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ORIGINAL RESEARCH

A novel cross-docking EOQ-based model to optimize a multi-item multi-supplier multi-retailer inventory management system

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Abstract

Nowadays, the retail industry accounts for a large share of the world's economy. Cross-docking is one of the most effective and smart inventory management systems used by retail companies to respond to demands efficiently. In this study, the aim is to develop a novel cross-docking EOQ-based model for a retail company. By considering a two-stage inventory procurement process, a new multi-item, multi-supplier, multi-retailer EOQ model is developed to minimize the total inventory costs. In the first stage, the required items are received from suppliers and are held in a central warehouse. In the second stage, these items are delivered to several retail stores. The total inventory costs include four main parts, i.e., holding costs at the central warehouse, holding costs at the retail stores, fixed ordering costs from the suppliers, and fixed ordering costs from the central warehouse. The optimal inventory policy is obtained by analyzing extrema, and a numerical example is used to confirm the efficiency of the proposed model. Based on the obtained results, it is evident that the proposed model produces the optimal policy for the cross-docking system. Furthermore, the model enables managers to analyze the effects of key factors on the costs of the system. Based on the obtained results, the annual demand of each retailer, the ordering cost by the central warehouse, the ordering cost at each retail store, and the holding cost at each retail store have a direct impact on the optimal cost. Furthermore, it is not possible to describe the effects of the holding cost at the central warehouse on the optimal cost of the system generally.

Keywords Inventory management · Cross-docking system · Retail industry · Multi-item multi-supplier multi-retailer EOQ · Central warehouse

1 Introduction

The retail industry has been one of the largest industries in the world for four decades. As reported by Statista (2022), worldwide retail sales reached 26.03 trillion USD in 2021, which is about 27 percent of world GDP in 2021 (World Bank, 2022). Furthermore, it has been forecasted by Statista (2022) that worldwide retail sales will reach 27.3 trillion USD at the end of 2022. Furthermore, based on Fortune (2022) Global 500 data, Walmart and Amazon

Extended author information available on the last page of the article

got the first and the second ranks in terms of revenue, respectively. Walmart and Amazon are among the best retail companies, and their revenues have been estimated at 573 and 470 billion USD, respectively, totaling 26.3% of the revenue of the top ten companies. In addition to high revenue generation, retail companies generally have a high rate of employment, as Walmart and Amazon in combination employ 39.6% of the employees of all top ten companies.

Every retail company usually has multiple warehouses to store inventory. Inventory constitutes the majority of retailers' costs while manufacturers mainly spend more on equipment and production facilities. Consequently, inventory management is of high significance for the retail companies (Get et al., 2019). Inventory management helps retail companies to cover the demands in the right place and time, with desirable quantity and quality, and at the lowest possible cost. Cross-docking is an effective and smart inventory management system that is used by many top retail companies such as Walmart, Amazon, Costco, Home Depot, and Target (Liu & Li, 2023). Cross-docking is an efficient way to get and hold different shipments in a central warehouse and then distribute them among smaller warehouses based on their demands (Rajabzadeh & Mousavi, 2023; Zhang et al., 2023). As frequently mentioned in the literature, the cross-docking system has many advantages. First, the number of travels is reduced by the use of this system, which consequently leads to a lower transportation cost. In addition, the cross-docking system reduces the amounts of inventories that are held over a long time, thus helping to minimize the holding cost. Furthermore, reduced storage space, optimized material handling, enhanced material quality, shorter delivery time, and improved service levels are other benefits of cross-docking systems.

Cross-docking is also known as one of the main elements in smart manufacturing and inventory systems by providing close synchronization of all inbound and outbound shipment movements (Lyu et al., 2020). The cross-docking process requires significant needs for information and material handling systems. The efficiency of such systems is dependent on inventory management policy which can be enhanced through digital technologies such as Internet-of-Things (IoT) (Ben-Daya et al., 2022). As shown in Fig. 1, a cross-docking system is a logistical strategy where goods are first unloaded from inbound trucks. Then, they are sorted and held in a central warehouse until they are needed by the retail stores.

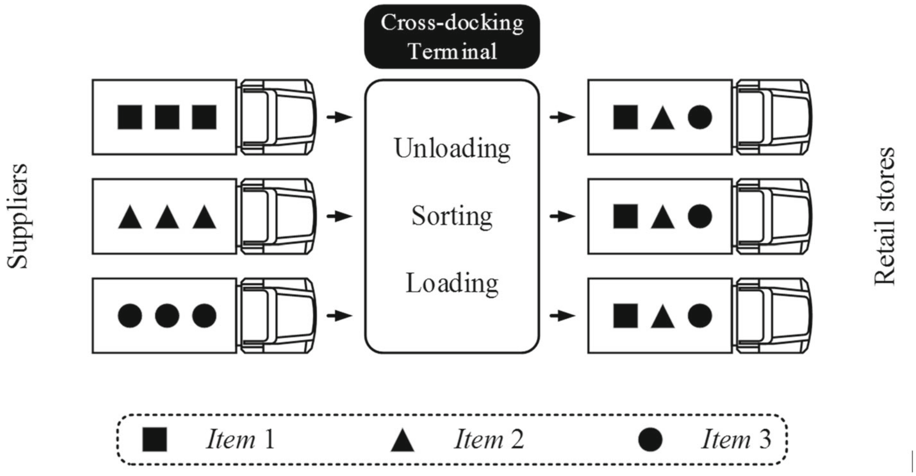

Fig. 1 A scheme of a cross-docking system

At last, they are loaded in outbound trucks to be delivered to retail stores. According to the above description, the performance of a cross-docking system highly depends on inventory planning.

The lack of proper planning leads to a reduction in service level and a rise in costs. This study aims to develop a novel Economic Order Quantity (EOQ) model for cross-docking systems that helps to find the optimal inventory policy, i.e., when to order and how much to order. Here, it is assumed that a retail company owns a central warehouse and several retail stores. The retail company aims to minimize inventory costs through a two-stage material procurement process. In the first stage, the optimal order size and cycle time of inbound flows are determined. In the second stage, the optimal order size and cycle time of outbound flows are similarly calculated. The specific contributions of the present study are as follows:

I. A novel multi-item EOQ model is developed for a cross-docking system that includes a central warehouse and several retail stores. In the developed model, a multi-supplier multi-retailer inventory system is investigated that aims to find the optimal replenishment policy. The objective function minimizes the sum of two cost types, namely ordering and holding costs. Ordering and holding costs are divided into two parts, i.e., at the central warehouse and retail stores.
II. The optimal inventory policy is determined through a two-stage process, i.e., inbound and outbound flows. In the first stage, the needed items are acquired from n suppliers and are held in one central warehouse, and in the second stage, these items are dispatched to m retail stores. The developed model optimizes the total cost of this two-stage process.
III. A numerical example is utilized to illustrate and analyze the efficiency of the developed model. Here, a retail company, which includes one central warehouse and ten retail stores, with twenty types of products is considered. Furthermore, input parameters are generated by uniform distributions.
IV. A sensitivity analysis is then executed to study the behavior of the system against the changes in the integer multiplier of cycle times.

Based on the above-mentioned contributions, the following questions are defined that should be investigated in this manuscript.

I. What is the best mathematical model for determining the optimal inventory policy of a multi-item multi-supplier multi-retailer cross-docking EOQ-based inventory management system?
II. How does the inventory behave through the two stages of the developed model?
III. How can the model be validated?
IV. How can we assess the impacts of key factors on the costs of the system?

The remaining sections are structured as follows. In Sect. 2, the literature review is performed to identify the research gaps. The mathematical model is formulated in Sect. 3, and the numerical example is presented in Sect. 4. The theoretical and practical insights are discussed in Sect. 5, and eventually, Sect. 6 gives the main concluding remarks and outlook for future studies.

2 Related works

So far, many studies have been done in the field of inventory management. Among them, inventory optimization models are important topics that have attracted considerable attention. In this section, we survey the EOQ models that concentrate on the retail industry. Moreover, we address the inventory models that have been introduced under the assumption of more than one warehouse. EOQ is one of the most traditional inventory models, which determines the economic order quantity by minimizing the inventory costs. It was first developed by Ford Whitman Harris in 1913 (Erlenkotter, 1990). Since then, different contributions have been made to formulate numerous EOQ-based models. Due to the importance of the retail industry, several attempts have been made to propose new EOQ models that are focused on this field. To the best of our knowledge, Bassin (1990) developed the first EOQ model that aims to meet the needs of small businesses such as retailers. Anily (1994) considered a multi-retailer EOQ problem with vehicle routing costs. In that study, the allocation of a single commodity from a central warehouse to *n* retailers was considered, and a simple heuristic approach for solving large-scale problems was presented.

Lin and Lin (2004) presented the joint inventory model between supplier and retailer while considering deteriorating items. They took into account a single producer and a single retailer, and the objective function was to minimize the total cost of both over the planning horizon. Sana (2008) presented a novel EOQ model with varying demands. They assumed a simple supply chain structure with only one supplier and one retailer and tried to determine the optimal replenishment period. In a supply chain consisting of two levels, featuring one supplier and one retailer, Huang et al (2010) formulated an EOQ model with retailer partial trade credit. Their model determined the retailer's optimal inventory policy to minimize the inventory costs. Mahata (2015) proposed an EOQ model for perishable items with expiration dates and trade credit, considering a supply chain consisting of a supplier and a retailer. The model identified the optimal replenishment and pricing policy of the retailer to maximize profit. Roy et al (2016) developed an EOQ model for a two-layer supply chain involving one manufacturer and one retailer. It was assumed that only one type of product is sold through advertising attempts and service level assurance. Given uncertain demands, the optimal inventory policy was found to maximize the expected profit. An EOQ model was offered by Huang and Wu (2016) to address the effect of backordering and batch demand on a wholesaler. Structural properties were defined for the wholesaler's cost function to find the optimal policy and ordering quantity. Malmberg and Marklund (2023) evaluated a one-warehouse multi-retailer inventory system considering quantity-based shipment consolidation for non-identical retailers under probabilistic demand. They offered a modified EOQ model to tackle the case of multiple consolidation groups and minimize average inventory/shipment cost. Recently, another one-warehouse multi-retailer inventory system was assessed by Andersson et al. (2023) considering deliveries with quantity limitations and stochastic inventory level distributions. Analyzing a numerical example demonstrated that taking into account the quantity-restricted deliveries employed in transportation and operations when determining the optimal reorder points is critical. Holzapfel et al. (2023) developed a Mixed-Integer Programming (MIP) model to minimize total cost by addressing the operations of warehouses, inbound and outbound transportation, inventory holding at distribution centers and instore logistics. Transshipment points were defined as demand nodes acting as cross-dock facilities in a case study of a major European retailer. An iterated decomposition-based local search algorithm was applied to treat the model.

To the best of our knowledge, the two-stage decision-making process has not been investigated in the previous models that have focused on the retail industry. Moreover, as shown in Table 1, only a few previous studies have considered central warehouses in their studies, such as Das et al. (2015) and Anily (1994). Furthermore, multiple retailers and multiple items have been analyzed in several studies, such as Paam et al. (2022), Kumar and Mahapatra, (2021), Das et al. (2015), and Zhao et al. (2006). In comparison to previous studies, this work

Table 1 Summary of the existing literature

References	Retail industry	EOQ model	No. of stages	Central warehouse	No. retailers (Warehouses)	No. of items	No. of suppliers
Bassin (1990)	✓	✓	1		Single	Single	N/A
Anily (1994)	✓	✓	1	✓	Multiple	Single	N/A
Zhou (2003)			1		Multiple	Single	N/A
Lin and Lin (2004)	✓	✓	1		Single	Single	Single
Zhao et al (2006)		✓	1		Multiple	Multiple	N/A
Huang and Hsu (2008)	✓	✓	1		Single	Single	Single
Sana (2008)	✓	✓	1		Single	Single	Single
Chung (2008)	✓	✓	1		Single	Single	Single
Huang et al. (2010)	✓	✓	1		Single	Single	Single
Darwish and Odah (2010)	✓		1		Multiple	Single	Single
Mahata and Mahata (2011)	✓	✓	1		Single	Single	Single
Guria et al. (2012)		✓	1		Two	Single	Single
Das et al. (2015)			1	✓	Multiple	Multiple	Single
Mahata (2015)	✓	✓	1		Single	Single	Single
Mahata and De (2016)	✓	✓	1		Single	Single	Single
Roy et al. (2016)	✓	✓	1		Single	Single	Single
Alfares and Attia (2017)	✓		1		Multiple	Single	Single
Shaikh et al. (2019)		✓	1		Two	Single	Single
Najafnejhad et al. (2021)	✓		1		Multiple	Single	Single
Esmaeili and Nasrabadi (2021)	✓		1		Multiple	Single	Single

Table 1 (continued)

References	Retail industry	EOQ model	No. of stages	Central warehouse	No. retailers (Warehouses)	No. of items	No. of suppliers
Kumar and Mahapatra (2021)			1		Multiple	Multiple	N/A
Paam et al. (2022)			1		Multiple	Multiple	N/A
Andersson et al. (2023)	✓	✓	1	✓	Multiple	Single	Single
Present study	✓	✓	2	✓	Multiple	Multiple	Multiple

makes significant contributions to the inventory management of the retail industry. By introducing a two-stage replenishment process, a novel multi-item multi-supplier multi-retailer EOQ model is designed to minimize the total inventory costs in the retail industry. In this two-stage process, the required items are first received from suppliers and are held in a central warehouse, and then they are delivered to several retail stores. Here, inventory costs are composed of four major terms: holding costs at the central warehouse, holding costs at the retail stores, ordering costs from the suppliers, and ordering costs from the central warehouse.

3 Mathematical model

In this section, a new multi-item, multi-supplier multi-retailer EOQ model for cross-docking systems is introduced. In the following, the assumptions and the notation of the model are presented, and then the mathematical model is formulated. As mentioned previously, the objective function is to minimize the sum of four main cost components, each of which will be described separately. At last, the optimal inventory policy is determined by analyzing extrema.

3.1 Assumptions

- The required items are procured through two consecutive stages. First, the required items are received from n suppliers and are held in one central warehouse. Next, these items are delivered to m retail stores (see Fig. 2),
- The retail company owns the central warehouse and all of the m retail stores,
- Each of the n suppliers provides only one type of items, and each item has only one supplier,
- In the first stage, all items have one identical cycle time, T,
- In the second stage, all items have an identical cycle time, T', where T is an integer multiple of T',
- The fixed ordering cost of each retailer j is equal for all items,
- Per unit annual holding costs at the central warehouse are identical and equal to h_c for all items,

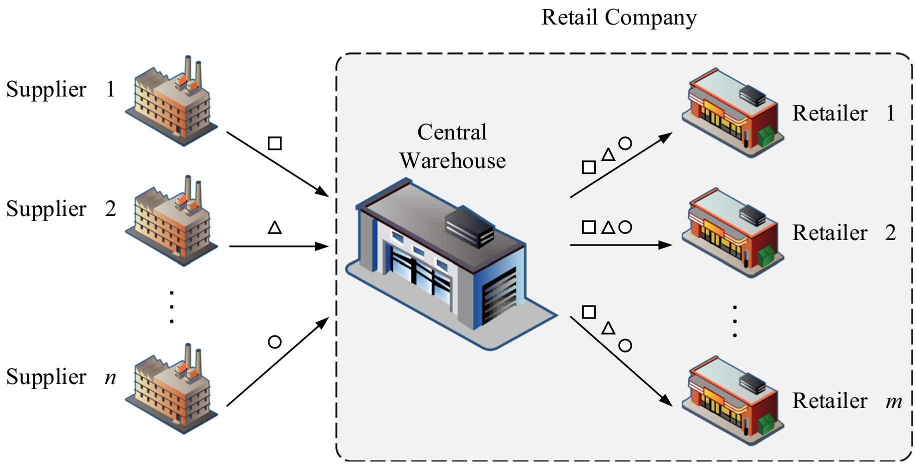

Fig. 2 Procurement scheme of the retail company

- Per unit annual holding costs at retailer j are identical and equal to h_{rj} for all items,
- Delivery of each order to the warehouse and each retailer is complete and instantaneous,
- Shortages are not allowed,
- All parameters are known and constant.

3.2 Parameters

n : Number of suppliers,
m : Number of retail stores,
D_{ij} : Annual demand for item i by retailer j ($i = 1, 2, \ldots, n$ & $j = 1, 2, \ldots, m$),
$D_{i.}$: Total annual demand for item i by all retailers ($D_{i.} = \sum_j D_{ij}$),
$D_{.j}$:Total annual demand of retailer j for all items ($D_{.j} = \sum_i D_{ij}$),
$D_{..}$:Total annual demand of all retail stores for all types of items ($D_{..} = \sum_i \sum_j D_{ij}$),
A_i :Stage 1 fixed ordering cost of item i by the central warehouse,
A'_j :Stage 2 fixed ordering cost of retailer j for any item i,
h_c : Per unit annual holding cost at the central warehouse for any item i,
h_{rj} :Per unit annual holding cost at retail store j for any item i.

3.3 Variables

N : Stage 1 number of yearly orders made by the central warehouse,
T : Stage 1 cycle time; i.e., the interval between successive replenishments at the warehouse,
Q_i : Stage 1 order quantity; i.e., size of order of item i made by the central warehouse,
T' : Stage 2 cycle time; i.e., the interval between successive replenishments at each retail store,
a : Integer multiplier related to the cycle times of Stages 1 and 2 ($T = aT'$).

3.4 Model development

As mentioned above, all items have an identical first-stage cycle time T. This means that:

$$T = \frac{Q_1}{D_{1.}} = \frac{Q_2}{D_{2.}} = \cdots = \frac{Q_n}{D_{n.}}. \tag{1}$$

Since $N = \frac{1}{T}$, then we have

$$N = \frac{D_{1.}}{Q_1} = \frac{D_{2.}}{Q_2} = \cdots = \frac{D_{n.}}{Q_n}. \tag{2}$$

Obviously, the annual input of the central warehouse, $N\sum_i Q_i$, must be equal to its annual output, $\sum_j D_{.j}$. Since $\sum_j D_{.j} = D_{..}$, then

$$N = \frac{D_{..}}{\sum_i Q_i}. \tag{3}$$

At the first stage, the objective is to minimize the sum of two cost types, namely the ordering cost and the holding cost. In the following, mathematical expressions are developed for the two cost components based on the above-mentioned assumptions. Since the first stage fixed ordering cost of item i is equal to A_i, the total first-stage ordering cost per cycle T is given by $\sum_i A_i$. Hence, the total first-stage ordering cost for the central warehouse per year (OC_c) is equal to

$$OC_c = N\sum_i A_i = \frac{\sum_i A_i}{T}. \tag{4}$$

Figure 3 shows the inventory levels at the central warehouse and retailer j. In the first stage, the required items are received from suppliers at cycle time T and are held in the central warehouse. In the second stage, these items are sent in several steps to retail stores at cycle time T', where $T = aT'$, and then they are sold to the customers.

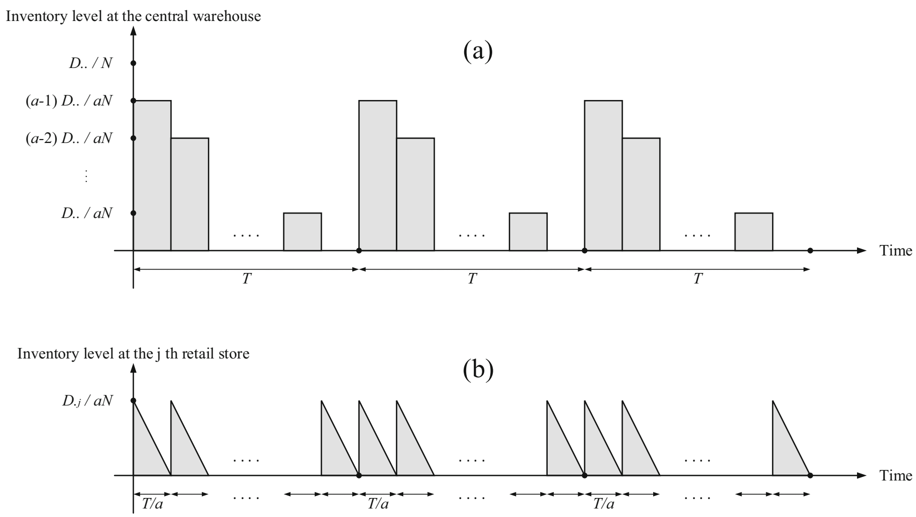

Fig. 3 Inventory levels at: **a** the central warehouse, and **b** retail store j

According to Fig. 3a, the total inventory held at the central warehouse during a cycle time T is computed as follows

$$I_c = \frac{(a-1)TD_{..}}{2aN}.$$

Dividing I_c by T, the average inventory level at the central warehouse per unit of time (i.e., per year) is given by

$$\overline{I}_c = \frac{(a-1)D_{..}}{2aN} = \frac{(a-1)TD_{..}}{2a}.$$

Multiplying $\overline{I}_c$ by the annual holding cost per unit at central warehouse h_c, the first stage holding cost at the central warehouse over a year (HC_c) is obtained by

$$HC_c = \frac{h_c(a-1)TD_{..}}{2a}. \tag{5}$$

In the second stage, the items are delivered from the central warehouse to the retail stores. Similar to the previous stage, both the ordering and the holding costs are considered in this stage. Since the fixed ordering cost of retailer j is equal to A'_j, the ordering cost of each cycle, T', is given by $\sum_j A'_j$. Hence, the second stage ordering cost for all retailers per year (OC_r) is expressed as

$$OC_r = aN\sum_j A'_j = \frac{a\sum_j A'_j}{T}. \tag{6}$$

According to Fig. 3b, the replenishment order size of retailer j is equal to $\frac{D_{.j}}{aN}$ or $\frac{D_{.j}T}{a}$. Hence, the average inventory of retailer j over a year is equal to $\overline{I}_{r,j} = \frac{D_{.j}T}{2a}$, and the annual holding cost of retailer j is $HC_{r,j} = \frac{h_{rj}D_{.j}T}{2a}$. Summing up the holding costs of all retailers, the second stage holding cost of all retail stores per year (HC_r) is expressed as follows

$$HC_r = \frac{\sum_j h_{rj}D_{.j}T}{2a} = \frac{T}{2a}\sum_j h_{rj}D_{.j}. \tag{7}$$

Adding up the cost components in Eqs. (4) to (7), the annual inventory cost $K(a, T)$ of the above-described cross-docking system is given by:

$$\begin{aligned} K(a, T) &= OC_c + OC_r + HC_c + HC_r \\ &= \frac{\sum_i A_i}{T} + \frac{a\sum_j A'_j}{T} + \frac{h_c(a-1)TD_{..}}{2a} + \frac{T}{2a}\sum_j h_{rj}D_{.j}. \end{aligned} \tag{8}$$

Now, let

$$\Delta = \frac{\sum_j h_{rj}D_{.j}}{2}, \tag{9}$$

$$\beta = \frac{h_c D_{..}}{2}. \tag{10}$$

The total cost $K(a, T)$ can be represented as the following function of a, T, Δ, and β:

$$K(a, T) = \frac{\sum_i A_i}{T} + \frac{a\sum_j A'_j}{T} + \beta T + \frac{(\Delta-\beta)T}{a}, \tag{11}$$

where $T > 0$ and $a \geq 1$, which takes integer values.

The first and the second partial derivatives of $K(a, T)$ with respect to both a and T are presented in Eqs. (12) to (15):

$$\frac{\partial K}{\partial a} = \frac{\sum_j A'_j}{T} - \frac{(\Delta - \beta)T}{a^2}, \tag{12}$$

$$\frac{\partial^2 K}{\partial a^2} = \frac{2(\Delta - \beta)T}{a^3}, \tag{13}$$

$$\frac{\partial K}{\partial T} = \frac{-(\sum_i A_i + a\sum_j A'_j)}{T^2} + \beta + \frac{(\Delta - \beta)}{a}, \tag{14}$$

$$\frac{\partial^2 K}{\partial T^2} = \frac{2(\sum_i A_i + a\sum_j A'_j)}{T^3}. \tag{15}$$

Based on Eq. (12), if $\frac{\partial K}{\partial a} = 0$ then

$$a = \sqrt{\frac{(\Delta - \beta)T^2}{\sum_j A'_j}}. \tag{16}$$

Similarly, according to Eq. (14), if $\frac{\partial K}{\partial T} = 0$ then

$$T = \sqrt{\frac{a(\sum_i A_i + a\sum_j A'_j)}{(\Delta - \beta) + a\beta}} \tag{17}$$

In the following, the total cost function $K(a, T)$ is minimized considering two possible cases:

i. $\Delta - \beta \leq 0$.
ii. $\Delta - \beta > 0$.

3.4.1 Case 1: $\Delta - \beta \leq 0$

In this case, it is obvious that $\frac{\partial K}{\partial a} > 0$. Since the slope of K is positive with respect to a, then the total cost K is a growing function of a. Accordingly, the minimum value of a, i.e., $a = 1$, is the best value of a that leads to the minimum value of K. Since $\frac{\partial^2 K}{\partial T^2} > 0$, then K is convex with respect to T. Setting $a = 1$ into Eq. (17) gives the best cycle time value T as follows:

$$T = \sqrt{\frac{\sum_i A_i + \sum_j A'_j}{\Delta}}. \tag{18}$$

Substituting $a = 1$ and the above value of T into Eq. (11) gives the minimum cost value $K(a, T)$.

$$K(a, T) = \frac{\sum_i A_i + \sum_j A'_j}{\sqrt{\frac{\sum_i A_i + \sum_j A'_j}{\Delta}}} + \Delta\sqrt{\frac{\sum_i A_i + \sum_j A'_j}{\Delta}}, \tag{19}$$

$$K(a, T) = \frac{(\sum_i A_i + \sum_j A'_j)}{\sqrt{\sum_i A_i + \sum_j A'_j}}\sqrt{\Delta} + \sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)}, \tag{20}$$

$$K(a, T) = \sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)} + \sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)}. \tag{21}$$

The complete optimum solution when $\Delta - \beta \leq 0$ is given by:

$$a = 1,\ T = \sqrt{\frac{\sum_i A_i + \sum_j A'_j}{\Delta}}, K(a,\ T) = 2\sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)}. \tag{22}$$

3.4.2 Case 2: $\Delta - \beta > 0$

Since both $\frac{\partial K}{\partial T}$ and $\frac{\partial K}{\partial a}$ can be either positive or negative, the optimum solution is found by setting $\frac{\partial K}{\partial T} = 0$ and $\frac{\partial K}{\partial a} = 0$ and solving for search for the optimum values of a and T. Setting $\frac{\partial K}{\partial a} = 0$ in Eq. (12), or alternatively using Eq. (16), leads to

$$T = \sqrt{\frac{a^2 \sum_j A'_j}{(\Delta - \beta)}}. \tag{23}$$

Setting $\frac{\partial K}{\partial a} = 0$ in Eq. (14) and substituting the above value of T into the equation gives

$$\frac{(\sum_i A_i + a \sum_j A'_j)}{\frac{a^2 \sum_j A'_j}{(\Delta - \beta)}} = \beta + \frac{(\Delta - \beta)}{a}, \tag{24}$$

$$\frac{(\sum_i A_i + a \sum_j A'_j)(\Delta - \beta)}{\sum_j A'_j} = \beta a^2 + (\Delta - \beta)a, \tag{25}$$

$$(\Delta - \beta)\sum_i A_i + (\Delta - \beta)a \sum_j A'_j = \beta a^2 \sum_j A'_j + (\Delta - \beta)a \sum_j A'_j, \tag{26}$$

$$(\Delta - \beta)\sum_i A_i = \beta a^2 \sum_j A'_j. \tag{27}$$

Solving the above equation for a, and then substituting into Eq. (23) to solve for T leads to the following results:

$$a = \sqrt{\frac{(\Delta - \beta)\sum_i A_i}{\beta \sum_j A'_j}}, \tag{28}$$

$$T = \sqrt{\frac{\sum_i A_i}{\beta}}. \tag{29}$$

Substituting the above optimum values of a and T into Eq. (11), we obtain the minimum total cost as shown below:

$$K(a,\ T) = \frac{\sum_i A_i}{\sqrt{\frac{\sum_i A_i}{\beta}}} + \frac{\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta \sum_j A'_j}} \sum_j A'_j}{\sqrt{\frac{\sum_i A_i}{\beta}}} + \beta\sqrt{\frac{\sum_i A_i}{\beta}} + \frac{(\Delta - \beta)\sqrt{\frac{\sum_i A_i}{\beta}}}{\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta \sum_j A'_j}}}, \tag{30}$$

$$K(a,\ T) = \sqrt{\beta \sum_i A_i} + \frac{\sqrt{(\Delta - \beta)\sum_i A_i \sum_j A'_j}}{\sqrt{\sum_i A_i}} + \sqrt{\beta \sum_i A_i} + \sqrt{(\Delta - \beta)\sum_j A'_j}, \tag{31}$$

$$K(a,\ T) = 2\sqrt{\beta \sum_i A_i} + 2\sqrt{(\Delta - \beta)\sum_j A'_j}. \tag{32}$$

Now, the Hessian matrix of the total cost function $K(a, T)$ must be used to determine whether the stationary (extreme) point specified by Eqs. (28) and (29) is an optimum (minimum) point. After computing the relevant partial derivatives with respect to a and T, we obtain the Hessian matrix of $K(a, T)$ as

$$H = \begin{bmatrix} \frac{2(\Delta-\beta)T}{a^3} & -\left[\frac{\sum_j A'_j}{T^2} + \frac{(\Delta-\beta)}{a^2}\right] \\ -\left[\frac{\sum_j A'_j}{T^2} + \frac{(\Delta-\beta)}{a^2}\right] & \frac{2(\sum_i A_i + a\sum_j A'_j)}{T^3} \end{bmatrix}. \tag{33}$$

The solution specified by Eqs. (28) and (29) is optimum (minimum) only if the 2×2 Hessian matrix H is positive definite, i.e., if both leading principal determinants of H are positive. The primary leading principal determinant of H is the element H_{11}, and it is evidently positive. The second leading principal determinant of H is the full matrix's determinant $|H|$, whose value at the extreme point defined by Eqs. (28) and (29) is calculated below:

$$|H| = \frac{2(\Delta-\beta)T}{a^3}\frac{2\left(\sum_i A_i + a\sum_j A'_j\right)}{T^3} - \left[\frac{\sum_j A'_j}{T^2} + \frac{(\Delta-\beta)}{a^2}\right]^2, \tag{34}$$

$$|H| = \frac{4(\Delta-\beta)\sum_i A_i + 4a(\Delta-\beta)\sum_j A'_j}{a^3T^2} - \left(\frac{\left(\sum_j A'_j\right)^2}{T^4} + \frac{2(\Delta-\beta)\sum_j A'_j}{a^2T^2} + \frac{(\Delta-\beta)^2}{a^4}\right), \tag{35}$$

$$|H| = \frac{4aT^2(\Delta-\beta)\sum_i A_i + 4a^2T^2(\Delta-\beta)\sum_j A'_j}{a^4T^4} - \left(\frac{a^4\left(\sum_j A'_j\right)^2 + 2a^2T^2(\Delta-\beta)\sum_j A'_j + T^4(\Delta-\beta)^2}{a^4T^4}\right), \tag{36}$$

$$|H| = \frac{1}{a^4T^4}\left[4aT^2(\Delta-\beta)\sum_i A_i + 2a^2T^2(\Delta-\beta)\sum_j A'_j - \left(a^4\left(\sum_j A'_j\right)^2 + T^4(\Delta-\beta)^2\right)\right]. \tag{37}$$

Substituting the values of a and T from Eqs. (28) and (29) into $|H|$, we obtain

$$|H| = \frac{1}{a^4T^4}\left[\begin{matrix} 4a\left(\frac{\sum_i A_i}{\beta}\right)(\Delta-\beta)\sum_i A_i + 2\left(\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}\right)\left(\frac{\sum_i A_i}{\beta}\right)(\Delta-\beta)\sum_j A'_j \\ -\left(\left(\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}\right)^2\left(\sum_j A'_j\right)^2 + \left(\frac{\sum_i A_i}{\beta}\right)^2(\Delta-\beta)^2\right) \end{matrix}\right], \tag{38}$$

$$|H| = \frac{1}{a^4T^4}\left[\frac{4a(\Delta-\beta)\left(\sum_i A_i\right)^2}{\beta} + 2\left(\frac{(\Delta-\beta)\sum_i A_i}{\beta}\right)^2 - \left(\frac{(\Delta-\beta)^2(\sum_i A_i)^2}{\beta^2} + \frac{(\Delta-\beta)^2\left(\sum_i A_i\right)^2}{\beta^2}\right)\right], \tag{39}$$

$$|H| = \frac{1}{a^4T^4}\left[\frac{4a(\Delta - \beta)\left(\sum_i A_i\right)^2}{\beta}\right], \tag{40}$$

$$|H| = \frac{4(\Delta - \beta)\left(\sum_i A_i\right)^2}{a^3T^4\beta}. \tag{41}$$

Since $\Delta - \beta > 0$, $a \geq 1$, and $\beta \geq 0$, the value of the determinant $|H|$ is positive. Hence, the Hessian matrix H is positive definite, and the extreme point defined by Eqs. (28) and (29) is a global minimum point.

As previously stated, a is a positive integer number ($a \geq 1$ and integer). If Eq. (28) produces an integer value of a, then the solution specified by Eqs. (28), (29), and (32) is optimal. Otherwise, if a fractional value of a is obtained from Eq. (28), then the two nearest integer values are alternatively tried and compared:

$(i)\lfloor a \rfloor =$ the floor of a, i.e., the value of a rounded *down* to the nearest integer.

(ii) $\lceil a \rceil =$ the ceiling of a, i.e., the value of a rounded *up* to the nearest integer.

Using Eq. (23), the cycle times corresponding to $\lfloor a \rfloor$ and $\lceil a \rceil$ are computed as $\lfloor T \rfloor$ and $\lceil T \rceil$, respectively. After that, Eq. (11) is used to calculate the total costs of both solutions: $K(a, T)$ and $K(a, T)$. Three cases are possible:

I. If $K(\lfloor a \rfloor, \lfloor T \rfloor) < K(\lceil a \rceil, \lceil T \rceil)$, then $\lfloor a \rfloor$ and $\lfloor T \rfloor$ specify the optimal inventory policy and $K(\lfloor a \rfloor, \lfloor T \rfloor)$ is the optimal inventory cost,

II. If $K(\lfloor a \rfloor, \lfloor T \rfloor) > K(\lceil a \rceil, \lceil T \rceil)$, then $\lceil a \rceil$ and $\lceil T \rceil$ specify the optimal inventory policy and $K(\lfloor a \rfloor, \lfloor T \rfloor)$ is the optimal inventory cost,

III. If $K(\lfloor a \rfloor, \lfloor T \rfloor) = K(\lceil a \rceil, \lceil T \rceil)$, then both policies are optimal, and $K(\lfloor a \rfloor, \lfloor T \rfloor) = K(\lceil a \rceil, \lceil T \rceil)$ is the optimal inventory cost.

4 Numerical example

4.1 Data

Here, a numerical example is used to demonstrate the applicability of the proposed model. The retail company utilizes one central warehouse and ten retail stores ($m = 10$) to purchase twenty types of products ($n = 20$). The given input parameters D_{ij}, A_i, A'_j, h_c, and h_{rj} are generated from uniform distributions as shown in Table 2. The detailed input values of D_{ij}, A_i, A'_j, and h_{rj} are listed in Tables 3, 4, 5, 6, respectively. Also, h_c is assumed equal

Table 2 Parameters setting

Parameter	Value
m	10
n	20
D_{ij}	Uniform(100, 900)
A_i	Uniform(50, 100)
A'_j	Uniform(5, 20)
h_c	Uniform(5, 10)
h_{rj}	Uniform(10, 30)

Table 3 Input values of D_{ij}, $D_{i.}$, $D_{.j}$, and $D_{..}$

D_{ij}		j										$D_{i.}$
		1	2	3	4	5	6	7	8	9	10	
i	1	548	862	655	846	388	679	551	461	640	118	5748
	2	562	842	277	443	210	528	669	833	127	818	5309
	3	689	207	155	367	878	533	522	309	362	444	4466
	4	841	236	190	290	246	783	425	713	817	754	5295
	5	177	513	212	303	765	531	625	229	618	483	4456
	6	714	847	868	897	828	176	598	473	817	567	6785
	7	705	263	450	853	608	771	213	845	705	201	5614
	8	351	713	757	484	638	162	619	814	482	530	5550
	9	846	620	184	346	762	212	217	585	536	222	4530
	10	343	794	304	384	597	797	286	716	679	484	5384
	11	245	859	106	407	147	696	334	296	839	686	4615
	12	278	459	133	752	419	852	499	380	180	877	4829
	13	354	289	327	654	238	103	329	380	686	146	3506
	14	825	604	671	306	295	424	296	397	606	438	4862
	15	859	866	364	696	503	790	224	225	883	398	5808
	16	757	570	278	894	758	381	689	608	844	234	6013
	17	631	849	252	434	692	565	239	316	333	115	4426
	18	136	768	291	385	252	709	619	323	232	387	4102
	19	482	464	368	339	552	578	232	131	265	861	4272
	20	409	778	100	578	228	759	249	529	366	558	4554
$D_{.j}$		10,752	12,403	6942	10,658	10,004	11,029	8435	9563	11,017	9321	$D_{..} = 100{,}124$

Table 4 Input values of A_i and $\sum_i A_i$

	i																				Sum
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
A_i	71	51	82	100	76	85	91	96	80	85	98	93	59	93	66	70	63	75	97	60	1591

Table 5 Input values of A'_j and $\sum_j A'_j$

	j										Sum
	1	2	3	4	5	6	7	8	9	10	
A'_j	5	14	8	15	19	16	10	10	17	13	127

Table 6 Input values of h_{rj}

	j									
	1	2	3	4	5	6	7	8	9	10
h_{rj}	20	26	29	16	27	23	29	20	28	28

to 5. The model was coded in the MATLAB software on a laptop with Intel Core i5 8250U 3.4 GHz (two cores) 8 GB RAM running Windows 10. In the following, first, the model is used to specify the optimal inventory policy, and then sensitivity analysis is performed on the total cost function $K(a, T)$.

4.2 Solution

Based on the input parameters, Eqs. (9) and (10) are respectively used to calculate the following values:

$\Delta = 1{,}219{,}239$,

$\beta = 250{,}310$.

Since $\Delta - \beta > 0$, T_1, Eqs. (28), (29), and (32) are respectively used to calculate the values below:

$a = 6.9637$,

$T = 0.0797$,

$K(a, T) = 62{,}098.00$.

As the above value of $a = 6.9$ is not an integer, it must be replaced by one of the two nearest integers . Eqs. (10) and (18) are used to calculate the relevant values for the two cases shown below:

$$\lfloor a \rfloor = 6, \quad \lfloor T \rfloor = 0.0756, \rightarrow K(\lfloor a \rfloor, \lfloor T \rfloor) = 62,\ 256.28;$$

$$\lceil a \rceil = 7, \quad \lceil T \rceil = 0.0799, \rightarrow K(\lceil a \rceil, \lceil T \rceil) = 62,\ 098.20.$$

Based on the above results, the optimal inventory policy is $a^* = 7$, $T^* = 0.0799$, and $K(a, T)^* = 62,\ 098.20$. According to the optimal policy, the cycle time is 0.0799 years (29 days), during which the central warehouse receives one order from each supplier and sends seven orders to each retail store. The order size of the warehouse (i.e., the sum of orders received from all suppliers) is equal to 8,000. The optimal replenishment order sizes for the 10 retail stores are shown in Table 7.

Table 7 Optimal replenishment order sizes of the retail stores

Retail store	1	2	3	4	5	6	7	8	9	10
Order size	122.9	141.7	79.3	121.8	114.3	126	96.4	109.3	125.9	106.5

4.3 Sensitivity analysis

Based on the given data, $\sum_i A_i = 1591$, $\sum_j A'_j = 127$, $\Delta = 1219239$, and $\beta = 250310$. Consequently, the total cost function can $K(a, T)$ be stated as follows:

$$K(a, T) = \frac{1591}{T} + \frac{127a}{T} + 250310T + \frac{968929T}{a}. \tag{42}$$

Figure 4 displays the behavior of $K(a, T)$ in terms of T for different values of a, i.e., $a \in \{1, 2, \ldots, 20\}$. As shown in this figure, each curve has a global minimum which can be found easily by using Eq. (42). For additional clarity, Fig. 5 shows the global minimum of all curves. Figure 5 confirms that the global minimum of $K(a, T)$ occurs indeed at $a^* = 7$.

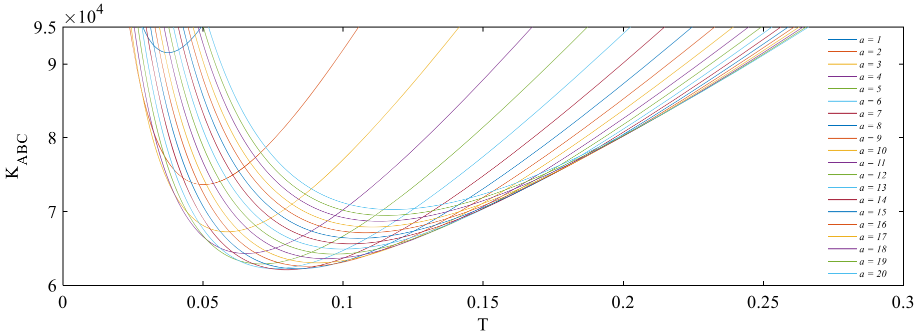

Fig. 4 Behavior of $K(a, T)$ in terms of T

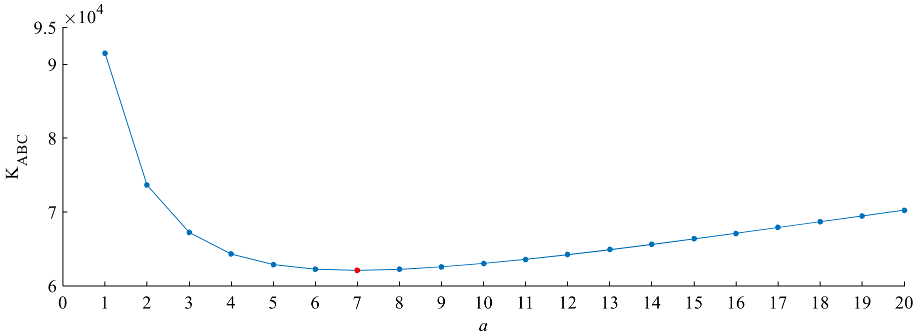

Fig. 5 Scatter plot of the global minimum of the curves

5 Discussion

As stated in Case 1, if $\Delta-\beta \leq 0$, the optimum solution is given by $a = 1, T = \sqrt{\frac{\sum_i A_i+\sum_j A'_j}{\Delta}}$, and $K(a, T) = 2\sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)}$, which the main results are listed in the following: (i) the optimal value of a is independent of all parameters, and it is always equal to 1, (ii) it is concluded that if $\sum_i A_i$ or $\sum_j A'_j$ increases, the optimal value of T increases, and vice versa. Moreover, if Δ gets larger, the optimal value of T decreases, and vice versa. In addition, it is observed that the optimal value of T is independent of β, and (iii) it is evident that the optimal value of K increases when $\sum_i A_i$, $\sum_j A'_j$, or Δ increases, and vice versa. Furthermore, the changes in β have no impact on the optimal value of K.

Based on Case 2, if $\Delta-\beta > 0$, the integer neighbors of $a = \sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}$ are examined to find the optimal solution. Here, the optimal values of T and K are attained by $\sqrt{\frac{a^2\sum_j A'_j}{(\Delta-\beta)}}$ and $\frac{\sum_i A_i}{T}+\frac{a\sum_j A'_j}{T}+\beta T+\frac{(\Delta-\beta)T}{a}$, respectively. As a special case, if $a = \sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}$ is an integer, the optimal values of T and K are obtained by $\sqrt{\frac{\sum_i A_i}{\beta}}$ and $2\sqrt{\beta\sum_i A_i}+2\sqrt{(\Delta-\beta)\sum_j A'_j}$, respectively. Here, the following arguments are concluded:

(1) a increases when $\sum_i A_i$ or Δ grows, and vice versa. In addition, a increases when $\sum_j A'_j$ or β decreases, and vice versa,
(2) If $\sum_i A_i$ grows, the optimal value of T increases, and if β increases, the optimal value of T decreases. In addition, it is observed that the optimal value of T is independent of $\sum_j A'_j$ and Δ, and,
(3) It is clear that the optimal value of K increases when $\sum_i A_i$, $\sum_j A'_j$, or Δ grows, and vice versa.

Based on the above-mentioned results and by considering $\Delta = \frac{\sum_j h_{rj} D_{.j}}{2}$ and $\beta = \frac{h_c D_{..}}{2}$, it is easy to analyze the effects of key factors, such as demand, ordering cost, and holding cost on the optimal solution. The most important analyses can be listed as follows: (i) if the annual demand of each retailer increases, the optimal cost increases (ii) the optimal cost increases when the ordering cost by the central warehouse or retailer increases (iii) if the holding cost at each retail store increases, the optimal cost increases, and (iv) the changes in the holding cost at the central warehouse has no impact on the optimal cost of case 1.

6 Conclusion and outlook

In this research, a novel EOQ-based model was presented for multi-item, multi-supplier, and multi-retailer cross-docking inventory systems. It is assumed that a retail company, which owns a central warehouse and several retail stores, supplies products via a two-stage procurement process. In the first stage, the required products are received from suppliers and are held in the central warehouse. In the second stage, products are delivered from the central warehouse to the retail stores in several smaller orders. The total annual cost of the cross-docking system was formulated by considering the principal costs, i.e., ordering and holding costs, during the two stages. Depending on the given data, the total cost can be optimized under two possible cases. For each case, a globally optimum solution was developed to minimize the total cost of the cross-docking system. Finally, a numerical example was solved to show the applicability and validity of the suggested model.

Based on the findings, the proposed model is considered effective in determining the optimal replenishment policy for the mentioned cross-docking system. The model enables managers in retail companies to set replenishment cycle times by considering the relevant costs, and it can help them evaluate the impacts of changes in the decision variables on the costs of the system. The offered model can be further developed in several ways. For example, the model can be extended by considering multiple central warehouses, multiple suppliers for each product, and multiple products for each supplier (Tirkolaee et al., 2021). In addition, shortage costs, purchase costs, and supplier capacities can be considered. Another future research direction is introducing probabilistic aspects, such as stochastic cost and demand parameters (Khakbaz, 2022). Sustainable development may be studied in the problem in order to enhance the greenness of the inventory system along with cost minimization (Barman et al., 2023a, 2023b; Paul et al., 2022; Pervin et al., 2023). Transportation decisions can be directly integrated with inventory control to make the model more practical (Das et al., 2021). On the other hand, heuristic algorithms can be developed to tackle the complexity of the problem in large scales (Amirteimoori et al., 2023; Golshahi-Roudbaneh et al., 2017). Finally, the application and contribution of digital technologies can be directly investigated in the model in order to compare traditional and smart inventory management systems in terms of cost and efficiency (Salehi-Amiri et al., 2022; Sharma et al., 2022; Simic et al., 2023).

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Data availability Given in Sect. 4.1.

Declarations

Conflict of interest The author does not have any pertinent financial or non-financial affiliations to report.

Human and animals rights Not applicable.

Informed consent Not applicable.

Appendix: Derivations

Derivation of Eq. (22)

$$K(a, T) = \frac{\sum_i A_i}{T} + \frac{a\sum_j A'_j}{T} + \beta T + \frac{(\Delta - \beta)T}{a},$$

$$K(1,\,T) = \frac{\sum_i A_i}{T} + \frac{\sum_j A'_j}{T} + \beta T + (\Delta - \beta)T,$$

$$K(1,\,T) = \frac{\sum_i A_i + \sum_j A'_j}{T} + \Delta T,$$

$$K\left(1,\,T^*\right) = \frac{\sum_i A_i + \sum_j A'_j}{\sqrt{\frac{\sum_i A_i + \sum_j A'_j}{\Delta}}} + \Delta\sqrt{\frac{\sum_i A_i + \sum_j A'_j}{\Delta}},$$

$$K\left(1,\,T^*\right) = \frac{(\sum_i A_i + \sum_j A'_j)}{\sqrt{\sum_i A_i + \sum_j A'_j}}\sqrt{\Delta} + \sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)},$$

$$K\left(1,\,T^*\right) = \sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)} + \sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)},$$

$$K\left(1,\,T^*\right) = 2\sqrt{\Delta\left(\sum_i A_i + \sum_j A'_j\right)}.$$

Derivation of Eqs. (28) and (29)

$$\frac{\partial K}{\partial a} = \frac{\sum_j A'_j}{T} - \frac{(\Delta - \beta)T}{a^2},$$

$$\frac{\partial K}{\partial T} = \frac{-(\sum_i A_i + a\sum_j A'_j)}{T^2} + \beta + \frac{(\Delta - \beta)}{a},$$

$$\frac{\sum_j A'_j}{T} = \frac{(\Delta - \beta)T}{a^2};\ \frac{\sum_j A'_j}{T^2} = \frac{(\Delta - \beta)}{a^2};\ T^2 = \frac{\sum_j A'_j}{(\Delta - \beta)}a^2.$$

Substituting $T^2 = \frac{\sum_j A'_j}{(\Delta-\beta)}a^2$ in

$$\frac{(\sum_i A_i + a\sum_j A'_j)}{T^2} = \beta + \frac{(\Delta - \beta)}{a},$$

$$\frac{(\sum_i A_i + a\sum_j A'_j)}{\frac{\sum_j A'_j}{(\Delta-\beta)}a^2} = \beta + \frac{(\Delta - \beta)}{a},$$

$$\frac{(\sum_i A_i + a\sum_j A'_j)(\Delta - \beta)}{a^2\sum_j A'_j} = \beta + \frac{(\Delta - \beta)}{a}.$$

Multiplying by $a.^2$

$$\frac{(\sum_i A_i + a\sum_j A'_j)(\Delta - \beta)}{\sum_j A'_j} = \beta a^2 + (\Delta - \beta)a,$$

$$\left(\sum_i A_i + a\sum_j A'_j\right)(\Delta - \beta) = \beta a^2\sum_j A'_j + (\Delta - \beta)a\sum_j A'_j,$$

$$(\Delta - \beta)\sum_i A_i + (\Delta - \beta)a\sum_j A'_j = \beta a^2\sum_j A'_j + (\Delta - \beta)a\sum_j A'_j,$$

$$(\Delta - \beta)\sum_i A_i = \beta a^2\sum_j A'_j,$$

$$a^2 = \frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j} \text{and} T^2 = \frac{\sum_j A'_j}{(\Delta-\beta)} a^2 = \frac{\sum_j A'_j}{(\Delta-\beta)} \frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j} = \frac{\sum_i A_i}{\beta}.$$

Alternative way to derive Eq. (29)

$$a^2 = \frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j},$$

$$T = \sqrt{\frac{a(\sum_i A_i + a\sum_j A'_j)}{(\Delta-\beta) + a\beta}},$$

$$T^2 = \frac{\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}\sum_i A_i + \frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}\sum_j A'_j}{(\Delta-\beta) + \beta\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}},$$

$$T^2 = \frac{\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}\sum_i A_i + \frac{(\Delta-\beta)\sum_i A_i}{\beta}}{(\Delta-\beta) + \beta\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}},$$

$$T^2 = \frac{\sum_i A_i\left(\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}} + \frac{(\Delta-\beta)}{\beta}\right)}{\beta\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}} + (\Delta-\beta)},$$

$$T^2 = \frac{\sum_i A_i\left(\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}} + \frac{(\Delta-\beta)}{\beta}\right)}{\beta\left(\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}} + \frac{(\Delta-\beta)}{\beta}\right)},$$

Where $T^2 = \frac{\sum_i A_i}{\beta}$ is obtained only if the optimum value of a is used.

Derivation of Eq. (32)

$$K(a, T) = \frac{\sum_i A_i}{T} + \frac{a\sum_j A'_j}{T} + \beta T + \frac{(\Delta-\beta)T}{a},$$

$$a = \sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}; T = \sqrt{\frac{\sum_i A_i}{\beta}},$$

$$K(a, T) = \frac{\sum_i A_i}{\sqrt{\frac{\sum_i A_i}{\beta}}} + \frac{\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}\sum_j A'_j}{\sqrt{\frac{\sum_i A_i}{\beta}}} + \beta\sqrt{\frac{\sum_i A_i}{\beta}} + \frac{(\Delta-\beta)\sqrt{\frac{\sum_i A_i}{\beta}}}{\sqrt{\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}}},$$

$$K(a,\,T) = \sqrt{\beta\sum\nolimits_i A_i} + \frac{\sqrt{\frac{(\Delta-\beta)\sum_i A_i \sum_j A'_j}{\beta}}}{\sqrt{\frac{\sum_i A_i}{\beta}}} + \sqrt{\beta\sum\nolimits_i A_i} + \frac{(\Delta-\beta)}{\sqrt{\frac{(\Delta-\beta)}{\sum_j A'_j}}},$$

$$K(a,\,T) = \sqrt{\beta\sum\nolimits_i A_i} + \frac{\sqrt{(\Delta-\beta)\sum_i A_i \sum_j A'_j}}{\sqrt{\sum_i A_i}} + \sqrt{\beta\sum\nolimits_i A_i} + \sqrt{(\Delta-\beta)\sum\nolimits_j A'_j},$$

$$K(a,\,T) = \sqrt{\beta\sum\nolimits_i A_i} + \sqrt{(\Delta-\beta)\sum\nolimits_j A'_j} + \sqrt{\beta\sum\nolimits_i A_i} + \sqrt{(\Delta-\beta)\sum\nolimits_j A'_j},$$

$$K(a,\,T) = 2\sqrt{\beta\sum\nolimits_i A_i} + 2\sqrt{(\Delta-\beta)\sum\nolimits_j A'_j},$$

Derivation of Eq. (41)

$$\frac{\partial K}{\partial T} = \frac{-\left(\sum_i A_i + a\sum_j A'_j\right)}{T^2} + \beta + \frac{(\Delta-\beta)}{a},$$

$$\frac{\partial K}{\partial a} = \frac{\sum_j A'_j}{T} - \frac{(\Delta-\beta)T}{a^2},$$

$$H_{11} = \frac{2(\Delta-\beta)T}{a^3};\, H_{12} = \frac{-\sum_j A'_j}{T^2} - \frac{(\Delta-\beta)}{a^2},$$

$$H_{21} = -\frac{\sum_j A'_j}{T^2} - \frac{(\Delta-\beta)}{a^2};\, H_{22} = \frac{2\left(\sum_i A_i + a\sum_j A'_j\right)}{T^3}.$$

Determinant of Hessian matrix H

$$H = \begin{bmatrix} \frac{2(\Delta-\beta)T}{a^3} & -\left[\frac{\sum_j A'_j}{T^2} + \frac{(\Delta-\beta)}{a^2}\right] \\ -\left[\frac{\sum_j A'_j}{T^2} + \frac{(\Delta-\beta)}{a^2}\right] & \frac{2\left(\sum_i A_i + a\sum_j A'_j\right)}{T^3} \end{bmatrix},$$

$$|H| = \frac{2(\Delta-\beta)T}{a^3}\frac{2\left(\sum_i A_i + a\sum_j A'_j\right)}{T^3} - \left[\frac{\sum_j A'_j}{T^2} + \frac{(\Delta-\beta)}{a^2}\right]^2,$$

$$|H| = \frac{4(\Delta-\beta)}{a^3}\frac{\left(\sum_i A_i + a\sum_j A'_j\right)}{T^2} - \left[\frac{\sum_j A'_j}{T^2} + \frac{(\Delta-\beta)}{a^2}\right]^2,$$

$$|H| = \frac{4(\Delta-\beta)\left(\sum_i A_i + a\sum_j A'_j\right)}{a^3T^2} - \left(\frac{\left(\sum_j A'_j\right)^2}{T^4} + \frac{2(\Delta-\beta)\sum_j A'_j}{a^2T^2} + \frac{(\Delta-\beta)^2}{a^4}\right),$$

$$|H| = \frac{4(\Delta-\beta)\sum_i A_i + 4a(\Delta-\beta)\sum_j A'_j}{a^3T^2} - \left(\frac{\left(\sum_j A'_j\right)^2}{T^4} + \frac{2(\Delta-\beta)\sum_j A'_j}{a^2T^2} + \frac{(\Delta-\beta)^2}{a^4}\right),$$

$$|H| = \frac{4aT^2(\Delta-\beta)\left(\sum_i A_i + a\sum_j A'_j\right)}{a^4T^4} - \left(\frac{a^4\left(\sum_j A'_j\right)^2 + 2a^2T^2(\Delta-\beta)\sum_j A'_j + T^4(\Delta-\beta)^2}{a^4T^4}\right),$$

$$|H| = \frac{4aT^2(\Delta-\beta)\sum_i A_i + 4a^2T^2(\Delta-\beta)\sum_j A'_j}{a^4T^4} - \left(\frac{a^4\left(\sum_j A'_j\right)^2 + 2a^2T^2(\Delta-\beta)\sum_j A'_j + T^4(\Delta-\beta)^2}{a^4T^4}\right),$$

$$|H| = \frac{4aT^2(\Delta-\beta)\sum_i A_i + 2a^2T^2(\Delta-\beta)\sum_j A'_j}{a^4T^4} - \left(\frac{a^4\left(\sum_j A'_j\right)^2 + T^4(\Delta-\beta)^2}{a^4T^4}\right).$$

Multiplying by $a^4T.^4$

$$a^4T^4|H| = 4aT^2(\Delta-\beta)\sum\nolimits_i A_i + 2a^2T^2(\Delta-\beta)\sum\nolimits_j A'_j - \left(a^4\left(\sum\nolimits_j A'_j\right)^2 + T^4(\Delta-\beta)^2\right),$$

$$a^4T^4|H| = 4aT^2(\Delta-\beta)\sum\nolimits_i A_i + 2a^2T^2(\Delta-\beta)\sum\nolimits_j A'_j - \left(a^4\left(\sum\nolimits_j A'_j\right)^2 + T^4(\Delta-\beta)^2\right),$$

$$a^4T^4|H| = 4a\left(\frac{\sum_i A_i}{\beta}\right)(\Delta-\beta)\sum\nolimits_i A_i + 2\left(\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}\right)\left(\frac{\sum_i A_i}{\beta}\right)(\Delta-\beta)\sum\nolimits_j A'_j - \left(\left(\frac{(\Delta-\beta)\sum_i A_i}{\beta\sum_j A'_j}\right)^2\left(\sum\nolimits_j A'_j\right)^2 + \left(\frac{\sum_i A_i}{\beta}\right)^2(\Delta-\beta)^2\right),$$

$$a^4T^4|H| = \frac{4a(\Delta-\beta)\left(\sum_i A_i\right)^2}{\beta} + 2\left(\frac{(\Delta-\beta)\sum_i A_i}{\beta}\right)^2 - \left(\frac{(\Delta-\beta)^2(\sum_i A_i)^2}{\beta^2} + \frac{(\Delta-\beta)^2\left(\sum_i A_i\right)^2}{\beta^2}\right),$$

$$a^4T^4|H| = \frac{4a(\Delta-\beta)\left(\sum_i A_i\right)^2}{\beta},$$

$$|H| = \frac{4(\Delta - \beta)\left(\sum_i A_i\right)^2}{a^3 T^4 \beta}.$$

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