}) \ , \tag{4.27}$$

are CCKY tensors of the M2-brane for any constant k-form φ on $\mathbb{R}^8$, where $d\mathrm{vol}(\mathbb{R}^{2,1})$ is the volume form of $\mathbb{R}^{2,1}$. As a result $\alpha(\varphi) = {}^*\beta(\varphi)$ are KY tensor and so spinning particle action (2.55) is invariant the under transformation (2.56) generated by $\alpha(\varphi)$. The associated constants of motion are given in (2.58). These KY tensors generate 2^7 linearly independent hidden symmetries for the action (2.55). .

M5-branes

The M5-brane solution [73] is

$$g = h^{-\frac{1}{3}} \eta_{ab} d\sigma^a d\sigma^b + h^{\frac{2}{3}} \delta_{ij} dy^i dy^j \ , \quad F = \pm \star_5 dh \ , \tag{4.28}$$

where σ^a, $a = 0, \dots, 5$, are the worldvolume coordinates, y^i, $i = 1, \dots, 5$, are the transverse coordinates, the Hodge duality operation has been taken with respect to the flat metric on the transverse space $\mathbb{R}^5$ and h is a harmonic function, $\partial_y^2 h = 0$, on $\mathbb{R}^5$. h is given in (4.15) with $n = 5$ and $q_0 = 1$.

For the spherically symmetric M5-brane solution that we consider here, h has one centre and so it can be arranged such that $h = 1 + \frac{q}{|y|^3}$. Such a solution admits a $SO(1,5) \ltimes \mathbb{R}^6 \times SO(5)$ isometry group. The Killing vector fields are $k_a = \partial_a$, $k_{ab} = \sigma_a \partial_b - \sigma_b \partial_a$ and $k_{ij} = y_i \partial_j - y_j \partial_i$, where $y_i = y^i$ and $\sigma_a = \eta_{ab} \sigma^b$. The transformations generated by these vector fields leave invariant the relativistic particle action (2.51) and the associated conserved charges are

$$Q_a = p_a \ , \quad Q_{ab} = \sigma_a p_b - \sigma_b p_a \ , \quad Q_{ij} = y_i p_j - y_j p_i \ . \tag{4.29}$$

As for M2-branes, relativistic particles propagating on the above M5-brane background admit additional symmetries associated with KS tensors. Adapting again an ansatz which is invariant under the worldvolume Poincaré symmetry and after some computation one finds that

$$d_{a_1\dots a_{2m}i_1\dots i_k} = h^{\frac{1}{3}(2k-m)} y^{j_1}\dots y^{j_q} a_{j_1\dots j_q,i_1\dots i_k}\eta_{(a_1a_2}\dots\eta_{a_{2m-1}a_{2m})} \ , \tag{4.30}$$

are KS tensors provided that the constant tensors a satisfy

$$a_{(j_1\dots j_q,i_1)\dots i_k} = a_{j_1\dots(j_q,i_1\dots i_k)} = 0 \ . \tag{4.31}$$

Clearly, these generate infinitely many hidden symmetries for the relativistic particle action (2.51). So the geodesic flow on the spherically symmetric M5-brane has infinite many conserved charges.

Furthermore, one can show that the dynamics of relativistic particles propagating on this M5-brane is completely integrable. Indeed one can verify that the 10 conserved charges

$$Q_a \ , \quad D_m = \frac{1}{4}\sum_{i,j\geq 6-m}(Q_{ij})^2 \ , \quad m\geq 2,\dots,5 \ , \tag{4.32}$$

are in involution. These together with the Hamiltonian of (2.51) yield 11 independent conserved charges in involution as required for complete integrability.

As for the M2-brane, the spinning particle action (2.55) admits, in addition to the symmetries generated by the Killing vectors field of the M5-brane, hidden symmetries generated by KY forms. To find these we adapt an ansatz which is invariant under the worldvolume Poincaré group of the M5-brane. Then after some computation, one can verify that

$$\beta(\varphi) = h^{\frac{1}{3}(k-1)} y_i dy^i\wedge\varphi\wedge d\mathrm{vol}(\mathbb{R}^{5,1}) \ , \tag{4.33}$$

are CCKY forms for any constant k-form φ on $\mathbb{R}^5$. As a result $\alpha(\varphi) = {}^*\beta(\varphi)$ are KY forms and so generate symmetries (2.56) for the spinning particle probe (2.55) with conserved charges (2.58). These KY forms generate 2^4 linearly independent hidden symmetries.

KK-monopoles

The KK-monopole solution is

$$g = \eta_{ab}d\sigma^a d\sigma^b + g_{(4)} \ , \quad g_{(4)} = h^{-1}(d\rho+\omega)^2 + h\delta_{ij}dy^i dy^j \ , \tag{4.34}$$

with $F=0$, where σ^a, $a=0,\dots,6$, are the worldvolume coordinates and $g_{(4)}$ is in general the Gibbons-Hawking hyper-Kähler metric with $\star_3 dh = d\omega$. h is a harmonic action on $\mathbb{R}^3$, $\partial_y^2 h = 0$. An expression for h can be found in (4.15) for $n=3$.

Here we shall consider the KK monopole solution with $g_{(4)}$ the Taub-NUT metric. In such a case h has one centre and so one can set without loss of generality $h = 1+\frac{q}{|y|}$. The isometry group of the solution is $SO(1,6)\ltimes\mathbb{R}^7\times SO(2)\times SO(3)$. As for the solutions investigated already, the Killing vector fields generated by the Poincaré subgroup acting on the worldvolume coordinates are $k_a = \partial_a$ and $k_{ab} = \sigma_a\partial_b - \sigma_b\partial_a$. To give the vector fields generated by the $SO(2) \times SO(3)$ subgroup, write the Taub-NUT metric $g_{(4)}$ is angular coordinates as

$$g_{(4)} = h^{-1}(d\rho + q\cos\theta d\phi)^2 + h\big(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\big) \;, \tag{4.35}$$

with $|y| = r$. Then the Killing vector fields generated by $SO(2) \times SO(3)$ are given by

$$\begin{aligned} &\tilde{k}_0 = \partial_\rho \;, \quad \tilde{k}_1 = \partial_\phi \;, \quad \tilde{k}_2 = -\sin\phi\cot\theta\partial_\phi + \cos\phi\partial_\theta + q\frac{\sin\phi}{\sin\theta}\partial_\rho \;, \\ &\tilde{k}_3 = \cos\phi\cot\theta\partial_\phi + \sin\phi\partial_\theta - q\frac{\cos\phi}{\sin\theta}\partial_\rho \;. \end{aligned} \tag{4.36}$$

The $SO(3)$ Killing vector fields are as in (4.5) with the addition of a component along ∂_ρ because ω is not invariant under (4.5) but instead it is invariant up to a gauge transformation.

As the relativistic particle action (2.51) is invariant under all these isometries, the associated conserved charges are $Q_a = p_a$, $Q_{ab} = \sigma_a p_b - \sigma_b p_a$ $\tilde{Q}_0 = p_\rho$ and $\tilde{Q}_r = \tilde{k}^i_r p_i$, $r = 1, 2, 3$, where $\sigma_a = \eta_{ab}\sigma^b$ and $\tilde{k}_r$ are given in (4.36). The background admits several KS tensors. As the solution is a product $\mathbb{R}^{6,1} \times N$, where N is the Taub-NUT manifold, one can consider the KS tensors of $\mathbb{R}^{6,1}$ and N separately. One can easily verify that the symmetric tensors

$$d_{a_1\dots a_k} = \sigma^{b_1} \dots \sigma^{b_q} c_{b_1\dots b_q, a_1\dots a_k} \;, \tag{4.37}$$

are KS tensors[7] provided that the constants c satisfy $c_{b_1\dots(b_q, a_1\dots a_k)} = 0$. N also admits three KS tensors given in [118] which we shall not explicitly state them here. They are constructed from the Kähler forms and the KY tensor of N given below. All these isometries and KS tensors generate infinite number of symmetries for the relativistic particle action (2.51).

The dynamics of the relativistic particle probe (2.51), or equivalently the geodesic flow, is completely integrable on this background. Indeed the commuting isometries of the KK-monopole are $k_a = \frac{\partial}{\partial\sigma^a}$, $\tilde{k}_0 = \partial_\rho$ and $\tilde{k}_1 = \partial_\phi$. These together with the Hamiltonian of the geodesic system give ten conserved charges in involution. There is an additional independent conserved charge in involution associated with the quadratic Casimir of $SO(3)$ and constructed using the Killing vector fields (4.36) as

$$D = \frac{1}{\sin^2\theta}(p_\phi - q\cos\theta p_\sigma)^2 + p_\theta^2 + q^2 p_\sigma^2 \;, \tag{4.38}$$

which proves the statement. The integrability of the geodesic flow on the Taub-NUT space has been known for some time, see [118].

The $SO(1,6) \ltimes \mathbb{R}^7 \times SO(2) \times SO(3)$ isometries mentioned above also generate symmetries for the spinning particle probe (2.55) propagating on the KK-monopole background. Such probes have additional hidden symmetries. For example, it is well known that g_{GH} is a hyper-Kähler metric for any (multi-centred) harmonic function h. The associated Kähler forms are

$$\kappa_{(i)} = (d\rho + \omega) \wedge dy^i - \frac{1}{2} h\,\epsilon^i{}_{jk} dy^j \wedge dy^k \;. \tag{4.39}$$

[7] There is a systematic investigation of KS tensors on Minkowski spacetime as well as on some black hole spacetimes. For example there is a 20 dimensional space of rank 2 conformal KS tensors on 4-dimensional Minkowski spacetime, see for a summary [117]. But the approach adopted here suffices.

These 2-forms are anti-self-dual on the transverse directions of the KK-monopole, parallel with respect to the Levi-Civita connection and the associated complex structures satisfy the algebra of imaginary unit quaternions. As a result, these Kähler forms can be thought as KY tensors and so generate symmetries (2.56) for the probe action (2.55) with conserved charges (2.58).

The KK monopole admits additional KY tensors. These are those of $\mathbb{R}^{6,1}$ and those of N. Observe that

$$\alpha = \frac{1}{k!}\big(\chi_{a_1\dots a_k} + \sigma^b \varphi_{b,a_1\dots a_k}\big)\, d\sigma^{a_1} \wedge \cdots \wedge d\sigma^{a_k} \ , \tag{4.40}$$

are KY tensors of $\mathbb{R}^{6,1}$ for any constant tensors χ, φ with the latter to satisfy $\varphi_{b,a_1\dots a_k} = \varphi_{[b,a_1\dots a_k]}$ [17]. If N is the Taub-Nut space, it is known [118], see also [119], that

$$\tilde{\alpha} = (d\rho + q\cos\theta d\phi) \wedge dr + r(2r+q)(1+\frac{r}{q})\sin\theta d\theta \wedge d\phi \ , \tag{4.41}$$

is the KY form. All these KY forms generate symmetries for the spinning probe action (2.55). Incidentally the three KS tensors mentioned above are constructed from squaring $\tilde{\alpha}$ with $\kappa_{(i)}$.

4.4 Hidden symmetries from the TCFH

4.4.1 Hidden symmetries and M-theory pp-waves

Assuming that the pp-wave propagates in the 5th direction[8] and allowing the pp-wave metric (4.14) to depend on a (multi-centred) harmonic function as in (4.15) with $q_0 = 0$ and $n = 9$, the Killing spinors of the background are constant, $\epsilon = \epsilon_0$ and satisfy the condition[9] $\Gamma_{05}\epsilon_0 = \pm\epsilon_0$. To solve this condition, we shall use spinorial geometry and write $\epsilon_0 = \eta + e_5 \wedge \lambda$, where η and λ are Majorana[10] $\mathfrak{spin}(9)$ spinors[11], i.e. $\eta, \lambda \in \Lambda^*(\mathbb{R}\langle e_1, \dots, e_4\rangle)$ with the reality condition imposed by the anti-linear operation $\Gamma_{6789}*$. Choosing the plus sign in the condition for ϵ_0, this can be solved to yield $\epsilon = \epsilon_0 = \eta$, i.e. $\Gamma_{05}\epsilon_0 = \epsilon_0$ implies that $\lambda = 0$.

Given the solution of the condition on ϵ implied by the KSE, it is straightforward to compute all the bilinears of the background. In particular one finds that $f^{rs} = 0$ for all Killing spinors and the rest of the form bilinears (4.2) can be written as

$$(e^0 - e^5) \wedge \phi^{rs} \tag{4.42}$$

where

$$\phi^{rs} = \frac{1}{k!}\, \langle \eta^r, \Gamma_{i_1\dots i_k}\eta^s\rangle_H\, e^{i_1} \wedge \cdots \wedge e^{i_k} \ , \quad k = 0,1,2,3,4 \ , \tag{4.43}$$

[8] This choice of worldvolume directions for the pp-wave, and those of the rest of M-branes below, may seem unconventional. But they are convenient as they are aligned with the basis used for the description of spinors in the context of spinorial geometry that we utilise to solve the conditions on the Killing spinors.

[9] All gamma matrices considered from in section 4 and appendix A.4 are in a frame basis.

[10] Note that the reality condition on ϵ in the spinorial geometry basis is $\Gamma_{6789} * \epsilon = \epsilon$ which in turn implies that η and λ are real as well.

[11] As before, more precisely these are pinors as we need to know how η and λ act on the odd part of the Clifford algebra.

$\langle\cdot,\cdot\rangle_H$ is the Hermitian inner product restricted on the Majorana representation of $\mathfrak{spin}(9)$ and $i_1,\dots,i_k = 1,2,3,4,6,7,8,9,\natural$. Moreover (e^0, e^5, e^i) is a pseudo-orthonormal frame such that $-e^0+e^5 = \sqrt{2}\,du$, $e^0+e^5 = \sqrt{2}\,(dv+\frac{1}{2}hdu)$ and $e^i = dy^i$ after a relabelling of the transverse coordinates y of the spacetime. For example $k^{rs} = \langle\eta^r,\eta^s\rangle_H\,(e^0-e^5)$ and so on.

It remains to specify the k-form bilinears ϕ^{rs}. It turns out that these span all the constant forms on the transverse space of the pp-wave up and including those of degree 4. To see this decompose $\phi^{rs} = e^\natural\wedge\alpha^{rs}+\beta^{rs}$, where α^{rs} and β^{rs} have components only along the directions transverse to $e^\natural$. The tensor product of two Majorana $\mathfrak{spin}(8)$ representations, $\Delta_{\mathbf{16}}$, can be decomposed as

$$\Delta_{\mathbf{16}}\otimes\Delta_{\mathbf{16}} = \oplus_{k=0}^{8}\Lambda^k(\mathbb{R}^8)\ . \tag{4.44}$$

Therefore the forms β^{rs} which are up to degree 4 span all forms of the same degree on $\mathbb{R}^8$ subspace transverse to $e^\natural$. On the other hand the Hodge duals of the forms α^{rs} span all forms of degree 5 and higher in $\mathbb{R}^8$. Thus the space of all bilinears of a pp-wave spans a 2^8-dimensional vector space.

As for the pp-waves we have been considering the 4-form field strength F vanishes, all the form bilinears are covariantly constant with respect to the Levi-Civita connection. As a result, all of them generate symmetries for the spinning particle probe action (2.55). The associated conserved charges are given in (2.58). They also generate symmetries for string probes as well similar to those investigated in [24]. The algebra of symmetries can be of W-type and has been described in [33, 32].

4.4.2 Hidden symmetries and the KK-monopole

Choosing the worldvolume directions of the KK-monopole along $012567\natural$ and allowing h in (4.34) to be any multi-centred harmonic function as in (4.15) with $n=3$, the Killing spinors $\epsilon=\epsilon_0$ of the background satisfy $\Gamma_{3489}\epsilon_0 = \pm\epsilon_0$, where ϵ_0 is a constant spinor. To solve this condition with the plus sign, we shall use spinorial geometry and write $\epsilon_0 = \eta^1 + e_{34}\wedge\lambda^1 + e_3\wedge\eta^2 + e_4\wedge\lambda^2$, where η and λ are Dirac spinors of $\mathfrak{spin}(6,1)$, i.e $\eta,\lambda\in\Lambda^*(\mathbb{C}\langle e_1,e_2,e_5\rangle)$. To begin, let us assume that ϵ_0 is a complex spinor and impose the reality condition at the end. Then $\Gamma_{3489}\epsilon_0 = \epsilon_0$ implies that $\eta^2=\lambda^2=0$ and so $\epsilon_0 = \eta + e_{34}\wedge\lambda$, where $\eta=\eta^1$ and $\lambda=\lambda^1$. The reality condition on ϵ_0, $\Gamma_{6789} * \epsilon_0 = \epsilon_0$, implies that $\lambda = -\Gamma_{67}\eta^*$. Therefore the spinors that solve the Killing spinor condition are

$$\epsilon_0 = \eta - e_{34}\wedge\Gamma_{67}\eta^*\ , \tag{4.45}$$

where η is any Dirac $\mathfrak{spin}(6,1)$ spinor.

The non-vanishing Killing spinors bilinears read

$$\begin{aligned}
f^{rs} &= 2\mathrm{Re}\langle\eta^r,\eta^s\rangle\ ,\quad k^{rs} = 2\mathrm{Re}\langle\eta^r,\Gamma_a\eta^s\rangle e^a\ ,\\
\omega^{rs} &= \frac{1}{2}\mathrm{Re}\langle\eta^r,\Gamma_{ab}\eta^s\rangle e^a\wedge e^b - 2\mathrm{Re}\langle\eta^r,\lambda^s\rangle(e^3\wedge e^4 - e^8\wedge e^9)\\
&\quad -2\mathrm{Im}\langle\eta^r,\eta^s\rangle(e^3\wedge e^8 + e^4\wedge e^9) - 2\mathrm{Im}\langle\eta^r,\lambda^s\rangle(e^3\wedge e^9 - e^4\wedge e^8)\ ,\\
\varphi^{rs} &= \frac{1}{3}\mathrm{Re}\langle\eta^r,\Gamma_{abc}\eta^s\rangle e^a\wedge e^b\wedge e^c - 2\mathrm{Im}\langle\eta^r,\Gamma_a\eta^s\rangle(e^3\wedge e^8 + e^4\wedge e^9)\wedge e^a\\
&\quad -2\mathrm{Im}\langle\eta^r,\Gamma_a\lambda^s\rangle(e^3\wedge e^9 - e^4\wedge e^8)\wedge e^a\ ,
\end{aligned}$$

$$\theta^{rs} = \frac{1}{12}\mathrm{Re}\langle\eta^r, \Gamma_{abcd}\eta^s\rangle e^a\wedge e^b\wedge e^c\wedge e^d - \mathrm{Re}\langle\eta^r, \Gamma_{ab}\lambda^s\rangle e^a\wedge e^b\wedge (e^3\wedge e^4 - e^8\wedge e^9)$$
$$-\mathrm{Im}\langle\eta^r, \Gamma_{ab}\eta^s\rangle e^a\wedge e^b\wedge (e^3\wedge e^8 + e^4\wedge e^9) - \mathrm{Im}\langle\eta^r, \Gamma_{ab}\lambda^s\rangle e^a\wedge e^b\wedge (e^3\wedge e^9 - e^4\wedge e^8)$$
$$+2\mathrm{Re}\langle\eta^r, \eta^s\rangle e^3\wedge e^4\wedge e^8\wedge e^9\ ,$$
$$\tau^{rs} = \frac{1}{60}\mathrm{Re}\langle\eta^r, \Gamma_{a_1\dots a_5}\eta^s\rangle e^{a_1}\wedge\cdots\wedge e^{a_5} + 2\mathrm{Re}\langle\eta^r, \Gamma_a\eta^s\rangle e^a\wedge e^3\wedge e^4\wedge e^8\wedge e^9$$
$$-\frac{1}{3}\mathrm{Re}\langle\eta^r, \Gamma_{abc}\lambda^s\rangle e^a\wedge e^b\wedge e^c\wedge (e^3\wedge e^4 - e^8\wedge e^9)$$
$$-\frac{1}{3}\mathrm{Im}\langle\eta^r, \Gamma_{abc}\eta^s\rangle e^a\wedge e^b\wedge e^c\wedge (e^3\wedge e^8 + e^4\wedge e^9)$$
$$-\frac{1}{3}\mathrm{Im}\langle\eta^r, \Gamma_{abc}\lambda^s\rangle e^a\wedge e^b\wedge e^c\wedge (e^3\wedge e^9 - e^4\wedge e^8)\ , \tag{4.46}$$

where $\langle\cdot,\cdot\rangle$ is the Dirac inner product, $a, b, c = 0, 1, 2, 5, 6, 7, \natural$, $e^a = dx^a$, and e^i, $i = 3, 4, 8, 9$, is an orthonormal frame of $g_{(4)}$ in (4.34), e.g.

$$e^3 = h^{-\frac{1}{2}}(d\rho + \omega)\ ,\quad e^4 = h^{\frac{1}{2}}dy^4\ ,\quad e^7 = h^{\frac{1}{2}}dy^7\ ,\quad e^8 = h^{\frac{1}{2}}dy^8\ , \tag{4.47}$$

after a relabelling of the coordinates of the spacetime. The bilinears of the spinors η span all real forms on the worldvolume $\mathbb{R}^{6,1}$ of the KK-monopole solution. The argument is similar to that produced for the pp-wave.

As for the KK-monopole solution, the 4-form field strength vanishes $F = 0$, a consequence of the TCFH is that all the form bilinears in (4.46) are covariantly constant with respect to the Levi-Civita connection. As a result, they generate symmetries (2.56) for the spinning particle probe (2.55). The conserved charges are given in (2.58). The algebra of symmetries can be a W-type of algebra [33, 32].

4.4.3 Hidden symmetries and the M2-brane

Choosing the M2-brane worldvolume directions along $05\natural$, the Killing spinors of the solution are $\epsilon = h^{-\frac{1}{6}}\epsilon_0$, where ϵ_0 is a constant spinor satisfying the condition $\Gamma_{05\natural}\epsilon_0 = \pm\epsilon_0$ and h is a (multi-centred) harmonic function as in (4.15) with $n = 8$. To solve the condition with the plus sign use spinorial geometry to write $\epsilon_0 = \eta + e_5\wedge\lambda$, where $\eta, \lambda \in \Lambda^*(\mathbb{R}\langle e_1, e_2, e_3, e_4\rangle)$. Then the condition $\Gamma_{05\natural}\epsilon_0 = \epsilon_0$ implies that $\eta, \lambda \in \Lambda^{\mathrm{ev}}(\mathbb{R}\langle e_1, e_2, e_3, e_4\rangle)$, i.e. η, λ are Majorana-Weyl $\mathfrak{spin}(8)$ spinors, where the reality condition is imposed with the anti-linear map $\Gamma_{6789}*$.

Using the solution of the condition on the Killing spinors and setting $\phi^{rs} = h^{-\frac{1}{3}}\mathring{\phi}^{rs}$ for for all bilinears ϕ^{rs}, one can easily find

$$\mathring{f}^{rs} = -\langle\eta^r, \lambda^s\rangle_H + \langle\lambda^r, \eta^s\rangle_H\ ,$$
$$\mathring{k}^{rs} = (\langle\eta^r, \eta^s\rangle_H + \langle\lambda^r, \lambda^s\rangle_H)e^0 + (-\langle\eta^r, \eta^s\rangle_H + \langle\lambda^r, \lambda^s\rangle_H)e^5 + (\langle\eta^r, \lambda^s\rangle_H + \langle\lambda^r, \eta^s\rangle_H)e^\natural\ ,$$
$$\mathring{\omega}^{rs} = (\langle\eta^r, \lambda^s\rangle_H + \langle\lambda^r, \eta^s\rangle_H)e^0\wedge e^5 + (\langle\eta^r, \eta^s\rangle_H - \langle\lambda^r, \lambda^s\rangle_H)e^0\wedge e^\natural$$
$$+(-\langle\eta^r, \eta^s\rangle_H - \langle\lambda^r, \lambda^s\rangle_H)e^5\wedge e^\natural + \frac{1}{2}(-\langle\eta^r, \Gamma_{ij}\lambda^s\rangle_H + \langle\lambda^r, \Gamma_{ij}\eta^s\rangle_H)e^i\wedge e^j\ ,$$
$$\mathring{\varphi}^{rs} = \frac{1}{2}(\langle\eta^r, \Gamma_{ij}\eta^s\rangle_H + \langle\lambda^r, \Gamma_{ij}\lambda^s\rangle_H)e^0\wedge e^i\wedge e^j + \frac{1}{2}(-\langle\eta^r, \Gamma_{ij}\eta^s\rangle_H + \langle\lambda^r, \Gamma_{ij}\lambda^s\rangle_H)e^5\wedge e^i\wedge e^j$$
$$+\frac{1}{2}(\langle\eta^r, \Gamma_{ij}\lambda^s\rangle_H + \langle\lambda^r, \Gamma_{ij}\eta^s\rangle_H)e^\natural\wedge e^i\wedge e^j + (-\langle\eta^r, \lambda^s\rangle_H + \langle\lambda^r, \eta^s\rangle_H)e^0\wedge e^5\wedge e^\natural\ ,$$
$$\mathring{\theta}^{rs} = \frac{1}{2}\Big((\langle\eta^r, \Gamma_{ij}\eta^s\rangle_H + \langle\lambda^r, \Gamma_{ij}\lambda^s\rangle_H)e^0\wedge e^5 + (\langle\eta^r, \Gamma_{ij}\eta^s\rangle_H - \langle\lambda^r, \Gamma_{ij}\lambda^s\rangle_H)e^0\wedge e^\natural$$
$$-(\langle\eta^r, \Gamma_{ij}\eta^s\rangle_H + \langle\lambda^r, \Gamma_{ij}\lambda^s\rangle_H)e^5\wedge e^\natural\Big)\wedge e^i\wedge e^j$$

$$-\frac{1}{4!}(\langle\eta^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle_H-\langle\lambda^r,\Gamma_{i_1\dots i_4}\eta^s\rangle_H)e^{i_1}\wedge\cdots\wedge e^{i_4}\ ,$$
$$\mathring{\tau}^{rs}=\frac{1}{2}(-\langle\eta^r,\Gamma_{ij}\lambda^s\rangle_H+\langle\lambda^r,\Gamma_{ij}\eta^s\rangle_H)e^0\wedge e^5\wedge e^\natural\wedge e^i\wedge e^j+\frac{1}{4!}\Big(\langle\eta^r,\Gamma_{i_1\dots i_4}\eta^s\rangle_H(e^0-e^5)$$
$$+\langle\lambda^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle_H(e^0+e^5)+(\langle\eta^r,\Gamma_{i_1\dots i_4}\lambda^s\rangle_H+\langle\lambda^r,\Gamma_{i_1\dots i_4}\eta^s\rangle_H)e^\natural\Big)\wedge e^{i_1}\wedge\cdots\wedge e^{i_4}\ ,\tag{4.48}$$

where (e^a,e^i) is the pseudo-orthonormal frame with $e^a=h^{-1/3}d\sigma^a$, $a=0,5,\natural$, and $e^i=h^{1/6}dy^i$ $i,j,k,\ell=1,2,3,4,6,7,8,9$, after an appropriate relabelling of the coordinates of the spacetime. As the product of two positive chirality Majorana-Weyl $\mathfrak{spin}(8)$ representations, $\Delta^+_{\mathbf{8}}$, is decomposed as

$$\otimes^2\Delta^+_{\mathbf{8}}=\Lambda^0(\mathbb{R}^8)\oplus\Lambda^2(\mathbb{R}^8)\oplus\Lambda^{4+}(\mathbb{R}^8)\ ,\tag{4.49}$$

it is expected that the form bilinears above span all the 0-, 2- and self-dual 4-forms along the transverse directions of the M2-brane.

It remains to find which of the above form bilinears are KY tensors with respect to the Levi-Civita connection so that generate symmetries for the spinning particle probe (2.55). To begin as the 1-form bilinears k are Killing they generate symmetries for the action (2.55) and the associated conserved charges are given in (2.58). For the bilinear ω to be a KY form, it is required that the terms in the TCFH connection that are proportional to F as well as those in the TCFH that contain explicitly the spacetime metric g must vanish. After some investigation, these terms vanish provided that the components, τ_{abcij}, of the form bilinear τ are zero, $\tau_{abcij}=0$. This in turn implies that $\omega_{ij}=0$. Setting $\omega_{ij}=0$, $\omega=\frac{1}{2}\omega_{ab}e^a\wedge e^b$ is a KY tensor and generates a symmetry transformation (2.56) for the action (2.55) with associated conserved charge given in (2.58). Note that ω has components only along the worldvolume directions of the M2-brane. There are Killing spinors such that $\omega\neq0$, even though $\omega_{ij}=0$, as a consequence of the decomposition (4.49). A similar investigation reveals that τ cannot be a KY form as the conditions arising from the analysis of the TCFH imply that $\tau=0$.

Next φ is a KY form with respect to the Levi-Civita connection provided that the terms proportional to F in the TCFH connection as well as those in the TCFH that contain explicitly the spacetime metric g vanishes. This is the case provided that the components, θ_{abij}, of θ vanish, $\theta_{abij}=0$. This in turn implies that $\varphi_{aij}=0$. Therefore $\varphi=\frac{1}{6}\varphi_{abc}e^a\wedge e^b\wedge e^c$ is a KY form and so generates a symmetry for the spinning particle probe action (2.55) with conserved charge (2.58). Note again that the KY form φ has components only along the worldvolume directions of M2-brane and that there are Killing spinors such that $\varphi\neq0$ even though $\varphi_{aij}=0$ as a consequence of (4.49). A similar investigation concludes that θ, as τ, cannot be a KY form.

4.4.4 Hidden symmetries and the M5-brane

Choosing the worldvolume directions of the M5-brane along 012567, the Killing spinors of the background are $\epsilon=h^{-\frac{1}{12}}\epsilon_0$, where the constant spinor ϵ_0 satisfies the condition $\Gamma_{3489\natural}\epsilon_0=\pm\epsilon_0$ and h is a multi-centred harmonic function as in (4.15) with $n=5$. To continue it is convenient to solve the condition on ϵ_0 with a plus sign by taking ϵ_0 to be complex and impose the reality condition on ϵ_0 at the end. Indeed for ϵ_0 complex, one can use spinorial geometry to write $\epsilon_0=\eta^1+e_{34}\wedge\lambda^1+e_3\wedge\eta^2+e_4\wedge\lambda^2$, where $\eta^1,\eta^2,\lambda^1,\lambda^2\in\Lambda^*(\mathbb{C}\langle e_1,e_2,e_5\rangle)$. Then the condition $\Gamma_{3489\natural}\epsilon_0=\epsilon_0$ implies that $\eta^1,\eta^2,\lambda^1,\lambda^2\in\Lambda^{\rm ev}(\mathbb{C}\langle e_1,e_2,e_5\rangle)$, i.e. $\eta^1,\eta^2,\lambda^1,\lambda^2$ are positive chirality spinors of

$\mathfrak{spin}(5,1)$. Next imposing the reality condition on ϵ_0, $\Gamma_{6789} * \epsilon_0 = \epsilon_0$, one finds that $\lambda^1 = -\Gamma_{67}(\eta^1)^*$ and $\lambda^2 = -\Gamma_{67}(\eta^2)^*$. Hence the spinors that solve the Killing spinor condition are

$$\epsilon_0 = \eta^1 - e_{34} \wedge \Gamma_{67}(\eta^1)^* + e_3 \wedge \eta^2 - e_4 \wedge \Gamma_{67}(\eta^2)^* \ , \tag{4.50}$$

where η^1, η^2 are any positive chirality $\mathfrak{spin}(5,1)$ spinors. The form bilinears of the M5-brane expressed in terms of the η^1 and η^2 spinors can be found in appendix A.4.

The 1-form bilinears k^{rs} are isometries and so generate symmetries for the spinning particle probe action (2.55). Next for the bilinear ω to be a KY tensor, and so generate a symmetry for the spinning particle probe (2.55), the term that contains F in the minimal TCFH connection $\mathcal{D}^{\mathcal{F}}$ and the term proportional to the spacetime metric g in the TCFH (4.3) must vanish. This is the case provided that the component, $\tau_{ijk\ell a}$, of τ vanishes. However this in turn implies that $\omega = 0$ and so ω does not generate a symmetry. It turns out θ, like ω, does not generate a symmetry for the probe (2.55) because the conditions required by the TCFH for θ to be a KY form are too restrictive and yield $\theta = 0$. Next φ is a KY form as a consequence of TCFH provided that $\theta_{iabc} = \theta_{aijk} = 0$. This implies $\varphi_{aij} = 0$ and leaves the possibility that the remaining component of φ $\varphi = \frac{1}{3!}\varphi_{abc} e^a \wedge e^b \wedge e^c$ is a KY form. However after some computation[12] one can verify that there are no Killing spinors such that $\varphi \neq 0$. A similar conclusion holds for the τ form bilinear.

4.5 Summary

We have presented the TCFH of 11-dimensional supergravity and we have demonstrated that the form bilinears of supersymmetric backgrounds of the theory satisfy a generalisation of the CKY equation with respect to a connection that depends on the 4-form field strength. We have also given the reduced holonomy of the minimal and maximal TCFH connections for generic backgrounds.

As KY forms with respect to the Levi-Civita connection generate symmetries for spinning particle actions, we investigated the question on whether the form bilinears of 11-dimensional supergravity generate symmetries for suitable particle probes propagating on supersymmetric backgrounds. For this we focused on M-branes which include the pp-wave, M2- and M5-brane, and KK-monopole solutions. As all the form bilinears of pp-wave and KK-monopole solutions are covariantly constant with respect to the Levi-Civita connection, they generate symmetries for the spinning particle action with only a metric coupling. For the M2-brane, there are Killing spinors such that the 1-form, 2-form and 3-form bilinears are KY tensors and therefore generate symmetries for the same spinning particle action. For the M5-brane only the 1-form bilinears generate symmetries for the spinning particle action.

We also took the opportunity to demonstrate the complete integrability of the geodesic flow of spherically symmetric pp-wave, M2- and M5-brane, and KK-monopole solutions. For this we presented a large class of KS and KY tensors on all these backgrounds. Relativistic particles on these solutions admit an infinite number of symmetries generated by KS tensors. We have also explicitly given all independent and in involution conserved charges of the geodesic flow on these backgrounds.

[12] This computation uses spinorial geometry to solve the conditions $\varphi_{aij} = 0$. Geometrically the solutions lie in the intersection of conics. Using these solutions one can verify that $\varphi = 0$.

Chapter 5

The TCFHs of D=11 AdS backgrounds and hidden symmetries

5.1 Introduction

This chapter intends to give the TCFHs on the internal space of all warped AdS backgrounds of 11-dimensional supergravity theory. This will put the conditions on the form bilinears implied by the KSEs in a firm geometric basis. For this, we shall use the solution of the (gravitino) KSE of the theory along the AdS subspace of a background presented in [94]. Then we shall explore some of the properties of the TCFH connections which include their (reduced) holonomy for generic internal spaces. In addition, we shall examine the conditions under which the form bilinears give rise to KY or CCKY forms[1]. The existence of such forms will imply in turn the presence of constants of motion in the propagation of spinning particles on the internal spaces of such backgrounds. We shall present several backgrounds with KY forms arising from the TCFH of their internal space. These include the maximally supersymmetric AdS solutions of the theory as well as the near-horizon geometries of some intersecting M-brane configurations.

This chapter is organised as follows. In sections 2, 3 and 4, we present the TCFH of warped AdS_2, AdS_3 and AdS_4 backgrounds of 11-dimensional supergravity, respectively, and investigate some of the properties of the TCFH connections. In section 5, we give the TCFHs and investigate their properties of the remaining AdS backgrounds. In section 6, we explore the hidden symmetries of probes that arise from the TCFH of some solutions that include the maximally supersymmetric AdS backgrounds as well as some AdS backgrounds which are the near horizon geometries of intersecting M-branes, and in section 7 we give our conclusions.

5.2 The TCFH of warped AdS_2 backgrounds

5.2.1 Fields and Killing spinors

The bosonic fields of 11-dimensional supergravity for warped AdS_2 backgrounds, $AdS_2 \times_w M^9$, can be written as

$$g = 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j \ , \quad F = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + X \ , \tag{5.1}$$

[1]The Hodge dual of a CCKY form is a KY form.

where Y and X are a 2-form and 4-form on the internal space M^9 with metric $g(M^9) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$, respectively,

$$\mathbf{e}^+ = du \ , \quad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\ell^{-2}A^{-2}du \ , \quad \mathbf{e}^i = e^i{}_J dy^J \ , \tag{5.2}$$

is a null pseudo-orthonormal frame on the spacetime with $\mathbf{e}^i$ an orthonormal frame on the internal space M^9, $h = -d\log A^2$ and A is the warp factor which is a function on M^9. The y^I are coordinates on M^9 and (u, r) are the remaining coordinates on the spacetime. It can be seen after a change of coordinates that the above metric can be rewritten in the standard warped product form, $g = A^2 g_\ell(AdS_2) + g(M^9)$, where $g_\ell(AdS_2)$ is the standard metric on AdS_2 of radius ℓ.

The KSE of 11-dimensional supergravity can be solved along the AdS_2 subspace of $\mathrm{AdS}_2 \times_w M^9$ [94] and the Killing spinors ϵ can be expressed as $\epsilon = \epsilon_1 + \epsilon_2$ with

$$\epsilon_1 = \phi_- + u\Gamma_+\Theta_-\phi_- + ru\Gamma_-\Theta_+\Gamma_+\Theta_-\phi_- \ , \quad \epsilon_2 = \phi_+ + r\Gamma_-\Theta_+\phi_+ \ . \tag{5.3}$$

where $\phi_\pm$ are spinors that depend only on the coordinates of M^9 and satisfy the light-cone projections $\Gamma_\pm\phi_\pm = 0$, where $\Gamma_\pm$ have been adapted to the frame[2] (5.2) and $\Theta_\pm$ are Clifford algebra elements that depend on the fields. For the explicit expressions of $\Theta_\pm$ as well as for the spinor notation we use below, see [94]. The dependence of Killing spinors in (5.3) on the (u, r) coordinates is explicit as it is that of the fields in (5.1). In addition, the (spacetime) gravitino KSE implies that $\phi_\pm$ satisfy the KSEs

$$\mathcal{D}_i^\pm\phi_\pm = 0 \ , \quad \mathcal{D}_i^\pm \equiv \nabla_i + \Psi_i^\pm \ , \tag{5.4}$$

on the internal space M^9, where ∇ is the Levi-Civita connection of the metric $g(M^9)$ induced on the spin bundle and

$$\Psi_i^\pm = \mp\frac{1}{4}h_i - \frac{1}{288}\Gamma\!\!\!/X_i + \frac{1}{36}X\!\!\!/_i \pm \frac{1}{24}\Gamma\!\!\!/Y_i \mp \frac{1}{6}Y\!\!\!/_i \ . \tag{5.5}$$

The TCFHs on M^9 that we shall explore below are associated with the supercovariant connections $\mathcal{D}^\pm$.

5.2.2 The TCFH on the internal space

To begin, the internal space M^9 of warped AdS_2 backgrounds is a Euclidean signature 9-dimensional manifold. Therefore, one has $n = 9$ and $\eta_{ij} = \delta_{ij}$. After integrating the gravitino KSE of 11-dimensional supergravity over the AdS_2 subspace, which has been summarised in the previous section, the Killing spinors $\phi_\pm$ on M^9 satisfy the KSEs (5.4). To proceed a basis in the space of form bilinears on the internal space M^9, up to Hodge duality, is

$$\begin{aligned}
f^{\pm r,s} &= \langle\phi_\pm^r, \phi_\pm^s\rangle \ , \quad k^{\pm r,s} = \langle\phi_\pm^r, \Gamma_i\phi_\pm^s\rangle\, \mathbf{e}^i \ , \\
\theta^{\pm r,s} &= \frac{1}{4!}\langle\phi_\pm^r, \Gamma_{ijk\ell}\phi_\pm^s\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k\wedge\mathbf{e}^\ell \ , \quad \omega^{\pm r,s} = \frac{1}{2}\langle\phi_\pm^r, \Gamma_{ij}\phi_\pm^s\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j \ , \\
\varphi^{\pm r,s} &= \frac{1}{3!}\langle\phi_\pm^r, \Gamma_{ijk}\phi_\pm^s\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k \ ,
\end{aligned} \tag{5.6}$$

[2]From here on, the gamma matrices are always adapted to a (pseudo-)orthonormal frame.

where all (time-like) space-like gamma matrices are (anti-)Hermitian with respect to the inner product $\langle\cdot,\cdot\rangle$. As the Killing spinors of 11-dimensional supergravity are real, note that the bilinears f, k and θ are symmetric in the exchange of spinors $\phi^r_\pm$ and $\phi^s_\pm$ and the rest are skew-symmetric.

After using that $\phi^r_\pm$ with $\phi^s_\pm$ are Killing spinors of the supercovariant connections $\mathcal{D}^\pm$, one finds that the TCFH of the form bilinears (5.6) that are symmetric in the exchange of $\phi^r_\pm$ with $\phi^s_\pm$ is

$$
\begin{aligned}
&\mathcal{D}^{(\pm)\mathcal{F}}_i f^\pm \equiv \nabla_i f^\pm \pm \partial_i \log A\, f^\pm = \pm\frac{1}{144}\,{}^*X_{i\ell_1\ell_2\ell_3\ell_4}\theta^{\pm\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{3}Y_{i\ell}k^{\pm\ell}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i k^\pm_j \equiv \nabla_i k^\pm_j \pm \partial_i \log A\, k^\pm_j - \frac{1}{12}X_{i\ell_1\ell_2\ell_3}\theta^{\pm\ell_1\ell_2\ell_3}{}_j\\
&= \frac{1}{18}X_{\ell_1\ell_2\ell_3[i}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j]} + \frac{1}{144}\delta_{ij}X_{\ell_1\ell_2\ell_3\ell_4}\theta^{\pm\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{3}Y_{ij}f^\pm \pm \frac{1}{12}Y_{\ell_1\ell_2}\theta^{\pm\ell_1\ell_2}{}_{ij}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \theta^\pm_{j_1j_2j_3j_4} \equiv \nabla_i\theta^\pm_{j_1j_2j_3j_4} \pm \partial_i \log A\,\theta^\pm_{j_1j_2j_3j_4} \pm \frac{3}{2}\,{}^*X_{\ell_1\ell_2 i[j_1j_2}\theta^{\pm\ell_1\ell_2}{}_{j_3j_4]}\\
&+2X_{i[j_1j_2j_3}k^\pm_{j_4]} + \frac{1}{3}\,{}^*Y_{\ell_1\ell_2\ell_3 i[j_1j_2j_3}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j_4]}\\
&= \pm\frac{5}{3}\,{}^*X_{\ell_1\ell_2[ij_1j_2}\theta^{\pm\ell_1\ell_2}{}_{j_3j_4]} + \frac{5}{6}X_{[ij_1j_2j_3}k^\pm_{j_4]} \pm \frac{1}{6}\,{}^*X_{ij_1j_2j_3j_4}f^\pm\\
&\mp\frac{2}{3}\delta_{i[j_1}\,{}^*X_{j_2j_3|k_1k_2k_3}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j_4]} + \frac{2}{3}\delta_{i[j_1}X_{j_2j_3j_4]\ell}k^{\pm\ell} - \frac{1}{18}\delta_{i[j_1}\,{}^*Y_{j_2j_3j_4]\ell_1\ell_2\ell_3\ell_4}\theta^{\pm\ell_1\ell_2\ell_3\ell_4}\\
&+\frac{5}{18}\,{}^*Y_{[ij_1j_2j_3|\ell_1\ell_2\ell_3|}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j_4]} \mp 2\delta_{i[j_1}Y_{j_2j_3}k^\pm_{j_4]}\ .
\end{aligned}
\tag{5.7}
$$

While the TCFH on the bilinears which are skew-symmetric in the exchange of $\phi^r_\pm$ with $\phi^s_\pm$ is

$$
\begin{aligned}
&\mathcal{D}^{(\pm)\mathcal{F}}_i \omega^\pm_{j_1j_2} \equiv \nabla_i\omega^\pm_{j_1j_2} \pm \partial_i \log A\,\omega^\pm_{j_1j_2} - \frac{1}{2}X_{\ell_1\ell_2 i[j_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]} \mp \frac{1}{2}Y_{i\ell}\varphi^{\pm\ell}{}_{j_1j_2}\\
&= -\frac{1}{4}X_{\ell_1\ell_2[ij_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]} \mp \frac{1}{12}\,{}^*X_{ij_1j_2\ell_1\ell_2}\omega^{\pm\ell_1\ell_2} - \frac{1}{18}\delta_{i[j_1}X_{j_2]\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3}\\
&\pm\frac{1}{6}\delta_{i[j_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]}Y_{\ell_1\ell_2} \pm \frac{1}{2}Y_{\ell[i}\varphi^{\pm\ell}{}_{j_1j_2]}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \varphi^\pm_{j_1j_2j_3} \equiv \nabla_i\varphi^\pm_{j_1j_2j_3} \pm \partial_i \log A\,\varphi^\pm_{j_1j_2j_3} \pm \frac{3}{4}\,{}^*X_{\ell_1\ell_2 i[j_1j_2}\varphi^{\pm\ell_1\ell_2}{}_{j_3]}\\
&-\frac{3}{2}X_{\ell i[j_1j_2}\omega^{\pm\ell}{}_{j_3]} \mp \frac{3}{2}Y_{i[j_1}\omega^\pm_{j_2j_3]}\\
&= \pm\frac{2}{3}\,{}^*X_{\ell_1\ell_2[ij_1j_2}\varphi^{\pm\ell_1\ell_2}{}_{j_3]} - \frac{2}{3}X_{\ell[ij_1j_2}\omega^{\pm\ell}{}_{j_3]} \pm \frac{1}{6}\delta_{i[j_1}\,{}^*X_{j_2j_3]\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3}\\
&-\frac{1}{4}\delta_{i[j_1}X_{j_2j_3]\ell_1\ell_2}\omega^{\pm\ell_1\ell_2} - \frac{1}{36}\,{}^*Y_{ij_1j_2j_3\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3} \mp Y_{[ij_1}\omega^\pm_{j_2j_3]} \mp \delta_{i[j_1}Y_{j_2|\ell|}\omega^{\pm\ell}{}_{j_3]}
\end{aligned}
\tag{5.8}
$$

where ${}^*X$ is the Hodge dual[3] of X and similarly for the other fields. In the TCFH above we have suppressed the r, s indices on the bilinears that count the number of Killing spinors.

The TCFH above has been expressed in terms of the minimal connection $\mathcal{D}^{(\pm)\mathcal{F}}$, see [22] for the definition. As expected, $\mathcal{D}^{(\pm)\mathcal{F}}$ is not form degree preserving connection. As the action of $\mathcal{D}^{(\pm)\mathcal{F}}_i$ on the space of forms preserves the subspaces of symmetric and skew-symmetric bilinears in the exchange of ϕ^r and ϕ^s Killing spinors and acts trivially on the scalars f, the reduced holonomy[4] of the connection is included in (the connected

[3]In our conventions $\Gamma_{i_1\dots i_9}\phi_\pm = \pm\epsilon_{i_1\dots i_9}\phi_\pm$ with $\epsilon^{123456789} = 1$.

[4]From here on with the term holonomy we shall always refer to the reduced holonomy of the TCFH connection, i.e. the connected component of the identity of the holonomy group.

component of the identity) $GL(135) \times GL(120)$. Note that the holonomy of the maximal TCFH connection, see again [22], is contained in $GL(136) \times GL(120)$ as it acts non-trivially on the scalars.

An alternative way to see that the holonomy of $\mathcal{D}^{(\pm)\mathcal{F}}$ is included in $GL(135)\times GL(120)$ is to observe that $\phi_\pm$ can be thought of as Majorana spinors of $\mathfrak{spin}(9)$. It is well known that the tensor product of two $\mathfrak{spin}(9)$ Majorana representations, $\Delta_{\mathbf{16}}$, decomposes as

$$\Delta_{\mathbf{16}} \otimes \Delta_{\mathbf{16}} = \oplus_{k=0}^{4} \Lambda^k(\mathbb{R}^9) \ , \tag{5.9}$$

where $\Lambda^k(\mathbb{R}^9)$ is the irreducible representation of $\mathfrak{spin}(9)$ on the space of k-degree forms on $\mathbb{R}^9$. The action of the supercovariant connection on the tensor product of two spin bundles, i.e. on a bispinor, preserves the symmetric and skew-symmetric subspaces. As the rank of the spin bundle is 16, these sub-bundles have rank 136 and 120, respectively. So the holonomy of all connections of the TCFH is included in $GL(136) \times GL(120)$.

Although the form bilinears are CKY forms with respect to the TCFH connection $\mathcal{D}^{(\pm)\mathcal{F}}$ as expected, it is clear from the TCFH (5.7) and (5.8) that they are neither KY nor CCKY forms for generic supersymmetric backgrounds. However, we shall demonstrate that for special solutions several terms in the TCFH vanish and as a result some bilinears become either KY or CCKY forms.

5.3 The TCFH of warped AdS$_3$ backgrounds

5.3.1 Fields and Killing spinors

The bosonic fields of 11-dimensional supergravity for a warped AdS$_3$ background, AdS$_3\times_w M^8$, can be written as

$$g = 2\mathbf{e}^+\mathbf{e}^- + (\mathbf{e}^z)^2 + \delta_{ij}\mathbf{e}^i\mathbf{e}^j \ , \quad F = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz \wedge Q + X \ , \tag{5.10}$$

where the metric, $g(M^8)$, on the internal space M^8 is $g(M^8) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$ and Q and X are a 1-form and a 4-form on M^8, respectively. Moreover,

$$\mathbf{e}^+ = du \ , \quad \mathbf{e}^- = dr - 2\ell^{-1}rdz - 2rd\ln A \ , \quad \mathbf{e}^z = Adz \ , \quad \mathbf{e}^i = e^i_J dy^J \ , \tag{5.11}$$

is a null pseudo-orthonormal frame on the spacetime with $\mathbf{e}^i$ an orthonormal frame on M^8, y are coordinates on M^8 and (u, r, z) are the remaining coordinates of the spacetime and A is the warp factor. As for the AdS$_2$ backgrounds in the previous section, there is a coordinate transformation such that the spacetime metric g can be put into the standard warped form, $g = A^2 g_\ell(AdS_3) + g(M^8)$, where $g_\ell(AdS_3)$ is the standard metric on AdS$_3$ with radius ℓ.

The gravitino KSE of 11-dimensional supergravity can be solved [94] along the AdS$_3$ subspace of a AdS$_3 \times_w M^8$ background with fields (5.10). The Killing spinors ϵ can be expressed as, $\epsilon = \epsilon(\sigma_\pm) + \epsilon(\tau_\pm)$, with

$$\begin{aligned}\epsilon(\sigma_\pm) &= \sigma_+ + \sigma_- - \ell^{-1}A^{-1}u\Gamma_{+z}\sigma_- \ , \\ \epsilon(\tau_\pm) &= e^{-\frac{z}{\ell}}\tau_+ - \ell^{-1}A^{-1}re^{-\frac{z}{\ell}}\Gamma_{-z}\tau_+ + e^{\frac{z}{\ell}}\tau_- \ , \end{aligned} \tag{5.12}$$

where $\sigma_\pm$ and $\tau_\pm$ spinors satisfy the lightcone projections $\Gamma_\pm\sigma_\pm = \Gamma_\pm\tau_\pm = 0$, and depend only on the coordinates of M^8. In addition, they satisfy the KSEs

$$\mathcal{D}^\pm_i \sigma_\pm = \mathcal{D}^\pm_i \tau_\pm = 0 \ , \tag{5.13}$$

on the internal space M^8, where

$$\mathcal{D}_i^{\pm} \equiv \nabla_i \pm \frac{1}{2}\partial_i \log A - \frac{1}{288}\Gamma\!\!\not{X}_i + \frac{1}{36}\not{X}_i \mp \frac{1}{12}A^{-1}\Gamma_z\Gamma\!\!\not{Q}_i \pm \frac{1}{6}A^{-1}\Gamma_z Q_i \ , \tag{5.14}$$

are the supercovariant connections on M^8 and ∇ is the connection induced on the spin bundle from the Levi-Civita connection of the metric $g(M^8)$. Furthermore, $\sigma_\pm$ and $\tau_\pm$ satisfy an additional algebraic KSE on M^8 arising from the integration of the gravitino KSE of 11-dimensional supergravity along the z coordinate. These algebraic KSEs have been explained in detail in [94] and they will be used in the examples below to produce the right counting for the number of Killing spinors of the AdS backgrounds but they do not contribute to the TCFH below.

5.3.2 The TCFH on the internal space

A basis in the space of form bilinear on the internal space M^8 is

$$\begin{aligned}
&f^{\pm r,s} = \langle \phi^r_\pm, \phi^s_\pm\rangle \ , \quad \tilde{f}^{\pm r,s} = \langle \phi^r_\pm, \Gamma_z\phi^s_\pm\rangle \ , \quad k^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i\phi^s_\pm\rangle\, \mathbf{e}^i \ , \\
&\theta^{\pm r,s} = \frac{1}{4!}\langle \phi^r_\pm, \Gamma_{ijk\ell}\phi^s_\pm\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k\wedge\mathbf{e}^\ell \ , \quad \tilde{\varphi}^{\pm r,s} = \frac{1}{3!}\langle \phi^r_\pm, \Gamma_{ijk}\Gamma_z\phi^s_\pm\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k \ , \\
&\tilde{k}^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i\Gamma_z\phi^s_\pm\rangle\, \mathbf{e}^i \ , \quad \omega^{\pm r,s} = \frac{1}{2}\langle \phi^r_\pm, \Gamma_{ij}\phi^s_\pm\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j \ , \\
&\tilde{\omega}^{\pm r,s} = \frac{1}{2}\langle \phi^r_\pm, \Gamma_{ij}\Gamma_z\phi^s_\pm\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j \ , \quad \varphi^{\pm r,s} = \frac{1}{3!}\langle \phi^r_\pm, \Gamma_{ijk}\phi^s_\pm\rangle\, \mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k \ ,
\end{aligned} \tag{5.15}$$

where $\phi_\pm$ stands[5] for either $\sigma_\pm$ or $\tau_\pm$. The first five form bilinears are symmetric in the exchange of $\phi^r_\pm$ with $\phi^s_\pm$ while the rest are skew-symmetric.

The TCFH expressed[6] in terms of the minimal connection is

$$\begin{aligned}
&\mathcal{D}_i^{(\pm)\mathcal{F}} f^\pm \equiv \nabla_i f^\pm \pm \partial_i \log A\, f^\pm = \pm\frac{1}{36}\,{}^*X_i{}^{\ell_1\ell_2\ell_3}\tilde{\varphi}^\pm_{\ell_1\ell_2\ell_3} \mp \frac{1}{3}A^{-1}Q_i\tilde{f}^\pm \ , \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} k^\pm_j \equiv \nabla_i k^\pm_j \pm \partial_i \log A\, k^\pm_j - \frac{1}{12}X_{i\ell_1\ell_2\ell_3}\theta^{\pm\ell_1\ell_2\ell_3}{}_j \\
&= -\frac{1}{18}X_{[i|\ell_1\ell_2\ell_3|}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j]} + \frac{1}{144}\delta_{ij}X_{\ell_1\ell_2\ell_3\ell_4}\theta^{\pm\ell_1\ell_2\ell_3\ell_4} \mp \frac{1}{6}A^{-1}Q_\ell\tilde{\varphi}^{\pm\ell}{}_{ij} \ , \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} \theta^\pm_{j_1j_2j_3j_4} \equiv \nabla_i\theta^\pm_{j_1j_2j_3j_4} \pm \partial_i \log A \mp 3\,{}^*X_{\ell i[j_1j_2}\tilde{\varphi}^{\pm\ell}{}_{j_3j_4]} + 2X_{i[j_1j_2j_3}k^\pm_{j_4]} \\
&+\frac{1}{3}A^{-1}\,{}^*Q_{\ell_1\ell_2\ell_3 i[j_1j_2j_3}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j_4]} \\
&= \mp\frac{10}{3}\,{}^*X_{\ell[ij_1j_2}\tilde{\varphi}^{\pm\ell}{}_{j_3j_4]} + \frac{5}{6}X_{[ij_1j_2j_3}k^\pm_{j_4]} \mp 2\delta_{i[j_1}\,{}^*X_{j_2j_3|\ell_1\ell_2|}\tilde{\varphi}^{\pm\ell_1\ell_2}{}_{j_4]} + \frac{2}{3}\delta_{i[j_1}X_{j_2j_3j_4]\ell}k^{\pm\ell} \\
&-\frac{1}{18}A^{-1}\delta_{i[j_1}\,{}^*Q_{j_2j_3j_4]\ell_1\ell_2\ell_3\ell_4}\theta^{\pm\ell_1\ell_2\ell_3\ell_4} + \frac{5}{18}A^{-1}\,{}^*Q_{\ell_1\ell_2\ell_3[ij_1j_2j_3}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j_4]} \ , \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} \tilde{\varphi}^\pm_{j_1j_2j_3} \equiv \nabla_i\tilde{\varphi}^\pm_{j_1j_2j_3} \pm \partial_i \log A\, \tilde{\varphi}^\pm_{j_1j_2j_3} \pm \frac{3}{4}\,{}^*X_{\ell_1\ell_2 i[j_1}\theta^{\pm\ell_1\ell_2}{}_{j_2j_3]} \\
&= \pm\,{}^*X_{\ell_1\ell_2[ij_1}\theta^{\pm\ell_1\ell_2}{}_{j_2j_3]} - \frac{1}{3}X_{ij_1j_2j_3}\tilde{f}^\pm \pm \frac{1}{6}\,{}^*X_{ij_1j_2j_3}f^\pm \pm \frac{1}{3}\delta_{i[j_1}\,{}^*X_{j2|\ell_1\ell_2\ell_3|}\theta^{\pm\ell_1\ell_2\ell_3}{}_{j_3]} \\
&-\frac{1}{36}A^{-1}\,{}^*Q_{ij_1j_2j_3\ell_1\ell_2\ell_3}\tilde{\varphi}^{\pm\ell_1\ell_2\ell_3} \mp A^{-1}\delta_{i[j_1}Q_{j_2}k^\pm_{j_3]} \ , \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} \tilde{f}^\pm \equiv \nabla_i\tilde{f}^\pm \pm \partial_i \log A\tilde{f}^\pm = \frac{1}{18}X_{i\ell_1\ell_2\ell_3}\tilde{\varphi}^{\pm\ell_1\ell_2\ell_3} \mp \frac{1}{3}A^{-1}Q_i f^\pm \ ,
\end{aligned}$$

[5]One can also consider mixed $\sigma_\pm$ and $\tau_\pm$ form bilinears. The TCFH is the same as the one stated below for the form bilinear basis above.

[6]In our conventions $\Gamma_z\Gamma_{i_1\dots i_8}\phi_\pm = \pm\epsilon_{i_1\dots i_8}\phi_\pm$ with $\epsilon^{12345678} = 1$.

$$
\begin{aligned}
&\mathcal{D}_i^{(\pm)\mathcal{F}}\omega^{\pm}_{j_1j_2} \equiv \nabla_i\omega^{\pm}_{j_1j_2} \pm \partial_i \log A\omega^{\pm}_{j_1j_2} - \frac{1}{2}X_{\ell_1\ell_2 i[j_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]} \pm \frac{1}{2}A^{-1}Q_i\tilde{\omega}^{\pm}_{j_1j_2}\\
&= -\frac{1}{4}X_{\ell_1\ell_2[ij_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]} - \frac{1}{18}\delta_{i[j_1}X_{j_2]\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3} \mp \frac{1}{6}\,{}^*X_{ij_1j_2\ell}\tilde{k}^{\pm\ell}\\
&\pm\frac{1}{3}A^{-1}\delta_{i[j_1}\tilde{\omega}^{\pm\ell}{}_{j_2]}Q_\ell \pm \frac{1}{2}A^{-1}Q_{[i}\tilde{\omega}^{\pm}_{j_1j_2]}\ ,\\
&\mathcal{D}_i^{(\pm)\mathcal{F}}\varphi^{\pm}_{j_1j_2j_3} \equiv \nabla_i\varphi^{\pm}_{j_1j_2j_3} \pm \partial_i \log A\varphi^{\pm}_{j_1j_2j_3} - \frac{3}{2}X_{\ell i[j_1j_2}\omega^{\pm\ell}{}_{j_3]} \pm \frac{3}{2}\,{}^*X_{\ell i[j_1j_2}\tilde{\omega}^{\pm\ell}{}_{j_3]}\\
&= \pm\frac{4}{3}\,{}^*X_{\ell[ij_1j_2}\tilde{\omega}^{\pm\ell}{}_{j_3]} - \frac{2}{3}X_{\ell[ij_1j_2}\omega^{\pm\ell}{}_{j_3]} \pm \frac{1}{2}\delta_{i[j_1}\,{}^*X_{j_2j_3]\ell_1\ell_2}\tilde{\omega}^{\pm\ell_1\ell_2} - \frac{1}{4}\delta_{i[j_1}X_{j_2j_3]\ell_1\ell_2}\omega^{\pm\ell_1\ell_2}\\
&-\frac{1}{36}A^{-1}\,{}^*Q_{ij_1j_2j_3\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3} \mp A^{-1}\delta_{i[j_1}Q_{j_2}\tilde{k}^{\pm}_{j_3]}\ ,\\
&\mathcal{D}_i^{(\pm)\mathcal{F}}\tilde{k}^{\pm}_j \equiv \nabla_i\tilde{k}^{\pm}_j \pm \partial_i \log A\tilde{k}^{\pm}_j = \frac{1}{6}X_{ij\ell_1\ell_2}\tilde{\omega}^{\pm\ell_1\ell_2} \mp \frac{1}{12}\,{}^*X_{ij\ell_1\ell_2}\omega^{\pm\ell_1\ell_2} \pm \frac{1}{6}A^{-1}Q_\ell\varphi^{\pm\ell}{}_{ij}\ ,\\
&\mathcal{D}_i^{(\pm)\mathcal{F}}\tilde{\omega}^{\pm}_{j_1j_2} \equiv \nabla_i\tilde{\omega}^{\pm}_{j_1j_2} \pm \partial_i \log A\tilde{\omega}^{\pm}_{j_1j_2} \mp \frac{1}{2}\,{}^*X_{\ell_1\ell_2 i[j_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]} \pm \frac{1}{2}A^{-1}Q_i\omega^{\pm}_{j_1j_2}\\
&= \mp\frac{1}{2}\,{}^*X_{\ell_1\ell_2[ij_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]} - \frac{1}{3}X_{ij_1j_2\ell}\tilde{k}^{\pm\ell} \mp \frac{1}{9}\delta_{i[j_1}\,{}^*X_{j_2]\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3} \pm \frac{1}{2}A^{-1}Q_{[i}\omega^{\pm}_{j_1j_2]}\\
&\pm\frac{1}{3}A^{-1}\delta_{i[j_1}\omega^{\pm\ell}{}_{j_2]}Q_\ell\ ,
\end{aligned}
\tag{5.16}
$$

where ${}^*X$ is the Hodge dual of X on the internal space M^8 and similarly for the other fields. We have also suppressed the r, s indices on the form bilinears that count the number of Killing spinors.

The action of the minimal TCFH connection on the space of forms preserves the subspaces of forms with are symmetric and skew-symmetric in the exchange of the Killing spinors $\phi^r_\pm$ and $\phi^s_\pm$. Furthermore, it preserves the subspaces of 1-forms $\tilde{k}^\pm$, where it acts as the Levi-Civita connection up to a rescaling with the warp factor A, and acts trivially on the scalars $f^\pm$ and $\tilde{f}^\pm$. Therefore the reduced holonomy is included in $GL(134) \times SO(8) \times GL(112)$. The reduced holonomy of the maximal TCFH connection instead is included in $GL(136) \times GL(120)$ because it preserves only the subspaces of forms which are symmetric and skew-symmetric in the exchange of the Killing spinors $\phi^r_\pm$ and $\phi^s_\pm$.

The holonomy of the TCFH connection can be understood in a way similar to that of AdS$_2$ backgrounds. As $\phi_\pm$ can be viewed as Majorana $\mathfrak{spin}(8)$ spinors, it is known that the product of two such Majorana representations, $\Delta_{\mathbf{16}}$, can be decomposed in terms of form representations, $\Lambda^k(\mathbb{R}^8)$, of $\mathfrak{spin}(8)$ as

$$
\Delta_{\mathbf{16}} \otimes \Delta_{\mathbf{16}} = \oplus_{k=0}^{8}\Lambda^k(\mathbb{R}^8)\ . \tag{5.17}
$$

As the supercovariant derivative preserves the space of symmetric and skew-symmetric bi-spinors, it is clear that the holonomy of all TCFH connections will be included in $GL(136) \times GL(120)$, where 136 is the rank of the sub-bundle of symmetric bi-spinors while 120 is the rank of the sub-bundle of skew-symmetric bi-spinors.

As expected all form bilinears are CKY forms with respect to the TCFH connections $\mathcal{D}^{(\pm)\mathcal{F}}$ in agreement with the general result in [22]. Apart from $A^{\pm1}\tilde{k}^\pm$ which is a Killing 1-form, the TCFH does not imply that the remaining form bilinears are KY forms for generic supersymmetric backgrounds. However, we shall demonstrate that many of them are either KY or CCKY forms for some AdS$_3$ solutions of 11-dimensional supergravity.

5.4 The TCFH of warped AdS$_4$ backgrounds

5.4.1 Fields and Killing spinors

The bosonic fields of 11-dimensional supergravity of warped AdS$_4$ backgrounds, $AdS_4 \times_w M^7$, can be written as

$$g = 2\mathbf{e}^+\mathbf{e}^- + (\mathbf{e}^z)^2 + (\mathbf{e}^x)^2 + g(M^7) \ , \quad F = S\,\mathbf{e}^+ \wedge \mathbf{e}^- \wedge \mathbf{e}^z \wedge \mathbf{e}^x + X \ , \tag{5.18}$$

with $g(M^7) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$, where S and X are a 0-form and 4-form on the internal space M^7, respectively. In addition,

$$\begin{aligned}&\mathbf{e}^+ = du \ , \quad \mathbf{e}^- = dr - 2\ell^{-1}rdz - 2rd\ln A \ , \quad \mathbf{e}^z = Adz \ , \quad \mathbf{e}^x = Ae^{\frac{z}{\ell}}dx \ , \\ &\mathbf{e}^i = e^i{}_J dy^J \ ,\end{aligned} \tag{5.19}$$

is a null pseudo-orthonormal frame on the spacetime with $\mathbf{e}^i$ an orthonormal frame on M^7, y are coordinates on M^7 and (u, r, z, x) are the remaining coordinates of the spacetime, and ℓ is the radius of AdS$_4$. As in previous cases, after a coordinate transformation, the spacetime metric can be written in the standard warped form with warp factor A^2.

The gravitino KSE of 11-dimensional supergravity can be explicitly integrated along the (u, r, z, x) coordinates and the Killing spinors can be written as, $\epsilon = \epsilon(\sigma_\pm) + \epsilon(\tau_\pm)$, see [94], with

$$\begin{aligned}\epsilon(\sigma_\pm) &= \sigma_+ + \sigma_- - \ell^{-1}e^{\frac{z}{\ell}}x\Gamma_{xz}\sigma_- - \ell^{-1}A^{-1}u\Gamma_{+z}\sigma_- \ , \\ \epsilon(\tau_\pm) &= e^{-\frac{z}{\ell}}\tau_+ - \ell^{-1}A^{-1}re^{\frac{z}{\ell}}\Gamma_{-z}\tau_+ - \ell^{-1}x\Gamma_{xz}\tau_+ + e^{\frac{z}{\ell}}\tau_- \ ,\end{aligned} \tag{5.20}$$

where the spinors $\sigma_\pm$ and $\tau_\pm$ depend only on the coordinates of M^7 and satisfy the light-cone projections $\Gamma_\pm\sigma_\pm = 0$ and $\Gamma_\pm\tau_\pm = 0$. Furthermore, these spinors satisfy the KSEs

$$\mathcal{D}_i^\pm\sigma_\pm = 0 \ , \quad \mathcal{D}_i^\pm\tau_\pm = 0 \ , \tag{5.21}$$

on the internal space M^7, where the supercovariant connection is

$$\mathcal{D}_i^\pm \equiv \nabla_i \pm \frac{1}{2}\partial_i \log A - \frac{1}{288}\Gamma\!\!\!/ X_i + \frac{1}{36}X\!\!\!/_i \pm \frac{1}{12}S\Gamma_{izx} \ , \tag{5.22}$$

with ∇ induced by the Levi-Civita connection of the metric $g(M^7)$ on M^7. The spinors $\sigma_\pm$ and $\tau_\pm$ satisfy an additional algebraic KSE which arises from the integration of the gravitino KSE of 11-dimensional supergravity along the z coordinate. These algebraic KSEs can be found in [94] and they are essential for the correct counting of Killing spinors for warped AdS backgrounds.

5.4.2 The TCFH on the internal manifold

A basis in the space of form bilinears on the internal space M^7 is

$$\begin{aligned}&f^{\pm r,s} = \langle\phi_\pm^r, \phi_\pm^s\rangle \ , \quad k^{\pm r,s} = \langle\phi_\pm^r, \Gamma_i\phi_\pm^s\rangle\,\mathbf{e}^i \ , \quad \tilde{\omega}^{\pm r,s} = \frac{1}{2}\langle\phi_\pm^r, \Gamma_{ij}\Gamma_{zx}\phi_\pm^s\rangle\,\mathbf{e}^i \wedge \mathbf{e}^j \ , \\ &\tilde{\varphi}^{\pm r,s} = \frac{1}{3!}\langle\phi_\pm^r, \Gamma_{ijk}\Gamma_{zx}\phi_\pm^s\rangle\,\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \ , \quad \tilde{k}^{\pm r,s} = \langle\phi_\pm^r, \Gamma_i\Gamma_{zx}\phi_\pm^s\rangle\,\mathbf{e}^i \ , \\ &\omega^{\pm r,s} = \frac{1}{2}\langle\phi_\pm^r, \Gamma_{ij}\phi_\pm^s\rangle\,\mathbf{e}^i \wedge \mathbf{e}^j \ , \quad \varphi^{\pm r,s} = \frac{1}{3!}\langle\phi_\pm^r, \Gamma_{ijk}\phi_\pm^s\rangle\,\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \ ,\end{aligned}$$

$$\tilde{f}^{\pm r,s} = \langle \phi^r_\pm, \Gamma_{zx}\phi^s_\pm\rangle \ . \tag{5.23}$$

where $\phi_\pm$ stands[7] for either $\sigma_\pm$ or $\tau_\pm$. Note that the bilinears f, k $\tilde{\omega}$ and $\tilde{\varphi}$ are symmetric in the exchange of spinors $\phi^r_\pm$ and $\phi^s_\pm$ while the rest are skew-symmetric.

The TCFH expressed in terms of the minimal connection can be written as[8]

$$\begin{aligned}
&\mathcal{D}^{(\pm)\mathcal{F}}_i f^\pm \equiv \nabla_i f^\pm \pm \partial_i \log A f^\pm = \pm\frac{1}{12}\,{}^*X_{i\ell_1\ell_2}\tilde{\omega}^{\pm\ell_1\ell_2}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i k^\pm_j \equiv \nabla_i k^\pm_j \pm \partial_i \log A k^\pm_j \mp \frac{1}{4}\,{}^*X_{i\ell_1\ell_2}\tilde{\varphi}^{\pm\ell_1\ell_2}{}_j\\
&= \pm\frac{1}{6}S\tilde{\omega}^\pm_{ij} \mp \frac{1}{3}{}^*X_{\ell_1\ell_2[i}\tilde{\varphi}^{\pm\ell_1\ell_2}{}_{j]} \mp \frac{1}{18}\delta_{ij}\,{}^*X_{\ell_1\ell_2\ell_3}\tilde{\varphi}^{\pm\ell_1\ell_2\ell_3}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{\omega}^\pm_{j_1j_2} \equiv \nabla_i\tilde{\omega}^\pm_{j_1j_2} \pm \partial_i \log A\tilde{\omega}^\pm_{j_1j_2} - \frac{1}{2}X_{\ell_1\ell_2 i[j_1}\tilde{\varphi}^{\pm\ell_1\ell_2}{}_{j_2]}\\
&= -\frac{1}{4}X_{\ell_1\ell_2[ij_1}\tilde{\varphi}^{\pm\ell_1\ell_2}{}_{j_2]} - \frac{1}{18}\delta_{i[j_1}X_{j_2]\ell_1\ell_2\ell_3}\tilde{\varphi}^{\pm\ell_1\ell_2\ell_3} \pm \frac{1}{6}\,{}^*X_{ij_1j_2}f^\pm \mp \frac{1}{3}S\delta_{i[j_1}k^\pm_{j_2]}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{\varphi}^\pm_{j_1j_2j_3} \equiv \nabla_i\tilde{\varphi}^\pm_{j_1j_2j_3} \pm \partial_i \log A\tilde{\varphi}^\pm_{j_1j_2j_3} - \frac{3}{2}X_{\ell i[j_1j_2}\tilde{\omega}^{\pm\ell}{}_{j_3]} \mp \frac{3}{2}\,{}^*X_{i[j_1j_2}k^\pm_{j_3]}\\
&= \mp\frac{4}{3}\,{}^*X_{[ij_1j_2}k^\pm_{j_3]} - \frac{2}{3}X_{\ell[ij_1j_2}\tilde{\omega}^{\pm\ell}{}_{j_3]} \mp \delta_{i[j_1}\,{}^*X_{j_2j_3]\ell}k^{\pm\ell}\\
&-\frac{1}{4}\delta_{i[j_1}X_{j_2j_3]\ell_1\ell_2}\tilde{\omega}^{\pm\ell_1\ell_2} - \frac{1}{36}\,{}^*S_{ij_1j_2j_3\ell_1\ell_2\ell_3}\tilde{\varphi}^{\pm\ell_1\ell_2\ell_3}\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \omega^\pm_{j_1j_2} \equiv \nabla_i\omega^\pm_{j_1j_2} \pm \partial_i \log A\omega^\pm_{j_1j_2} - \frac{1}{2}X_{\ell_1\ell_2 i[j_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]}\\
&= -\frac{1}{4}X_{\ell_1\ell_2[ij_1}\varphi^{\pm\ell_1\ell_2}{}_{j_2]} - \frac{1}{18}\delta_{i[j_1}X_{j_2]\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3} \mp \frac{1}{6}\,{}^*X_{ij_1j_2}\tilde{f}^\pm \pm \frac{1}{3}S\delta_{i[j_1}\tilde{k}^\pm_{j_2]}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \varphi^\pm_{j_1j_2j_3} \equiv \nabla_i\varphi^\pm_{j_1j_2j_3} \pm \partial_i \log A\varphi^\pm_{j_1j_2j_3} - \frac{3}{2}X_{\ell i[j_1j_2}\omega^{\pm\ell}{}_{j_3]} \pm \frac{3}{2}\,{}^*X_{i[j_1j_2}\tilde{k}^\pm_{j_3]}\\
&= \pm\frac{4}{3}\,{}^*X_{[ij_1j_2}\tilde{k}^\pm_{j_3]} - \frac{2}{3}X_{\ell[ij_1j_2}\omega^{\pm\ell}{}_{j_3]} \pm \delta_{i[j_1}\,{}^*X_{j_2j_3]\ell}\tilde{k}^{\pm\ell}\\
&-\frac{1}{4}\delta_{i[j_1}X_{j_2j_3]\ell_1\ell_2}\omega^{\pm\ell_1\ell_2} - \frac{1}{36}\,{}^*S_{ij_1j_2j_3\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{f}^\pm \equiv \nabla_i \tilde{f}^\pm \pm \partial_i \log A \tilde{f}^\pm = \mp\frac{1}{12}\,{}^*X_{i\ell_1\ell_2}\omega^{\pm\ell_1\ell_2}\ ,\\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{k}^\pm_j \equiv \nabla_i \tilde{k}^\pm_j \pm \partial_i \log A \tilde{k}^\pm_j \pm \frac{1}{4}\,{}^*X_{i\ell_1\ell_2}\varphi^{\pm\ell_1\ell_2}{}_j\\
&= \pm\frac{1}{3}\,{}^*X_{\ell_1\ell_2[i}\varphi^{\pm\ell_1\ell_2}{}_{j]} \pm \frac{1}{18}\delta_{ij}\,{}^*X_{\ell_1\ell_2\ell_3}\varphi^{\pm\ell_1\ell_2\ell_3} \mp \frac{1}{6}S\omega^\pm_{ij}\ ,
\end{aligned} \tag{5.24}$$

where ${}^*X$ and ${}^*S$ are the Hodge duals of X and S on the internal space M^7, respectively, and we have suppressed the r, s indices on the form bilinears that label the number of Killing spinors.

The action of the minimal TCFH connection on the space of forms preserves the subspaces of symmetric and skew-symmetric bilinears in the exchange of $\phi^r_\pm$ and $\phi^s_\pm$ Killing spinors and acts trivially on the scalars f and $\tilde{f}$. As a consequence, the holonomy of the connection is included in $GL(63)\times GL(63)$. Note that the TCFH on the 0-, 1- and 3-form bilinears which are symmetric in the exchange of $\phi^r_\pm$ and $\phi^s_\pm$ is almost identical to that of the corresponding form bilinears which are skew-symmetric in the exchange of the same spinors. The difference is a sign in the terms containing the fluxes S and ${}^*F$. The holonomy of the maximal TCFH connection is included in $GL(64)\times GL(64)$ since it acts non-trivially on the scalars.

[7]Unlike the case of warped AdS$_3$ backgrounds, the $\sigma_\pm$ and $\tau_\pm$ Killing spinors of all warped AdS$_k$, $k>3$, backgrounds are related with Clifford algebra operations.

[8]In our conventions $\Gamma_{zx}\Gamma_{i_1i_2i_3i_4i_5i_6i_7}\phi_\pm = \pm\epsilon_{i_1i_2i_3i_4i_5i_6i_7}\phi_\pm$ with $\epsilon^{1234567}=1$

The spinors $\sigma_\pm$ and $\tau_\pm$ are associated with the (reducible) Majorana representation $\Delta_{\mathbf{16}}$ of $\mathfrak{spin}(8)$. This decomposes under $\mathfrak{spin}(7)$ as $\Delta_{\mathbf{16}} = \Delta_{\mathbf{8}} \oplus \Delta_{\mathbf{8}}$, where $\Delta_{\mathbf{8}}$ is the (irreducible) Majorana representation of $\mathfrak{spin}(7)$. Moreover, the tensor product of two such representations, $\Delta_{\mathbf{8}}$, decomposes in terms of form representations as

$$\Delta_{\mathbf{8}} \otimes \Delta_{\mathbf{8}} = \sum_{k=0}^{3} \Lambda^k(\mathbb{R}^7) \ . \tag{5.25}$$

Clearly the TCFH includes two copies of the forms that appear in the above decomposition. This implies that the holonomy of the maximal TCFH connection to be included in $GL(64) \times GL(64)$.

5.5 The TCFH of warped AdS$_k$, $k = 5, 6, 7$, backgrounds

5.5.1 Fields and Killing spinors

The fields of 11-dimensional supergravity for warped AdS$_k \times_w M^{11-k}$, $k = 5, 6, 7$, backgrounds can be written as

$$g = 2\mathbf{e}^+\mathbf{e}^- + (\mathbf{e}^z)^2 + \sum_a (\mathbf{e}^a)^2 + g(M^{11-k}) \ , \quad F = X \ , \tag{5.26}$$

with $g(M^{11-k}) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$, where X is a 4-form on the internal space M^{11-k}. The null pseudo-orthonormal frame $(\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^z, \mathbf{e}^a, \mathbf{e}^i)$ is expressed as

$$\begin{aligned}
&\mathbf{e}^+ = du \ , \quad \mathbf{e}^- = dr - 2\ell^{-1} r dz - 2r d\ln A \ , \quad \mathbf{e}^z = A dz \ , \\
&\mathbf{e}^a = A e^{\frac{z}{\ell}} dx^a \ , \quad \mathbf{e}^i = e^i_J dy^j \ ,
\end{aligned} \tag{5.27}$$

where y are coordinates of the internal space M^{11-k} and (u, r, z, x^a) are the rest of the coordinates of the spacetime, ℓ is the radius of AdS subspace and A is the warp factor.

As in all previous cases, the KSEs of 11-dimensional supergravity can be integrated over the AdS subspace and the Killing spinors can be expressed as, $\epsilon = \epsilon(\sigma_\pm) + \epsilon(\tau_\pm)$, see [94], with

$$\begin{aligned}
\epsilon(\sigma_\pm) &= \sigma_+ + \sigma_- - \ell^{-1} e^{\frac{z}{\ell}} x^a \Gamma_{az}\sigma_- - \ell^{-1} A^{-1} u \Gamma_{+z}\sigma_- \ , \\
\epsilon(\tau_\pm) &= e^{-\frac{z}{\ell}}\tau_+ - \ell^{-1} A^{-1} r e^{-\frac{z}{\ell}} \Gamma_{-z}\tau_+ - \ell^{-1} x^a \Gamma_{az}\tau_+ + e^{\frac{z}{\ell}}\tau_- \ ,
\end{aligned} \tag{5.28}$$

where the $\sigma_\pm$ and $\tau_\pm$ spinors satisfy the lightcone projections $\Gamma_\pm\sigma_\pm = \Gamma_\pm\tau_\pm = 0$, and depend only on the coordinates of M^{11-k}. In addition, they satisfy the KSEs

$$\mathcal{D}^\pm_i \phi_\pm = 0 \ , \tag{5.29}$$

along the internal space M^{11-k}, where the supercovariant connection is

$$\mathcal{D}^\pm_i \equiv \nabla_i \pm \frac{1}{2}\partial_i \log A - \frac{1}{288}\Gamma\!\!\!/ X_i + \frac{1}{36} X\!\!\!/_i \ , \tag{5.30}$$

∇ is the connection on the spin bundle of M^{11-k} induced from the metric $g(M^{11-k})$ and $\phi_\pm$ stands from either $\sigma_\pm$ or $\tau_\pm$. Note that for warped AdS$_7$ backgrounds the term $\Gamma\!\!\!/ X$ in the supercovariant connection vanishes. As in previous cases, $\sigma_\pm$ and $\tau_\pm$ satisfy an additional algebraic KSE which arises from the integration of the gravitino KSE of 11-dimensional supergravity along the z coordinate and can be found in [94]. It will be used to determine the number of Killing spinors in some examples below.

5.5.2 The TCFH of warped AdS$_5$ backgrounds

A basis in the space of form bilinears on the internal space M^6 is

$$
\begin{aligned}
&f^{\pm r,s} = \langle \phi^r_\pm, \phi^s_\pm \rangle \ , \quad k^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i \phi^s_\pm \rangle \, \mathbf{e}^i \ , \quad \tilde k^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i \Gamma_{(3)} \phi^s_\pm \rangle \, \mathbf{e}^i \ , \\
&\tilde\omega^{\pm r,s} = \frac{1}{2} \langle \phi^r_\pm, \Gamma_{ij} \Gamma_{(3)} \phi^s_\pm \rangle \, \mathbf{e}^i \wedge \mathbf{e}^j \ , \quad \omega^{\pm r,s} = \frac{1}{2} \langle \phi^r_\pm, \Gamma_{ij} \phi^s_\pm \rangle \, \mathbf{e}^i \wedge \mathbf{e}^j \ , \\
&\varphi^{\pm r,s}_{ijk} = \frac{1}{3!} \langle \phi^r_\pm, \Gamma_{ijk} \phi^s_\pm \rangle \, \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \ , \quad \tilde f^{\pm r,s} = \langle \phi^r_\pm, \Gamma_{(3)} \phi^s_\pm \rangle \ ,
\end{aligned}
\tag{5.31}
$$

where $\Gamma_{(3)} = \Gamma_{zx^1x^2}$, i.e. it is the product of gamma matrices along the directions $\mathbf{e}^z$ and $\mathbf{e}^a$ for $a = 1, 2$. The first four bilinears are symmetric in the exchange of $\phi^r_\pm$ and $\phi^s_\pm$ spinors while the rest are skew-symmetric.

The TCFH expressed[9] in terms of the minimal connection is

$$
\begin{aligned}
&\mathcal{D}^{(\pm)\mathcal{F}}_i f^\pm \equiv \nabla_i f^\pm \pm \partial_i \log A f^\pm = \pm \frac{1}{6} {}^*X_{i\ell} \tilde k^{\pm\ell} \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i k^\pm_j \equiv \nabla_i k^\pm_j \pm \partial_i \log A k^\pm_j \pm \frac{1}{2} {}^*X_{i\ell} \tilde\omega^{\pm\ell}{}_j \\
&= \pm \frac{2}{3} {}^*X_{[i|\ell|} \tilde\omega^{\pm\ell}{}_{j]} \mp \frac{1}{6} \delta_{ij} \, {}^*X_{\ell_1\ell_2} \tilde\omega^{\pm\ell_1\ell_2} \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde k^\pm_j \equiv \nabla_i \tilde k^\pm_j \pm \partial_i \log A \tilde k^\pm_j = \frac{1}{6} X_{ij\ell_1\ell_2} \tilde\omega^{\pm\ell_1\ell_2} \pm \frac{1}{6} {}^*X_{ij} f^\pm \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde\omega^\pm_{j_1j_2} \equiv \nabla_i \tilde\omega^\pm_{j_1j_2} \pm \partial_i \log A \tilde\omega^\pm_{j_1j_2} \pm {}^*X_{i[j_1} k^\pm_{j_2]} \\
&= -\frac{1}{3} X_{ij_1j_2\ell} \tilde k^{\pm\ell} \pm \frac{2}{3} \delta_{i[j_1} {}^*X_{j_2]\ell} k^{\pm\ell} \pm {}^*X_{[ij_1} k^\pm_{j_2]} \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \omega^\pm_{j_1j_2} \equiv \nabla_i \omega^\pm_{j_1j_2} \pm \partial_i \log A \omega^\pm_{j_1j_2} - \frac{1}{2} X_{\ell_1\ell_2 i[j_1} \varphi^{\pm\ell_1\ell_2}{}_{j_2]} \\
&= -\frac{1}{4} X_{\ell_1\ell_2[ij_1} \varphi^{\pm\ell_1\ell_2}{}_{j_2]} - \frac{1}{18} \delta_{i[j_1} X_{j_2]\ell_1\ell_2\ell_3} \varphi^{\pm\ell_1\ell_2\ell_3} \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \varphi^\pm_{\ell_1\ell_2\ell_3} \equiv \nabla_i \varphi^\pm_{\ell_1\ell_2\ell_3} \pm \partial_i \log A \varphi^\pm_{\ell_1\ell_2\ell_3} - \frac{3}{2} X_{\ell i[j_1j_2} \omega^{\pm\ell}{}_{j_3]} \\
&= -\frac{1}{4} \delta_{i[j_1} X_{j_2j_3]\ell_1\ell_2} \omega^{\pm\ell_1\ell_2} - \frac{2}{3} X_{\ell[ij_1j_2} \omega^{\pm\ell}{}_{j_3]} \pm \delta_{i[j_1} {}^*X_{j_2j_3]} \tilde f^\pm \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde f^\pm \equiv \nabla_i \tilde f^\pm \pm \partial_i \log A \tilde f^\pm = \mp \frac{1}{6} {}^*X_{\ell_1\ell_2} \varphi^{\pm\ell_1\ell_2}{}_i \ ,
\end{aligned}
\tag{5.32}
$$

where ${}^*X$ is the Hodge dual of X on M^6 and we have suppressed the indices r, s of the form bilinears.

As expected, the minimal connection of the TCFH $\mathcal{D}^{(\pm)\mathcal{F}}$ is not form degree preserving. On the other hand, its action closes on the form bilinears which are either symmetric or skew-symmetric in the interchange of spinors $\phi^r_\pm$ and $\phi^s_\pm$. The holonomy of $\mathcal{D}^{(\pm)\mathcal{F}}$ is contained in $SO(6) \times GL(21) \times GL(35)$ as in addition it acts with the Levi-Civita connection on the 1-form bilinear $A\,\tilde k$ and trivially on the scalar bilinears f and $\tilde f$. Note that the holonomy of the maximal TCFH connection is contained in $GL(28) \times GL(35)$ as it only closes on the symmetric and skew-symmetric form bilinears under the exchange of spinors $\phi^r_\pm$ and $\phi^s_\pm$.

5.5.3 The TCFH of warped AdS$_6$ backgrounds

A basis in the space of form bilinears on the internal space M^5 is

$$
f^{\pm r,s} = \langle \phi^r_\pm, \phi^s_\pm \rangle \ , \quad k^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i \phi^s_\pm \rangle \, \mathbf{e}^i \ , \quad \tilde f^{\pm r,s} = \langle \phi^r_\pm, \Gamma_{(4)} \phi^s_\pm \rangle \ ,
$$

[9] In our conventions $\Gamma_{(3)} \Gamma_{i_1 \dots i_6} \phi_\pm = \pm \epsilon_{i_1 \dots i_6} \phi_\pm$ and $\epsilon^{123456} = 1$.

$$\tilde{k}^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i \Gamma_{(4)} \phi^s_\pm \rangle \, \mathbf{e}^i \ ,$$
$$\omega^{\pm r,s} = \frac{1}{2} \langle \phi^r_\pm, \Gamma_{ij} \phi^s_\pm \rangle \, \mathbf{e}^i \wedge \mathbf{e}^j \ , \quad \tilde{\omega}^{\pm r,s} = \frac{1}{2} \langle \phi^r_\pm, \Gamma_{ij} \Gamma_{(4)} \phi^s_\pm \rangle \, \mathbf{e}^i \wedge \mathbf{e}^j \ , \tag{5.33}$$

where the first four form bilinears are symmetric in the exchange of $\phi^r_\pm$ and $\phi^s_\pm$ spinors while the rest are skew-symmetric and $\Gamma_{(4)} = \Gamma_{zx^1x^2x^3}$.

The TCFH expressed[10] in terms of the minimal connection, $\mathcal{D}^{(\pm)\mathcal{F}}$, is

$$\mathcal{D}^{(\pm)\mathcal{F}}_i f^\pm \equiv \nabla_i f^\pm \pm \partial_i \log A f^\pm = \pm \frac{1}{6} {}^*X_i \tilde{f}^\pm \ ,$$
$$\mathcal{D}^{(\pm)\mathcal{F}}_i k^\pm_j \equiv \nabla_i k^\pm_j \pm \partial_i \log A k^\pm_j \mp \frac{1}{2} {}^*X_i \tilde{k}^\pm_j = \mp \frac{2}{3} {}^*X_{[i} \tilde{k}^\pm_{j]} \mp \frac{1}{3} \delta_{ij} {}^*X_\ell \tilde{k}^{\pm\ell} \ ,$$
$$\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{f}^\pm \equiv \nabla_i \tilde{f}^\pm \pm \partial_i \log A \tilde{f}^\pm = \pm \frac{1}{6} {}^*X_i f^\pm \ ,$$
$$\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{k}^\pm_j \equiv \nabla_i \tilde{k}^\pm_j \pm \partial_i \log A \tilde{k}^\pm_j \mp \frac{1}{2} {}^*X_i k^\pm_j = \mp \frac{2}{3} {}^*X_{[i} k^\pm_{j]} \mp \frac{1}{3} \delta_{ij} {}^*X_\ell k^{\pm\ell} \ ,$$
$$\mathcal{D}^{(\pm)\mathcal{F}}_i \omega^\pm_{j_1j_2} \equiv \nabla_i \omega^\pm_{j_1j_2} \pm \partial_i \log A \omega^\pm_{j_1j_2} \mp \frac{1}{2} {}^*X_i \tilde{\omega}^\pm_{j_1j_2} = \mp {}^*X_{[i} \tilde{\omega}^\pm_{j_1j_2]} \mp \frac{2}{3} \delta_{i[j_1} \tilde{\omega}^{\pm\ell}{}_{j_2]} {}^*X_\ell \ ,$$
$$\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{\omega}^\pm_{j_1j_2} \equiv \nabla_i \tilde{\omega}^\pm_{j_1j_2} \pm \partial_i \log A \tilde{\omega}^\pm_{j_1j_2} \mp \frac{1}{2} {}^*X_i \omega^\pm_{j_1j_2} = \mp {}^*X_{[i} \omega^\pm_{j_1j_2]} \mp \frac{2}{3} \delta_{i[j_1} \omega^{\pm\ell}{}_{j_2]} {}^*X_\ell \ , \tag{5.34}$$

where ${}^*X$ is the Hodge dual of X and we have suppressed the r, s indices on the form bilinears as in previous cases.

Unlike previous cases, the minimal TCFH connection $\mathcal{D}^{(\pm)\mathcal{F}}$ for AdS$_6$ backgrounds is form degree preserving. Furthermore, its action can be diagonalised on the forms

$$\zeta^\pm_{(+)} = \zeta^\pm + \tilde{\zeta}^\pm \ , \quad \zeta^\pm_{(-)} = \zeta^\pm - \tilde{\zeta}^\pm \ , \tag{5.35}$$

where $\zeta^\pm$ stands for either $k^\pm$ or $\omega^\pm$, i.e. one has that

$$\mathcal{D}^{(\pm)\mathcal{F}}_i \zeta^\pm_{(+)} = \nabla_i \zeta^\pm_{(+)} \pm \partial_i \log A \zeta^\pm_{(+)} \mp \frac{1}{2} {}^*X_i \zeta^\pm_{(+)} \ ,$$
$$\mathcal{D}^{(\pm)\mathcal{F}}_i \zeta^\pm_{(-)} = \nabla_i \zeta^\pm_{(-)} \pm \partial_i \log A \zeta^\pm_{(-)} \pm \frac{1}{2} {}^*X_i \zeta^\pm_{(-)} \ . \tag{5.36}$$

Such a connection arises provided one gauges the scale transformation $\zeta \to s\,\zeta$ accompanied with ${}^*X \to {}^*X \pm 2s^{-1}ds$, where the sign is plus for $\zeta^+_{(+)}$ and $\zeta^-_{(-)}$ while it is minus for the rest of the form bilinears. Clearly, there are two sectors and the holonomy of the connection in each sector is $SO(5) \times (\mathbb{R} - \{0\})$.

5.5.4 The TCFH of AdS$_7$ backgrounds

A basis in the space of spinor bilinears on the internal space M^4 is

$$f^{\pm r,s} = \langle \phi^r_\pm, \phi^s_\pm \rangle \ , \quad k^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i \phi^s_\pm \rangle \, \mathbf{e}^i \ , \quad \tilde{f}^{\pm r,s} = \langle \phi^r_\pm, \Gamma_{(5)} \phi^s_\pm \rangle \ ,$$
$$\tilde{k}^{\pm r,s} = \langle \phi^r_\pm, \Gamma_i \Gamma_{(5)} \phi^s_\pm \rangle \, \mathbf{e}^i \ , \quad \omega^{\pm r,s} = \frac{1}{2} \langle \phi^r_\pm, \Gamma_{ij} \phi^s_\pm \rangle \, \mathbf{e}^i \wedge \mathbf{e}^j \ , \tag{5.37}$$

where the first three are symmetric in the exchange of spinors $\phi^r_\pm$ and $\phi^s_\pm$ while the rest are skew-symmetric and $\Gamma_{(5)} = \Gamma_{zx^1x^2x^3x^4}$.

The TCFH expressed[11] in terms of the minimal connection $\mathcal{D}^{(\pm)\mathcal{F}}$ is

[10] In our conventions $\Gamma_{(4)} \Gamma_{i_1 \dots i_5} \phi_\pm = \pm \epsilon_{i_1 \dots i_5} \phi_\pm$ with $\epsilon^{12345} = 1$.

[11] In our conventions $\Gamma_{(5)} \Gamma_{i_1 \dots i_4} \phi_\pm = \pm \epsilon_{i_1 \dots i_4} \phi_\pm$ and $\epsilon^{1234} = 1$.

$$
\begin{aligned}
&\mathcal{D}_i^{(\pm)\mathcal{F}} f^{\pm} \equiv \nabla_i f^{\pm} \pm \partial_i \log A f^{\pm} = 0 \;, \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} k_j^{\pm} \equiv \nabla_i k_j^{\pm} \pm \partial_i \log A k_j^{\pm} = \mp\frac{1}{3}\delta_{ij}\, {}^{*}X \tilde{f}^{\pm} \;, \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} \tilde{f}^{\pm} \equiv \nabla_i \tilde{f}^{\pm} \pm \partial_i \log A \tilde{f}^{\pm} = \pm\frac{1}{3}\, {}^{*}X k_i^{\pm} \;, \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} \omega_{j_1 j_2}^{\pm} \equiv \nabla_i \omega_{j_1 j_2}^{\pm} \pm \partial_i \log A \omega_{j_1 j_2}^{\pm} = \pm\frac{2}{3}\delta_{i[j_1} \tilde{k}_{j_2]}^{\pm}\, {}^{*}X \;, \\
&\mathcal{D}_i^{(\pm)\mathcal{F}} \tilde{k}_j^{\pm} \equiv \nabla_i \tilde{k}_j^{\pm} \pm \partial_i \log A \tilde{k}_j^{\pm} = \mp\frac{1}{3}\, {}^{*}X \omega_{ij}^{\pm} \;,
\end{aligned} \tag{5.38}
$$

where ${}^{*}X$ is the Hodge dual of the 4-form X on the internal space M^4.

It is clear that the (reduced) holonomy of $\mathcal{D}^{(\pm)\mathcal{F}}$ is contained in $SO(4)$. Furthermore, $A^{\pm 1}k^{\pm}$ and $A^{\pm 1}\omega^{\pm}$ are CCKY forms. Therefore, their dual in M^4 are KY. In addition, $A^{\pm 1}\tilde{k}^{\pm}$ are KY tensors. It is well-known KY tensors generate symmetries in spinning particle actions.

5.6 Probes and symmetries

5.6.1 Relativistic and spinning particles

We have integrated the KSE of 11-dimensional supergravity along the AdS_k subspace of a warped spacetime, $\mathrm{AdS}_k \times_w M^{11-k}$, and found the TCFHs on the internal space M^{11-k}. To investigate whether the form bilinears of the TCFHs on the internal space generate symmetries for spinning particle actions, we have to integrate the dynamics of the spinning particle along the AdS_k subspace and describe the effective dynamics of the system on the internal space M^{11-k}.

For this consider first the dynamics of a relativistic particle on a warped spacetime, $N \times_w M$, with metric $g = A^2 h + \gamma$, where h is a metric on N and γ is a metric on M and A is the warped factor. Varying the action

$$
S = \frac{1}{2}\int dt\, g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \;, \tag{5.39}
$$

one finds that the equations of motion are

$$
\nabla_t^h (A^2 \dot{\rho}^a) = 0 \;, \quad \nabla_t^{\gamma} \dot{y}^I - \frac{1}{2}\gamma^{IJ}\partial_J A^2\, h_{ab}\dot{\rho}^a\dot{\rho}^b = 0 \;, \tag{5.40}
$$

where ∇^h and ∇^{γ} denote the Levi-Civita connections of h and γ, respectively, ρ^a are coordinates on N and y^I are coordinates on M. It is clear that

$$
Q^2 = \frac{1}{2}A^4 h_{ab}\dot{\rho}^a\dot{\rho}^b \;, \tag{5.41}
$$

is conserved as a consequence of the field equation on N. Then notice that the dynamics of the relativistic particle on M can be described by the effective action

$$
S_M = \frac{1}{2}\int dt\, \big(\frac{1}{2}\gamma_{IJ}\dot{y}^I\dot{y}^J - Q^2 A^{-2}\big) \;. \tag{5.42}
$$

The action apart from the usual kinetic term exhibits a potential depending on the warped factor. There are various sectors to consider parameterised by the value of Q^2. If either

$Q^2 = 0$, which is the case for ρ constant, or A^2 is constant, A_M becomes the standard action for geodesic motion on M possibly shifted by an ignorable constant.

A similar analysis can be performed for a spinning particle probe [13] propagating on a spacetime with metric g described by the action

$$S = -\frac{i}{2} \int dt\, d\theta\; g_{\mu\nu}\, DX^\mu\, \dot{X}^\nu \ , \tag{5.43}$$

where (t, θ) are worldline superspace coordinates, X are worldline superfields $X = X(t, \theta)$ and D is a worldline superspace derivative with $D^2 = i\partial_t$.

The equations of motion of the spinning particle (5.43) propagating on a warped spacetime $N \times_w M$, as for the relativistic particle above, are

$$\nabla^h (A^2 \dot{\rho}^a) + \nabla^h_t (A^2 D\rho^a) = 0 \ , \quad \nabla^\gamma \dot{y}^I = \frac{1}{2} \gamma^{IJ} \partial_J A^2 h_{ab} D\rho^a \dot{\rho}^b \ . \tag{5.44}$$

In this case, there is not a simple description of the effective dynamics on M as for the relativistic particle described by the action (5.42). However, note that ρ equals to a constant is a solution of the equations of motion above. So if either ρ is constant or the warp factor A is constant, the effective dynamics of the spinning particle on M is described by the action

$$S_M = -\frac{i}{2} \int dt\, d\theta\; \gamma_{IJ}\, Dy^I\, \dot{y}^J \ . \tag{5.45}$$

It is well known that the action above is invariant under an infinitesimal transformation

$$\delta y^I = \epsilon\, \alpha^I{}_{J_1 \cdots J_{m-1}} Dy^{J_1} \cdots Dy^{J_{m-1}} \ , \tag{5.46}$$

provided that α is a KY form [16], where ϵ is the infinitesimal parameter.

There is an extensive list of 11-dimensional supersymmetric AdS solutions, see e.g. [120, 121, 122, 123, 124, 125]. The purpose here is to give some examples of TCFHs and investigate their properties instead of being comprehensive. So we shall focus below on the TCFH of the maximally supersymmetric AdS solutions and some AdS solutions that arise as near-horizon geometries of intersecting M-branes.

5.6.2 Maximally supersymmetric AdS backgrounds

$\mathbf{AdS}_4 \times S^7$

The TCFH of warped AdS_4 backgrounds with only electric flux, i.e. $X = 0$ in (5.18), can be written as

$$\begin{aligned}
&\mathcal{D}^{(\pm)\mathcal{F}}_i f^\pm \equiv \nabla_i f^\pm \pm \partial_i \log A\, f^\pm = 0 \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i k^\pm_j \equiv \nabla_i k^\pm_j \pm \partial_i \log A\, k^\pm_j = \pm\frac{1}{6} S \tilde{\omega}^\pm_{ij} \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{\omega}^\pm_{j_1 j_2} \equiv \nabla_i \tilde{\omega}^\pm_{j_1 j_2} \pm \partial_i \log A\, \tilde{\omega}^\pm_{j_1 j_2} = \mp\frac{1}{3} S \delta_{i[j_1} k^\pm_{j_2]} \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \tilde{\varphi}^\pm_{j_1 j_2 j_3} \equiv \nabla_i \tilde{\varphi}^\pm_{j_1 j_2 j_3} \pm \partial_i \log A\, \tilde{\varphi}^\pm_{j_1 j_2 j_3} = -\frac{1}{36}\, {}^*S_{i j_1 j_2 j_3 \ell_1 \ell_2 \ell_3} \tilde{\varphi}^{\pm \ell_1 \ell_2 \ell_3} \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \omega^\pm_{j_1 j_2} \equiv \nabla_i \omega^\pm_{j_1 j_2} \pm \partial_i \log A\, \omega^\pm_{j_1 j_2} = \pm\frac{1}{3} S \delta_{i[j_1} \tilde{k}^\pm_{j_2]} \ , \\
&\mathcal{D}^{(\pm)\mathcal{F}}_i \varphi^\pm_{j_1 j_2 j_3} \equiv \nabla_i \varphi^\pm_{j_1 j_2 j_3} \pm \partial_i \log A\, \varphi^\pm_{j_1 j_2 j_3} = -\frac{1}{36}\, {}^*S_{i j_1 j_2 j_3 \ell_1 \ell_2 \ell_3} \varphi^{\pm \ell_1 \ell_2 \ell_3} \ ,
\end{aligned}$$

$$\mathcal{D}_i^{(\pm)\mathcal{F}} \tilde{f}^{\pm} \equiv \nabla_i \tilde{f} \pm \partial_i \log A\, \tilde{f}^{\pm} = 0 \ ,$$
$$\mathcal{D}_i^{(\pm)\mathcal{F}} \tilde{k}_j^{\pm} \equiv \nabla_i \tilde{k}_j^{\pm} \pm \partial_i \log A\, \tilde{k}_j^{\pm} = \mp\frac{1}{6} S\omega_{ij}^{\pm} \ . \tag{5.47}$$

It is clear from the TCFH above that $A^{\pm1}k$, $A^{\pm1}\tilde{k}$, $A^{\pm1}\varphi$ and $A^{\pm1}\tilde{\varphi}$ are KY forms, and $A^{\pm1}\omega$ and $A^{\pm1}\tilde{\omega}$ are CCKY forms. As $A^{\pm1}\omega$ and $A^{\pm1}\tilde{\omega}$ are CCKY forms their duals in the internal space are KY forms. The holonomy of the TCFH connection is included in $SO(7)$.

The maximally supersymmetric AdS$_4$ solution is a Freund-Rubin type of background with internal space M^7 the round 7-sphere, $M^7 = S^7$, and the warp factor A constant. All the forms bilinears above generate symmetries for the spinning particle probe action (5.45). Note that the form bilinears on S^7 are not necessarily invariant forms under the $SO(8)$ isometry group of S^7.

AdS$_7 \times S^4$

The maximally supersymmetric AdS$_7$ solution is again a Freund-Rubin type of background with internal space M^4 the round 4-sphere, $M^4 = S^4$, and the warp factor A constant. An inspection of the TCFH of warped AdS$_7$ backgrounds in (5.38) reveals that the bilinear $\tilde{k}^{\pm}$ is a KY form, and $k^{\pm}$ and $\omega^{\pm}$ are CCKY forms. Again the duals ${}^*k^{\pm}$ and ${}^*\omega^{\pm}$ of $k^{\pm}$ and $\omega^{\pm}$ in S^4, respectively, are KY forms and so $\tilde{k}^{\pm}$, ${}^*k^{\pm}$ and ${}^*\omega^{\pm}$ generate symmetries for the spinning particle probe action (5.45).

5.6.3 AdS backgrounds from intersecting branes

More examples of AdS backgrounds emerge as near-horizon geometries of intersecting M-branes [126]. We shall not explore all the possibilities, see [127] for more examples. Instead we shall focus on the AdS$_2$ and AdS$_3$ solutions that arise as near-horizon geometries of the intersection of three M2-branes on an 0-brane, the intersection of an M2-brane and M5-brane on a 1-brane and the intersection of three M5-branes on a 1-brane configurations.

AdS$_2$ solution from intersecting M2-branes

One could take the near horizon geometry of the three intersecting M2-brane solution on a 0-brane and proceed to examine the associated TCFH. Instead, we shall write an ansatz for the fields which includes the solution. In particular, we set

$$g = g_\ell(AdS_2) + g(S^3) + g(\mathbb{R}^6) \ , \quad F = d\mathrm{vol}_\ell(AdS_2) \wedge Y \ , \tag{5.48}$$

where $g_\ell(AdS_2)$ and $d\mathrm{vol}_\ell(AdS_2)$ is the standard metric and volume 2-form on AdS$_2$ with radius ℓ, respectively, $g(S^3)$ is the round metric on S^3 of unit radius, $g(\mathbb{R}^6)$ is the Euclidean metric on $\mathbb{R}^6$ and Y is a constant 2-form on $\mathbb{R}^6$. Using the scale symmetry $g \to \Omega^2 g$ and $F \to \Omega^3 F$ of 11-dimensional supergravity as well as some coordinate transformations, one can show that the near horizon geometry of three M2-branes, with arbitrary charge densities, intersecting on a 0-brane solution can be cast into the above form. Clearly for this ansatz $X = 0$ in (5.1) and we have set $A = 1$.

Focusing on the ϕ_+ Killing spinors, the gravitino KSE along the directions of $\mathbb{R}^6$ implies that Y is a non-degenerate 2-form and proportional to a Kähler form λ on $\mathbb{R}^6$ associated with the Euclidean metric, i.e. $Y = \gamma\lambda$, $\gamma \in \mathbb{R}$. Furthermore ϕ_+ has to satisfy the conditions $\Gamma_{1234}\phi_+ = \eta\phi_+$ and $\Gamma_{1256}\phi_+ = \zeta\phi_+$, where $\eta, \zeta = \pm 1$ and we have arranged such that $\mathbb{R}^6$ lies in the directions $1, \dots, 6$. Then the warp factor field equation in [94] implies that $\gamma^2 = \ell^{-2}$. Next the field equation along S^3, which is the round unit sphere, gives $\ell^{-2} = 4$. All the remaining KSEs and field equations are satisfied without further conditions. Therefore, the background (5.48) with the above choice of parameters admits 4 ϕ_+ Killing spinors. The solution also admits 4 more ϕ_- Killing spinor and so it preserves 1/4 of the supersymmetry.

Next notice that the supercovariant derivative along S^3 can be written as

$$\mathcal{D}^{\pm}_{\alpha} = \nabla^{S^3}_{\alpha} \pm \frac{1}{24}\Gamma_{\alpha}\not{Y} \ , \tag{5.49}$$

where α here labels the three orthonormal directions tangential to S^3. Note that $\Gamma_\alpha \not{Y} = \not{Y}\Gamma_\alpha$. Moreover, considering only those components of the form bilinears that lie on S^3, i.e.

$$k^{\pm rs}_{\alpha} = \langle \phi^r_{\pm}, \Gamma_\alpha \phi^s_{\pm}\rangle \ , \quad \omega^{\pm rs}_{\alpha\beta} = \langle \phi^r_{\pm}, \Gamma_{\alpha\beta} \phi^s_{\pm}\rangle \ , \quad \varphi^{\pm rs}_{\alpha\beta\delta} = \langle \phi^r_{\pm}, \Gamma_{\alpha\beta\delta} \phi^s_{\pm}\rangle \ , \tag{5.50}$$

one can demonstrate that k and φ are KY forms while ω is CCKY form. Therefore, all of them or their Hodge duals on S^3 generate symmetries for the probe action (5.45).

AdS$_3$ solution from M2- and M5-branes

An ansatz which includes the near horizon geometry of an M2-brane intersecting an M5-brane on a 1-brane is

$$g = g_\ell(AdS_3) + g(S^3) + g(\mathbb{R}^5) \ , \quad F = d\mathrm{vol}_\ell(AdS_3) \wedge Q + d\mathrm{vol}(S^3) \wedge P \ , \tag{5.51}$$

i.e. $Q, X \neq 0$ in (5.10), where $g_\ell(AdS_3)$ and $d\mathrm{vol}_\ell(AdS_3)$ are the standard metric and volume 3-form of AdS$_3$ with radius ℓ, respectively, and $g(\mathbb{R}^5)$ is the Euclidean metric on $\mathbb{R}^5$. Similarly, $g(S^3)$ and $d\mathrm{vol}(S^3)$ are the metric and volume 3-form of unit round 3-sphere, respectively, and the 1-forms P and Q are constant and lie along the same direction in $\mathbb{R}^5$, e.g. $P = p\,dw$ and $Q = q\,dw$ with p, q constants.

Focusing on the KSEs on σ_+ and setting without loss of generality $A = 1$, the integrability of the gravitino KSE along the $\mathbb{R}^5$ directions implies that $p^2 = q^2$. Moreover, one has to also consider the algebraic KSE $\Xi^+\sigma_+ = 0$ which arises from the integration of the KSE of 11-dimensional supergravity along the z direction of AdS$_3$. As $\Xi^+ = -(2\ell)^{-1} + \frac{1}{288}\Gamma_z \not{X} + \frac{1}{6}\not{Q}$, see [94], the algebraic KSE can be arranged as

$$\Big(-\frac{1}{\ell}\Gamma_w + \frac{p}{6}\Gamma_z\Gamma_{(3)} + \frac{1}{3}q\Big)\sigma_+ = 0 \ , \tag{5.52}$$

where $\Gamma_{(3)}$ is the product of the three gamma matrices along orthonormal directions tangential to S^3.

As $p^2 = q^2$ to solve (5.52) let us set $p = q$. The other case $p = -q$ can be treated in a similar way. Then decompose (5.52) into eigenspaces of Γ_w and $\Gamma_z\Gamma_{(3)}$. Using $\Gamma^2_w = 1$ and $(\Gamma_z\Gamma_{(3)})^2 = 1$ and $\Gamma_w\Gamma_z\Gamma_{(3)} = \Gamma_z\Gamma_{(3)}\Gamma_w$, we have that

$$-\eta\frac{1}{\ell} + \zeta\frac{q}{6} + \frac{1}{3}q = 0 \ , \tag{5.53}$$

where $\Gamma_w\sigma_+ = \eta\sigma_+$ and $\Gamma_z\Gamma_{(3)}\sigma_+ = \zeta\sigma_+$ with $\eta, \zeta = \pm 1$. There are four cases to consider leading to $q = \pm 2\ell^{-1}$ and $q = \pm 6\ell^{-1}$. Two of these solutions are related to the other two by a change of the overall sign of the 4-form field strength F. So there are only two remaining independent solutions. Furthermore the $q = \pm 6\ell^{-1}$ solution is ruled out by the warp factor field equation [94]. In addition, the field equation along S^3 implies that $p^2 = 4$. As $q = \pm 2\ell^{-1}$, one finds that $\ell^2 = 1$, which is the near-horizon geometry of the M2- and M5-brane intersection on a 1-brane solution. This solution preserves 1/2 of supersymmetry as each of the KSEs on $\sigma_\pm$ and $\tau_\pm$ give 4 independent solutions.

As in the previous AdS_2 case, we next consider the KSE along the S^3 directions whose supercovariant derivative can be put in the form

$$\mathcal{D}^+_\alpha = \nabla^{S^3}_\alpha + \Gamma_\alpha \Gamma_z (\frac{1}{6} p\zeta + \frac{1}{12} q)\eta \ , \tag{5.54}$$

where α labels the three othonormal directions tangential to S^3. Considering the form bilinears (5.50) with $\phi_+ = \sigma_+$, it is easy to show that k and ω are CCKY forms on S^3. As a result, their Hodge duals on S^3 are KY forms and generate symmetries for the probe action (5.45). The bilinear φ is also a CCKY form but its dual is a scalar.

5.6.4 AdS_3 solution for intersecting M5-branes

An ansatz which includes the near horizon geometry of three M5-branes intersecting on a 1-brane is

$$g = g_\ell(AdS_3) + g(S^2) + g(\mathbb{R}^6) \ , \quad F = d\mathrm{vol}(S^2) \wedge W \ , \tag{5.55}$$

i.e. $Q = 0$ in (5.10), where $g_\ell(AdS_3)$ is the metric of AdS_3 with radius ℓ, $g(S^2)$ and $d\mathrm{vol}(S^2)$ are the metric and volume 2-form of round 2-sphere with unit radius, respectively, $g(\mathbb{R}^6)$ is the Euclidean metric on $\mathbb{R}^6$ and W is a constant non-degenerate 2-form on $\mathbb{R}^6$.

To continue let us focus on the gravitino KSE on σ_+. The integrability condition of this equation along the $\mathbb{R}^6$ directions implies that $W = \gamma\lambda$ and that $\Gamma_{1234}\sigma_+ = \eta\sigma_+$ and $\Gamma_{1256}\sigma_+ = \zeta\sigma_+$, where λ is a Kähler form of the Euclidean metric on $\mathbb{R}^6$ and we have chosen $\mathbb{R}^6$ along the 123456 directions. Without loss of generality, one can always choose $\lambda = \lambda_1 dx^1 \wedge dx^2 + \lambda_2 dx^3 \wedge dx^4 + \lambda_3 dx^5 \wedge dx^6$ with $\lambda_1, \lambda_2, \lambda_3 = \pm 1$. Then the algebraic KSE, $\Xi^+\sigma_+ = \big(-(2\ell)^{-1} + (288)^{-1}\Gamma_z \not{X}\big)\sigma_+ = 0$, implies, after imposing $\Gamma_z\Gamma_{12}\Gamma_{(2)}\sigma_+ = \theta\sigma_+$, $\theta = \pm 1$, that either $\gamma = \pm 6\ell^{-1}$ or $\gamma = \pm 2\ell^{-1}$, where $\Gamma_{(2)}$ is the product of two gamma matrices along two orthonormal directions tangential to S^2. The warp factor field equation is not satisfied for $\gamma = \pm 6\ell^{-1}$. While for $\gamma = \pm 2\ell^{-1}$, the Einstein equation along S^2 gives $\ell = 2$. This is the solution that describes the near horizon geometry of three intersecting M5-branes and preserves 1/4 of supersymmetry.

After imposing all the conditions above on σ_+ appropriate for this solution, the gravitino KSE along S^2 can be written as

$$\mathcal{D}^+_\alpha = \nabla_\alpha \pm \frac{1}{2}\epsilon_{\alpha\beta}\Gamma^\beta\Gamma_{12} \ , \tag{5.56}$$

where a, b are restricted along two orthonormal tangential directions of S^2. Next it is straightforward to show that the 1-form bilinears restricted on S^2 are KY forms and the 2-form bilinears restricted on S^2 are CCKY forms. So the 1-form bilinears generate symmetries for the probe action (5.45).

5.7 Summary

We have presented the TCFH of all supersymetric AdS backgrounds of 11-dimensional supergravity. Therefore, we have demonstrated that all the form bilinears on the internal space of these backgrounds are CKY forms with respect to the TCFH connection. This provides a geometric interpretation for all the conditions on these form bilinears implied by the KSE of the theory. We have also given the reduced holonomy of the TCFH connections for generic supersymmetric backgrounds and we have found that it factorises on the space of symmetric and skew-symmetric form bilinears under the exchange of the two Killing spinors. We have illustrated our results with some examples that include the maximally supersymmetric AdS backgrounds of 11-dimensional supergravity as well as some other AdS backgrounds that arise as near horizon geometries of intersecting M-branes. We have found that some of the form bilinears on these backgrounds are KY forms and so generate symmetries for spinning particle probes propagating on the internal spaces.

Chapter 6

W-symmetries, anomalies and heterotic backgrounds with SU holonomy

It is well known that 2-dimensional supersymmetric sigma models with couplings a metric, g, and a Wess-Zumino term, b, are invariant under symmetries generated by $\hat{\nabla}$-covariantly constant forms [128, 129, 106, 130, 131], where $\hat{\nabla}$ is a metric connection with torsion, $H = db$. The sigma model target spaces considered so far are manifolds N^n that admit such a connection $\hat{\nabla}$ whose holonomy is included in the groups $U(\frac{n}{2})$, $SU(\frac{n}{2})$, $Sp(\frac{n}{4})$, $Sp(\frac{n}{4}) \cdot Sp(1)$, $G_2(n = 7)$ and $\mathrm{Spin}(7)(n = 8)$. We call these symmetries, *holonomy symmetries*, since they arise from the reduction of the holonomy of $\hat{\nabla}$ to a subgroup of $O(n)$. It has been found that the algebra of holonomy symmetries is a W-algebra [106, 130, 131], i.e. the structure constants of the algebra depend on the conserved currents of the theory. The structure of these algebras has been explored both in the classical and quantum theory, see [106, 130, 131, 132, 133, 134, 135]. More recently in [136], some of these backgrounds have been considered as target spaces of heterotic sigma models and the chiral anomalies of these symmetries have been investigated.

Symmetries of heterotic sigma models are expected to be anomalous in the quantum theory because of the presence of chiral worldsheet fermions in the sigma model actions [137, 138, 139]. Preservation of the geometric interpretation of these theories requires the anomaly cancellation of some of these symmetries. Certainly, after assigning an anomalous variation to the sigma model coupling constant b [140], the anomalies of spacetime frame rotations and the gauge sector transformations are cancelled. A consequence of this variation, is a non-tensorial transformation law for H which appears as a coupling in many vertices in the background field method of quantising the theory. The covariance in the quantum theory is restored by modifying H with appropriate Chern-Simons terms [141, 142] at all loops.

According to the classification of supersymmetric heterotic backgrounds [143, 144], a more general class of spacetimes can be used as target spaces of heterotic sigma models than those previously considered in the literature and stated above, for a review see [107]. These backgrounds exhibit a variety of $\hat{\nabla}$-covariantly constant forms constructed as Killing spinor bilinears all of which have been identified. All these forms generate holonomy symmetries in heterotic sigma models. The purpose of this work is to identify the algebra of symmetries generated by the covariantly constant form bilinears, as well as, to find and investigate their chiral anomalies using Wess-Zumino consistency conditions.

We shall demonstrate that the anomalies are consistent at one loop in the sigma model perturbation theory. Furthermore, they cancel at the same order by either the addition of plausible finite local counterterms in the sigma model effective action or by the assumption that the transformations are appropriately quantum mechanically corrected.

In this chapter, we shall focus on two classes of heterotic supersymmetric backgrounds those for which the connection with skew-symmetric torsion, $\hat{\nabla}$, has holonomy included in $SU(2)$ and in $SU(3)$. The spacetime, M^{10}, of the former backgrounds can be locally described as a principal bundle whose fibre is a 6-dimensional Lorentzian Lie group G with self-dual structure constants and base space a 4-dimensional conformally hyper-Kähler manifold [143, 144, 107]. The holonomy symmetries of sigma models on $SU(2)$ holonomy backgrounds are generated by 1-forms and 2-forms. The 1-forms are those associated with vector fields generated by the action of G on M^{10} with respect to the spacetime metric g. In the $SU(3)$ holonomy case, the spacetime is locally a principal bundle with fibre a 4-dimensional Lorentzian Lie group G and base space a conformally balanced Kähler manifold with torsion [143, 144, 107]. The holonomy symmetries are generated by four 1-forms which are associated with the vector fields generated by G on M^{10} with respect to g as well as one 2-form, I, and two 3-forms, L_1 and L_2.

We find that the closure of the algebra of holonomy symmetries in both cases requires the inclusion of additional generators. The incorporation of right-handed worldsheet translations and supersymmetry transformations generated by the right-handed energy-momentum tensor, T, as well as the symmetries generated by the second Casimir operator of the Lie algebra of G are required for the closure of the algebra of symmetries since the sigma models we shall consider manifestly exhibit a (1,0) worldsheet supersymmetry. In addition, the closure of the algebra of symmetries of the sigma model on $SU(3)$ holonomy backgrounds requires the symmetries generated by the conserved current which is the product, TJ_I, of T with the conserved current, J_I, of the symmetry generated by the 2-form I. In both cases, the symmetry algebra is a W-algebra because the structure constants of the algebra depend on the symmetries' currents.

To analyse the chiral anomalies of the holonomy symmetries, we assume that there is a regularisation scheme which manifestly preserves the (1,0) quantum-mechanically worldsheet supersymmetry of the theory. The fact that the perturbation theory for the model can be done in (1,0) superfields justifies this. The anomalous part of the effective action for spacetime frame rotations and gauge transformations has been computed in [101, 145]. After possibly including appropriate finite local counterterms [145] in the effective action of the theory, the anomaly of spacetime frame rotations and transformations of the gauge sector can be brought into the standard form given here in (6.30) and (6.31), respectively. Then, the use of Wess-Zumino consistency conditions allows to show that the anomalies of the holonomy symmetries of the sigma model generated by the $\hat{\nabla}$-covariantly constant forms can be expressed in terms of the Chern-Simons form of an appropriate connection as in (6.33) up to possibly spacetime frame rotation and gauge transformation invariant terms, see also [136]. Similar arguments apply to the anomalies of the additional symmetries required for the closure of the algebra of holonomy transformations.

We show that if the associated Chern-Simons terms are expressed in terms of the frame connection associated with the connection with torsion $-H$, $\check{\nabla}$, and the connection that appears in the classical action of the gauge sector, all these anomalies are consistent at one loop. In fact, the anomalous part of the effective action is naturally expressed in terms of these connections [145]. The cancellation of anomalies is done in two different ways. One way assumes that the form generators of the holonomy symmetries receive quantum mechanical corrections such that to the given order in perturbation theory they are covariantly constant with respect to a new connection $\hat{\nabla}^{\hbar}$ with skew-symmetric torsion $H^{\hbar}$ which includes the Chern-Simons form. Such a modification of the torsion is also justified as part of the anomaly cancellation mechanism for the spacetime frame rotation and gauge anomalies of the theory mentioned above. It is also consistent with the fact that the Killing spinor equations of heterotic supergravity retain their form [146] up to and including two loops in the sigma model perturbation theory provided that the 3-form field strength H is replaced by $H^{\hbar}$. Such a replacement is a consequence of the cancellation of gravitational anomalies for heterotic supergravity [141]. For certain symmetries, the anomaly cancellation is based on the existence of finite local counterterms which can be added to the effective action. Under particular assumptions, we show that these finite local terms can be used to remove the anomalies of the symmetries generated by 1-forms and 2-forms.

This chapter is organised as follows. In section 1, we introduce the action of the heterotic sigma model and present its sigma model and holonomy symmetries. We also give the main formulae for the anomalies, present the consistency conditions and discuss the mechanisms for anomaly cancellation. In section 2, we present the commutators of the symmetries of the sigma model with target space the heterotic backgrounds with $SU(2)$ holonomy. We also give the anomalies of these symmetries, prove that they are consistent and describe their cancellation. In section 3, we describe similar results for the symmetries and anomalies of sigma models with target space heterotic backgrounds with $SU(3)$ holonomy. In section 4, we provide a summary.

6.1 Classical symmetries of chiral sigma models

6.1.1 Action and sigma model symmetries

The classical fields of the (1,0)-supersymmetric 2-dimensional sigma models that we shall investigate are maps, X, from the worldsheet superspace $\Xi^{2|1}$ with coordinates $(\sigma^{=}, \sigma^{\ddagger}, \theta^{+})$ into a spacetime M, $X: \Xi^{2|1} \to M$, and Grassmannian odd sections ψ of a vector bundle $S_{-} \otimes X^{*}E$ over $\Xi^{2|1}$, where S_{-} is the anti-chiral spinor bundle over $\Xi^{2|1}$ and E is a vector bundle over M. An action[1] for these fields [140] is

$$S = -i \int d^2\sigma d\theta^{+} \Big((g_{\mu\nu} + b_{\mu\nu}) D_{+} X^{\mu} \partial_{=} X^{\nu} + i h_{\mathrm{ab}} \psi_{-}^{\mathrm{a}} \mathcal{D}_{+} \psi_{-}^{\mathrm{b}} \Big) , \tag{6.1}$$

where g is a spacetime metric, b is a locally defined 2-form on M such that $H = db$ is a globally defined 3-form, $D_{+}^2 = i\partial_{\ddagger}$, h is a fibre metric on E and

$$\mathcal{D}_{+} \psi_{-}^{\mathrm{a}} = D_{+} \psi_{-}^{\mathrm{a}} + D_{+} X^{\mu} \Omega_{\mu}{}^{\mathrm{a}}{}_{\mathrm{b}} \psi_{-}^{\mathrm{b}} , \tag{6.2}$$

with Ω a connection on E with curvature F. We take without loss of generality that $\mathcal{D}_{\mu} h_{\mathrm{ab}} = 0$. We shall refer to the part of the action with couplings h and $\mathcal{D}$ as the gauge sector of the theory. Note that

$$\delta S = -i \int d^2\sigma d\theta^{+} \big(\delta X^{\mu} \mathcal{S}_{\mu} + \Delta \psi_{-}^{\mathrm{a}} \mathcal{S}_{\mathrm{a}} \big) , \tag{6.3}$$

[1]The action can include a potential term [147]. But we shall not consider this here.

where

$$\Delta\psi_-^{\mathrm{a}} \equiv \delta\psi_-^{\mathrm{a}} + \delta X^\mu \Omega_\mu{}^{\mathrm{a}}{}_{\mathrm{b}}\psi_-^{\mathrm{b}} \ , \tag{6.4}$$

is the covariantisation of $\delta\psi_-$ and

$$\begin{aligned} \mathcal{S}_\mu &= -2g_{\mu\nu}\hat{\nabla}_= D_+ X^\nu - i\psi_-^{\mathrm{a}}\psi_-^{\mathrm{b}} D_+ X^\nu F_{\mu\nu\mathrm{ab}} \ , \\ \mathcal{S}^{\mathrm{a}} &= 2i\mathcal{D}_+\psi_-^{\mathrm{a}} \ , \end{aligned} \tag{6.5}$$

are the field equations. In addition, the connection, $\hat{\nabla}$, is

$$\hat{\nabla}_\nu Z^\mu = \nabla_\nu Z^\mu + \frac{1}{2}H^\mu{}_{\nu\lambda}Z^\lambda \ , \tag{6.6}$$

where ∇ the Levi-Civita connection on M with respect to the metric g and H is the torsion of $\hat{\nabla}$ which is skew-symmetric.

The transformations of the fields X and ψ, as well as the coupling constants g, b, h and Ω, that leave the action (6.1) invariant, are known as *sigma model symmetries.* This is to distinguish them from the standard symmetries of a field theory which act only on the fields and leave the action invariant. Clearly, such transformations are the diffeomorphisms of the target space M as well as the gauge transformations of the gauge sector. The fields and coupling constants under infinitesimal diffeomorphisms generated by the vector field v transform as

$$\begin{aligned} &\delta X^\mu = v^\mu \ , \quad \delta g_{\mu\nu} = -\mathcal{L}_v g_{\mu\nu} \ , \quad \delta b_{\mu\nu} = -\mathcal{L}_v b_{\mu\nu} \ , \\ &\delta\psi_-^{\mathrm{a}} = w^{\mathrm{a}}{}_{\mathrm{b}}\psi_-^{\mathrm{b}} \ , \quad \delta\Omega_\mu{}^{\mathrm{a}}{}_{\mathrm{b}} = -\mathcal{L}_v\Omega_\mu{}^{\mathrm{a}}{}_{\mathrm{b}} - \partial_\mu w^{\mathrm{a}}{}_{\mathrm{b}} + w^{\mathrm{a}}{}_{\mathrm{c}}\,\Omega_\mu{}^{\mathrm{c}}{}_{\mathrm{b}} - \Omega_\mu{}^{\mathrm{a}}{}_{\mathrm{c}}\,w^{\mathrm{c}}{}_{\mathrm{b}} \ , \\ &\delta h_{\mathrm{ab}} = -\mathcal{L}_v h_{\mathrm{ab}} - w^{\mathrm{c}}{}_{\mathrm{a}}h_{\mathrm{cb}} - h_{\mathrm{ac}}w^{\mathrm{c}}{}_{\mathrm{b}} \ , \end{aligned} \tag{6.7}$$

where it is assumed that the diffeomophisms generated by the vector field v lift to the vector bundle E and generate a fibre rotation[2] $w = w(v)$. The fields and coupling constants under the infinitesimal gauge transformations of the gauge sector transform as

$$\begin{aligned} &\delta_u\psi_-^{\mathrm{a}} = u^{\mathrm{a}}{}_{\mathrm{b}}\psi_-^{\mathrm{b}} \ , \quad \delta_u\Omega_\mu{}^{\mathrm{a}}{}_{\mathrm{b}} = -\partial_\mu u^{\mathrm{a}}{}_{\mathrm{b}} + u^{\mathrm{a}}{}_{\mathrm{c}}\,\Omega_\mu{}^{\mathrm{c}}{}_{\mathrm{b}} - \Omega_\mu{}^{\mathrm{a}}{}_{\mathrm{c}}\,u^{\mathrm{c}}{}_{\mathrm{b}} \ , \\ &\delta_u h_{\mathrm{ab}} = -u^{\mathrm{c}}{}_{\mathrm{a}}h_{\mathrm{cb}} - h_{\mathrm{ac}}u^{\mathrm{c}}{}_{\mathrm{b}} \ , \end{aligned} \tag{6.8}$$

where u is the infinitesimal parameter, and the remaining fields and couplings of the theory remain inert.

In quantum theory, it is practical to introduce a frame on the tangent bundle of the spacetime. This is because it is convenient to express the quantum fiield in a frame basis when computing the effective action using the background field method, see section 6.1.3 for more details. In such a case, if we write the metric as $g_{\mu\nu} = \eta_{AB}\mathbf{e}_\mu^A\mathbf{e}_\nu^B$, then the action of infinitesimal spacetime frame rotations will be

$$\delta_\ell \mathbf{e}_\mu^A = \ell^A{}_B\mathbf{e}_\mu^B \ , \quad \delta_\ell\omega_\mu{}^A{}_B = -\partial_\mu\ell^A{}_B + \ell^A{}_C\,\omega_\mu{}^C{}_B - \omega_\mu{}^A{}_C\,\ell^C{}_B \ , \tag{6.9}$$

where ℓ is the infinitesimal parameter and ω is a frame connection of the tangent bundle which we shall always assume preserves the spacetime metric. Of course, ω transforms under diffeomorphisms as Ω in (6.7), as well as the spacetime (co-)frame $\mathbf{e}^A$.

The commutator of two gauge transformations (6.8) is $[\delta_u, \delta_{u'}] = \delta_{[u,u']}$, where $[\cdot,\cdot]$ is the usual commutator of two matrices. A similar result holds for the commutator of two spacetime frame rotations (6.9). In addition to these, there is a gauge symmetry

$$\delta b_{\mu\nu} = (dm)_{\mu\nu} \ , \tag{6.10}$$

associated with 2-form gauge potential, where m is a 1-form on the spacetime.

[2]Typically, one sets $w(v) = 0$ as $w(v)$s generate gauge transformations which are independent sigma model symmetries of the system.

6.1.2 Holonomy symmetries

Consider the infinitesimal transformation generated by the vector ℓ-form, L,on the sigma model target space.

$$\delta_L X^\mu = a_L L^\mu{}_{\lambda_1\dots\lambda_\ell} D_+X^{\lambda_1}\dots D_+X^{\lambda_\ell} \equiv a_L L^\mu{}_L D_+X^L \ , \quad \Delta_L\psi_-^{\rm a} = 0 \ , \tag{6.11}$$

where a_L is the parameter chosen such that $\delta_L X^\mu$ is even under Grassmannian parity, the index L is the multi-index $L = \lambda_1 \dots \lambda_\ell$ and $D_+X^L = D_+X^{\lambda_1}\cdots D_+X^{\lambda_\ell}$. The action (6.1) is invariant under such transformations [106, 131] if L is an $(\ell+1)$-form, after lowering the vector index with the metric g, and

$$\hat\nabla_\nu L_{\lambda_1\dots\lambda_{\ell+1}} = 0 \ , \quad F_{\nu[\lambda_1}L^\nu{}_{\lambda_2\dots\lambda_{\ell+1}]} = 0 \ . \tag{6.12}$$

The second condition[3] above has appeared in [136] and generalises the condition required for the model to admit (2,0) worldsheet supersymmetry, see [147]. Moreover, the parameter a_L satisfies $\partial_= a_L = 0$, i.e. $a_L = a_L(x^{\ddagger}, \theta^+)$. It is straightforward to observe that for $\ell = 0$, $L = K$ is a Killing vector field and $i_K H = dK$. As H is a closed 3-form, $\mathcal{L}_K H = 0$.

The existence of $\hat\nabla$-covariantly constant forms implies the reduction[4] of the holonomy group of $\hat\nabla$ to a proper subgroup of SO. This occurs in several scenarios that include string compactifications on special holonomy manifolds. The geometry of all supersymmetric heterotic backgrounds has recently been shown in [143, 144] to be characterised by the reduction of the holonomy group of the $\hat\nabla$ connection to a subgroup of $SO(9,1)$. As a result of this reduction, $\hat\nabla$-covariantly constant forms on the spacetime which are constructed as Killing spinor bilinears exist. In addition, the second condition in (6.12) is a consequence of the gaugino Killing spinor equation. Therefore, the action (6.1) exhibits symmetries associated with all these covariantly constant forms which will be investigated below.

The commutator of two transformations (6.11) on the field X has been explored in detail in [134]. Here, we shall summarise some of the formulae and emphasise some differences as some assumptions made previously on the algebraic properties of the $\hat\nabla$-covariantly constant tensors are not valid for those of general supersymmetric heterotic backgrounds. Moreover, we shall describe the commutator of the transformations on the field ψ. The commutator of two transformations (6.11) on the field X generated by the vector ℓ-form L and the vector m-form M can be written as

$$[\delta_L, \delta_M]X^\mu = \delta^{(1)}_{LM}X^\mu + \delta^{(2)}_{LM}X^\mu + \delta^{(3)}_{LM}X^\mu \ , \tag{6.13}$$

with

$$\delta^{(1)}_{LM}X^\mu = a_M a_L N(L,M)^\mu{}_{LM} D_+X^{LM} \ , \tag{6.14}$$

$$(\delta^{(2)}_{LM}X)_\mu \ = \ \big(- m a_M D_+ a_L (L\cdot M)_{\nu L_2, \mu M_2}$$

[3]This can also be written as $i_L F = 0$, where i_L is the inner derivation of F with respect to the vector ℓ-form L. In general, the inner derivation of a p-form, P, with respect to L is $i_L P = \frac{1}{\ell!(p-1)!} L^\nu{}_L P_{\nu P_2} dx^{LP_2}$, where $P_2 = \mu_2 \dots \mu_p$.

[4]The structure group of the spacetime reduces as well. The structure group is a subgroup of the holonomy group of any connection.

$$+\ell(-1)^{(\ell+1)(m+1)}a_L D_+ a_M (L\cdot M)_{\mu L_2,\nu M_2}\Big) D_+ X^{\nu L_2 M_2} \ ,\tag{6.15}$$

and

$$(\delta^{(3)}_{LM}X)_\mu = -2i\ell m(-1)^\ell a_M a_L (L\cdot M)_{(\mu|L_2|,\nu)M_2}\partial_{\ddagger}X^\nu D_+ X^{L_2M_2} \ ,\tag{6.16}$$

where

$$(L\cdot M)_{\lambda L_2,\mu M_2} = L_{\rho\lambda[L_2}M^\rho{}_{|\mu|M_2]} \ ,\tag{6.17}$$

and

$$\begin{aligned} N_{\mu LM}dx^{LM} &= \Big(- H_{\mu\nu\rho}L^\nu{}_L M^\rho{}_M + mL^\nu{}_L H_{\nu\mu_1}{}^\rho M_{\mu\rho M_2} - \ell M^\nu{}_M H_{\nu\lambda_1}{}^\rho L_{\mu\rho L_2} \\ &\quad + \ell m H^\rho{}_{\lambda_1\mu_1}(L\cdot M)_{(\mu|L_2|,\rho)M_2}\Big)\, dx^{LM} \\ &= \Big(- (\ell+m+1)H_{[\mu|\nu\rho|}L^\nu{}_L M^\rho{}_{M]} + \ell m H^\rho{}_{\lambda_1\mu_1}(L\cdot M)_{(\mu|L_2|,\rho)M_2}\Big)\, dx^{LM} \end{aligned}\tag{6.18}$$

The multi-index M stands for $M = \mu_1 \dots \mu_m$ while the multi-indices L_2 and M_2 stand for $L_2 = \lambda_2 \dots \lambda_\ell$ and $M_2 = \mu_2 \dots \mu_m$, respectively. The tensor $N(L, M)$ is the Nijenhuis tensor of the vector-forms L and M which has been rewritten using that L and M are $\hat\nabla$-covariantly constant. This concludes the description of the commutator of two holonomy symmetries. The conserved current of a symmetry generated by the $(\ell + 1)$-form L is

$$J_L = L_{\mu_1\dots\mu_{\ell+1}} D_+ X^{\mu_1\dots\mu_{\ell+1}} \ .\tag{6.19}$$

In can be easily seen that $\partial_= J_L = 0$ subject to field equations (6.5).

The investigation of the commutator (6.13), requires to include the symmetries generated by the vector (q+1)-form

$$S^\mu{}_{\nu\rho_1\dots\rho_q} = g^{\mu\lambda}S_{\lambda,\nu\rho_1\dots\rho_q} = \delta^\mu{}_{[\nu}Q_{\rho_1\dots\rho_q]} \ .\tag{6.20}$$

It turns out that if Q is a $\hat\nabla$-parallel q-form and $i_Q F = 0$, i.e. it satisfies (6.12), one can show that the infinitesimal transformation

$$\begin{aligned} \delta_S X_\mu &= \alpha_S \hat\nabla_+ D_+ X^\nu S_{\nu,\mu Q} D_+ X^Q + \frac{(-1)^q}{q+1}\hat\nabla_+(\alpha_S S_{\mu,\nu Q} D_+ X^{\nu Q}) \\ &- \frac{q+3}{3(q+1)}\alpha_S H_{[\mu\nu\rho}Q_{Q]} D_+ X^{\nu\rho Q} \ , \\ \Delta\psi^{\rm a}_- &= -\frac{(-1)^q}{q+1}\alpha_S Q_Q F_{\mu\nu}{}^{\rm a}{}_{\rm b}\psi^{\rm b}_- D_+ X^{Q\mu\nu} \ , \end{aligned}\tag{6.21}$$

is a symmetry of the action. To prove the invariance of the action (6.1) under the transformations (6.21), the Bianchi identity below is needed. This identity will be important later for the analysis of the anomaly consistency conditions.

$$\hat R_{\mu[\nu,\rho\sigma]} = -\frac{1}{3}\hat\nabla_\mu H_{\nu\rho\sigma} \ ,\tag{6.22}$$

where H is taken to be a closed form, $dH = 0$. The associated conserved current is TJ_Q, where T is the right-handed (super) energy-momentum tensor

$$T_{+\ddagger} = g_{\mu\nu}D_+X^\mu\hat\nabla_+ D_+ X^\nu - \frac{1}{3}H_{\mu\nu\rho}D_+X^{\mu\nu\rho} \ ,\tag{6.23}$$

which does not depend on the left-handed fermionic superfield ψ. It can be easily demonstrated using (6.22) that $\partial_= T = 0$ subject to the field equations (6.5). Note that the infinitesimal transformation generated by T is

$$\delta_T X^\mu = 2i\alpha_{{}_T}\partial_{\ddagger} X^\mu + D_+\alpha_{{}_T} D_+ X^\mu \ , \quad \Delta_T \psi_-^{\mathrm{a}} = -\alpha_{{}_T} F_{\mu\nu}{}^{\mathrm{a}}{}_{\mathrm{b}} \psi_-^{\mathrm{b}} D_+ X^{\mu\nu} \ , \tag{6.24}$$

which is a right-handed worldsheet translation followed by a right-handed supersymmetry transformation. The proof that the action (6.1) is invariant under (6.24) again requires the use of the Bianchi identity (6.22).

To proceed, the commutator (6.13) has to be re-organised as a sum of variations where each variation is independently a symmetry of the action. Such a re-organisation has first been proposed for the models with $H = 0$ in [148] and later in the models with $H \neq 0$ in [134]. However, the $\hat{\nabla}$-covariantly constant forms of general supersymmetric heterotic backgrounds that we shall be discussing, do not satisfy the necessary conditions mentioned in these references. Locally, all supersymmetric heterotic backgrounds with compact holonomy group are fibrations with fibre a group manifold[5] [143, 144, 107]. The conditions mentioned in [148, 134] fail along the fibre directions but still apply provided that the right-hand-side of the commutator (6.13) is restricted along orthogonal directions to those of the fibres. Because of this, the formulae in [134] are still useful for the analysis. Eventually, the whole commutator (6.13) of holonomy symmetries generated by the $\hat{\nabla}$-covariantly constant forms of general supersymmetric heterotic backgrounds can be rewritten as a sum of symmetries. But as we shall demonstrate, this will require the addition of new generators which will be investigated on a case-by-case basis.

Next, let us consider the commutator of two (6.11) transformations on the field ψ. As $\Delta_L \psi = \Delta_M \psi = 0$, one finds that

$$[\delta_L, \delta_M]\psi_-^{\mathrm{a}} = -\Omega_\mu{}^{\mathrm{a}}{}_{\mathrm{b}}[\delta_L, \delta_M]X^\mu \psi^{\mathrm{b}} - F_{\mu\nu}{}^{\mathrm{a}}{}_{\mathrm{b}} \delta_L X^\mu \delta_M X^\nu \psi_-^{\mathrm{b}} \ . \tag{6.25}$$

Therefore, the commutator may give rise to a non-trivial transformation, Δ_{LM}, on ψ given by

$$\Delta_{LM}\psi_-^{\mathrm{a}} = -F_{\mu\nu}{}^{\mathrm{a}}{}_{\mathrm{b}} \delta_L X^\mu \delta_M X^\nu \psi_-^{\mathrm{b}} \ . \tag{6.26}$$

Such a transformation, which appears on the right-hand side of the commutator of two holonomy transformations, may have been expected for consistency as the invariance of the action under (6.24) and (6.21) require such a contribution.

For the analysis of anomalies described below, it is necessary to find the commutators of sigma model and holonomy symmetries. As it can always be arranged for the diffeomorphism sigma model symmetries not to be anomalous up to possibly the addition of a finite local counterterm in the effective action [101], it remains to describe the commutator of gauge transformations (6.8) and (6.9) with the holonomy symmetries (6.11). It is straightforward to see that they commute

$$[\delta_\ell, \delta_L] = [\delta_u, \delta_L] = 0 \ , \tag{6.27}$$

on both X and ψ fields.

[5]The spacetime of these backgrounds is a principal bundle, P, with fibre a Lorentzian Lie group, G, and base space, B, a suitable Riemannian manifold with skew-symmetric torsion. In general, the total space, E, may not be necessarily of product form, $G \times B$, either topologically or metrically.

6.1.3 Anomaly consistency conditions

The background field method [149, 150, 151] allows to retain most of the geometric properties of sigma models in the quantum theory. This involves the splitting of the (total) field of the theory into a background field, that is treated classically, and a quantum field that is quantised. However, the understanding of the quantum theory presents several challenges as this splitting is non-linear for sigma models. These challenges include the non-linear split symmetry [152] needed to control the counterterms to correctly subtract the ultraviolet infinities and determine the effective action, Γ, from the 1PI diagrams with only external background lines[6]. After writing the theory in terms of background and quantum fields and considering the effective action constructed from 1PI diagrams with external background lines only, some of the symmetries of the theory, like those of spacetime frame rotations and gauge sector transformations, act linearly on the quantum fields, for a detailed discussion see [154]. Investigation of such symmetries is significantly simpler. Unfortunately, this does not apply to the holonomy symmetries, where the induced transformations on the quantum fields are non-linear, and a much more in-depth analysis is required [148]. To proceed, we shall consider the effective action, Γ, computed from 1PI diagrams with only external background lines. Then, after stating the spacetime frame rotations and gauge transformation anomalies, which we take to be expressed in terms of the background fields, we shall use Wess-Zumino consistency conditions to determine the anomalies of the holonomy symmetries. From now on, it will be assumed that all fields that are included in the expression for the anomalies as well as those that appear in the various transformations required for the investigation are the background fields.

Suppose that the classical theory is invariant under the symmetry algebra whose variations on the fields satisfy the commutation relations

$$[\delta_A, \delta_B] = \delta_{[A,B]} \ , \tag{6.28}$$

where δ_A (δ_B) is a transformation on the (background) fields generated by A (B) generator with parameter $a_{_A}$ ($a_{_B}$) and $[A, B]$ is the commutator of the two generators. If these symmetries are anomalous in the quantum theory, i.e. $\delta_A\Gamma = \Delta(a_{_A})$, then applying the commutator (6.28) on Γ, one finds that

$$\delta_A\Delta(a_{_B}) - \delta_B\Delta(a_{_A}) = \Delta(a_{_{[A,B]}}) \ . \tag{6.29}$$

These relations between anomalies are known as Wess-Zumino anomaly consistency conditions[7]. A solution of these conditions will yield an expression for the anomaly of a symmetry in terms of the fields.

It is well known that the anomaly associated with the gauge transformations (6.9) is determined by the descent equations [101] starting from a 4-form, $P_4(R) = \mathrm{tr}(R(\omega) \wedge R(\omega))$, which is proportional to the first Pontryagin form of the manifold, where R is the curvature of a connection ω. As this is closed, one can locally write $P_4(R) = dQ^0_3(\omega)$, where Q^0_3 is the Chern-Simon form. As P_4 is invariant under the gauge transformations (6.9), one has that $d\delta_\ell Q^0_3(\omega) = 0$ and so $\delta_\ell Q^0_3(\omega) = dQ^1_2(\ell, \omega)$. The gauge anomaly[8] is

[6] Potential anomalies in the shift symmetry have been examined in [153].

[7] It is customary in the investigation of Wess-Zumino consistency conditions for anomalies to use the BRST formalism. We shall not do this here. Instead, we shall use the commutators as these emphasise the geometry structure of the theory.

[8] For applications to string theory, replace in the formulae below $\hbar$ with α'.

given by

$$\Delta(\ell) = \frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ Q_2^1(\omega, \ell)_{\mu\nu} D_+ X^\mu \partial_= X^\nu \ , \tag{6.30}$$

where the numerical coefficient in front is determined after an explicit computation of the term in the effective action that contributes to the anomaly. A similar calculation reveals that the anomaly of the gauge transformation (6.8) is

$$\Delta(u) = -\frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ Q_2^1(\Omega, u)_{\mu\nu} D_+ X^\mu \partial_= X^\nu \ . \tag{6.31}$$

The connection that appears in the expressions for the anomalies is not uniquely defined since it can be altered upon adding a finite local counterterm in the effective action [101]. Later, this flexibility in choosing the connection that appears in the expressions for the anomaly will be used to demonstrate the consistency of Wess-Zumino conditions. Furthermore, notice that despite the Chern-Simons form having a standard expression, Q_3^0 is specified up to an exact form, $Q_3^0 \to Q_3^0 + dW$. It turns out that this can be used to cancel some of the anomalies by adding an appropriate finite local counterterm constructed from W in the effective action.

As the commutator of gauge symmetries with the holonomy symmetries vanishes (6.27), the anomaly consistency conditions (6.29) in this case imply that

$$\delta_\ell \Delta(a_{_L}) - \delta_L \Delta(\ell) = 0 \ , \tag{6.32}$$

and similarly for the gauge transformations (6.8), where $\Delta(a_{_L})$ is the anomaly of the holonomy symmetry generated by L. A solution to both consistency conditions is

$$\Delta(a_{_L}) = \frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ \, Q_3^0(\omega, \Omega)_{\mu\nu\rho} \delta_L X^\mu D_+ X^\nu \partial_= X^\rho + \Delta_{\rm inv}(a_{_L}) \ , \tag{6.33}$$

up to possibly δ_ℓ- and δ_u-invariant terms, $\Delta_{inv}(a_{_L})$, where $Q_3^0(\omega, \Omega) = Q_3^0(\omega) - Q_3^0(\Omega)$. This form of the holonomy symmetry anomaly is also consistent with the second commutator in (6.27). From here on, we shall take $\Delta_{inv}(a_{_L}) = 0$.

In the case that L is a Killing vector field K, $L = K$ such that $i_K P_4 = 0$, one has that $\mathcal{L}_K P_4 = 0$ and so $d\mathcal{L}_K Q_3^0 = 0$. Thus $\mathcal{L}_K Q_3^0(\omega) = dQ_2^1(a_{_K}, \omega)$. Then it is straightforward to show that the anomaly can be wirtten as

$$\Delta(a_K) = \frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ Q_2^1(\omega, a_K)_{\mu\nu} D_+ X^\mu \partial_= X^\nu \ . \tag{6.34}$$

$\Delta(a_K)$ satisfies the consistency conditions that arise from the commutator of isometries of the sigma model target space.

It remains to investigate the consistency of (6.33) with respect to the commutators of holonomy symmetries. Consider two holonomy symmetries generated by the forms L and M. After a direct computation, one finds that

$$\begin{aligned}\delta_L \Delta(a_{_M}) - \delta_M \Delta(a_{_L}) &= \frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ P_4(R, F)_{\mu\nu\rho\sigma} \delta_L X^\mu \delta_M X^\nu D_+ X^\rho \partial_= X^\sigma \\ &+ \frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ Q_3^0(\omega, \Omega)_{\mu\nu\rho} [\delta_L, \delta_M] X^\mu D_+ X^\nu \partial_= X^\rho \ ,\end{aligned} \tag{6.35}$$

where $P_4(R,F) = P_4(R) - P_4(F)$. It turns out that the above consistency condition is more general. If the anomaly of two transformations δ_1 and δ_2 is given as in (6.33), then their mutual consistency condition will be given as in (6.35) with $\delta_L = \delta_1$ and $\delta_M = \delta_2$.

A comparison of (6.35) with the anomaly consistency condition (6.29) suggests the terms that contain $P_4(R,F)$ in the right-hand side of (6.35) may potentially give rise to an inconsistency. In fact, the consistency condition is more subtle as it also depends on whether the individual symmetries that appear on the right-hand side of the commutator $[\delta_L, \delta_M]$ are anomalous. If they are not anomalous, then the whole right-hand side of (6.35) must vanish for consistency. It is clear that the study of these consistency conditions depends on the symmetries that arise in the commutator $[\delta_L, \delta_M]$. This in turn depends on the geometry of the heterotic backgrounds, particularly the details of the properties of L and M forms.

6.1.4 Anomaly cancellation and consistency conditions revisited

The number of Killing spinors and the holonomy of $\hat{\nabla}$ for heterotic backgrounds are thought to be preserved in some form under quantum corrections to possibly all orders in perturbation theory. There are two scenarios of how this can happen. Let us focus on the anomaly cancellation at one loop to explain them. We shall comment in the concluding remarks about anomaly cancellation in higher orders. First, one may expect that finite local counterterms could remove the anomalies. This is mostly the case whenever there is a renormalisation scheme that prevents L, the symmetry generator, from receiving quantum corrections. To illustrate this, consider a spacetime $\mathbb{R}^k \times N^{10-k}$ and $L = I$ a complex structure on N^{10-k}. In this scenario, the part of the sigma model action on N^{10-k} has a second supersymmetry generated by I and the theory is (2,0) supersymmetric. In complex coordinates that I is constant, the (2,0) supersymmetry transformations are linear in the fields. Therefore, one should not expect these transformations to be corrected in the quantum theory. Furthermore, since the perturbation theory can be set up using (2,0) superfields, one should not expect a (2,0) supersymmetry anomaly. These manifestly preserve the symmetry. Thus, there must be a renormalisation scheme that manifestly preserves the (2,0) supersymmetry. Indeed, it has been shown that if the perturbation theory is set up in (1,0) superfields, then the anomaly (6.33) of the symmetry generated by I is cancelled by a finite local counterterm [142, 154].

Secondly, it is plausible that L will receive quantum corrections. This scenario is consistent with the fact that the Killing spinor equations of heterotic supergravity retain their form up to and including two loops[9] in the sigma model perturbation theory [146]. These quantum corrections require to replace H with $H^\hbar$. Let us denote the quantum corrected L with $L^\hbar$. Then note that (6.33) in terms of $L^\hbar$ can be rewritten as

$$\begin{aligned}
&\delta^\hbar_L \Gamma = \delta^\hbar_L(\Gamma^{(0)} + \hbar \Gamma^{(1)}) = \Delta_L(a_L) \Longrightarrow \\
&\quad -i \int d^2\sigma d\theta^+ \big(a_L \frac{2(-1)^\ell}{\ell+1} \hat{\nabla}^\hbar_\mu L^\hbar_{L+1} \partial_= X^\mu D_+ X^{L+1} - i a_L L^{\hbar\mu}{}_L F^\hbar_{\mu\nu \mathrm{ab}} \psi^{\mathrm{a}} \psi^{\mathrm{b}} D_+ X^{L\nu} \\
&\quad + 2i \Delta^\hbar_L \psi^{\mathrm{a}}_- \mathcal{D}^\hbar_+ \psi_{-\mathrm{a}} \big) = 0 + \mathcal{O}(\hbar^2) \ ,
\end{aligned} \tag{6.36}$$

[9] Therefore, from the supergravity perspective, the anomalies cancel up to and including two loops in the sigma model perturbation theory.

where $\hat{\nabla}^\hbar$ is the quantum corrected connection[10] with skew-symmetric torsion

$$H^\hbar = H - \frac{\hbar}{4\pi} Q_3^0(\omega, \Omega) + \mathcal{O}(\hbar^2) \ . \tag{6.37}$$

Similarly, $\mathcal{D}^\hbar$ is the quantum corrected connection of the gauge sector and for the holonomy symmetries $\Delta_L^\hbar \psi = 0$. Nevertheless, as we shall see later, this term must be added as it contributes to the commutator of two holonomy symmetries. Observe that $dH^\hbar = -\frac{\hbar}{4\pi} P_4(\omega, \Omega)$, i.e is not closed. It is evident that the anomaly is cancelled, $\delta_L^\hbar \Gamma = 0 + \mathcal{O}(\hbar^2)$, provided that $L^\hbar$ is covariantly constant with respect to $\hat{\nabla}^\hbar$, $\hat{\nabla}^\hbar L^\hbar = 0$, and $i_{L^\hbar} F^\hbar = 0$, i.e. the second condition in (6.12) is satisfied with $F = F^\hbar$ and $L = L^\hbar$.

Note that the correction of H as in (6.37) is also required to restore the tensorial properties of the 3-form coupling H. A non-trivial transformation on H at one loop arises as a consequence of the anomalous variations to b [140] at the same loop order, $\delta_\ell b = \frac{\hbar}{4\pi} Q_2^1(\ell, \omega)$ and $\delta_u b = -\frac{\hbar}{4\pi} Q_2^1(u, \Omega)$, which are needed to cancel the frame rotations and gauge anomalies, respectively. The transformation of H is cancelled in $H^\hbar$ by the zeroth order variation of the Chern-Simons term. Hence $H^\hbar$ is invariant under such a transformation up to order $\mathcal{O}(\hbar^2)$. This appropriately persists to all loop orders [142, 154].

The cancellation of holonomy anomalies described above is also consistent with the corrections to the heterotic supergravity up and including two loops in the sigma model perturbation theory. It is known that the Killing spinor equations of the theory remain unaltered to this loop order provided one replaces the 3-form field strength H with $H^\hbar$ [146]. The Killing spinors ϵ are consequently parallel with respect to the spin connection of $\hat{\nabla}^\hbar$, $\hat{\nabla}^\hbar \epsilon = 0$. Thus, the Killing spinor bilinears can automatically be identified with $L^\hbar$ and they are covariantly constant with respect to $\hat{\nabla}^\hbar$. Finally, the gaugino Killing spinor equation of the theory implies that $i_{L^\hbar} F^\hbar = 0$.

To continue, let us review the consistency condition (6.35) considering the implications of corrections on L and M. It can be re-derived by varying (6.36) with $\delta_M^\hbar$ and taking the commutator. After assuming that $i_{L^\hbar} F^\hbar = i_{M^\hbar} F^\hbar = 0$, the final expression can be cast into the form

$$\begin{aligned} -i \int d^2\sigma d\theta^+ \big(& -2 g_{\mu\nu} [\delta_L^\hbar, \delta_M^\hbar] X^\mu \hat{\nabla}^\hbar_{=} D_+ X^\nu + dH^\hbar_{\mu\nu\rho\sigma} \delta_L^\hbar X^\mu \delta_M^\hbar X^\nu D_+ X^\rho \partial_= X^\sigma \\ & + 2i (\Delta^\hbar_{LM} \psi_-^{\mathrm{a}} + F^\hbar_{\mu\nu}{}^{\mathrm{a}}{}_{\mathrm{b}} \psi_-^{\mathrm{b}} \delta_L^\hbar X^\mu \delta_M^\hbar X^\nu) \mathcal{D}_+^\hbar \psi_{-\mathrm{a}} \big) = 0 + \mathcal{O}(\hbar^2) \ , \end{aligned} \tag{6.38}$$

where $\Delta^\hbar_{LM} \psi_-^{\mathrm{a}} \equiv \delta^\hbar_{LM} \psi_-^{\mathrm{a}} + \Omega^{\hbar\,\mathrm{a}}_{\mu\ \mathrm{b}} \delta^\hbar_{LM} X^\mu \psi_-^{\mathrm{b}}$ and it should be read as the covariantisation of the right-hand side of the commutator on ψ (6.25) after it has been decomposed as a sum of individual symmetries.

The symmetries appearing in the right-hand side of the commutator $[\delta_L^\hbar, \delta_M^\hbar]$ will determine how we may proceed. It turns out that, provided our previous assumption that $i_{L^\hbar} F^\hbar = i_{M^\hbar} F^\hbar = 0$, the term in (6.38) involving variations of the field ψ is always satisfied in all cases. In all examples, we shall be considering, the commutator on X will read

$$[\delta_L^\hbar, \delta_M^\hbar] = \delta_N^\hbar + \delta_S^\hbar + \delta_{JP}^\hbar \ , \tag{6.39}$$

[10] It is expected that both g and b are also corrected in quantum theory to $g^\hbar$ and $b^\hbar$, respectively. This is especially the case whenever one searches for a scheme to make the theory manifestly superconformal, i.e. a scheme that the beta function vanishes. So $\hat{\nabla}^\hbar$ should be taken with respect to $g^\hbar$ and $b^\hbar$. But for simplicity in what follows, we shall drop the $\hbar$ superscript from g and b.

where $\delta^{\hbar}_N$ is a symmetry generated by a $\hat{\nabla}^{\hbar}$-covariantly constant form N with parameter a_N constructed from those of $\delta^{\hbar}_L$ and $\delta^{\hbar}_M$, and $\delta^{\hbar}_S$ is a transformation given in (6.21) with a parameter α_S constructed again from those of $\delta^{\hbar}_L$ and $\delta^{\hbar}_M$. Next $\delta^{\hbar}_{JP}$ is a transformation again generated by $\hat{\nabla}^{\hbar}$-covariantly constant forms collectively denoted by P but now with parameters constructed from those $\delta^{\hbar}_L$ and $\delta^{\hbar}_M$ and some conserved currents J of the theory. The structure of $\delta^{\hbar}_{JP}$ will be explained below. We will label, type I, type II, and type III, the three different types of transformations that take place on the right side of a commutator. Naturally, a typical commutator will close to a linear combination of all 3 types of transformations. The consistency condition in each case will be separately treated.

Type I: If a commutator closes to a $\delta^{\hbar}_N$ type of symmetry, then the consistency condition (6.38) is

$$P(\omega,\Omega)_{\mu\nu[\rho|\sigma|}L^{\mu}{}_{L}M^{\nu}{}_{M]} = 0 \ , \tag{6.40}$$

as the first term vanishes because N is $\hat{\nabla}^{\hbar}$ covariantly constant and $\partial_{=}a_N = 0$. Note that the condition is expressed in terms of L and M as $P(\omega,\Omega)$ is first order in $\hbar$.

Type II: If the commutator closes to a $\delta^{\hbar}_S$ symmetry, the first term in (6.38) contributes to the consistency condition since the Bianchi identity (6.22) is involved to prove that a S type symmetry leaves the classical action invariant. Note that the Bianchi identity is modified to

$$\hat{R}_{\mu[\nu,\rho\sigma]} = -\frac{1}{3}\hat{\nabla}_{\mu}H_{\nu\rho\sigma} - \frac{1}{6}dH_{\mu\nu\rho\sigma} \ , \tag{6.41}$$

for $dH \neq 0$. This is the case here as $dH^{\hbar} \neq 0$. Therefore the consistency condition now reads

$$\alpha_S\frac{(-1)^q}{3(q+1)}P(\omega,\Omega)_{\sigma[\rho\lambda\tau}Q_{Q]} + a_M a_L P(\omega,\Omega)_{\mu\nu[\rho|\sigma|}L^{\mu}{}_{L}M^{\nu}{}_{M]} = 0 \ , \tag{6.42}$$

where α_S is expressed in terms of a_L and a_M, and the indices satisfy $\lambda\tau Q = LM$.

Type III: Finally, suppose that a commutator closes to a $\delta^{\hbar}_{JP}$ type of symmetry which has to be analysed as a straightforward modification of a transformation generated by a a covariantly constant form P by allowing the parameter a_P to depend on $\sigma^{=}$, $\partial_{=}a_P \neq 0$, is not always[11] a symmetry of the classical action. Thus, it is not expected to be a symmetry of the quantum theory. However, since the commutator of two symmetries δ_L and δ_M of a classical action is also a symmetry of the theory, δ_{JP} must be a symmetry as well. To achieve this, there should exist L' and M' $\hat{\nabla}$-covariantly constant forms that satisfy (6.12) such that

$$\delta_{JP} = (m'+1)c_{L'}J_{L'}\delta_{M'} + (\ell'+1)c_{M'}J_{M'}\delta_{L'} \ , \tag{6.43}$$

for some constants $c_{L'}$ and $c_{M'}$ and with parameters related[12] as $(-1)^{(\ell'+1)(m'+1)}c_{L'}a_{M'} = c_{M'}a_{L'}$, where $J_{L'}$ is the current associated to L' as in (6.19) and similarly for $J_{M'}$. Indeed, one has that

$$\delta_{JP}S \ = \ -i\int d^2\sigma d\theta^{+}\left(\delta_{JP}X^{\mu}\mathcal{S}_{\mu}\right)$$

[11] Though in some cases, a_P depends on the currents of the theory in such a way that $\delta^{\hbar}_{JP}$ is a symmetry, see SU examples below.

[12] As the transformation δ_{JP} arises in the right hand side of the commutator $[\delta_L,\delta_M]$, the parameters $a_{L'}$ and $a_{M'}$ are expressed in terms of a_L and a_M and so they are related.

$$= \quad -i\int d^2\sigma d\theta^+\big(-2(-1)^{\ell' m'}c_{M'}a_{L'}\partial_=(J_{M'}J_{L'})\big) = 0 \ , \tag{6.44}$$

where we have used the condition on F in (6.12) for both the forms L' and M' Repeating this computation in the quantum theory reveals that the first term in the consistency condition vanishes and (6.40) must be satisfied.

The treatment of type III commutators discussed above can be extended to include cases with currents, J_A and J_B, associated with the symmetries, δ_A and δ_B, respectively, which are not necessarily generated by a $\hat{\nabla}$-covariantly constant form, i.e. $\delta_{JP} = c_B J_A \delta_B + c_A J_B \delta_A$. Note that one may let the parameter a_B to depend on $\sigma^=$ in order to calculate the current for a symmetry generated by the variation δ_B. Then it is known that

$$\delta_B S \sim \int d^2\sigma d\theta^+ \partial_= a_B J_B \ . \tag{6.45}$$

Using the above formula of calculating a current and after replacing a_B with $J_A a_B$, where now $\partial_= a_B = 0$, and similarly for J_A, one finds that

$$\delta_{JP} S \sim \int d^2\sigma d\theta^+ a_A \partial_=(J_A J_B) = 0 \ , \tag{6.46}$$

after an appropriate choice of constants c_A and c_B and a relation amongst the parameters a_A and a_B. However if the Bianchi identity (6.22) is used to arrange the δ_{JP} variation of the action as above, then the consistency of anomalies will require the condition (6.42) instead of (6.40).

6.2 Anomalies and holonomy $SU(2)$ backgrounds

6.2.1 Summary of the Geometry

The spacetime of supersymmetric heterotic backgrounds for which the holonomy of $\hat{\nabla}$ is included in $SU(2)$ admits six $\hat{\nabla}$-parallel 1-forms $\mathbf{e}^a$, $a = 0, \dots, 5$ and three $\hat{\nabla}$-parallel 2-forms I_r such that the Lie bracket algebra of the associated vector fields $\mathbf{e}_a$ to $\mathbf{e}^a$ is a 6-dimensional Lorentzian Lie algebra with self-dual structure constants $H^a{}_{bc}$. As a result $\mathbf{e}_a$ are Killing vector fields[13]. In addition, we have that $\mathcal{L}_{\mathbf{e}_a} H = 0$. Moreover,

$$i_{\mathbf{e}_a} I_r = 0 \ , \quad \mathcal{L}_{\mathbf{e}_a} I_r = 0 \ . \tag{6.47}$$

Furthermore, the endomorphisms (vector 1-forms) I_r, $g(X, I_r Y) = I_r(X, Y)$, satisfy

$$I_r I_s = -\delta_{rs}(\mathbf{1} - \mathbf{e}_a \otimes \mathbf{e}^a) + \epsilon_{rs}{}^t\, I_t \ . \tag{6.48}$$

These backgrounds admit 8 Killing spinors and all these forms arise as Killing spinor bilinears.

The metric and 3-form field strength of the backgrounds can be written as

$$g = \eta_{ab}\mathbf{e}^a \mathbf{e}^b + \tilde{g} \ , \quad H = \frac{1}{3}\eta_{ab}\mathbf{e}^a \wedge d\mathbf{e}^b + \frac{2}{3}\mathbf{e}^a \wedge \mathcal{F}^b + \tilde{H} \ , \tag{6.49}$$

[13]So far, we have used K to denote the Killing vector fields and we shall continue to do so in the analysis of the anomalies that will follow. But here, we stress with $\mathbf{e}^a$ that these 1-forms can be used as part of a pseudo-orthonormal (co-)frame on the spacetime.

where η is the Minkowski space metric and $\tilde{g} = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$ with $\mathbf{e}^i$ an orthonormal (co)-frame orthogonal to $\mathbf{e}^a$. Moreover, $\mathcal{F}^a = d\mathbf{e}^a - \frac{1}{2}H^a{}_{bc}\mathbf{e}^b\wedge\mathbf{e}^c = \frac{1}{2}H^a{}_{ij}\mathbf{e}^i\wedge\mathbf{e}^j$ and $i_{\mathbf{e}_a}\tilde{H} = i_{\mathbf{e}_a}\mathcal{F}^b = 0$. Furthermore, $\mathcal{L}_{\mathbf{e}_a}\tilde{H} = 0$ and $\mathcal{F}^a$ is an (1,1)-form[14] with respect to all three endomorphisms I_r, i.e. $\mathcal{F}(I_rX, I_rY) = \mathcal{F}(X,Y)$ (no summation over r).

The Killing spinor equations also restrict the curvature of connection, F, of the gauge sector. The conditions are that $i_{\mathbf{e}_a}F = 0$ and $F(I_rX, I_rY) = F(X,Y)$ (no summation over r). Therefore, F is anti-self dual in the directions orthogonal to the orbits of the isometry group.

The geometry of such spacetime, M^{10}, locally can be modelled as that of a principal bundle over and HKT 4-dimensional manifold N^4 with metric $\tilde{g}$ and torsion $\tilde{H}$, principal bundle connection $\mathbf{e}^a$ whose curvature is $\mathcal{F}$ and fibre a group manifold, G, whose (Lorentzian) Lie algebra is $\mathbb{R}^6$, $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)$ or $\mathfrak{cw}_6$ with self-dual structure constants. Moreover N^4 is conformally hyper-Kähler, i.e there is a hyper-Kähler metric on N^4, $\mathring{g}$, such that $\tilde{g} = e^{2\Phi}\mathring{g}$, where Φ is the dilaton. The hypercomplex structure on N^4 is spanned by the three endomorphisms $\tilde{I}_r$ and the associated Kähler forms $\mathring{I}_r$ are closed. For more details on the geometry of supersymmetric heterotic backgrounds with $SU(2)$ holonomy, see [143, 144, 107]. Note that M^{10} may not be a product $G\times N^4$ either topologically or metrically. The curvature $\mathcal{F}$ may not be zero.

6.2.2 $SU(2)$ holonomy symmetries and their commutators

The symmetries generated by the form bilinears $\mathbf{e}_a$ and I_r are

$$\delta_K X^\mu = a^a_{{}_K}\mathbf{e}^\mu_a\ , \qquad \delta_I X^\mu = a^r_I (I_r)^\mu{}_\nu D_+X^\nu\ , \tag{6.50}$$

with $\Delta_K\psi = \Delta_I\psi = 0$. Let us clarify our notation, we use the pseudo-orthonormal frame $(\mathbf{e}^A) = (\mathbf{e}^a, \mathbf{e}^i)$ and define $\delta X^A = \mathbf{e}^A_\mu \delta X^\mu$, $D_+X^A = \mathbf{e}^A_\mu D_+X^\mu$ and $\partial_{\mp}X^A = \mathbf{e}^A_\mu\partial_{\mp}X^\mu$. In this notation $\delta_K X^a = a^a_{{}_K}$ and $\delta_I X^i = a^r_I(I_r)^i{}_jD_+X^j$ with all the other components of the variations to vanish. The closure of the algebra of these transformations requires the following symmetry[15]

$$\delta_C X^a = \alpha_{{}_C}\hat{\nabla}_+ D_+X^a + \hat{\nabla}_+(\alpha_{{}_C} D_+X^a)\ , \tag{6.51}$$

with $\delta_C X^i = \Delta_C\psi = 0$. The δ_C symmetry is associated with the quadratic Casimir operator of the Lie algebra of isometries and the conserved current is

$$C = \eta_{ab}\mathbf{e}^a_\mu\mathbf{e}^b_\nu D_+X^\mu\hat{\nabla}_+D_+X^\nu\ . \tag{6.52}$$

It is straightforward to verify using the algebraic properties of the form bilinears that

$$\begin{aligned}
&[\delta_K, \delta'_K]X^\mu = a^a_{{}_K} a'^b_{{}_K}[\mathbf{e}_a, \mathbf{e}_b]^\mu = -a^a_{{}_K}a'^b_{{}_K}H_{ab}{}^c\mathbf{e}^\mu_c = \delta''_K X^\mu\ ,\\
&[\delta_K, \delta_I]X^\mu = a^a_{{}_K}a^r_I(\mathcal{L}_{\mathbf{e}^a}I_r)^\mu{}_\nu D_+X^\nu = 0\ ,
\end{aligned} \tag{6.53}$$

where $(a''_{{}_K})^c = -a^a_{{}_K}a'^b_{{}_K}H_{ab}{}^c$.

[14] Here, we have adopted the terminology of hypercomplex geometry to assign a holomorphic and anti-holomorphic degree for forms even though I_r is not a hypercomplex structure over the whole spacetime.

[15] Although the variations of the symmetries on the fields are given in frame indices, for the computations of the commutators below it is convenient to re-express them in spacetime indices as those in section 6.1.2.

It remains to compute the commutator of two symmetries generated by I_r. To carry out the above computation, it is helpful to notice that $N(I_r, I_s)^i{}_{jk} = 0$. After some computation, one finds that

$$[\delta_I, \delta'_I] = \delta_T + \delta_C + \delta_K + \delta''_I \ , \tag{6.54}$$

where $\alpha_{_T} = a'^s_I a^r_I \delta_{rs}$, $\alpha_{_C} = -a'^s_I a^r_I \delta_{rs}$, $a^a_{_K} = a'^s_I a^r_I \delta_{rs} H^a{}_{bc} J^b_K J^c_K$ and $a''^t_I = -(a'^s_I D_+ a^r_I + a^r_I D_+ a'^s_I)\epsilon_{rs}{}^t$. Notice that the parameter of the transformation δ_K in the right-hand side of the commutator depends quadratically on the currents J_K associated with isometries. Nevertheless, δ_K with the above field-dependent parameter is a symmetry of the theory because it can be rewritten as

$$\delta_{\bar{H}} X^a = a_{_{\bar{H}}} H^a{}_{bc} D_+ X^{bc} \ , \tag{6.55}$$

for some parameter $a_{_{\bar{H}}}$, $\partial_= a_{_{\bar{H}}} = 0$, generated by

$$\bar{H} = \frac{1}{3!} H_{abc} \mathbf{e}^{abc} \ . \tag{6.56}$$

$\bar{H}$ is a $\hat{\nabla}$-covariantly constant form as a consequence of the Bianchi identity (6.22) and the $SU(2)$ holonomy of $\hat{\nabla}$. In principle, we could have introduced (6.55) as an independent symmetry and compute its commutators. The advantage of this approach would be a standard Lie algebra as the symmetry algebra of the sigma model, instead of the W-type of algebra with current dependent structure constants that emerges in (6.54). Initially, we followed this path, but we later came to the conclusion that it would be more economical for the presentation that follows not to consider (6.55) as an independent symmetry and express the commutator of two δ_I transformations as (6.54).

The commutator of two δ_I transformations on ψ (6.25) can be expressed as (6.54), where

$$\Delta_{II} \psi^{\mathrm{a}}_- = \Delta_T \psi^{\mathrm{a}}_- = -a'^s_I a^r_I \delta_{rs} F_{\mu\nu}{}^{\mathrm{a}}{}_{\mathrm{b}} \psi^{\mathrm{b}}_- DX^{\mu\nu} \ , \tag{6.57}$$

and $\Delta_C \psi = \Delta''_I \psi = 0$.

The remaining commutators are

$$[\delta_C, \delta_I] = 0 \ , \quad [\delta_K, \delta_C] = \delta'_K \ , \quad [\delta_C, \delta'_C] = \delta''_C + \delta''_K \ , \tag{6.58}$$

where $a'^a_K = 2i\alpha_C \partial_{\ddagger} a^a_K + D_+\alpha_C D_+ a^a_K + 2H^a{}_{bc}\alpha_C D_+ a^b_K J^c_K$, $a''_{\bar{H}} = \alpha''_C = D_+\alpha'_C D_+\alpha_C + 2i(\alpha'_C \partial_{\ddagger}\alpha_C - \alpha_C \partial_{\ddagger}\alpha'_C)$ and $a''^a_K = \alpha''_C H^a{}_{bc} J^b_K J^c_K$. Note again that the parameters of the δ'_K and δ''_K transformations are field dependent via the currents of the theory and so the algebra of symmetries is a W-algebra. As we have explained the field dependent part of the δ''_K transformation is a symmetry of the action as it can be interpreted as a $\delta_{\bar{H}}$ symmetry. Similarly, for the field dependent part of δ'_K transformation since $\delta_{\vec{H}} X^a = a^b_{\vec{H}} H^a{}_{bc} D_+ X^c$ is a symmetry of the sigma model action generated by the $\hat{\nabla}$-covariantly constant 2-forms $\vec{H}_a = -\frac{1}{2} H_{abc} \mathbf{e}^{bc}$. But as we have already mentioned, we have not proceeded in this way. Note also that for all transformations in (6.58) $\Delta\psi = 0$ and $\Delta_{CI} = \Delta_{KC} = \Delta_{CC} = 0$. Below, we shall demonstrate the symmetries generated by K and C are not anomalous as their anomalies can be cancelled with the addition of a finite local counterterm in the effective action of the theory.

6.2.3 Anomalies and consistency conditions

The analysis of the anomalies of these models is similar to that of the standard (2,0)-supersymmetric chiral sigma models in [142, 154]. However, there are some key differences. There are potentially anomalous isometries present in this class of sigma models. In addition, I is not associated with a second supersymmetry as it is not a complex structure over the whole spacetime – the commutator of two transformations generated by I given in (6.54) also reflects this fact. Nevertheless, the analysis presented in [154] for (2,0)-supersymmetric sigma models can be appropriately adjusted to apply to this case as follows.

So far, the connection on M^{10} that contributes in the anomalies (6.30) and (6.33) has been arbitrary. From now on, we shall set $\omega = \breve{\omega}$, where $\breve{\omega}$ is the frame connection associated with $\breve{\nabla}$, whose torsion is $-H$. It is well known that if $dH = 0$, then $\hat{R}_{\mu\nu,\rho\sigma} = \breve{R}_{\rho\sigma,\mu\nu}$. Observe that $i_K P_4(\breve{R}, F) = 0$ as a consequence of the holonomy of $\hat{\nabla}$ being contained in $SU(2)$, the curvature 2-form $\breve{R}$ satisfies $i_K\breve{R} = 0$ and $\breve{R}(I_rX, I_rY) = \breve{R}(X, Y)$ (no summation over $r = 1, 2, 3$). This in turn implies that $i_K P_4(\breve{R}) = 0$ and as a result of the gaugino KSE, $i_K P_4(F) = 0$.

The potential anomalies associated with δ_K, δ_I, and δ_C are given as in (6.33) with δ_L replaced with the symmetry under investigation. This is the case as those transformations commute with both frame rotations δ_ℓ and δ_u gauge transformations.

The commutators of δ_K with δ_K, δ_I and δ_C either vanish or close to a type I and a type III transformation, see (6.53) and (6.58). The type III transformation is a symmetry with the current dependent parameter stated. The condition (6.40) required for the consistency of the anomalies is satisfied as a result of $i_K P_4(\breve{R}, F) = 0$. Similarly, for the commutator of δ_C with δ_I in (6.58).

The commutator of two δ_C transformations in (6.58) closes to a type II and a type III transformation generated by C and K, respectively. The latter is a symmetry regardless of the current dependent parameters given in (6.54). It is straightforward to check that the consistency condition (6.42) is also satisfied because $i_K P_4(\breve{R}, F) = 0$.

The commutator of two δ_I symmetries, (6.54), closes to a type I transformation generated by I, two type II transformations generated by T and C, respectively, and a type III transformation generated by K. The latter is a symmetry with the current dependent parameter indicated. In this scenario, the consistency condition needed is given in (6.42), with the first term associated with the transformation generated by T. The contribution of C and K transformations vanishes as $i_K P_4(\breve{R}, F) = 0$. Setting $L = I_r$ and $M = I_s$ in the second term in (6.42) demands that $P_4(\breve{R}, F)$ should be a (2,2)-form with respect to all three endomorphisms I_r in order to be satisfied. It turns out that this is the case as a consequence of the conditions $\breve{R}(I_rX, I_rY) = \breve{R}(X, Y)$ and $F(I_rX, I_rY) = F(X, Y)$ (no summation over r) of the curvature 2-forms.

Therefore, we have shown that the anomalies of all symmetries are consistent at least at one loop. Following the analysis of section 6.1.4, it can be argued that the anomalies of these symmetries are removed if the forms which generate the holonomy symmetries get quantum corrected as explained before. In the next subsection we shall examine the cancellation of some of these anomalies by adding finite local counterterms in the effective action.

6.2.4 Anomaly cancellation and finite local counterterms

The consistency and cancellation of anomalies for holonomy symmetries have already been discussed in section 6.1.4. Here we shall explore whether the anomalies of the symmetries generated K, C and I can be cancelled under certain conditions with the addition of finite local counterterms in the effective action. The global anomaly is cancelled provided that $P_4(\check{R}, F)$ is an exact 4-form. Next $P_4(\check{R}, F)$ satisfies $i_K P_4(\check{R}, F) = 0$ and $\mathcal{L}_K P_4(\check{R}, F) = 0$. Therefore, there is a $\tilde{P}_4$ on N^4 such that $P_4(\check{R}, F) = \pi^* \tilde{P}_4$, where π is the projection from the spacetime M^{10} to the orbit space, N^4, of the group of isometries. As $d\tilde{P}_4 = 0$, there is $\tilde{Q}^0_3$ such that $\tilde{P}_4 = d\tilde{Q}^0_3$. Therefore, one has

$$Q^0_3(\breve{\omega}, \Omega) = \pi^* \tilde{Q}^0_3 + dW \ , \tag{6.59}$$

where W is a 2-form on M^{10}. This allows to add the finite local counterterm

$$\Gamma^{\rm fl}_{(1)} = -\frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ \, W_{\mu\nu} D_+ X^\mu \partial_= X^\nu \ , \tag{6.60}$$

in the effective action. Adding this finite local couterterm implies that the anomalies of all the symmetries written as (6.33) now will be given with $\pi^* \tilde{Q}^0_3$ in place of $Q^0_3(\breve{\omega}, \Omega)$. This finite local counterterm removes the anomalies of the symmetries δ_K and δ_C since $i_K \pi^* \tilde{Q}^0_3 = 0$. Thus, these transformations are not anomalous. The same would have been the case, if we had considered the symmetries generated by $\bar{H}$ and $\vec{H}$ as independent symmetries.

It remains to investigate the cancellation of the remaining anomalies of the theory. For this notice that the Hodge dual of $\tilde{P}_4$ on N^4 taken with respect to the hyper-Kähler metric $\mathring{g}$, $\mathring{\star}\tilde{P}_4$, is a scalar. As $\tilde{P}_4$ is exact, $\mathring{\star}\tilde{P}_4$ is not harmonic. Therefore, there exists[16] a function $\tilde{f}$ on N^4 such that $\mathring{\star}\tilde{P}_4 = \mathring{\nabla}^2 \tilde{f}$, where $\mathring{\nabla}$ is the Levi-Civita connection on N^4 with respect to $\mathring{g}$. As a result, one can write

$$\tilde{Q}^0_3 = -\mathring{\star} d\tilde{f} + d\tilde{X} \ , \tag{6.61}$$

where $\tilde{X}$ is a 2-form on N^4. Observe that $\tilde{Q}^0_3$ can also be written as

$$\tilde{Q}^0_3 = d_r \mathring{Y}_r + d\tilde{X} \ , \quad \text{no summation over } r \ , \tag{6.62}$$

where $\mathring{Y}_r = \mathring{I}_r \tilde{f}$ and $\mathring{I}_r$ is the Kähler form of the hyper-Kähler metric $\mathring{g}$ on N^4 associated with the complex structure I_r, and $d_r = i_{I_r} d - d i_{I_r}$.

Next, adding the following finite local counterterm in the effective action

$$\Gamma^{\rm fl}_{(2)} = -\frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ \left((\pi^* \tilde{X})_{\mu\nu} + f \, (\pi^* \mathring{g})_{\mu\nu} \right) D_+ X^\mu \partial_= X^\nu \ , \tag{6.63}$$

where $f = \pi^* \tilde{f}$. It can be shown that the anomalies, $\Delta(a_I)$, associated with the symmetries generated by I_r endomorphisms cancel as well.

The frame rotations and gauge transformations anomalies can be modified with the inclusion of the finite local counterterms $\Gamma^{\rm fl}_{(1)}$ and $\Gamma^{\rm fl}_{(2)}$ as

$$\delta_\ell (\Gamma + \Gamma^{\rm fl}_{(1)} + \Gamma^{\rm fl}_{(2)}) = \Delta(\ell) + \delta_\ell \Gamma^{\rm fl}_{(1)} + \delta_\ell \Gamma^{\rm fl}_{(2)} \ , \tag{6.64}$$

[16]This is the case provided that N^4 is compact. The same applies for N^4 non-compact provided that the operator $\mathring{\nabla}^2$ has an inverse and $\mathring{\star}\tilde{P}_4$ is in the range of the operator. For a non-compact example take $N^4 = \mathbb{R}^4$ with the flat metric and $\tilde{P}_4$ constructed using anti-self-dual instantons, see [155].

and similarly for the gauge anomaly $\Delta(u)$. The cancellation of these anomalies is done after assigning an anomalous variation to both the spacetime metric g and b coupling of the sigma model action (6.1).

We can refine this approach if we assume the parameters of the frame rotations and gauge transformations depend only on the coordinates of N^4. In such a case notice that Q^0_3 can be written as

$$Q^0_3(\breve{\omega}, \Omega) = \pi^*(\mathring{\delta}\mathring{\star}\tilde{f}) + d\pi^*\tilde{X} + dW \ , \tag{6.65}$$

where $\mathring{\delta}$ is the adjoint of d on N^4 with respect to the hyper-Kähler metric. If ℓ depends only on the coordinates of N^4, then

$$d(Q^1_2(\ell, \breve{\omega}) - \delta_\ell \pi^* \tilde{X} - \delta_\ell W) = \pi^*(\mathring{\delta}\mathring{\star}\delta_\ell \tilde{f}) \ , \tag{6.66}$$

where we have used that ℓ does not depend on the fibre coordinates in the term on the right-hand side.

As a consequence of the right-hand side vanishing along the orbits of the isometry group, the same should hold for the left-hand side in the expression above. In addition, the Lie derivative of the left-hand side vanishes along the directions of the orbits of the isometry group. Therefore, $d(Q^1_2(\ell, \breve{\omega}) - \delta_\ell \pi^* X - \delta_\ell W)$ is the pull-back of a 3-form on N^4. Assuming that it is the pull-back of an exact 3-form on N^4 and using the orthogonality of exact and co-exact forms on N^4, one concludes that

$$Q^1_2(\ell, \breve{\omega}) = \delta_\ell \pi^* X + \delta_\ell W + dL \ , \quad \delta_\ell \tilde{f} = 0 \ , \tag{6.67}$$

for some 1-form L on the spacetime. As a result, the finite local counterterms $\Gamma^{\text{fl}}_{(1)}$ and $\Gamma^{\text{fl}}_{(2)}$ cancel also the spacetime frame rotation anomaly $\Delta(\ell)$. The terms dL can be interpreted as new anomalies, such as the holomophic anomalies in [142, 154]. These anomalies can be cancelled by a gauge transformation (6.10) of the coupling b.

6.3 Anomalies and $SU(3)$ holonomy backgrounds

6.3.1 Summary of geometry

The spacetime M^{10} of supersymmetric heterotic backgrounds for which the holonomy of $\hat{\nabla}$ is included in $SU(3)$ admits four $\hat{\nabla}$-parallel 1-forms $\mathbf{e}^a$, $a = 0, \ldots, 3$, one $\hat{\nabla}$-parallel 2-form I and a $\hat{\nabla}$-parallel complex 3-form L such that the Lie bracket algebra of the associated vector fields $\mathbf{e}_a$ of $\mathbf{e}^a$ is a 4-dimensional Lorentzian Lie algebra with structure constants $H^a{}_{bc}$. As a result $\mathbf{e}_a$ are Killing vector fields and $i_{\mathbf{e}_a} H = d\mathbf{e}^a$, and so $\mathcal{L}_{\mathbf{e}_a} H = 0$ as $dH = 0$. Moreover, one has that

$$i_{\mathbf{e}_a} I = 0 \ , \quad \mathcal{L}_{\mathbf{e}_a} I = 0 \ ; \quad i_{\mathbf{e}_a} L = 0 \ , \quad \mathcal{L}_{\mathbf{e}_a} L = -\frac{i}{6} \epsilon_a{}^{bcd} H_{bcd} L = -\frac{i}{2} H_{aij} I^{ij} \ . \tag{6.68}$$

Notice that if the Lie algebra of the isometry group is not abelian, L is not invariant under the action of the isometry group.

Furthermore, the algebraic properties of the $\hat{\nabla}$ covariantly constant forms include

$$I^2 = -(\mathbf{1} - \mathbf{e}_a \otimes \mathbf{e}^a) \ , \quad i_I L = 3iL \ , \tag{6.69}$$

where $g(X, IY) = I(X, Y)$. Therefore, L is a (3,0)-form[17] with respect to the endomorphism I.

The metric and 3-form field strength of the backgrounds can be written as

$$g = \eta_{ab}\mathbf{e}^a\mathbf{e}^b + \tilde{g} \ , \quad H = \frac{1}{3}\eta_{ab}\mathbf{e}^a \wedge d\mathbf{e}^b + \frac{2}{3}\mathbf{e}^a \wedge \mathcal{F}^b + \tilde{H} \ , \tag{6.70}$$

where $\tilde{g} = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$, $\mathcal{F}^a = d\mathbf{e}^a - \frac{1}{2}H^a{}_{bc}\mathbf{e}^b \wedge \mathbf{e}^c = \frac{1}{2}H^a{}_{ij}\mathbf{e}^i \wedge \mathbf{e}^j$ and $\tilde{g}(\mathbf{e}_a, \cdot) = 0$, $i_{\mathbf{e}_a}\tilde{H} = i_{\mathbf{e}_a}\mathcal{F}^b = 0$. Moreover, $\mathcal{L}_{\mathbf{e}_a}\tilde{H} = 0$ and $\mathcal{F}^a$ is an (1,1)-form with respect to the endomorphism I, i.e. $\mathcal{F}(IX, IY) = \mathcal{F}(X, Y)$. $\tilde{H}$ is a (1,2)- and (2,1)-form with respect to the endomorphism I, i.e.

$$H_{ijk} - 3H_{pq[i}I^p{}_jI^q{}_{k]} = 0 \ . \tag{6.71}$$

The Killing spinor equations also restrict the curvature of connection, F, of the gauge sector. The conditions are that $i_{\mathbf{e}_a}F = 0$ and $F(IX, IY) = F(X, Y)$ and $i_L F = 0$. Therefore, F has non-vanishing components only along the directions orthogonal to the orbits of the isometry group and it is a (1,1)- and traceless-form with respect to the endomorphism I, i.e. F satisfies a generalisation of the Hermitian-Einstein conditions of an $SU(3)$ instanton.

Locally, the geometry of the spacetime, M^{10}, can be modelled as that of a principal bundle over and 6-dimensional KT manifold N^6 with metric $\tilde{g}$ and torsion $\tilde{H}$, principal bundle connection $\mathbf{e}^a$ whose curvature is $\mathcal{F}$ and fibre a group manifold G whose (Lorentzian) Lie algebra is $\mathbb{R}^4$, $\mathbb{R} \oplus \mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)$ or $\mathfrak{cw}_4$, for more details see [143, 144, 107]. The spacetime M^{10} may not be necessarily of product form $G \times N^6$ either topologically or metrically.

6.3.2 $SU(3)$-structure symmetries and their commutators

The symmetries generated by the $\hat{\nabla}$-covariantly constant form bilinears $\mathbf{e}_a$, I and L are

$$\begin{aligned} &\delta_K X^\mu = a^a_K \mathbf{e}^\mu_a \ , \quad \delta_I X^\mu = a_I I^\mu{}_\nu D_+ X^\nu \ , \\ &\delta_L X^\mu = a^r_L (L_r)^\mu{}_{\nu_1\nu_2} D_+ X^{\nu_1\nu_2} \ , \end{aligned} \tag{6.72}$$

where $L_1 = \operatorname{Re} L$ and $L_2 = \operatorname{Im} L$, $r = 1, 2$. We have normalised the form bilinears such that in holomorphic frame indices $I_{\alpha\bar{\beta}} = -i\delta_{\alpha\bar{\beta}}$ and $(L_1) = (\epsilon_{\alpha\beta\gamma}, \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}})$ and $(L_2) = (-i\epsilon_{\alpha\beta\gamma}, i\epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}})$.

It is straightforward to verify using the algebraic properties of the form bilinears that

$$[\delta_K, \delta'_K] = \delta''_K \ , \quad [\delta_K, \delta_I] = 0 \ , \tag{6.73}$$

where $(a''_K)^c = -a^a_K a'^b_K H_{ab}{}^c$. The vanishing of the second commutator is a consequence of $\mathcal{L}_K I = 0$ for heterotic backgrounds with $SU(3)$ holonomy. For all transformations in (6.73) $\Delta\psi = 0$ and $\Delta_{KK} = \Delta_{KI} = 0$.

The commutator of two symmetries generated by I on X is

$$[\delta_I, \delta'_I] = \delta_T + \delta_C + \delta_K \ , \tag{6.74}$$

[17] Here, we have adopted the terminology of complex geometry to assign a holomorphic and anti-holomorphic degree for forms even though I is not a complex structure over the whole spacetime.

where δ_C is defined as in (6.51) after appropriately adapting the formulae for backgrounds with holonomy $SU(3)$, and $\alpha_{_T} = a'_I a_I$, $\alpha_{_C} = -a'_I a_I$ and $a^a_K = a'_I a_I H^a{}_{bc} J^b_K J^c_K$. The δ_K transformation in the above commutator's right-hand side could be written as a $\delta_{\bar{H}}$ transformation (6.55), which would result in field dependent structure constants and a Lie algebra rather than the expected W-algebra structure. However, it is more practical to present the commutator as a W-algebra. The commutator of two symmetries generated by I on ψ can be given as in (6.74) provided that $\Delta_{II}\psi^{\mathrm{a}}_- = \Delta_T\psi^{\mathrm{a}}_- = -a'_I a_I F_{\mu\nu}{}^{\mathrm{a}}{}_{\mathrm{b}}\psi^{\mathrm{b}}_- DX^{\mu\nu}$ and $\Delta_C\psi = \Delta_K\psi = 0$, where we have used that F is a (1,1)-form with respect to the endomorhism I and $i_K F = 0$.

Next, consider the commutator of δ_K and δ_L symmetries. After using the conditions on the geometry of $SU(3)$ holonomy backgrounds, one finds that

$$[\delta_K, \delta_L] = \delta'_L \ , \tag{6.75}$$

where $a'^r_L = -\frac{1}{3} a^a_{_K} \epsilon^r{}_s a^s_{_L} H_{apq} I^{pq}$ and ϵ is the Levi-Civita symbol with $\epsilon^1{}_2 = 1$. For all transformations in (6.75) $\Delta\psi = 0$ and $\Delta_{KL} = 0$.

The commutator of the symmetries generated by I and L reads

$$[\delta_I, \delta_L] = \delta'_L + \delta_K \ , \tag{6.76}$$

where

$$a'^r_L = \frac{1}{2} a_I a^r_{_L} H_{apq} I^{pq} J^a_K + 2a^s_{_L} D_+ a_I \epsilon^r{}_s - a_I D_+ a^s_{_L} \epsilon^r{}_s \ , \quad a^a_K = -\frac{1}{6} a_I a^r_{_L} H^a{}_{pq} I^{pq} J_{L_r} \ . \tag{6.77}$$

To compute this commutator, we have used that $N(I, L_r)^i{}_{jkp} = 0$ and

$$\begin{aligned} &I^m{}_i (L_1)_{mjk} = -(L_2)_{ijk} \ , \quad I^m{}_i (L_2)_{mjk} = (L_1)_{ijk} \ , \\ &H_{a[i|m|}(L_1)^m{}_{jk]} = \frac{1}{6} H_{apq} I^{pq} (L_2)_{ijk} \ , \quad H_{a[i|m|}(L_2)^m{}_{jk]} = -\frac{1}{6} H_{apq} I^{pq} (L_1)_{ijk} \ . \end{aligned} \tag{6.78}$$

The commutator of these transformations on ψ is as in (6.76). Note that $\Delta_{IL} = 0$ as a consequence of $i_L F = 0$ which is consistent with the vanishing of $\Delta\psi$ for all transformations contributing to (6.76).

The symmetry algebra is W-type since its structure constants depend on the conserved currents J_K and J_L of the theory.

It remains to compute the commutator of two transformations generated by L. In particular, one finds that

$$[\delta_{L_1}, \delta_{L_2}] = \delta_S + \delta_I + \delta_K + \delta_C \ , \tag{6.79}$$

where the transformation δ_S is given in (6.20) and (6.21) for $Q = -I$, i.e.

$$S^\mu{}_{\nu\rho\sigma} = -\delta^\mu{}_{[\nu} I_{\rho\sigma]} \ , \tag{6.80}$$

$$\alpha_S = -6a^1_{_L} a^2_{_L} \ , \quad a_{_I} = 4a^1_{_L} a^2_{_L} C - \frac{4}{3} a^1_{_L} a^2_{_L} H_{abc} J^a_K J^b_K J^c_K - 2a^1_{_L} a^2_{_L} J_I J^H_K \ , \tag{6.81}$$

where J^H_K in $a_{_I}$ is $J^H_K = H_{aij} I^{ij} D_+ X^a$, and

$$a_{_C} = -2a^1_{_L} a^2_{_L} J_I \ , \quad a^b_{_K} = \frac{1}{2} a^1_{_L} a^2_{_L} H^b{}_{ij} I^{ij} J^2_I + 2H^b{}_{cd} J^c_K J^d_K a^1_{_L} a^2_{_L} J_I \ . \tag{6.82}$$

Notice that the parameters of the transformations that appear in the right-hand side of the commutator depend on the currents J_I and J_K of the theory. Similarly, one finds that

$$[\delta_{L_1}, \delta'_{L_1}] = \delta_I \ , \quad [\delta_{L_2}, \delta'_{L_2}] = \delta_I \ , \tag{6.83}$$

where $a_I = 2(a^1_L D_+ a'^1_L - a'^1_L D_+ a^1_L) J_I$ and $a_I = -2(a^2_L D_+ a'^2_L - a'^2_L D_+ a^2_L) J_I$, respectively. This summarises the calculation of the commutators of the original symmetries (6.72) of the theory.

The commutators of δ_C with δ_K and δ_I are given as in (6.58) for the $SU(2)$ case, i.e. $[\delta_C, \delta_I] = 0$ and $[\delta_K, \delta_C] = \delta'_k$. In addition,

$$[\delta_L, \delta_C] = \delta'_L + \delta_K \ , \tag{6.84}$$

where

$$\begin{aligned} a'^r_L &= \epsilon^r{}_s a^s_L \big(\alpha_C D_+ J^a_K + \frac{1}{2} D_+ \alpha_C J^a_K\big) H_{aij} I^{ij} \ , \\ a^a_K &= H^a{}_{ij} I^{ij} \epsilon^r{}_s \big(\frac{1}{3} a^s_L \alpha_C D_+ J_{L^r} + \frac{1}{3} D_+ a^s_L \alpha_C J_{L^r} + \frac{1}{6} a^s_L D_+ \alpha_C J_{L^r}\big) \ . \end{aligned} \tag{6.85}$$

This concludes the computation of all commutators of the symmetries of sigma models on heterotic backgrounds with holonomy $SU(3)$.

6.3.3 Anomalies and consistency conditions

Similar to the analysis for $SU(2)$ holonomy backgrounds, the expression for the anomaly of frame rotations $\Delta(\ell)$ is given in terms of the connection $\breve{\omega}$ after possibly adding an appropriate finite local counterterm in the effective action of the theory. Next, as the commutators of frame rotations δ_ℓ and gauge transformations δ_u with δ_K, δ_I and δ_L vanish, the anomalies of the holonomy symmetries, $\Delta(a_K)$, $\Delta(a_I)$ and $\Delta(a_L)$, are given as in (6.33) for $L = K, I, L_1, L_2$. The commutator algebra of δ_K, δ_I, δ_{L_1} and δ_{L_2} impose additional consistency conditions on these anomalies. Ihe commutators of δ_K with δ_K, δ_I, δ_{L_1} and δ_{L_2}, eqns (6.73) and (6.75), are easily shown to either vanish or close to a type I transformation. Thus, the condition (6.40) is necessary for the anomalies to be consistent. This is satisfied as $i_K P_4(\breve{R}, F) = 0$.

The commutator of two δ_I transformations in (6.74) closes to two type II transformations generated by T and C, respectively, and a type III transformation generated by K. The latter is a symmetry with the indicated current dependence of the parameter. The consistency of the anomaly requires that (6.42) for $S = T$ and $S = C$ is satisfied. It turns out this is the case since $i_K P_4(\breve{R}, F) = 0$ and $P_4(\breve{R}, F)$ is (2,2)-form with respect to the endomorphism I.

Next, consider the consistency condition on the anomalies that arise from the commutators $[\delta_I, \delta_{L_1}]$ and $[\delta_I, \delta_{L_2}]$ in (6.76). In the former case, the commutator closes in a type I transformation generated by L_2 and a type III transformation, (6.43), generated by $L' = K$ and $M' = L_1$. For both cases, the consistency condition is given in (6.40) for $L = I$ and $M = L_1$ which is satisfied as $P_4(\breve{R}, F)$, I and L_1 are (2,2)-, (1,1)- and (3,0)+(0,3)-forms with respect to the endomorphism I, respectively. The commutator $[\delta_I, \delta_{L_2}]$ can be treated in a similar way.

The commutators $[\delta_{L_1}, \delta_{L_1}]$ and $[\delta_{L_2}, \delta_{L_2}]$ in (6.83) close to a type III transformation, (6.43), generated by $L' = M' = I$. In both scenarios, the consistency on the anomalies requires that (6.40) for $L = M = L_1$ and $L = M = L_2$, is satisfied. This is the case because of the skew-symmetric properties of the above forms in the condition (6.40).

The commutator $[\delta_{L_1}, \delta_{L_2}]$ in (6.79) closes to type II transformations generated by S in (6.80) and C, and type III transformations. The latter are associated with the symmetries generated by the pair of tensors $(A, B) = (I, C)$ and $(L', M') = (I, K)$, see (6.46) and (6.43), respectively. After some computation, the consistency condition (6.42) can be written as

$$-\frac{2}{3} P_4(\check{R}, F)_{[\mu_1\mu_2\mu_3|\sigma|} I_{\mu_4\mu_5]} + P_4(\check{R}, F)_{\lambda\rho[\mu_1|\sigma|} (L_1)^\lambda{}_{\mu_2\mu_3} (L_2)^\rho{}_{\mu_4\mu_5]} = 0 \ . \tag{6.86}$$

All $P_4(\check{R}, F)$ which are (2,2)-forms with respect to the endomorphism I satisfy the expression above. This holds since $\check{R}$ and F are (1,1)-forms with respect to I. The connections $\check{R}$ and F do not need to satisfy the traceless condition with respect to I despite what might have been expected. We have shown that all the anomalies are consistent at least at one loop. As a result, all of the anomalies cancel, assuming that the forms that generate the holonomy symmetries are corrected as stated in section 6.1.4. Similar to the $SU(2)$ case, some of these anomalies can also be removed with the addition of finite local counterterms in the effective action and this will be described below.

6.3.4 Anomaly cancellation and finite local counterterms

Following the arguments used for the SU(2) case in section 6.2.4, it can be shown that the anomalies of the symmetries generated by K, C and I cancel when appropriate finite local counterterms are added to the effective action of the theory. First, $P_4(\check{R}, F)$ can be seen as the pull-back of a form $\tilde{P}_4(\check{R}, F)$ on the orbit space N^6 of the isometry group. Then $Q_3^0(\breve{\omega}, \Omega)$ can be written as in (6.59), i.e. $Q_3^0(\breve{\omega}, \Omega) = \pi^* \tilde{Q}_3^0 + dW$, where now $\tilde{Q}_3^0$ is a 3-form on N^6. Next, one can argue that both anomalies generated by K and C cancel by including a finite local counterterm constructed from W in the effective action as in (6.60).

The calculation of the finite local counterterms that are required to cancel the anomaly associated with the symmetry generated by I can be done as in [154, 155] for the cancellation of (2,0) supersymmetry anomaly in sigma models. Take $\tilde{P}_4(\check{R}, F)$ as a 4-form on N^6 and note it is (2,2) with respect to the complex structure, I, on N^6. Then, using the local $\partial\bar{\partial}$-lemma P_4 can be written as $\tilde{P}_4(\check{R}, F) = dd_I \tilde{Y}$. Therefore, $\tilde{Q}_3^0 = d\tilde{X} + d_I \tilde{Y}$ for some 2-form X on N^6. Using this, one can construct a finite local counterterm

$$\Gamma^{\text{fl}} = -\frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ \, (\pi^* \tilde{Z}_{\mu\nu} + \pi^* \tilde{X}_{\mu\nu}) D_+ X^\mu \partial_= X^\nu \ , \tag{6.87}$$

that cancels the $\Delta(a_{_I})$ anomaly, where $\tilde{Y}(\cdot, \cdot) = \tilde{Z}(\cdot, I\cdot)$. It can be demonstrated that using

$$\delta_\ell \tilde{X} = \tilde{Q}_2^1(\ell) + d\tilde{A} + d_I \tilde{V} \ , \quad \delta_\ell \tilde{Y} = d\tilde{V} + d_I i_I \tilde{V} \ , \tag{6.88}$$

and after varying the finite local counterterm with δ_ℓ, one finds that the $\Delta(\ell)$ anomaly is cancelled. The $d\tilde{A}$ terms can be absorbed in a gauge transformation, (6.10), of b while the remaining terms can be removed by assigning anomalous variations to both g and b. Note that the remaining holomorphic frame rotations give rise to $\tilde{V}$ terms, see [142, 154]. A similar analysis can be done for the anomaly $\Delta(u)$ of the gauge sector transformations.

6.4 Summary

We have presented all the commutators of the symmetries, which are generated by the $\hat{\nabla}$-covariantly constant forms, of sigma models with supersymmetric heterotic backgrounds with $SU(2)$ and $SU(3)$ holonomy as target spaces. In both cases the algebra of transformations is a W-algebra and its closure requires additional generators which we have described. We have also given the Wess-Zumino consistency conditions of the anomalies of these symmetries arising in quantum theory due to the presence of worldsheet chiral fermions in the sigma model action. We have shown that these anomalies are consistent up to at least one loop in perturbation theory. In addition, we have argued that these anomalies can be cancelled by either adding finite local counterterms followed by suitable anomalous variations of the sigma model couplings or by an appropriate quantum correction of the tensors that generate the associated symmetries in the quantum theory. The latter is consistent with both the anomaly cancellation mechanism for the spacetime frame rotation and gauge sector anomalies [140], as well as the preservation of the form of the Killing spinor equations of heterotic supergravity up to and including two loops in the sigma model perturbation theory [146].

Chapter 7

Conclusion

Studying hidden symmetries in physical systems allows us to get new insights and a better understanding of them. Chapters 3-5 of this thesis analysed the (hidden) symmetries in spinning particle probes propagating on supersymmetric backgrounds generated by Killing-Yano forms arising as a consequence of the TCFH. This in turn raises the question of whether supersymmetry is closely related to integrability. This statement is too general to be approached. First of all, there are many probe actions that can be consider, see [61], and a choice has to be made. Moreover, as we have mentioned in chapter 3, there is no clear straightforward generalisation of Liouville's theorem to analyse the integrability of the geodesic flow of spinning particles[1] Thus, below we will address this question by looking at the geodesic flow of free relativistic particles propagating in M-theory backgrounds. Another issue raised by looking at the relation between supersymmetry and integrability is whether one can determine the amount of supersymmetry required for integrability.

Based on our work there are some partial answers to these questions but it is required to examine case by case. We start by reflecting on some results found in the literature. There are no known examples of supersymmetric backgrounds preserving more than $\frac{1}{2}$ supersymmetry, whose geodesic flows are not integrable in 11D supergravity. All $\mathrm{Ad}S_n \times S^{11-n}$ and plane wave solutions have integrable geodesic flows due to the large amount of isometries. Moreover, it is known that all supersymmetric backgrounds which preserve more than $\frac{1}{2}$ supersymmetry are homogeneous spaces [157]. However, not all homogeneous spaces have integrable geodesic flows. Therefore, it remains inconclusive whether the geodesic flow of all the solutions preserving more than $\frac{1}{2}$ supersymmetry in 11d supergravity is integrable.

The geodesic flow of many backgrounds preserving $\frac{1}{2}$ supersymmetry is not integrable. To illustrate this, consider the geodesic flow of systems of M-branes, in chapter 4, we proved that the geodesic flow of solutions with one centre y_k is integrable since these solutions are spherically symmetric. Although, it is not expected that this remains true for multi-centred solutions with centres at generic points as the large amount of symmetries exhibited will be lost. For spherically symmetric solutions, the transverse space has isometry group, $SO(10-p)$, as the harmonic function, h, depends on $|y|$, which is not invariant under translations. Then following the approach in [114], we have con-

[1] In [156] it was argued that the bosonic part of the motion of a spinning particle propagating in higher dimensional rotating black holes is completely integrable due to the existence of hidden symmetries constructed from a principal CKY. However, the CKY form emerging from the TCFH is not expected to be principal as this requires that the form possess the maximal number of functionally independent eigenvalues.

structed the Casimir charges which together with the translations along the M-brane directions proved the Liouville integrability. On the other hand, for multi-centre cases, the transverse space loses the $SO(10-p)$ isometry group, This implies that one cannot construct the Casimir charges required to prove Liouville integrability. Suppose we have a multi-M-brane system at the same centre, then if we move one apart, the isometry group becomes $SO(9-p)$. If we continue setting M-branes apart, the isometry group keeps being reduced to one of its subgroups according to the relative positions of the centres. As a consequence, the amount of Casimir charges that can be constructed is continually decreasing. However, the total number of charges required for Liouville integrability stays the same. For this reason, we do not expect such cases to have integrable geodesic flows[2].

Comparing the symmetries required for the integrability of the geodesic flow on M-brane backgrounds with the symmetries generated by the form bilinears, one can see that these contribute to different sectors in the probe dynamics. If a form bilinear generates a symmetry for a particle probe, it will generate a symmetry on all M-brane backgrounds including those that depend on multi-centred harmonic functions. It was found that the KY forms responsible for the integrability of the geodesic flow of all spherical symmetric M-branes differ from those constructed as Killing spinor bilinears. In addition to the KY forms associated with the TCFH, we computed the most general solution of all the KS, KY and CCKY tensors of M-branes using their standard definition of general relativity, i.e. with respect to the Levi-Civita connection. In particular, the KS expressions we have found should contain the rank 2 KS tensors related to the Casimir charges used to proved the integrability of the spherically symmetric M-branes. Then, as KY forms "square" to KS tensors, it is clear that the KY forms that yield the KS tensors associated with the Casimir charges should be considered in the general KY solution derived. On the other hand, the expressions obtained after squaring those KY forms constructed from Killing spinors do not correspond to the KS tensors used to prove the integrability of the geodesic flow. Therefore, generically form bilinears of supersymmetric backgrounds are not responsible for the integrability properties of a probe, but generate additional symmetries for probes, e.g. additional worldvolume supersymmetries, which characterise the dynamics. In this context, supersymmetry and integrability seem to point to different directions.

We have found that the Killing spinor bilinears generate symmetries in a variety of probe actions:

- For minimal 4D, N=2 supergravity we have found probes for which all particle systems propagating on backgrounds with a null Killing spinor admit symmetries generated by the form bilinears. For backgrounds with a timelike Killing spinor all solutions are locally isometric to Minkowski spacetimes.

- Similarly, we have found backgrounds of 5D, N=1 minimal supergravity admitting a null or timelike Killing spinor for which the form bilinears generate symmetries in the probe actions.

- Many of the form bilinears of M-brane backgrounds are either KY or CCKY.

- The form bilinears of some internal spaces of AdS backgrounds of 11D supergravity are KY or CCKY.

[2]It might be possible that a two-centre M-brane system still has integrable geodesic flow. It remains open to check this case explicitly.

Clearly, from these results and those of [22], the TCFHs can be constructed for all supersymmetric theories that exhibit a gravitino KSE. As a consequence, all form bilinears are generalised CKY forms with respect to some connection. However, given a TCFH of a supersymmetric theory, there is not systematic way to construct a probe action which exhibits symmetries generated by the form bilinears. In the minimal four dimensional supergravity theory, the matching of the conditions to leave the spinning particle action invariant with those arising from the TCFH required to consider the scenarios where the right-hand side was vanishing as they only coincide in the connection side. Moreover, in appendix A.2 we took the opportunity to construct an explicit example for which all the bilinears are parallel with respect to the TCFH connection in D=4, N=2 minimal supergravity and explain how the matching of the TCFH with the conditions for invariance of the probe action leads to severe constraints on the backgrounds. Similar type of constraints arise in appendix A.3 where we compute the TCFHs of D=5, N=2 and D=4, N=1 gauged supergravity theories. Furthermore, in 11-dimensional supergravity, one should also consider spinning particle probes that exhibit a 4-form coupling, as the TCFH connection depends on the 4-form field strength of the theory. The expectation would be that in this way one can better match the TCFH with the conditions for invariance of the probe action under transformations generated by the form bilinears. Such a probe action has been presented in appendix A.5. However under some reasonable assumptions on the couplings and on the transformations constructed from the form bilinears, one finds that the conditions for invariance of the probe action are too strong for M-brane backgrounds and they do not match with those of TCFH. These cases do not exhaust all possibilities of matching the TCFHs computed in this work with the conditions required for the form bilinear to generate a symmetry for a spinning particle probe constructed using the results of [61]. One can choose different TCFHs associated with the same supergravity theory as well as different probe actions. Hence, it remains an open question whether such a matching of conditions can be achieved in general.

Next, moving to generic heterotic backgrounds, the TCFH connection corresponds to a connection with torsion[3], $\nabla + \frac{1}{2}H$, for which all Killing spinor bilinears are parallel. This allows for a more straightforward application since probes that are left invariant under symmetries generated by CKY forms have not been found yet.[4] For the two-dimensional sigma models analysed in this work, the covariantly constant form bilinears generate additional worldsheet symmetries.

Overall, we have concluded:

1. All spacetimes admitting a $\nabla^{\mathcal{F}}$-parallel spinor are associated with a TCFH.

2. The Killing spinor bilinears generate symmetries in many probe actions propagating on supersymmetric backgrounds. However, their contribution to the integrability of the dynamics of probes is subtle.

3. There are different TCFHs associated with the same supergravity theory according to the basis of Killing spinor bilinears chosen and a plethora of probe actions. Thus, it remains an open question whether the conditions for invariance of the probe actions can be matched with those of the TCFHs in most of the theories analysed.

[3]Note that the TCFH expressions of heterotic backgrounds act on a single bilinear.

[4]In most of the cases investigated in chapters 3-5, we look at sectors where the terms involving the metric were zero, and hence, the bilinears satisfied a KY equation rather than the CKY one.

4. All the $\hat{\nabla}$-covariantly constant forms of non-linear supersymmetric two-dimensional sigma models, where the target space is identified with heterotic backgrounds with $SU(2)$ and $SU(3)$ holonomy, satisfy a W-algebra and its closure required additional generators.

5. The anomalies of these symmetries are consistent up to at least one loop in perturbation theory. The cancellation of these anomalies can be done either with the addition of finite local counterterms followed by suitable anomalous variations of the sigma model couplings or with plausible quantum corrections of the symmetry generators in the quantum theory.

7.0.1 Future directions

A more detailed picture of this work could emerge from the investigation of probe actions that remain invariant under CKY forms or by finding some cases where the conditions for invariance of the probe actions can be matched with those of the TCFHs.

One can keep exploring the role of hidden symmetries, KS, CKY and KY tensors in supergravity and the interplay between separability/integrability in supergravity solutions. For instance, recently an infinite family of asymptotically $AdS_3 \times S^3$ supergravity solutions was found in which the null geodesic problem is completely integrable because of the existence of a non-trivial conformal KS tensor [158]. Turning back to the TCFH, all the solutions of the generalized type CKY equation (2.78) are solutions of the TCFH connection but the converse is not necessarily true. It will be interesting to explore some of the latter solutions and their applications.

In [159] possible Lie (super)-algebraic structures on the space of parallel forms restricted to the cone metric were discussed. The approach was to extend the Killing superalgebras by considering higher-rank differential forms. As these forms obey complicated partial differential equations which involve all forms at once, one way to overcome this difficulty is to use the cone construction which establishes a one-to-one correspondence between geometric Killing spinors on a spin manifold and parallel spinors on its metric cone. In our case, it will be interesting to analyse whether our conformal Killing-Yano forms satisfy an algebraic structure and whether they form a Lie superalgebra together with the Killing spinors.

In addition, one can explore whether the symmetries associated with the TCFH satisfy a W-symmetry algebra following the work done for heterotic backgrounds.

Moreover, in chapter 6, we have not considered the possibility that the spacetime frame rotations and gauge transformation invariant terms of the effective action contribute to the anomalies associated with the holonomy symmetries. These transformations can potentially give rise to a $\Delta_{\text{inv}}(a_L)$ term in (6.33). One can extend the work done to include these terms, as well as sigma models with target spaces heterotic backgrounds with G_2 holonomy and non-compact holonomy groups such as $SU(4) \ltimes \mathbb{R}^8, Sp(2) \ltimes \mathbb{R}^8$ or $\mathbb{R}^8$.

Appendix A

A.1 Notation for forms

Let M be a manifold with a (local) coframe $\mathbf{e}^i$ and coordinates y^I

$$\chi = \frac{1}{k!}\chi_{I_1\cdots I_k} dy^{I_1} \wedge \cdots \wedge dy^{I_k} = \frac{1}{k!}\chi_{i_1\cdots i_k}\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_k} \ , \tag{A.1}$$

Then, the exterior derivative and inner derivation read

$$d\chi := \frac{1}{k!}\partial_{I_1}\chi_{I_2\cdots I_{k+1}} dy^{I_1} \wedge \cdots \wedge dy^{I_{k+1}} \ , \tag{A.2}$$

$$i_X\chi := \frac{1}{(k-1)!} X^j \chi_{j i_1\cdots i_{k-1}}\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_{k-1}} \ , \tag{A.3}$$

respectively. One can define a Clifford algebra element $\not{\chi}$

$$\not{\chi} := \chi_{i_1\cdots i_k}\Gamma^{i_1\cdots i_k} \tag{A.4}$$

where Γ^i, $i = 1, \cdots n$ are the Dirac gamma matrices. In addition, we have introduced the notation

$$\not{\chi}_{i_1} := \chi_{i_1\cdots i_k}\Gamma^{i_2\cdots i_k} \ , \quad \Gamma\!\!\!\not{\chi}_{i_1} := \Gamma_{i_1}{}^{i_2\cdots i_{k+1}}\chi_{i_2\cdots i_{k+1}} \tag{A.5}$$

A.2 Construction of a covariantly constant TCFH of D=4, N=2 minimal supergravity

The supercovariant connection of minimal $D = 4$, $N = 2$ supergravity can be written as

$$\mathcal{D}_\mu \equiv \nabla_\mu + \frac{1}{2}{}^*F_{MA}\Gamma^A\Gamma_5 - \frac{i}{2}F_{MA}\Gamma^A \ . \tag{A.6}$$

Now it is linear in gamma matrices and depends only on terms of the type $i_X F$. There is a choice of the Killing spinor bilinears such that the associated TCFH is a parallel transport equation with respect to a connection $\nabla^{\mathcal{F}}$. The basis of bilinears in order to construct such connection is

$$\begin{aligned} &f = \langle\epsilon, \epsilon\rangle_D \ , \quad h = \langle\epsilon, \Gamma_5\epsilon\rangle_D \ , \quad k = \langle\epsilon, \Gamma_a\epsilon\rangle_D \, e^a \ , \\ &Y = \langle\epsilon, \Gamma_a\Gamma_5\epsilon\rangle_D \, e^a \ , \quad Y^3 + iY^2 = \langle\tilde{\epsilon}, \Gamma_a\Gamma_5\epsilon\rangle_D \, e^a \ , \end{aligned}$$

$$\omega^1 = \frac{1}{2}\langle\epsilon, \Gamma_{ab}\epsilon\rangle_D\, e^a \wedge e^b\ , \quad \omega^3 + i\omega^2 = \frac{1}{2}\langle\tilde{\epsilon}, \Gamma_{ab}\epsilon\rangle_D\, e^a \wedge e^b\ ,$$
$$\chi^1 = \frac{1}{2}\langle\epsilon, \Gamma_{ab}\Gamma_5\epsilon\rangle_D\, e^a \wedge e^b\ , \quad \chi^3 + i\chi^2 = \frac{1}{2}\langle\tilde{\epsilon}, \Gamma_{ab}\Gamma_5\epsilon\rangle_D\, e^a \wedge e^b\ .$$
$$\varphi^1 = \frac{1}{3!}\langle\epsilon, \Gamma_{abc}\epsilon\rangle_D\, e^a \wedge e^b \wedge e^c\ , \quad \varphi^3 + i\varphi^2 = \frac{1}{3!}\langle\tilde{\epsilon}, \Gamma_{abc}\epsilon\rangle_D\, e^a \wedge e^b \wedge e^c\ , \qquad \text{(A.7)}$$

where the spacetime metric $g = \eta_{ab}e^a e^b$ with $e^a = e^a_\mu dx^\mu$ a local co-frame, $\langle\cdot,\cdot\rangle_D$ is the Dirac inner product, C is a charge conjugation matrix such that $C * \Gamma_a = -\Gamma_a C*$ and $C * C* = -1$, and $\tilde{\epsilon} = C * \epsilon$. $C = \Gamma_3$. Observe that if ϵ is a Killing spinor so is $\tilde{\epsilon}$. The TCFH of the theory reads

$$\begin{aligned}
&\nabla^{\mathcal{F}}_\mu f = \nabla_\mu f - iF_{\mu\nu}k^\nu = 0\ , \quad \nabla^{\mathcal{F}}_\mu h = \nabla_\mu h - {}^*F_{\mu\nu}k^\nu = 0\ ,\\
&\nabla^{\mathcal{F}}_\mu k_\nu = \nabla_\mu k_\nu - ifF_{\mu\nu} + h\,{}^*F_{\mu\nu} = 0\ ,\\
&\nabla^{\mathcal{F}}_\mu Y^r_\nu = \nabla_\mu Y^r_\nu + {}^*F_{\mu\rho}\omega^{r\rho}{}_\nu - iF_{\mu\rho}\chi^{r\rho}{}_\nu = 0\ , \quad r = 1,2,3\ ,\\
&\nabla^{\mathcal{F}}_\mu \omega^r_{\nu\rho} = \nabla_\mu\omega^r_{\nu\rho} - 2\,{}^*F_{\mu[\nu}Y^r_{\rho]} - iF_{\mu\lambda}\varphi^{r\lambda}{}_{\nu\rho} = 0\ , \quad r = 1,2,3\\
&\nabla^{\mathcal{F}}_\mu \chi^r_{\nu\rho} = \nabla_\mu\chi^r_{\nu\rho} - 2i\,F_{\mu[\nu}Y^r_{\rho]} - {}^*F_{\mu\lambda}\varphi^{r\lambda}{}_{\nu\rho} = 0\ , \quad r = 1,2,3\\
&\nabla^{\mathcal{F}}_\mu \varphi^r_{\nu\rho\lambda} = \nabla_\mu\varphi^r_{\nu\rho\lambda} + 3{}^*F_{\mu[\nu}\chi^r_{\rho\lambda]} - 3iF_{\mu[\nu}\omega^r_{\rho\lambda]} = 0\ , \quad r = 1,2,3\ .
\end{aligned} \qquad \text{(A.8)}$$

The TCFH derived is parallel in all Killing spinor bilinears. However, finding a probe action and suitable transformations which reproduce the TCFH remains a complicated task. Below we provide two attempts

$$S = \int dt d\theta \left(-\frac{i}{2} g_{\mu\nu} DX^\mu \partial_t X^\nu + F_{\mu\nu}\psi^\mu_1 \partial_t \phi^\nu + i{}^*F_{\mu\nu}\psi^\mu_2 \partial_t X^\nu \right)\ , \qquad \text{(A.9)}$$

under the variations

$$\begin{aligned}
&\delta X^\mu = aY^\mu + \alpha\omega^\mu{}_\lambda DX^\lambda + \beta\chi^\mu{}_\lambda DX^\lambda + b\varphi^\mu{}_{\lambda\rho}DX^{\lambda\rho}\ ,\\
&\delta\psi^\mu_1 = a\chi^\mu{}_\lambda DX^\lambda - \frac{1}{2}\alpha\varphi^\mu{}_{\lambda\rho}DX^{\lambda\rho} + \beta Y_\lambda DX^{\lambda\mu} + b\omega_{\lambda\rho}DX^{\lambda\rho\mu}\ ,\\
&\delta\psi^\mu_2 = a\omega^\mu{}_\lambda DX^\lambda - \alpha Y_\lambda DX^{\lambda\mu} + \frac{1}{2}\beta\varphi^\mu{}_{\lambda\rho}DX^{\lambda\rho} + b\chi_{\lambda\rho}DX^{\lambda\rho\mu}\ ,
\end{aligned} \qquad \text{(A.10)}$$

where a, b are commuting infinitesimal parameters whereas α, β are anti-commuting. The conditions for the invariance of the action (A.9) under the variations (A.10) can be expressed as

$$\begin{aligned}
&\nabla_\mu Y_\nu + {}^*F_{\mu\lambda}\omega^\lambda{}_\nu - iF_{\mu\lambda}\chi^\lambda{}_\nu = 0\ , \quad \nabla_\mu\omega_{\nu\rho} - 2\,{}^*F_{\mu[\nu}Y_{\rho]} - iF_{\mu\lambda}\varphi^\lambda{}_{\nu\rho} = 0\ ,\\
&\nabla_\mu\chi_{\nu\rho} - 2i\,F_{\mu[\nu}Y_{\rho]} - {}^*F_{\mu\lambda}\varphi^\lambda{}_{\nu\rho} = 0\ , \quad \nabla_\mu\varphi_{\nu\rho\lambda} + 3{}^*F_{\mu[\nu}\chi_{\rho\lambda]} - 3iF_{\mu[\nu}\omega_{\rho\lambda]} = 0\ ,\\
&\nabla_{[\mu}F_{|\nu|\lambda]}Y^\lambda = 0\ , \quad \nabla_{[\mu}{}^*F_{|\nu|\lambda]}Y^\lambda = 0\ , \quad \nabla_{[\mu}F_{|\nu|\lambda]}\omega^\lambda{}_\rho = 0\ , \quad \nabla_{[\mu}{}^*F_{|\nu|\lambda]}\omega^\lambda{}_\rho = 0\ ,\\
&\nabla_{[\mu}F_{|\nu|\lambda]}\chi^\lambda{}_\rho = 0\ , \quad \nabla_{[\mu}{}^*F_{|\nu|\lambda]}\chi^\lambda{}_\rho = 0\ , \quad \nabla_{[\mu}F_{|\nu|\lambda]}\varphi^\lambda{}_{\rho\sigma} = 0\ , \quad \nabla_{[\mu}{}^*F_{|\nu|\lambda]}\varphi^\lambda{}_{\rho\sigma} = 0\ ,\\
&F_{\mu\nu}Y^\nu = 0\ , \quad {}^*F_{\mu\nu}Y^\nu = 0\ , \quad F_{\mu\nu}\omega^\nu{}_\lambda = 0\ , \quad {}^*F_{\mu\nu}\omega^\nu{}_\lambda = 0\ ,\\
&F_{\mu\nu}\chi^\nu{}_\lambda = 0\ , \quad {}^*F_{\mu\nu}\chi^\nu{}_\lambda = 0\ , \quad F_{\mu\nu}\varphi^\nu{}_{\lambda\sigma} = 0\ , \quad {}^*F_{\mu\nu}\varphi^\nu{}_{\lambda\sigma} = 0\ .
\end{aligned} \qquad \text{(A.11)}$$

The first two lines match exactly the TCFH connection for each bilinear. Nevertheless, the last two lines are too severe on the backgrounds to admit non-trivial solutions. One could try to use the following action

$$S = -\frac{i}{2}\int dt d\theta \left(g_{\mu\nu}DX^\mu\partial_t X^\nu + c_1 F_{\mu\nu}DX^\mu DX^\nu\psi_1 + c_2{}^*F_{\mu\nu}DX^\mu DX^\nu\psi_2\right)\ . \quad \text{(A.12)}$$

In this case, we found a problem defining the transformation of the superfields. Let us focus on the two-form bilinear, ω. The corresponding connection is twisted by two terms proportional to ${}^*FY$ and $F\varphi$. But the latter term cannot be achieved by a simple transformation as the ones consider below. This can be seen immediately since none of the conditions to leave the action invariant will have a term proportional to $\phi\partial_t DXDX$ and such terms are used to match the TCFH.

$$\delta X^\mu = \alpha\omega^\mu{}_\lambda DX^\lambda \ , \quad \delta\psi_1 = \alpha\varphi_{\lambda\rho\sigma}DX^{\lambda\rho\sigma} \ , \quad \delta\psi_2 = \alpha Y_\lambda \partial_t X^\lambda \ . \tag{A.13}$$

A.3 TCFH of gauged supergravities

The construction of the TCFH for gauged supergravities holds the same strategy explained in Chapter 2. We will provide two examples which we will relate with the probe actions 3.6 annd 3.19.

A.3.1 D=5, $\mathcal{N}$=2 Gauged Supergravity

The supercovariant derivative is given by [160]

$$\mathcal{D}_M \equiv \nabla_M + \frac{i}{8}X_I(\Gamma_M{}^{AB} - 4\delta_M{}^A\Gamma^B)F^I_{AB} - \frac{3}{2}i\chi V_I A^I{}_M + \frac{1}{2}\chi V_I X^I \Gamma_M \ . \tag{A.14}$$

Consider the following bi-linears constructed from the Dirac inner product.

$$f = \langle\epsilon,\epsilon\rangle_D \ , \quad k = \langle\epsilon,\Gamma_A\epsilon\rangle_D\, e^A \ , \quad \omega = \frac{1}{2}\langle\epsilon,\Gamma_{AB}\epsilon\rangle_D\, e^A\wedge e^B \ . \tag{A.15}$$

Our conventions are $\epsilon^{01234} = 1$, $\Gamma_4 = i\Gamma_{0123}$, ${}^*F_{MN} = \frac{1}{2}\epsilon_{MNAB}F^{AB}$.

Putting them into the TCFH form with respect to the minimal connection, one finds that

$$\begin{aligned}
&\mathcal{D}^{\mathcal{F}}_M f \equiv \nabla_M f = iX_I F^I_{MA}k^A \ , \\
&\mathcal{D}^{\mathcal{F}}_M k_N \equiv \nabla_M k_N - \chi V_I X^I\omega_{MN} = -\frac{1}{2}X_I\,{}^*F^I_{MNA}k^A + iX_I F^I_{MN} f \ , \\
&\mathcal{D}^{\mathcal{F}}_M \omega_{NR} \equiv \nabla_M\omega_{NR} + 3X_I\,{}^*F^I_{M[N|P|}\omega^P{}_{R]} \\
&= 3X_I\,{}^*F^I_{P[NR}\omega^P{}_{M]} - X_I g_{M[N}\,{}^*F^I_{R]PQ}\omega^{PQ} + 2\chi V_I X^I g_{M[N}k_{R]} \ , \quad ,
\end{aligned} \tag{A.16}$$

Note that the TCFH for the one form k includes a term where the index M is on the two form ω, this contributes to the $\mathcal{P}$ term. Otherwise, we would have to include extra bilinears which would be unnecessary since the remaining bilinears are dual to the ones already defined.

For the TCFH associated with the two-form, one can use the same action (3.19) and transformation laws as in the non-gauged case but now the three-form coupling c is identified with $c = 3X_I{}^*F^I$ and one can follow with a similar analysis.

A.3.2 D=4, N=2 gauged supergravity

The supercovariant connection is [161]

$$\mathcal{D}_M \ = \ \nabla_M + \frac{i}{2}A_M\Gamma_5 + i\ell\,\xi_I A^I_M + \ell\,\Gamma_M\xi_I(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I)$$

$$+\frac{i}{4}\Gamma^{AB}\Big(\mathrm{Im}\big(F^{-I}_{AB}X^J\big)-i\Gamma_5\mathrm{Re}\big(F^{-I}_{AB}X^J\big)\Big)\mathrm{Im}\mathcal{N}_{IJ}\Gamma_M\ , \tag{A.17}$$

where

$$F^{\pm I}_{AB}=\frac{1}{2}(F^I_{AB}\pm\tilde{F}^I_{AB})\ ,\quad \tilde{F}^I_{AB}=-\frac{i}{2}\epsilon_{AB}{}^{CD}F^I_{CD}\ , \tag{A.18}$$

with $\epsilon_{0123}=1$ and $\Gamma_5=i\Gamma_{0123}$.

From now on we set

$$F^-_{MN}=F^{-I}_{MN}X^J\mathrm{Im}\mathcal{N}_{IJ}\ ,\quad X=\xi_IX^I\ , \tag{A.19}$$

to simplify the notation.

The bilinears are

$$\begin{aligned}&f=\langle\epsilon,\epsilon\rangle_D\ ,\quad k=\langle\epsilon,\Gamma_A\epsilon\rangle_D\,e^A\ ,\quad \omega=\frac{1}{2}\langle\epsilon,\Gamma_{AB}\epsilon\rangle_D\,e^A\wedge e^B\ ,\\ &Y=\langle\epsilon,\Gamma_A\Gamma_5\epsilon\rangle_D\,e^A\ ,\quad h=\langle\epsilon,\Gamma_5\epsilon\rangle_D\ .\end{aligned} \tag{A.20}$$

Putting them into the TCFH form with respect to the minimal connection, one finds that

$$\begin{aligned}&\mathcal{D}^{\mathcal{F}}_Mf\equiv\nabla_Mf+iA_Mh=-2i\ell\mathrm{Re}XY_M+2i\mathrm{Im}(F^-_{MA})k^A\ ,\\ &\mathcal{D}^{\mathcal{F}}_Mk_N\equiv\nabla_Mk_N+\ell\mathrm{Re}X\epsilon_{MN}{}^{PQ}\omega_{PQ}\\ &=2\ell\mathrm{Im}X\omega_{MN}+2i\mathrm{Im}(F^-_{MN})f-2\mathrm{Re}(F^-_{MN})h\ ,\\ &\mathcal{D}^{\mathcal{F}}_MY_N\equiv\nabla_MY_N+4\mathrm{Re}(F^-_{MA})\omega^A{}_N\\ &=2\ell g_{MN}\mathrm{Im}Xh+2i\ell g_{MN}\mathrm{Re}Xf-g_{MN}\mathrm{Re}(F^-_{AB})\omega^{AB}+4\mathrm{Re}(F^-_{[M|A|})\omega^A{}_{N]}\ ,\\ &\mathcal{D}^{\mathcal{F}}_M\omega_{NR}\equiv\nabla_M\omega_{NR}-8\mathrm{Re}(F^-_{M[N})Y_{R]}-\frac{1}{2}A_M\epsilon_{NRPQ}\omega^{PQ}\\ &=+4\ell\mathrm{Im}Xg_{M[N}k_{R]}+2\ell\mathrm{Re}\,{}^*X_{MNRP}k^P\\ &-4g_{M[N}\mathrm{Re}(F^-_{R]A})Y^A-6\mathrm{Re}(F^-_{[MN})Y_{R]}\ ,\\ &\mathcal{D}^{\mathcal{F}}_Mh\equiv\nabla_Mh+iA_Mf=2\ell\mathrm{Im}XY_M+2\mathrm{Re}(F^-_{MA})k^A\ .\end{aligned} \tag{A.21}$$

Note that on the right-hand side for the one form k and the zero form h there are terms where the index M appears on bilinears instead of the fluxes. This is because it is interpreted as the $\mathcal{P}$ term. Since the remaining bilinears can be obtained by computing the Hodge duals of the bilinears defined, it is not required to consider more bilinears beyond the two-form.

One can use probe 3.6 and the same transformations. However, one needs to look at sectors where the terms that do not resemble those which appear in (A.8) vanish. As a consequence, this together with the conditions for invariance (3.10) impose severe constraints on the backgrounds to admit non-trivial solutions. The task to find suitable probe actions and transformations to match the TCFH with the conditions to leave the action invariant for generic backgrounds remains open.

A.4 M5-brane bilinears

Using the solution (4.50) of the condition on the Killing spinors and setting $\phi^{rs}=h^{-\frac{1}{6}}\mathring{\phi}^{rs}$ for all bilinears ϕ^{rs}, one can easily find

$$\mathring{f}^{rs}=0\ ,\quad \mathring{k}^{rs}=2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle+\mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle)\,e^a\ ,$$

$$
\begin{aligned}
\mathring{\omega}^{rs} &= 2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle + \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)\, e^a\wedge e^3\\
&\quad +2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle - \mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)\, e^a\wedge e^4\\
&\quad +2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle - \mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)\, e^a\wedge e^8\\
&\quad +2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle - \mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)\, e^a\wedge e^9\\
&\quad +2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle - \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle)\, e^a\wedge e^\natural\ ,\\
\mathring{\varphi}^{rs} &= \frac{1}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle + \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle)\, e^a\wedge e^b\wedge e^c\\
&\quad -2\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle(e^3\wedge e^4 - e^8\wedge e^9)\wedge e^a\\
&\quad +2\mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle(e^3\wedge e^4 + e^8\wedge e^9)\wedge e^a\\
&\quad -2\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle(e^3\wedge e^8 + e^4\wedge e^9)\wedge e^a\\
&\quad +2\mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle(e^3\wedge e^8 - e^4\wedge e^9)\wedge e^a\\
&\quad -2\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle(e^3\wedge e^9 - e^4\wedge e^8)\wedge e^a\\
&\quad +2\mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle(e^3\wedge e^9 + e^4\wedge e^8)\wedge e^a\\
&\quad -2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle - \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)e^3\wedge e^\natural\wedge e^a\\
&\quad -2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle + \mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)e^4\wedge e^\natural\wedge e^a\\
&\quad -2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle + \mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)e^8\wedge e^\natural\wedge e^a\\
&\quad -2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle + \mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)e^9\wedge e^\natural\wedge e^a\ ,\\
\mathring{\theta}^{rs} &= \frac{1}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle + \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\eta^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^3\\
&\quad +\frac{1}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle - \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\lambda^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^4\\
&\quad +\frac{1}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle - \mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\eta^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^8\\
&\quad +\frac{1}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle - \mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\lambda^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^9\\
&\quad +\frac{1}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle - \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^\natural\\
&\quad +2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle + \mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)e^a\wedge e^3\wedge e^4\wedge e^8\\
&\quad -2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle + \mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)e^a\wedge e^3\wedge e^4\wedge e^9\\
&\quad +2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle + \mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)e^a\wedge e^3\wedge e^8\wedge e^9\\
&\quad -2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle - \mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)e^a\wedge e^4\wedge e^8\wedge e^9\\
&\quad -2\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle\, e^a\wedge(e^3\wedge e^4\wedge e^\natural - e^8\wedge e^9\wedge e^\natural)\\
&\quad -2\mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle\, e^a\wedge(e^3\wedge e^4\wedge e^\natural + e^8\wedge e^9\wedge e^\natural)\\
&\quad -2\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle\, e^a\wedge(e^3\wedge e^8\wedge e^\natural + e^4\wedge e^9\wedge e^\natural)\\
&\quad -2\mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle\, e^a\wedge(e^3\wedge e^8\wedge e^\natural - e^4\wedge e^9\wedge e^\natural)\\
&\quad -2\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{1s}\rangle\, e^a\wedge(e^3\wedge e^9\wedge e^\natural - e^4\wedge e^8\wedge e^\natural)\\
&\quad -2\mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{2s}\rangle\, e^a\wedge(e^3\wedge e^9\wedge e^\natural + e^4\wedge e^8\wedge e^\natural)\ ,\\
\mathring{\tau}^{rs} &= -\frac{1}{3}\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^4 - e^8\wedge e^9)\\
&\quad +\frac{1}{3}\mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^4 + e^8\wedge e^9)\\
&\quad -\frac{1}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{1s}\rangle e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^8 + e^4\wedge e^9)\\
&\quad +\frac{1}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\eta^{2s}\rangle\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^8 - e^4\wedge e^9)\\
&\quad -\frac{1}{3}\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{1s}\rangle e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^9 - e^4\wedge e^8)\\
&\quad +\frac{1}{3}\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\lambda^{2s}\rangle\, e^a\wedge e^b\wedge e^c\wedge(e^3\wedge e^9 + e^4\wedge e^8)\\
&\quad -\frac{1}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle - \mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\eta^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^3\wedge e^\natural
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{3}(\mathrm{Re}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle+\mathrm{Re}\langle\eta^{2r},\Gamma_{abc}\lambda^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^4\wedge e^\natural\\
&-\frac{1}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\eta^{2s}\rangle+\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\eta^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^8\wedge e^\natural\\
&-\frac{1}{3}(\mathrm{Im}\langle\eta^{1r},\Gamma_{abc}\lambda^{2s}\rangle+\mathrm{Im}\langle\eta^{2r},\Gamma_{abc}\lambda^{1s}\rangle)e^a\wedge e^b\wedge e^c\wedge e^9\wedge e^\natural\\
&+\frac{2}{5!}\big(\mathrm{Re}\langle\eta^{1r},\Gamma_{a_1\dots a_5}\eta^{1s}\rangle+\mathrm{Re}\langle\eta^{2r},\Gamma_{a_1\dots a_5}\eta^{2s}\rangle\big)\; e^{a_1}\wedge\cdots\wedge e^{a_5}\\
&+2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{1s}\rangle-\mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{2s}\rangle)e^a\wedge e^3\wedge e^4\wedge e^8\wedge e^9\\
&-2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle-\mathrm{Im}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)e^a\wedge e^3\wedge e^4\wedge e^8\wedge e^\natural\\
&+2(\mathrm{Im}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle-\mathrm{Im}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)e^a\wedge e^3\wedge e^4\wedge e^9\wedge e^\natural\\
&-2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\lambda^{2s}\rangle-\mathrm{Re}\langle\eta^{2r},\Gamma_a\lambda^{1s}\rangle)e^a\wedge e^3\wedge e^8\wedge e^9\wedge e^\natural\\
&+2(\mathrm{Re}\langle\eta^{1r},\Gamma_a\eta^{2s}\rangle+\mathrm{Re}\langle\eta^{2r},\Gamma_a\eta^{1s}\rangle)e^a\wedge e^4\wedge e^8\wedge e^9\wedge e^\natural\;, \qquad \text{(A.22)}
\end{aligned}
$$

where after a relabelling of the spacetime coordinates $e^a = h^{-1/6}d\sigma^a$, $a = 0, 1, 2, 5, 6, 7$, and $e^i = h^{1/3}dy^i$, $i = 3, 4, 8, 9, \natural$, is a pseudo-orthonormal frame of the M5-brane metric (4.28).

A.5 Spinning particle probes with 4-form couplings

Following the use of spinning particle probes on 4- and 5-dimensional minimal supergravity backgrounds that exhibit 2-form couplings [27], one may be tempted to generalise these to spinning particle probes that exhibit 4-form couplings. Such a generalisation is desirable as the TCFH connection of 11-dimensional supergravity exhibits terms that depend of the 4-form field strength F. One way to generalise (2.55) is to adapt the general construction of [61] and introduce a fermonic superfield ψ of mass dimension [1/2]. Insisting for the couplings of the action to be dimensionless, a minimal choice for an action with a 4-form coupling is

$$S = -\frac{1}{2}\int dtd\theta\,[ig_{\mu\nu}Dx^\mu\partial_t x^\nu - \frac{i}{12}F_{\mu\nu\rho\sigma}\psi^{\mu\nu\rho}\partial_t x^\sigma + \beta\psi_{\mu\nu\rho}\nabla\psi^{\mu\nu\rho}]\;, \qquad \text{(A.23)}$$

with β a constant which will be specified later,

$$\nabla\psi^{\mu\nu\rho} = D\psi^{\mu\nu\rho} + 3Dx^\lambda\Gamma^{[\mu}_{\lambda\mu'}\psi^{|\mu'|\nu\rho]}\;, \qquad \text{(A.24)}$$

and Γ are the Christoffel symbols of the spacetime metric g. The numerical coefficient of the coupling $F\psi\partial_t x$ could be arbitrary but the above choice will suffice. Also one could add additional terms in the action like $F\nabla\psi Dx$ which we shall explore later. Other terms include couplings of the type $\nabla F\psi Dx$. After a superspace partial integration these can be re-expressed in terms of the $F\psi\partial_t x$ and $F\nabla\psi Dx$ couplings.

The variation of the action (A.23) can be expressed as

$$\delta S = -\int dtd\theta\,[g_{\mu\nu}\delta x^\mu\mathcal{S}^\nu + \Delta\psi_{\mu\nu\rho}\mathcal{S}^{\mu\nu\rho}]\;, \qquad \text{(A.25)}$$

where

$$\Delta\psi^{\mu\nu\rho} = \delta\psi^{\mu\nu\rho} + 3\delta x^\lambda\Gamma^{[\mu}_{\lambda\mu'}\psi^{|\mu'|\nu\rho]}\;, \qquad \text{(A.26)}$$

δx and $\delta\psi$ are arbitrary variations of the fields and

$$\mathcal{S}^\mu = -i\nabla_t Dx^\mu - \frac{i}{24}\nabla^\mu F_{\nu\rho\sigma\lambda}\psi^{\nu\rho\sigma}\partial_t x^\lambda - \frac{i}{24}\nabla_\lambda F^\mu{}_{\nu\rho\sigma}\psi^{\nu\rho\sigma}\partial_t x^\lambda$$

$$
\begin{aligned}
&-\frac{i}{24}F^{\mu}{}_{\nu\rho\sigma}\nabla_t\psi^{\nu\rho\sigma}+\frac{3}{2}\beta\psi_{\nu\rho\sigma}Dx^{\lambda}R^{\mu}{}_{\lambda,}{}^{\nu}{}_{\tau}\psi^{\tau\rho\sigma}\ ,\\
\mathcal{S}^{\mu\nu\rho}&=\beta\nabla\psi^{\mu\nu\rho}-\frac{i}{24}F^{\mu\nu\rho}{}_{\lambda}\partial_t x^{\lambda}\ ,
\end{aligned}
\tag{A.27}
$$

are the equations of motion of x and ψ, respectively.

The action (A.23) is manifestly invariant under one supersymmetry. For a probe described by the action (A.23) propagating on an M-brane background with spacetime metric g and 4-form field strength F to exhibit additional symmetries that are generated by the form bilinears ω and τ of 11-dimensional supergravity, one can consider the infinitesimal transformations

$$
\begin{aligned}
\delta x^{\mu}&=\alpha\,\omega^{\mu}{}_{\nu}Dx^{\nu}+\alpha\,c_1\,\omega_{\rho\sigma}\psi^{\mu\rho\sigma}\ ,\\
\delta\psi^{\mu\nu\rho}&=\alpha\,\tau^{\mu\nu\rho}{}_{\sigma\lambda}Dx^{\sigma}Dx^{\lambda}+\alpha\,c_2\,\tau^{\mu\nu\rho}{}_{\sigma\lambda}\psi^{\sigma\lambda}{}_{\kappa}Dx^{\kappa}\ ,
\end{aligned}
\tag{A.28}
$$

where α is the supersymmetry parameter assigned mass dimension $[-1/2]$ and c_1, c_2 are constants. These transformations are the most general ones allowed such that the infinitesimal variations have the same mass dimension as those of the associated fields, and ω and τ are dimensionless.

For the TCFH on ω to be interpreted as an invariance condition for the probe action (A.23), the conditions that arise for the invariance of this action under the infinitesimal variations (A.28) should match the TCFH. For this first notice that the equations of motion (A.27) contain the spacetime curvature R. As such terms do not arise in the TCFH, these terms in the invariance conditions must vanish. This requires that $\beta=0$. Moreover, if the action had contained a $F\nabla\psi Dx$ coupling, this would have given rise to a FR term in the equations of motion. Because the TCFH does not contain such a term, the $F\nabla\psi Dx$ coupling was neglected from the beginning. The remaining conditions that arise from the invariance of the action (A.23) with $\beta=0$ under the infinitesimal transformations (A.28) read

$$
\begin{aligned}
&\nabla_{\mu}\omega_{\nu\rho}-\frac{1}{12}F_{\mu\lambda\kappa\sigma}\tau^{\lambda\kappa\sigma}{}_{\nu\rho}=0\ ,\quad c_1\omega_{[\rho\sigma}g_{\nu]\mu}+\frac{1}{24}F_{\rho\sigma\nu\kappa}\omega^{\kappa}{}_{\mu}=0\ ,\\
&(2c_1+c_2)F_{\lambda\kappa_1\kappa_2\kappa_3}\tau^{\kappa_1\kappa_2\kappa_3}{}_{[\rho\sigma}g_{\nu]\mu}+\nabla_{\kappa}F_{\nu\rho\sigma\lambda}\omega^{\kappa}{}_{\mu}-\nabla_{\lambda}F_{\nu\rho\sigma\kappa}\omega^{\kappa}{}_{\mu}=0\ ,\\
&F_{\mu_1\mu_2\mu_3[\nu_1}\omega_{\nu_2\nu_3]}=0\ ,\quad -\omega_{[\mu_1\mu_2}\nabla_{\mu_3]}F_{\nu_1\nu_2\nu_3\lambda}+\nabla_{\lambda}F_{\nu_1\nu_2\nu_3[\mu_1}\omega_{\mu_2\mu_3]}=0\ .
\end{aligned}
\tag{A.29}
$$

The first condition matches the expression of the TCFH connection on ω. However, the second condition is rather strong on both M2- and M5-brane backgrounds to admit non-trivial solutions. Moreover, this condition persists even if $\beta\neq 0$ and the curvature terms are included. This does not exclude the possibility that there may be backgrounds such that the TCFH matches with the conditions (A.29) but if this is the case, such examples will be restricted.

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