The Royal Society: Mathematical Futures Programme

Project 6 (RFQ 537-7)

EDUCATIONAL TECHNOLOGIES IN MATHEMATICS EDUCATION

FINAL REPORT

-prepared for submission on 26 April 2023-

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Educational Technologies (EdTech) vs. Digital Technologies (DT)

The Royal Society has commissioned UCLC Consultants to support the Mathematical Futures Programme by undertaking work for Project 6: **EDUCATIONAL TECHNOLOGIES IN MATHEMATICS EDUCATION.** 'Educational technologies' is the terminology used in the brief of Project 6. In general, 'Educational Technologies' are any technologies used for educational purposes at all levels (from early years to higher education). These technologies include but are not limited to hardware, software and digital resources and which can be used to facilitate and enhance the doing, learning and teaching (of any subject) in a variety of modes (face-to-face, online, hybrid or blended). For example, 'Educational Technologies' term has also been used to describe non-digital technologies such as drawing instruments, measuring devices etc that are already embedded in school mathematics.

For the reasons above, in this report we are proposing to use instead the '**Digital Technologies (DT)**' terminology**,** to include hardware, software and digital resources which can be used to facilitate and enhance the doing, learning and teaching of mathematics in a variety of modes (face-to-face, online, hybrid or blended). Like us, the experts we involved and consulted in the Delphi like study for the purpose of carrying out this project also indicated a strong preference for the 'Digital Technology' terminology (see Appendix A). It is worth mentioning that in the mathematics education literature, the commonly used term is 'Digital technology, and the brief of Project 6 also makes use of such terminology, too (e.g., 'the background to Royal Society's programme of education, which aims to equip young students, citizens of the 21st century with the knowledge […] world that is being transformed by <u>digital technologies</u> that are expected to change the types of jobs that will be available in future. [….] <u>Digital technologies</u> are transforming our lives […]).

Section 1: Our vision for digital technologies in mathematics education

The introduction of DT has transformed mathematics and data science. Mathematics, already a powerful tool, has become more powerful with the aid of computers and other computational devices. Recent developments, such as AI, seem likely to accelerate this transformation. As a result, the citizens of the future will need to be both techno-mathematically literate (Hoyles et al., 2010) and data literate (Royal Society report, 2018). However, despite a plethora of calls (e.g., Joint Mathematical Council, 2011; Royal Society report, 2019; Wolfram, 2020), mathematics education has yet to rise to this challenge and the use of digital technologies in the mathematics classroom lags considerably behind the world outside the classroom. Technologies, such as word processing, email and spreadsheets, are widely used in education and there is widespread use of presentational software in mathematics lessons. Yet, in other respects, digital technologies have made little impact on the practices of teaching, learning and doing mathematics in the classroom.

There is a substantial body of research demonstrating the benefits, implementation challenges and constraints in using technology for school mathematical teaching and learning (e.g., Clark-Wilson & Hoyles, 2017; Geraniou & Jankvist, 2019; Crisan et al., 2021) and, moreover, Young's meta-analysis demonstrates that technology enhanced mathematics instruction can contribute significantly to pupils' achievements (Young, 2017).

Research has evidenced that the role of the teacher is crucial in the successful integration of DT in the mathematics classrooms (e.g., Clark-Wilson, Robutti, & Sinclair, 2022; Kissane, McConney, & Ho, 2015). To date, we have witnessed the inspirational use of DT by teachers who are 'strong' believers of the potential and value of DT towards supporting teaching practice and student learning and understanding. These teachers rely on their **intrinsic motivation** to learn how to integrate DT in their practice, e.g., by searching online for resources, watching YouTube videos, speaking to other colleagues who may have used DT too, etc. They become better at it with time and personal effort and dedication. These 'keen to use DT' teachers provide exemplars of how DT can be used to transform the practices of teaching, learning and doing mathematics in the classroom, but they are relatively few in number and our focus needs to be on the many teachers who do not feel they have the time, energy, capability and skills to learn how to use DT in their practice, or those teachers who simply do not believe DT has benefits for mathematics education. But, while professional development (PD) opportunities will be necessary, this will not be sufficient. We cannot rely on teachers' intrinsic motivation to change. They, and the schools that they work in, will need to be incentivised to make these changes through changes to high-stakes assessments and other accountability measures.

Enabling these changes will not be easy and change of this nature takes time. Students and teachers will need ready access to well-maintained and up-to-date digital devices. Teachers will need access to high-quality curriculum materials as well as professional development (PD) opportunities. Awarding bodies will need to develop robust assessments that involve the use of DT. We envisage that such changes are likely to take a generation and will require substantial funding. In order to achieve this, our recommendations focus on actions in three stages: work that needs to be carried out over the next five years, work that will build on this to be completed over the next 10 years and the next 20 years. Therefore, our key recommendation is the development and implementation of a **Maths EdTech Strategy**, to ensure that education providers and the technology industry help improve and increase the effective use of DT in mathematics education.

Below we present a summary of our recommendations. It is important to note here that the numbered recommendations are not presented in a particular order (e.g., of importance), but rather the order follows the structure of this report (see Table of Contents).

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Recommendation 3 **All mathematics teachers are required to engage in a minimum level of DT related PD**	All teachers should have access to CPD opportunities on how DT can be used in their practice and therefore be trained and certified in using maths-specific DT. The government and school leaders should ensure sufficient PD opportunities for all teachers to develop the competencies needed. There should be a requirement for all teachers to engage in a minimum level of DT related PD.
Recommendation 4 **Establish mechanisms for incentives and accountability measures: *Assessment, National Curriculum, Ofsted, Teacher Training Curriculum***	*Assessment:* High stakes assessments and assessments in school mathematics should change to integrate the use of DT for mathematical purposes. DT needs to be fully integrated into the assessment and the assessment need to signal to teachers and students the value of DT. *National Curriculum:* The NC should stipulate and explicitly require the use of DT in mathematics. *Ofsted:* Ofsted inspection framework targets the inspection of DT in schools. *Teacher Training Curriculum:* There needs to be a technology-strand in the Teacher Standards and ECT criteria for QTS qualification, as a technology-enriched ITE maths curriculum.
Recommendation 5 **Support schools to improve their technology infrastructure and provision of DT devices, tools and apps**	All schools have adequate internet connectivity, adequate IT support and service. Funding should be provided to enable all students and teachers to have access to sufficiently up-to-date and well-maintained DT devices, tools and apps.
Recommendation 6 **Improve and support the working environment with access to DT and quality didactical resources**	The working environment (classrooms, schools, home) should be set-up to accommodate the smooth integration of DT in maths teaching and learning. All teachers should have access to high quality didactical resources that enable the integration of DT within mathematics teaching together with guidance on how to use DT.

Section 2: Background

The Royal Society has commissioned UCLC Consultants to support the Mathematical Futures Programme by undertaking work for Project 6: **EDUCATIONAL TECHNOLOGIES IN MATHEMATICS EDUCATION.** The proposed project aims at making recommendations to support educators in recognising which DT are best and for what purpose in mainstream mathematics education (including primary and secondary school level education and higher education) by suggesting strategies for their successful integration. The researchers working on this consultancy project are: Dr Cosette Crisan, Dr Eirini Geraniou and Prof Jeremy Hodgen.

Our approach taken to carry out this project

In carrying out this project and producing this integrated report, we used various sources of information drawing on the results from desk research to explore existing information and knowledge and dialogues with a range of stakeholders, including staff in primary and secondary schools, colleges and universities.

In **Stage I: Developing a Classification** we built on existing work in the area of DT in mathematics education, on the body of research and literature we are very familiar with, hence well-placed to review, adapt and synthesise it and on our academic expertise on DT in mathematics education to classify different DT for teaching, learning and doing mathematics.

Our review of the literature enabled us to **document and classify existing DT that are used in pedagogy, curricula, and assessment against maths outcomes** and put forward a number of recommendations for which technologies should be adopted into mainstream education and provide suggestions for how they should be successfully integrated to support teaching and learning.

In **Stage II: Dialogue with stakeholders** Following our initial classifications and recommendations, **i**n order to further refine and validate our initial classification and recommendations, we undertook a Delphi-like study approach (e.g., as previously used in science education, Osborne et al., 2001 or a literature informed Delphi study in mathematics education and technology by Kallia et al., 2021) with a group consisting of 5 individuals drawn from the following groups of mathematics educators – primary school teachers, secondary school teachers, university tutors, educational technology designers and teacher educators.

The Delphi method is a research tool for establishing consensus among experts in any given field, and so the members of our group will be recruited based on their national and international reputation as experts in integrating DT in their practices and working at different stages of education. Ethical approval for the survey was granted by the UCL IOE Faculty of Education and Society's Ethics Committee (REC1774), and the data collected and included from the participants has been pseudonymised. The participants contributed to this project by i) rating our initial classifications on a 6 – point Likert scale survey and commenting on our suggested recommendations, and ii) taking part in a follow up 30 – 45 min focus group interview, where further conversations about their rating took place. During the focus group interviews, the interview questions are intended to encourage the participants' critical reflection and explicitness about their answers and justifications they provided in the survey. They were also invited to provide specific examples to justify and illustrate the basis for their rating. These examples were then checked against those yielded by our literature review and added to our tables populated with examples which identify technologies and their value to mathematics teaching, learning, and doing in relation to the competences that young people need. The main findings from our Delphi study have been captured in Appendix A, and specific comments of our participants are included at various points in our report.

In **Stage III: Refine the Classification and Recommendations,** we continued to work on the initial classifications and recommendations in light of data from the Delphi study and our literature review, which are included in this report.

A guide to this report

What does this report cover?

In this report we review and classify existing DT with potential for the doing, learning and teaching mathematics.

In Section 3 we present several **classifications** of the DT (devices, tools and apps) intrinsic to mathematics in response to *RQ1.1 What tools are intrinsic to mathematics?* and we also put forward some **recommendations** about which technologies should be adopted into mainstream education, as well as providing suggestions for how they should be successfully integrated.

In Section 4, we engage with RQ2 *How should the system change to adopt these technologies*? And we suggest strategies for addressing the barriers to using new technologies as we identified in our work, answering thus RQ2.2 in particular.

In Section 5 we present another classification, where the potential contribution of those DT identified in section 3 is discussed with respect to the doing, learning and teaching mathematics, and also provide suggestions for how they should be successfully integrated. We address RQ2.1 in particular, namely *What are the professional development needs?* Aspects such as students' engagement or motivation in doing and learning mathematics, and how DT could support assessment are also looked into.

The report ends with the list of References and the Appendices section.

Who is this report for?

The findings of this report are intended to inform the Royal Society's considerations of which DT have the potential to enhance the teaching, learning and doing of mathematics and enable 21st century citizens develop mathematical competency. It is hoped that these findings will also be used to guide the Mathematics Futures Programme's policy recommendations for increasing the use of DT in mathematics education in schools and colleges where it is proved to be beneficial.

We also envisage that the report will be of interest to a wider readership, including policy-makers, educational researchers, school senior leaders, EdTech developers and most importantly mathematics class teachers will benefit from reading this report. We note that the report provides a number of comprehensive classifications devised on the best available evidence. One needs to be cautious and fully aware of the diversity of contexts in which the studies reviewed were carried out. However, some common messages about DT' potential and limitations have emerged over the last approximately 30 years since DT started infiltrating the educational system at all levels.

Section 3: Digital technologies for mathematics education

In this section we are introducing our classification of the types of DT for mathematics education, with the aim of answering *RQ 1.1. What tools are intrinsic to mathematics?*

Note on terminology: devices/tools/apps

Since **Digital Technologies** could include hardware, software and digital resources, tools and applications, also giving rise to confusion. In this report, we also differentiate between **DT devices, tools** and **apps**, terminology we will define and clarify in Section 3 of this report

Our proposed classification differentiates between i) Devices (Table 1) and ii) Tools & Apps that are generic in nature with potential for mathematics education (Table 2), and that are also mathematics specific (Table 3) and these terms are explained below and examples provided to clarify how we propose to use this terminology. An over encompassing table subsuming Tables 1, 2 and 3 is included in the Appendix B – **Classification I: Types of Digital Technologies for mathematics education.**

Classification I: Types of DT for mathematics education

In this report, we use **DT devices** to refer to hardware equipment as exemplified in the column 2 of the Table 1 below. These devices could be used on their own (calculators, graphics calculators, digital measuring devices), for their presentational properties (IWB, laptops, tablets, visualisers), or for accessing information and maths apps (Desmos app. GeoGebra app, AI-calculator).

Table 1: DT devices

Types of DT - Devices	Examples of such DT devices available to students/teachers/schools
DT Devices that could also be used for maths education	Calculators; Graphics Calculators; iPads/tablets; Laptops/desktops; IWB; smart phones; visualisers, graphics tablets and display devices; measuring devices such as ruler app measure; AR ruler app;

Such DT devices are usually used in combinations with a 'software', or an 'app', or a 'platform' or a 'language' with the aim of facilitating learning and interaction between the user and the device.

In our report we will use the terminology **DT apps** to refer to applications which provide an interactive digital environment for processing and analysing, sending, and displaying information. Such DT apps enable connected classrooms, where students can share their work, reflections, solutions with their peers, but also with the teacher. To note here that 'apps' is a shorthand for 'applications', but not in the sense of a process of applying.

In mathematics, DT apps are hosted by various websites, and are designed to perform a specific task. DT apps have various aims and design features, and a number of maths specific apps commonly used in by learners in schools or at home are included in the table below:

Table 2: DT apps

Types of DT- Apps	Examples of such DT apps available to students/teachers/schools
Generic DT apps	Email, MS Teams, VLEs, Google classrooms, Zoom videoconferencing, Mentimeter, Poll, Twitter, YouTube
Apps for mathematics education	Practice-based apps; Game-based apps; Stick 'n' Split – times table game; Geoboard; Khan Academy;

In our report we will also use the terminology **DT tools**, also frequently used in the mathematics education field (e.g., Clark-Wilson et al., 2020).

In the table below, we provide a classification of DT tools

- that differentiates between mathematics specific and generic tools (column 1);
- drawing on our expertise and knowledge of research into the potential and limitations for DT for mathematics education, in column 2 we categorise DT tools according to how they could be used to support/enhance/facilitate the learning and teaching of mathematics;
- in column 3 we provide a brief description of the potential the use of such tools has for enhancing/supporting/facilitating mathematics education. The key term here is *potential*, as DT has the potential to improve the teaching and learning of mathematics in many ways, however the degree and the type of use in maths lessons differs widely and is dependent on conditions such as availability, ease of access, levels of technical and pedagogical expertise;
- Relying on our knowledge of the most widely used DT for mathematics education purposes, but also on our experiences as former classroom teachers, and currently as teacher educators working with partnerships with over 100 schools annually, in column 4 we provide specific examples of DT tools and apps in each category of DT. These are examples (and not a comprehensive list) of tools and apps that are currently available to and used in (some) schools.

Throughout this report we will be referring to the main categories of DT tools (column 2) and make little or no references at all to specific tools and apps; in other words we won't single out a specific tool or apps such as GeoGebra, Desmos or Hegarty Maths platform, as their access and availability over the years could change (e.g., GeoGebra is currently free to use but this may not be the case in a few years' time, which will then affect whether or not teachers will use/continue to use it) or other new applications will be made available to schools in the future.

Table 3. DT tools

Types of DT - Tools	Categories	A brief description of their potential for maths education	Examples of such tools
Mathematics-specific DT tools	**Tools for outsourcing the maths**	Tools that outsource processing power and are capable of performing numerical calculations and algorithmic processes and of producing accurate 'answers' quickly, and in some cases handling vast amount of routine algebraic manipulations (CAS).	Four-function, Scientific, and Graphic calculators Excel CAS applications (Maple, Mathematica, *Derive*, Wolfram Alpha)
	Dynamic Mathematical tools	Software with specific in-built tools (e.g., for sketching graphs, drawing shapes, typing algebraic expressions, representing data) which can be manipulated dynamically.	GeoGebra, Geometer's Sketchpad, Cabri, Graph, Autograph, Omnigraph, Desmos, CAS with dynamic features
	Data analysis software	Tools that allow the teachers and students to capture, collate and analyse data.	Excel, Python, CODAP, R
	Programming tools and languages	Software programs that aid the users in creating, editing, debugging, maintaining and/or performing any programming task.	Coding environments such as: Scratch, Logo, 3D Logo Programming languages such as: Python, MATLAB
	Extended Reality (XR) technology tools	Tools that enable students experience and interact with the 3D world virtually, which involve the human body through movement and gestures.	Augmented reality such as GeoGebra's 3D Calculator (with AR) Virtual reality (VR) such as GeoGebra 3D
	Mathematics content learning platforms	**Learning platforms** enable users access a variety of learning resources such as worksheets, homework tasks, supporting information, links to useful sources as information. These platforms (also known as integrated learning systems (ILS) record all the work that each individual students do. These are interactive and exploratory tools with personalised and adaptive student learning.	MyMaths, Dr Frost Maths, Hegarty Maths, Integral Maths, Purple Maths, Headstart, Times Tables Rock Stars, AtomLearning, Mathletics
	AI tools	AI-based calculators take input as text, translates it into mathematical notation, and returns the answers in numerical form. ChatGPT is an OpenAI which could be used to answer parts of questions, to check, and offer support for learners' mathematical reasoning.	AI-based calculators (Photomath), AI tutoring systems (Chat GPT)
Generic DT Tools	Generic tools that enable the communication of mathematics through creating worksheets, power point presentations, converting handwritten scripts, etc. Such generic tools significantly transform the processes of writing, and in some cases, the reading of mathematics.		LatTeX, MyScript, Equation Editor, Mathpix Snip, MS Word, MS PowerPoint, MS Forms

Recommendations for the successful integration of DT

Table 3 above documents and classifies existing DT that are used in mathematics education at *various* extents, by *a few* teachers and students in *some* schools, showcasing variation in use of DT.

Several studies found that differential access to technology was a significant factor in different levels of participation and engagement in mathematics in remote learning during the COVID pandemic and that this particularly affected disadvantaged students (e.g., Anders et al., 2022; Taylor et al., 2022). To ensure that technology does not exacerbate these inequalities, we are proposing that DT are adopted into mainstream education by all schools and are readily and easily available to all teachers and students. These technologies need to be not only up-to-date, but also well-maintained. Considerable funding will be required to provide these technologies in the first place, then ongoing funding will be required to maintain and replace them as necessary.

- **Recommendation 1 – Make mathematics-specific DT easily available and accessible in mainstream education**

School should provide teachers and students with access to the mathematics-specific DT according to the timescale suggested below:

The adoption and integration of maths-specific DT	
Now – 5years	**5-10 years**
• **Tools for outsourcing the maths;** • **Dynamic Mathematical tools;** • **Data analysis software;**	• **Programming tools and languages;** • **AI tools;** • **Extended Reality (XR) tools**

Currently, the **Mathematics-specific DT** have been widely researched and empirical evidence exists for their potential as well as limitations for mathematics education (e.g., calculator use can actually improve pupils' mental and written calculation skills – in Hodgen et al. (2018); "there is a moderate cumulative effect of technology enhanced instruction on mathematics achievement" (Young, 2017, p. 27); there is a need for a knowledgeable other (like the teacher) to guide students' interaction in order to promote conceptual knowledge when Dynamic Maths tools (like Dynamic Geometry Software) are used for presentation and modelling mathematics - in Young (2017)). To note here that the participants in our Delphi-like study rated the **Mathematics-specific DT tools: Tools for outsourcing maths**; **Dynamic Mathematical tools**; **Data analysis software**; and **Programming and language tools** very highly for their potential to enhance each of the following doing/learning/teaching mathematics, and all agreed that teachers have a huge influence on the learning potential of these tools, especially in the case of Dynamic Mathematical tools, which are *"very powerful but might be tricky for learners to use themselves, at least initially"*.

The participants on one of the two focus groups discussed about the **Programming and language tools** in greater detail and commented on and rated highly the **difficulty in integrating** these tools in maths doing and learning lessons, as they are *"quite dependent on teacher's confidence in coding*", which should be part of the maths syllabus. Regarding the **Data plotting and analysis software,** one participant commented on the challenges in realising the potential for learning mathematics *"because it has a narrow maths focus, but for teaching data and analysis, they are now, I would say, absolutely necessary".*

The other tools such as the **Mathematics content learning platforms** and **Apps for mathematics education** also have potential for mathematics education (e.g., mathematics content learning platforms, including intelligent tutoring systems (ITS) which 'works' in some cases (Cheung & Slavin, 2013; Steenbergen-Hu & Cooper, 2013). In particular, such technologies could be used to free up teacher time, and thus improve mathematics outcomes this way – Hodgen et al. (2018). However, these tools are varied in design and scope, and some have a narrow focus of exam/test preparation, but nevertheless "*Content platforms are useful for extra practice and catch up"* according to one of our participants, but overall, the Delphi participants acknowledged the limited research and empirical evidence on the factors that contribute to the efficacy of these tools (Outhwaite et al., 2022).

There are also the more recent tools with potential for mathematics education such as the **AI tools**, which have recently become the focus of educational research. There is currently almost no empirical evidence about whether, and how, these tools can be used to support mathematics education, and while in this report we will comment on these tools, we do not claim to have done so based on 'hard evidence' for our comments and recommendations. Our Delphi participants found rating these tools difficult, since "*These tools are rather untested."; "Hard to say about AI – haven't explored it enough to know its potential for doing/learning/teaching)";* Similarly, the participants did not rate XR tools, as *"These tools are rather untested".*

Availability of such DT tools and apps is a pre-requisite for DT to fulfil their potential for mathematics education. However, despite research findings that have shown that DT can be a powerful tool in mathematics education, further research is needed to support teachers on how to best integrate them into their teaching and for what purposes. Similarly, despite the growing evidence which points to a positive correlation between students' engagement with DT and education outcomes, the OECD (2020) reports on a lack of consensus on the contribution DT make to students' educational attainment or cognitive performance in general.

- × **Barriers - The lack of research on how DT can be implemented in maths education and the lack of empirical evidence on their potential contribution and limitations to doing/learning/teaching mathematics. It is worth noting that in particular, the impact of *AI tools* and also *XR tools* in maths education has not been researched yet.**

- ➢ **Recommendation 2 – Continue to research and develop the implementation and application of mathematics-specific DT in mathematics education**

Further research on the implementation and application of specific DT tools such as, *Data analysis software; Programming tools and languages; AI tools* and also *XR technology* in the teaching and learning of mathematics to be carried out according to the time scale suggested below:

Now	In 10 years	In 20 years
To continue to research and develop implementation of: • **Tools for outsourcing the maths;** • **Dynamic Mathematical tools;**	Further research needed on the implementation and application of: • **Data analysis software; Programming tools and languages;** • **AI tools;** • **Extended Reality (XR) technology**	Research-Informed implementation of Maths EdTech strategy is launched.

Section 4: System needs for integrating digital technologies

In Section 4, we answer RQ2 *How should the system change to adopt these technologies*? and we suggest strategies for addressing the barriers to using new technologies as we identified in our work, answering thus RQ2.2 in particular.

As mentioned in Section 1, our key recommendation is the creation and implementation of a **Maths EdTech strategy** in the UK requiring actions from government and policymakers, schools and those involved in teacher training to ensure future students and teaching force become digitally literate.

System needs for effective integration and use of DT in mathematics education: Barriers and Recommendations

We have identified four main **'system needs'** challenges associated with use and implementation of an Maths EdTech strategy in mathematics education, namely: **Mathematics Teachers' Knowledge and Competency with DT, Incentives and accountability measures, Infrastructure and Working environment and resources**, which include teachers professional development needs, and which we will discuss in turn below.

System Needs - Mathematics teachers' knowledge and competency with DT

Availability of DT is a pre-requisite for digital technologies to fulfil their potential for mathematics education. However, despite research findings that have shown that digital technologies can be a powerful tool in mathematics education, use of such technologies in maths classrooms is sparse. A survey of English mathematics teachers' use of ICT carried out in 2014 by Bretscher revealed high frequency usage of certain types of hardware, e.g., IWBs, and software, e.g., tools for IWBs, PowerPoint and web-based mathematics resources, e.g., HegartyMaths. More recent findings (Crisan et al., 2021; Clark-Wilson et al., 2023; Jankvist & Geraniou, 2022) show that there still is a tendency among teachers to assimilate the new technology into traditional practice, instead of using it in a transformative manner, which encourages more exploration, inquiry and collaboration, through capitalising on the visualization at the symbolic, graphical and numerical level facilitated by technologies. As a result, not much has changed in the way mathematics is taught and how students learn and do mathematics in the digital era, despite the plethora of educational technologies available to teachers and students.

Teachers rely on their own skills to select and adapt resources involving the use of technology, irrespective of their access to training (e.g., Clark-Wilson & Noss, 2015). Given the evidence that teachers' skills are positively correlated to teaching quality, which in turn affects pupil outcomes (Kunter et al., 2013), one of the challenges/issues? is how to help mathematics teachers use DT effectively. A recent report by Oxford University Press released online in 2021 revealed that, there is "a lack of digital skills, in both learners and teachers, preventing their full engagement with education" and therefore acts as a barrier to learning. In mathematics education in particular, research (e.g., Geraniou & Jankvist, 2019; Geraniou et al., 2022) indicates the need for teachers to develop mathematical digital competency for doing maths with DT, as well as mathematical digital competency for teaching with DT, as shown in the table 4 below. In order to understand the table, in the following we explain the terminology.

A mathematical competency is an individual's well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge (Niss & Højgaard, 2011, p. 49). Regarding digital competency, Hatlevik and Christophersen (2013) claimed: "a concept such as digital skills focusses on dealing with the technical conditions, whereas digital competence and literacy are broader terms that emphasise what kind of skills, understandings, and critical reflections students are able to use" (p. 241). Mathematics students of today are expected to engage deeply in a techno-mathematical discourse when interacting with EdTech, and their understanding of the mathematical concepts involved is almost inseparable from the digital tools and their digital competency. Therefore, we should not look separately at mathematical and digital competency, but instead at the interplay of the two, defined as 'mathematical digital competency' by Geraniou and Jankvist (2019). Regarding teachers, MDCT is the competency "teachers need (or have) to select and implement technology in their practice in pedagogically productive ways" (Geraniou et al., 2022, p. 7). Possessing MDC involves three elements as shown in the table below, and similarly possessing MDCT also involves three elements as shown in the second column of table 2 below.

Table 4: MDC and MDTC

Development of Mathematical Digital Competency (MDC) for teachers and learners	**Development of Mathematical Digital Competency for Teaching (MDCT)**
[MDC1]: ***Being able to engage in a techno-mathematical discourse***. In particular, this involves being fluent in using any DT as a tool for doing/learning/teaching mathematics, and therefore becoming techno-mathematically fluent. [MDC2]: ***Being aware of which DT tools to apply within different mathematical situations and contexts, and being aware of the different tools' capabilities and limitations.*** [MDC3]: ***Being able to use DT reflectively in problem solving and when learning mathematics***. This involves knowing how and knowing why specific features of any DT tool to solve a mathematics problem.	[MDCT1]: Being able to engage in a techno-mathematical discourse not own use, but for **pedagogically** supporting learners to become techno-mathematically fluent. [MDCT2]: Being aware of which digital tools to apply within different mathematical situations and context, and being aware of the different tools' capabilities and limitations, **so as to think, and act, pedagogically with these tools,** while considering the benefits and limitations of these. [MDCT3]: Being able to use digital technology reflectively in problem solving and when doing (learning **or teaching**) mathematics.

× **Barriers - Teachers' limited competency in doing and teaching maths with DT, as well as access to resources, including professional development**.

➢ **Recommendation 3 - All mathematics teachers are required to engage in a minimum level of DT related PD**

The government and school leaders should ensure sufficient PD opportunities for all teachers to develop the competencies needed. There should be a requirement for all teachers to engage in a minimum level of DT related PD. All teachers should have access to CPD opportunities on how DT can be used in their practice and therefore be trained and certified in using maths specific DT tools.
In order for the maths specific DT to be used by teachers effectively in their practice, the following should happen:

	Digital competencies	**CPD provision and certification**	**School leadership / Resources**
in 5 years	Teachers become aware of the DT tools intrinsic to maths (Classification I) and become familiar with their capabilities.	Develop, pilot and evaluate a universal, on-demand 'DT in Mathematics passport' certification.	All mathematics teachers will have an entitlement (at least once a year) to access training opportunities in using the DT tools for teaching maths.
in 10 years	Each school will have an DT Maths Lead to run CPD for developing maths teachers' MDC and MDCT and to offer support.	The self-study asynchronous e-CPD is rolled out nationally and EdTech Maths Lead in each school gains the 'DT in Maths passport' certification.	All mathematics teachers will have an entitlement (three times in a year) to access training opportunities in using the DT tools for teaching maths.
in 20 years	All maths teachers become proficient with maths specific DT tools and can make the right choices in selecting DT tools for teaching various maths topics.	All mathematics teachers in schools in England will have gained 'DT in Maths passport', and a requirement for updating the training every 3 years (as part of the EdTech Maths strategy).	All mathematics teachers will have an entitlement to enable them access *continuous* professional development opportunities on how DT can be used to improve doing/learning/teaching of mathematics.

System Needs - Incentives and accountability measures: *Assessment, National Curriculum, Ofsted, Teacher Training Curriculum*

Research on the uptake of digital technologies in maths teachers' practices (e.g., Thomas and Palmer (2014) in Clark-Wilson et al., 2020; Fauth et al., 2020; Thurm&Barzel, 2022) also highlighted the important personal component of the teachers' "personal orientations", that is their beliefs, motivations and attitudes towards teaching and learning with EdTech tools. For example, recent studies showed that teachers' inexperience and low confidence in their skills for using digital technology for assessment purposes (Drijvers et al., 2021) is a barrier to EdTech implementation. When using maths learning platforms for remote mathematics teaching and learning (e.g., online homework in MyMaths), formative assessment could be an issue due to limited opportunities to engage in mathematical talk, meta-cognitive activities or receive formative feedback (Hodgen et al., 2020).

Chopin et al. (2014) documented two distinct digital maths curriculum types, individualized learning programs and digitized versions of traditional textbooks. While the programs offered some of the features identified as transformative, particularly with respect to assessment systems that rapidly and visually report student performance, there were many features that did not take full advantage of the digital medium. Subsequently, teachers may need to be supported to better understand the assessment reports of students' performance offered by these digital programs.

Drijvers (2018) concluded that "digital assessment of mathematics is a phenomenon that will play an increasingly important role in mathematics education." (p. 63). Indeed, how to design tests with and through digital technology that assess student knowledge and mathematics, and not their DT usage competencies and skills is an under-researched area. Current attempts at integrating DT into assessment of mathematics rarely go beyond multiple-choice types of questions, an aspect that is more related to the *digitisation of the assessment*, where the approaches to assessing maths are the same as they have always been apart from assessment 'happening' via a digital tool. There is some research on how Mathematics-Specific DT tools *DT* could be *integrated into assessment* to "make students really "do mathematics" in a digital test, to express themselves mathematically, to show, to produce." (Frenken et al., 2022, p. 61). For example, Frenken et al. (2022) proposed how students could interact with GeoGebra within standardized testing, as part of an initiative in Germany aiming at providing a standardized assessment instrument by digital means, in order to assess students' mathematical competencies.

Influential in this respect could be an explicit mention of DT in the Ofsted inspection framework, while schools need support in the form of external guidance to understand how to prepare for Maths EdTech strategy.

- × **Barriers - Need for guidance and professional development opportunities for teachers in understanding how to use DT for *assessment purposes*; DT often perceived by schools and school leadership as an add-on to doing and learning mathematics, not clearly embedded into *schemes of work*; no explicit mention of DT in the NC for mathematics; DT use in mathematics currently missing from criteria of Ofsted inspection framework.**

- ➢ **Recommendation 4 - Establish mechanisms for incentives and accountability measures: *Assessment, National Curriculum, Ofsted, Teacher Training Curriculum***

	Assessment
in 5 years	Assessment practices, both formative (e.g., informing teachers about students' progress) and summative (e.g., high-stake assessments) motivate? the use of DT tools and devices; high stake assessment includes exam questions that require interaction with maths-specific DT tool for solving and/or submitting answers to questions.
in 10 years	A review of the assessment policy is undertaken to incorporate the interaction with maths specific DT tools; Considerable resources need to be devoted towards developing good assessments with DT as an integral feature, that need to be piloted and evaluated; Assessment processes become more effective and efficient.

in 20 years	Assessment practices (both formative and summative), including high-stake assessment include activities that require the interaction with DT; Awarding bodies develop robust assessments that involve the use of DT.

	Curriculum
in 5 years	A revision of the current National Curriculum for Mathematics should be undertaken, to include specification of the knowledge and skills required to use DT for doing and learning various mathematics topics - e.g., embed DT into textbooks
in 10 years	A new National Curriculum for Mathematics is developed, with opportunities to use DT tools as programming language and environments.
in 20 years	The content of the National Curriculum for Mathematics is developed and implemented in all schools, that has the use of DT at its heart, to acknowledge the significant use of DT in mathematical practices in schools and in the everyday lives of young people.

	Ofsted
in 5 years	Ofsted inspection framework used to regulate 5 years PD targets for schools
in 10 years	Ofsted inspection framework used to regulate 10 years PD targets for schools
in 20 years	Ofsted inspection framework used to regulate 20 years PD targets for schools

	Teacher Training Curriculum
in 5 years	Teachers Standards should make specific references to the need for use of DT in mathematics education, hence the curriculum for ITE programmes will be developed to address the development of student-teachers DT needs.
in 10 years	All initial teacher education providers will be implementing the renewed programme and the assessment for awarding QTS should entail an assessment component for DT usage in their maths teaching practice.
in 20 years	Teachers need to dedicate time to familiarise themselves with the DT in question, both by freely exploring the functionalities of the DT, doing some mathematics themselves.

System Needs - Infrastructure

The DT tools availability to learners and teachers varies from one school to another, and even from one teacher to another in the same school. Digital divide shows pupils themselves have varied access to such tools and devices (e.g., Oxford University Press Report, 2021). In order for DT to be an integral part of the education of young people, and therefore support the development of 21st century skills, it is imperative that availability and access to such tools are prioritised. However, availability and accessibility do not suffice for DT tools and applications to change current pedagogic understanding and practices. Joyce and Showers (1995) (as cited in Smith et al., 2006) argue that teachers need extended opportunities to think through new ideas and to try out new practices, supported by more expert practitioners in order to change or refine their use of DT.

- **Barriers - differential internet connectivity at school and home; inadequate IT support and service in schools; differential access to DT tools, devices and apps for students and teachers.**

  A very simplistic modelling approach taking into account the number of schools (primary and secondary) and the number of students 3-19 (about 10million pupils) suggests that a substantial investment is needed, assuming one device per student; costs incurred as a result of schools needing to maintain and support these devices via e.g., one EdTech support-person on average per 5 schools). However, such an investment will support not just the implementation of the **Maths EdTech strategy, but also of an EdTech strategy for ALL other SCHOOL SUBJECTS**.

- **Recommendation 5 - Support schools to improve their technology infrastructure and provision of DT devices, tools and apps**

	Infrastructure needs	Schools 'preparedness'
in 5 years	A Maths EdTech strategy developed and agreed, with a clear and costed plan for support and implementation of DT tools and devices, for realising the infrastructure needs. Schools assess their EdTech needs and preparedness.	Elicit the beliefs & needs of school leaders and teachers with regards to the potential of DT to improve maths education. School leaders to receive professional development on why change is needed and how to do it. Government needs to commission work on costs to support implementation of a Maths EdTech strategy.
in 10 years	All mathematics teachers and learners should have access to DT tools and devices and have an entitlement to learn how to use them; DT devices provisions must be made as equitable as possible.	All school leaders to have access to established (research & evidence informed) professional development opportunities; Government is supporting financially all schools to provide good access to well-maintained devices for all students and teachers.
in 20 years	All mathematics teachers and learners should have access to EdTech devices and tools to enable access of maths specific tools and applications during maths lessons and beyond classrooms; Access to the DT tools and devices should be free at point of use; All teachers and learners will have internet access at home and school. IT support available (one person per 5 schools).	

System Needs - Working environment and resources

Ruthven (2009) developed a conceptual framework that acknowledges the challenges teachers are faced with and identifies strategies for overcoming them. More specifically, Ruthven identifies the key structuring features of classroom practice, showing how they relate to technology integration, amongst which are the working environment and resource system. An example of such a resource is the interactive white boards (IWBs) which are not expensive and have been found to be a useful presentational tool to have in the classroom. In itself, this device will not bring about fundamental change in the traditional patterns of whole class teaching shown (Smith et al., 2006), but if research informed guidance as part of the Maths EdTech strategy exist, its potential can be realised.

Cajella et al. (2021) found that teachers, at all levels, need 'just-in-time' assistance as they struggle to adapt new curricula and new instructional practices to their unique classroom contexts. There is potential for teacher guidance on how to integrate DT into their practices to be incorporated within the curriculum materials.

× **Barriers - The existing working environment (classrooms, schools, home) set-up is not designed to accommodate the smooth integration of DT in maths teaching and learning. Currently, there is an un-coordinated collection of didactical resources left to teachers' own decision on how to combine and adapt for subject teaching, where the schemes of work are fairly rigid with no explicit references to opportunities for DT use.**

➢ **Recommendation 6 - Improve and support the working environment with access to DT and quality didactical resources**

	Working environment
in 5 years	There needs to be a Maths EdTech strategy developed and approved, with a clear and costed plan for support and implementation of DT and for realising the infrastructure needs.
in 10 years	All mathematics teachers and learners should have access to DT tools and devices and have an entitlement to learn how to use them. Teacher guidance to be incorporated within the curriculum materials, so that teachers are informed on best practices with DT; DT devices provisions must be made as equitable as possible.
in 20 years	All mathematics teachers and learners should have access to DT devices and tools to enable access of maths specific tools and applications during maths lessons and beyond classrooms; Access to the DT tools and devices should be free at point of use.

Section 5: The potential of digital technologies for mathematics education

In this section we present another classification, where the potential contribution of the digital technologies we identified as intrinsic for mathematics education (Classification I) is discussed with respect to the doing, learning and teaching mathematics. We also provide suggestions for how these technologies could be successfully integrated. We address in particular RQ2.1 *What are the professional development needs?*, as well as aspects such as students' engagement and motivation in, and assessment of learning mathematics benefits from DT.

Using DT for doing, learning and teaching mathematics

We start by clarifying our interpretation of using DT for various 'activities' (to use the terminology of the brief), namely doing, learning and teaching mathematics.

Students' learning of mathematics is a result of their teachers teaching it and happens through their actually doing mathematics and digital technology can play a role in all three of activities. Since this project requires to *"identify technologies and their value to mathematics teaching, learning, and doing in relation to the competences that young people need*", below we share our 'definitions' of using EdTech for doing/learning/teaching mathematics:

- **Using DT for *doing mathematics*** – it refers to the functionalities offered by DT that enable learners (pupils/students) to 'do' mathematics, i.e., how DT could be used to support learners in searching for and finding solutions to problems, by using and applying the maths learned.

- **Using DT for *learning mathematics*** – it refers to how DT could support the practising of mathematical skills and the development of conceptual knowledge to support to support the understanding of mathematics and the development of mathematical and computational thinking*.

  * **Computational Thinking** is a term defined in variable, yet similar, ways in the literature. We align our work with the definition offered by Kallia et al. (2021): "A structured problem-solving approach in which one is able to solve and/or transfer the solution of a mathematical problem to other people or a machine by employing thinking processes that include abstraction, decomposition, pattern recognition, algorithmic thinking, modelling, logical and analytical thinking, generalisation and evaluation of solutions and strategies" (p.28).

- **Using DT *for teaching mathematics*** – it refers to how teachers employ DT to plan for and organise the teaching, together with the pedagogical strategies for using DT productively to support the learning of mathematics and its assessment.

Classification II: Contributions of DT to mathematics education 'activities'

There are several well-developed existing taxonomies that have been used by other researchers in their attempt to classify educational technologies in mathematics education (e.g., Drijver's (2012), 'do mathematics' and 'learn mathematics'; Pierce & Stacey's (2010) map of 'pedagogical opportunities' with respect to 'mathematics analysis software; Chopin et al. (2014) 'typology for analyzing digital curricula in mathematics education'). Given our 'definition' of terms introduced earlier, and based on our review, analysis and synthesis of these taxonomies, but also on our very own classification of using educational technology for understanding mathematics, managing interactions and communication in mathematics, and assessing mathematics in Bretscher, Geraniou, Clark-Wilson, and Crisan (2022), we propose the following classification of the contributions of DT tools for the doing, learning, and teaching of mathematics (Table 5, also included in Appendix B).

Table 5. Classification II

Contributions of DT to maths 'activities':	Functionalities of using DT
Doing mathematics	**Carrying out procedures (**computations and procedures are carried out efficiently and with accuracy by the tools) and **communicating mathematically**
	Visualisations of representations and **dynamic manipulations** of mathematical objects enabled
	Supporting reasoning and problem-solving
Learning mathematics	**Practising skills** to support acquisition of maths knowledge and skills through targeted practice and to communicate mathematically
	Developing conceptual knowledge through:
	• ***Interacting with representations of mathematical ideas/concepts/objects***
	• ***Dynamic exploration of mathematical ideas/concepts/objects***
	• ***Developing strategies for problem-solving***
	Developing reasoning skills and problem-solving skills
	Developing productive dispositions to learning maths with DT
Teaching mathematics	**Planning and preparing teaching**
	Making decisions for exploiting the chosen tools' 'functionalities'
	'Real time' teaching
	Subsequent planning of teaching informed by data provided by DT

In the following, we will expand Table 5 (also included in Appendix C) by considering 'Using DT for each activity', i.e., doing (table 6a), learning (table 6b) and teaching Table 6c) mathematics.

These tables share the same headings, whereby:

- column 1 stipulates the specific **activity,** i.e., doing, learning and teaching mathematics;
- column 2 lists the **functionalities** of DT tools to support each 'activity'; functionalities of a DT tool do not refer to the technical aspects/capabilities of a tool (e.g., the menu of the dynamic tool GeoGebra which enables users to draw shapes), but rather they refer to what the DT tool can do to support a maths 'activity' (doing, learning and teaching) in ways that may not be available without such technologies (e.g., Dynamic tools allow for many examples to be generated with speed and accuracy);
- column 3 lists the **values** of using DT; the value of using a DT tool states how each 'activity' benefits from using a DT tool. In this column we also indicate how the values of DT tools enable learners to develop different mathematical competences (C1, C2 and C3) required by 21st century citizens*.
- column 4 exemplifies both the functionalities and values of a DT tool for the specific 'activity', and finally,
- column 5 returns to the categories of DT tools in Classification I, and provides **examples of DT tools** that could be used to enrich maths activities mentioned in column 4.

*Note: Building on the RS citizen typology (1 to 5), we are proposing a 'simpler' categorisation of mathematical competences for as follows:

C1: 'functional mathematics' competences (by collapsing 1 and 2)

1. Basic functional numeracy for personal decision making
2. Mathematical literacy to understand and critique numerical claims

C2: competences for 'traditional non-quantitative professions' (3),

3. Traditionally non-quantitative jobs/professions, but requiring mathematical expertise: e.g., lawyers, journalists, civil servants, politicians, health practitioners, administrators, teaching practitioners (not mathematics), technical and vocational professions – including technicians

C3: competences for 'mathematics-based professions' (by collapsing 4 and 5)

4. Jobs/professions where mathematical competences are a core component, e.g., analysts, engineers, financial professionals, scientists, social scientists.
5. Mathematical sciences, e.g., mathematicians, mathematics teachers.

Tables 6a, 6b and 6c are very detailed and built based on our expertise and extensive experience, and review of over 30 years of literature on DT in mathematics education.
These tables provide guidance and suggestions for how DT could be successfully integrated in maths education for various 'activities' (doing/learning/teaching).

Table 6a: Using DT for doing mathematics

Activity focus	Functionalities	Value	Examples of DT enriching maths 'activity'	Examples of DT tools
Doing maths	**Carrying out procedures** and **communicating mathematically**	DT tools afford the learner to perform mathematics tasks quickly and accurately. Most tasks that students outsource to these tools can be completed by hand, although sometimes these tasks may be extremely arduous. (**C1, C2, C3**)	**Maths activities are outsourced to DT such as:** • carrying out numerical calculations by hand, such as long division, or performing calculations involving large numbers; • evaluating numerical expressions; • carrying out calculations with other than 'tidy' numbers (e.g., decimal numbers not just natural numbers), which can be accessed at earlier levels of education	**Tools for outsourcing the maths** (e.g., four-function calculators, scientific calculators)
			• drawing graphs, especially if sketching graph is not the learning objective of the task • performing operations on functions • solving equations graphically • measuring sides, angles, perimeter, areas of geometric figures with precision	**Tools for outsourcing the maths** (e.g., graphics calculators) **CAS applications** (Maple, Mathematica, *Derive*) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre)
			• 'controlling' devices and developing deep procedural knowledge • creating algorithms or codes, promoting computational thinking	**Programming tools and languages**

Doing maths **(cont.)**	**Carrying out procedures** and **communicating mathematically**	By releasing students from the more 'such' tedious activities, DT tools allow students more time to think about the mathematical ideas and concepts involved and to explore (**C1, C2, C3**) and focus more on conceptual knowledge (**C2, C3**) and communicate mathematically (**C1, C2, C3**)	**DT tools can support students engage in more complex maths activities such as:** • management of large number of computations performed with accuracy, which support a focus on generalisation, spotting and testing patterns. • number reasoning (e.g., Pattern spotting among digits of large numbers) • constructing shapes for defining and testing properties of shapes • using the almost instantaneous feedback of the tools and enabling the successive refinement of informal conjectures and solutions. • handling (large) data by entering it into a spreadsheet and pre-analysing the data by producing a table of output quickly, then analysing it for visual patterns, or quickly rearrange information and re-engage with data and re-analyse from this new perspective. • coding, which leads to developing programming skills such as problem-solving, creativity, debugging, observing patterns, seeing connections, identifying algorithms, etc. • refining steps in their solutions or code based on instant feedback via the 'output' of the tool.	**Tools for outsourcing the maths** (e.g., graphics calculators) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Programming tools and languages** (e.g., Scratch, Python) **Data analysis software** (e.g., Excel) **Programming tools and languages** (e.g., Scratch, Python)

Doing maths (cont.)	**Visualising** representations and **dynamically manipulating** mathematical objects	DT tools allow more examples to be generated, which in turn allow and encourage students to explore mathematical ideas, concepts and problems and visualise, spot (**C1, C2, C3**) and focus on underlying relationships. (**C2, C3**)	**DT tools can support doing mathematics through activities such as:** • Displaying of representations of mathematical objects (shapes, graphs, but also numerical representations) • Changing size, positions and orientations • Representing transformations on objects (shapes and graphs) • Construct multiple representations to explore maths representations dynamically and how they are connected • Such tools can afford teachers and students to dynamically link multiple mathematical representations, to create models of real situations, and link symbolic, algebraic and geometrical representational simultaneously • Using coding to build a model to offer a mathematical representation or mathematical solution to a problem.	**Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Data analysis software** (e.g. , Excel **CAS applications** (Maple, Mathematica, *Derive*) **Programming tools and languages** (e.g., Scratch, Python)
	Supporting more advanced reasoning and problem-solving skills	DT tools support students' mathematical reasoning through encouraging exploratory and constructionist approaches to mathematics. (**C2, C3**)	**DT tools can support mathematics reasoning through activities such as:** • checking, modifying, providing an interactive interface to explore what happens if parameters are changed/stay the same, linking the general to the specific • instant (tailored) feedback; taking risks in a safe digital environment • investigate maths problems/relationships in digital environments • using logic and reasoning to evaluate and interpret situations before modifying their investigative process • using DT to model mathematically; applying their programming skills to relatively sophisticated mathematics problems • giving students a measure of autonomy in their investigation • creating codes or a programme, hence developing computational thinking • use a programming language to solve a variety of computational problems.	**Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Programming tools and languages** (e.g., Scratch, Python) **Data analysis software** (e.g. , Excel **CAS applications** (Maple, Mathematica, *Derive*) **AI tools**

Table 6b: Using DT for learning mathematics				
Activity focus	**Functionalities**	**Value**	**Examples of DT enriching maths 'activity'**	**Examples of DT tools**
Learning maths	**Practising skills** to support acquisition of maths knowledge and skills through targeted practice and to communicate mathematically	DT tools can quickly and accurately perform calculations, measurements and transformations of shapes. Such tools provide an almost instantaneous feedback to changing parameters, shape, position, and could be used to rapidly generate more examples to provide more persistent practice until a degree of competency and understanding is achieved (**C1, C2, C3**)	• Practicing at own pace, and repeating certain types exercises with feedback (repeating as often as needed - e.g., calculus differentiation, integration, solving equations, estimation of shape of graphs of functions before sketching)	**Tools for outsourcing the maths** (e.g., graphics calculators) **CAS applications** (Maple, Mathematica, *Derive*) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Mathematics content learning platforms** (e.g, Dr Frost Maths, Hegarty Maths, Integral Maths, Times Tables Rock Stars)
	Developing conceptual knowledge through:			
	➢ ***Interacting with representations of mathematical ideas/concepts/objects and communication mathematically***	DT tools offer students access to and interaction with multiple representations of mathematical concepts; they enable students to explore abstract mathematical ideas, for example the concept of a variable or a constant. (**C1, C2, C3**)	• Creating multiple mathematical representations (e.g., Desmos) • 'Seeing' the connections between multiple representations (e.g., CornerstoneMaths) • Constructing dynamic models and interacting with virtual manipulatives • Creating charts of survey results or large data sets. • Computing possibilities to solve problems and carry out solutions • Articulating verbally the rapidly generated output and discuss the connections, what they 'see' with others	**CAS applications** (Maple, Mathematica, *Derive*) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Data analysis software** (e.g., Excel)

Learning maths (cont.)	➢ ***Dynamic exploration of mathematical ideas/concepts/objects***	DT tools promote self-reflection on the mathematical relations between different representations, e.g., symbolic, algebraic, numerical and graphical; DT tools support a shift in the focus of students' mathematical activity and thinking – e.g. from drawing and measuring, to finding patterns, making and testing conjectures rather than on drawing and measuring triangles. (**C1, C2, C3**)	• Manipulating dynamic representations of objects and acting/reacting on 'immediate feedback' • Analyzing and identify objects' variant and invariant features under change • Exploring connections between multiple mathematical representations; students could alter the symbolic and observe the effect on the other representations • Noticing and examining conjectures empirically by trialling different cases and validating properties of the mathematical objects • Arguing and discuss conjectures with others • Using a slider to vary a parameter to see the effect on the graph of a function	**CAS applications** (Maple, Mathematica, *Derive*) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Data analysis software** (e.g., Excel)

Learning maths (cont.)	➢ ***Developing strategies for problem-solving***	DT tools can help students to create a diagram, or a numerical expression, or a graphical representation, or a dynamic manipulation to support them towards getting a sense of the problem and finding the solution DT provides instant feedback to inform students of the correctness of a solution, or to support students with problem solving strategies such as trial-and-improvement. (**C1, C2, C3**)	• Estimating, approximating and rounding • Use a calculator to find an answer • Using spreadsheets to observe "what if..." • Dragging a corner of a triangle in dynamic geometry to see what happens, what changes, what stays the same • Formulating conjectures, for example through trial and improvement • Formulating, advancing and supporting conjectures • Analysing and moving from particular to general by quickly generating multiple examples • Making generalisations about mathematical ideas/concepts/objects • Writing a program/code e.g., for performing maths procedure or for mathematical modelling • All of the above can happen independently and/or collaboratively	**Tools for outsourcing the maths** (e.g., graphics calculators) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Data analysis software** (e.g., Excel) **Programming tools and languages** (e.g., Scratch, Python)

Learning maths (cont.)	**Developing reasoning skills and problem-solving skills**	DT tools support the development of students' mathematical reasoning through encouraging exploratory and constructionist approaches to mathematics. DT provides instant feedback to support students in reflecting on their solving strategies, making decisions based on the feedback, and provide evidence for students to use in their justification of the solving steps and answers. (**C1, C2, C3**)	• checking, modifying, providing an interactive interface to explore what happens if parameters are changed/stay the same, linking the general to the specific • using the most appropriate representations to justify • mathematical statements and convince oneself and others • investigate maths problems/relationships in digital environments • using logic and reasoning to evaluate and interpret situations before modifying their investigative process • using DT to model mathematically; applying their programming skills to relatively sophisticated mathematics problems • instant (tailored) feedback; taking risks in a safe digital environment • reflect on the strategies used and also to debug and/or improve them • giving students a measure of autonomy in their investigation	**Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Programming tools and languages** (e.g., Scratch, Python) **Data analysis software** (e.g. , Excel **CAS applications** (Maple, Mathematica, *Derive*) **AI tools**

Learning maths (cont.)	**Developing productive dispositions to learning maths with DT**	DT tool could help students see maths more fun and interesting through the interactive and visual nature of their learning experiences; providing intrinsic motivation to investigate complex problems in a reflective manner Pursue topics independently, through reading about mathematical topics of interest, watching video tutorials, finding interesting problems and puzzles, manipulating virtual objects (**C1, C2, C3**)	• Seeing maths in different ways, e.g.,visualising representations of data simultaneously with symbolic forms • Increasing willingness to test out ideas in a non-judgemental environment, hence more willing to take risks. • Measurement and spatial thinking facilitated by 3D images makes modelling more realistic	**Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Programming tools and languages** (e.g., Scratch, Python) **Data analysis software** (e.g. , Excel **CAS applications** (Maple, Mathematica, *Derive*)

Table 6c: Using DT for teaching mathematics				
Activity focus	**Functionalities**	**Value**	**Examples of DT enriching maths 'activity'**	**Examples of DT tools**
Teaching maths	**Planning and preparing teaching**	• Deciding on digital resources to use based on learning objective/outcomes, considering the pedagogical benefits and limitations of the digital resource • Planning the use of a particular digital resource based on students' prior knowledge and skills required • Design and choice of sequence of maths activities • Considering independent and collaborative learning activities • Deciding and planning for formative and summative assessment of students' progress	Teachers can use DT for **demonstration and modelling of the mathematics**. For example, they can: • select tasks that focus on generality and variation • select DT that enables linking representations • access many correct and varied examples through varying parameters • give time to students to explore and conjecture and share outcomes	**Tools for demonstrating the maths:** **Graphics calculators** **CAS applications** (Maple, Mathematica, *Derive)* **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Programming tools and languages** (e.g., Scratch) **Data analysis software** (e.g., Excel)

Teaching maths (cont.)	**Making decisions for exploiting the chosen tools' 'functionalities'**	• Deciding on how to use chosen digital resources based on pedagogical benefits and limitations for teaching • Deciding on which representation embedded offered by the digital resource to use, why, and how and in what order (e.g., start with graphical representation, then the table of values, then the algebraic representation of a function) • Based on the teacher's didactical intentions, deciding on when, how and why to use certain functionalities • Based on the teacher's didactical intentions, deciding on when, how and why to assess students' learning enabled by the tools' functionalities (e.g., GeoGebra classroom, Learning analytics)	• Using features such as dragging, sliders, animation, etc., enables teachers to teach challenging concepts, such as generalisation, variability, transformations, etc. • Teachers need to decide when, how and why to: - create a picture/graphical representation of an object/concept/idea/ - dynamically manipulate objects (dragging), - link multiple representations, - use sliders to move objects continuously and observe an immediate effect or change, - zoom in and out, - move objects around the screen, - try different numerical values for variables and constants, - explore particular cases - write a program/code - create various representations of large data sets	**Tools for modelling the maths:** **Graphics calculators** **CAS applications** (Maple, Mathematica, *Derive*) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Programming tools and languages** (e.g., Scratch) **Data analysis software** (e.g., Excel)

Teaching maths (cont.)	**'Real time' teaching**	• Modelling the use of digital resources for supporting learning and communicate mathematically (e.g., the seven instrumental orchestrations: technical-demo, link-screen-board, discuss-the-screen, explain-the-screen, spot-and-show, sherpa-at-work, work-and-walk-by, manipulating dynamic objects systematically and productively in order to test a conjecture, draw attention to special cases) • Supporting students develop practices of doing and learning maths afforded by the chosen tools (e.g., reading and getting a sense of the screen, dragging, changing shapes and size, identifying what changes, what stays the same, trying out examples, testing conjectures) • Making explicit links between the functionalities of the tool and the mathematical concepts/idea/object it represents. • Enabling the formative assessment of students' work, progress during lessons and learning outcomes.	• Modelling the mathematics 'live' by describing and being explicit about each step in a solution of a mathematical problem • Explaining and justifying the feedback offered by DT instantly to a student or the whole class • Using DT to show how and why an answer is wrong • Using DT to assess students' learning outcomes, e.g., summative assessments or diagnostic questions	**Tools for demonstrating and outsourcing, as well as assessing the maths:** **Graphics calculators** **CAS applications** (Maple, Mathematica, *Derive*) **Dynamic Maths Tools** (e.g., GeoGebra, Cabri Geometre) **Programming tools and languages** (e.g., Scratch) **Data analysis software** (e.g., Excel) **Mathematics content learning platforms (**e.g., Dr Frost Maths, Hegarty Maths, Integral Maths, Times Tables Rock Stars)

Teaching maths (cont.)	**Subsequent planning of teaching informed by data provided by DT**	• Interpreting information offered by the tool (e.g., students log files, students' productions) to assess students' progress and make informed decisions for planning subsequent lessons)	• Teachers look at student log files to assess progress, evaluate lessons, decide on differentiated work for future lessons, but also to inform future lesson plans	**Tools for assessing students' outcomes:** **Mathematics content learning platforms (**e.g., Dr Frost Maths, Hegarty Maths, Integral Maths, Atom Learning, Times Tables Rock Stars)

Mathematics teachers' professional developmental needs in the digital era

One of the restraining factors in the successful integration of DT in education seems to be the difficulty of changing teachers' practices. Dalby and Swan's (2019) study suggests "that the greatest challenge for teachers in using technology in the classroom is not the technology but an understanding of the process by which it can enhance student learning." (p. 843). Therefore, teachers need to be supported in implementing appropriate pedagogical approaches enabled by the DT functionalities, which requires a focus on (improving) teachers' skills with DT, as well as teachers understanding the representation of concepts and pedagogy when using DT alongside knowledge of how technology can address students' conceptual difficulties.

Mathematics teachers are not equipped to incorporate the use of the mathematics-Specific DT tools into their day-to-day teaching practice. Well known factors identified in the literature (Assude et al., 2009; Crisan et al., 2007; Hennessy et al., 2005; Mumtaz, 2000; McCulloch, 2018) which contribute to this current situation are:

× Teachers may be familiar with a limited range of DT devices and tools (e.g., calculators, graphics calculators, Desmos);
× Even if teachers are familiar with some DT devices and tools, they may not have had any training or CPD on how best to use them for mathematical teaching and learning;
× Teachers lack mathematical digital competency for teaching (MDCT): they may not be techno-mathematically literate (Hoyles et al., 2010), data literate (Royal Society report, 2018); they may not be aware of which DT tools to apply within different mathematical situations and contexts and they may not be aware of the different tools' capabilities and limitations, and therefore may not act in an effective pedagogical manner when using DT; they may not be able to use DT reflectively when supporting students' problem solving and learning mathematics in general;
× Teachers may not have the subject knowledge to understand how the DT affords perspectives on mathematics that are different from non-dynamic teaching materials;
× Teachers' lack of experience in using DT prevents them from trialling DT with students in their practice;
× Teachers' lack expertise and experience in designing activities with DT and are not confident in integrating DT in their lessons.

We are thus proposing that in order for DT to become an integral part of mathematics teachers' practices:

Teachers need to have techno-mathematical fluency, defined by Jacinto and Carreira (2017) as "the ability to combine two types of background knowledge and skills—mathematical and technological—constantly being intertwined to develop techno-mathematical thinking" (p. 1122). Subsequently, they need to develop a techno-mathematical discourse, which is an element of Mathematical Digital Competency for Teaching (MDCT1, Table 4, column 2);

➢ Teachers need to be able to use DT with flexibility and adaptivity to the various classroom situations, as well as model mathematics with the use of DT;

- Teachers should be trained through CPD courses to recognise and select tasks that may focus on generality and variation; linking representations; access many correct and varied examples through varying parameters, for example; guide students' explorations and conjecturing, while encouraging discussions to share outcomes; interpret the instant feedback offered by DT, etc;
- Teachers should be allowed time to practice with students and gain confidence and competence in using the dynamic aspects of DT to teach a 'mathematics of change' (as opposed to a 'static maths' on paper/board). Otherwise, if students are asked "to do and learn mathematics in a qualitatively different manner than what they are accustomed to, then the inclusion of a technology that can support that activity will likely have little effect." (Sherman, 2014, p. 240);
- Teachers should develop Mathematical Digital Competency for Teaching (MDCT) (as per Table 4, column 2);
- Teachers should receive training on designing effective bridging activities (Geraniou & Mavrikis, 2015) that support learners with making links between their interactions with tools and mathematical knowledge and understanding the tool is designed to support them with, should be created and used by teachers in their teaching.

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Appendices

Appendix A

Our Delphi study main findings

Classification of DT tools	• A hard task to classify EdTech, as constantly emerging • DT terminology is preferred • Agreement with our classifications (I and II), in particular with our emphasis on the potential of the Dynamic Tools for doing/learning/teaching mathematics.
Value of maths specific DT tools	• AI and XR tools not much used and hence needed to be researched for their potential in education • Dynamic Maths Tools rated the highest • Programming and language tools should be incorporate in the maths curriculum • Data plotting and analysis tools absolutely necessary for teaching students manage and interpret large data sets
Role of the maths teachers	• Teachers have huge influence on learning potential of EdTech tools • Teachers' confidence and competency are crucial factors for EdTech tools integration and implementation
Emerging DT practices	• Doing/Learning/Teaching maths differently when using DT
System needs	• Infrastructure and Resources - Ed Tech use to become mandatory, and the issue of access to EdTech devices needs to be addressed • Assessment - EdTech to be used in national assessments (e.g. SQA Higher Applications of Mathematics in Scotland already require the use of spreadsheets and statistical software such as R Studio).

Appendix B

Classification I: Types of DT for mathematics education

Types of DT - Tools	Categories	A brief description of their potential for maths education	Examples of such tools
Mathematics-specific DT Tools	**Tools for outsourcing the maths**	Tools that outsource processing power and are capable of performing numerical calculations and algorithmic processes and of producing accurate 'answers' quickly, and in some cases handling vast amount of routine algebraic manipulations (CAS).	Four-function, Scientific, and Graphic calculators Excel CAS applications (Maple, Mathematica, *Derive*, Wolfram Alpha)
	Dynamic Mathematical tools	Software with specific in-built tools (e.g., for sketching graphs, drawing shapes, typing algebraic expressions, representing data) which can be manipulated dynamically.	GeoGebra, Geometer's Sketchpad, Cabri, Graph, Autograph, Omnigraph, Desmos, CAS with dynamic features
	Data analysis software	Tools that allow the teachers and students to capture, collate and analyse data.	Excel, Python, CODAP, R
	Programming tools and languages	Software programs that aid the users in creating, editing, debugging, maintaining and/or performing any programming task.	Coding environments such as: Scratch, Logo, 3D Logo Programming languages such as: Python, MATLAB
	Extended Reality (XR) technology tools for learning mathematics	Tools that enable students experience and interact with the 3D world virtually, which involve the human body through movement and gestures.	Augmented reality such as GeoGebra's 3D Calculator (with AR) Virtual reality (VR) such as GeoGebra 3D
	Mathematics content learning platforms	**Learning platforms** enable users access a variety of learning resources such as worksheets, homework tasks, supporting information, links to useful sources as information. These platforms (also known as integrated learning systems (ILS) record all the work that each individual students do. These are interactive and exploratory tools with personalised and adaptive student learning.	MyMaths, Dr Frost Maths, Hegarty Maths, Integral Maths, Purple Maths, Headstart, Times Tables Rock Stars, AtomLearning, Mathletics, etc
	Apps for mathematics education	Educational tools/applications hosted by various websites that have been designed to perform a specific task, which have different aims and design features.	Practice-based apps; Game-based apps; Stick 'n' Split – times table game; Geoboard; Khan Academy;
	AI tools	AI-based calculators take input as text, translates it into mathematical notation, and returns the answers in numerical form. ChatGPT is an OpenAI which could be used to answer parts of questions, to check, and offer support for leaners' mathematical reasoning.	AI-based calculators (Photomath), AI tutoring systems (Chat GPT)

Generic DT Tools	Generic tools that enable the communication of mathematics through creating worksheets, power point presentations, converting handwritten scripts, etc. Such generic tools significantly transform the processes of writing, and in some cases, the reading of mathematics.	LatTeX, MyScript, Equation Editor, Mathpix Snip, MS Word, MS PowerPoint, MS Forms
Generic DT apps	Such tools provide an interactive environment, processing and analysing, sending and displaying information. They enable connected classrooms, where students can share their work, reflections, solutions with their peers, but also with the teacher.	Email, MS Teams, VLEs, Google classrooms Zoom videoconferencing, Mentimeter, Poll Twitter, Youtube
DT devices that could be used for maths education.	These devices could be used on their own (calculators, graphics calculators, digital measuring devices), for their presentational properties (IWB, laptops, tablets, visualisers), for accessing information and maths apps (Desmos app. GeoGebra app, AI-calculator).	Calculators; Graphics Calculators; iPads/tablets; Laptops/desktops; IWB; smart phones; visualisers, graphics tablets and display devices; measuring devices such as ruler app measure; AR ruler app;

Appendix C

Classification II: Contributions of DT to mathematics education 'activities'

Contributions of DT to maths 'activities':	Functionalities of using DT
Doing mathematics	**Carrying out procedures** (computations and procedures are carried out efficiently and with accuracy by the tools) and **communicating mathematically**
	Visualisations of representations and **dynamic manipulations** of mathematical objects enabled
	Supporting reasoning and problem-solving
Learning mathematics	**Practising skills** to support acquisition of maths knowledge and skills through targeted practice and to communicate mathematically
	Developing conceptual knowledge through:
	• ***Interacting with representations of mathematical ideas/concepts/objects***
	• ***Dynamic exploration of mathematical ideas/concepts/objects***
	• ***Developing strategies for problem-solving***
	Developing reasoning skills and problem-solving skills
	Developing productive dispositions to learning maths with DT
Teaching mathematics	**Planning and preparing teaching**
	Making decisions for exploiting the chosen tools' 'functionalities'
	'Real time' teaching
	Subsequent planning of teaching informed by data provided by DT