Shared drawings in a mathematical modelling activity: An exploratory study

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Recently, scholars have argued that drawings have a crucial role in the students' modelling performance. Nevertheless, such a correlation cannot be extended to complex modelling problems. In this work, we consider the students' drawing activities when dealing with a complex geometry problem. In particular, this paper reports an exploratory study from 11-grade students exposed to a realistic scenario from the sport context on which they have worked for 4 hours. The analysed data, which focus on one of the class discussions, show that drawings help students in going throughout the modelling process and that even shared inaccurate drawings play a crucial role in this sense.

Keywords: Mathematical modelling, qualitative analysis, drawing.

Introduction and theoretical background

Mathematical modelling (MM) is relevant to prepare students for addressing problems arising in everyday and professional activities (OECD, 2019). The term *mathematical modelling* indicates the process of translating between the real world and mathematics in both directions (Blum, 2015), with the purpose of tackling problems arising in the real world. Such a process can be schematised by the Blum's (2015) modelling cycle, which is composed of seven steps within and between reality and mathematics. This scheme is not meant to be a process through which students go sequentially. Despite its importance, MM is demanding because it requires mathematical and extra-mathematical knowledge, as well as appropriate beliefs and attitudes, especially for more complex modelling activities (Blum, 2015, Niss, 2003). One strategy to support students in overcoming these difficulties can be the use of drawings (Rellensmann et al., 2022). In particular, Rellensmann et al. (2022) point out a positive correlation between the drawings and the modelling performance in the context of geometry. Namely, accurate drawings allow students to better achieve the solution of geometric tasks, while inaccurate ones can prevent students from building a correct mathematical model. A drawing is defined as accurate if it contains all the relevant elements and relationships between elements (Rellensman et al., 2022). Nevertheless, Rellensman et al. (2022, p. 415) state that their "results cannot be transferred to more complex modeling problems that students work on for days or weeks." We share the idea of complex modelling activity as a "realistic, authentic modelling problem", in which "students will experience this feeling of helplessness and insecurity, which is characteristic during the process of solving real world problems" (Stender & Kaiser, 2015, p. 1255). We refer to drawings as belonging to the paramount of resources that students and teachers may resort to during mathematical activities (Arzarello et al., 2009). According to Arzarello et al. (2009), we distinguish various types of resources: words (written and oral), expressions (gestures glances, etc.), inscriptions (drawings, sketches, etc.) and instruments (paper and pencil, digital devices, etc.). The system made by the former three resources and their relationship produced by a student or by a group of students while solving a problem and/or discussing a mathematical question is defined as semiotic bundle (Arzarello et al., 2009). Moreover, Arzarello et al. (2009) point out how students can successfully work together by sharing the semiotic bundle, which helps them to bridge the gap between the wordily experience and the more formal mathematics. On the path of such works, we aim at presenting findings about the employment of drawings in dealing with a complex modelling activity to contribute to this rarely investigated topic, as evidenced in Rellensmann et al. (2022). Moreover, we will show how even inaccurate drawings can help students in going through the mathematical modelling process. More precisely we address the following research question: Do accurate and inaccurate drawings help students in going throughout the modelling cycle when dealing with complex MM activities? If so, where and how? To that end, we will analyze a MM activity carried out in high school.

Research context

The present study involved two researchers (the authors), 18 grade 11 students (16-17 years old) and their teacher, fictionally named Veronica. The mathematical modelling activity stems from a real-world scenario in the context of sport. A peculiarity of this activity is that the modelling problem has not been directly presented to the students. The students have instead been gradually introduced to the complex scenario such that the problem has emerged from them. The activity has been carried out through two meetings of two hours each and is framed in the context of rugby. More precisely, a particular moment of the match is considered: the conversion kick. After scoring a try, which is worth 5 points, the team can score 2 additional points if a player, the kicker, can get the ball through the posts, shooting the kick from an imaginary line, perpendicular to the try line and intersecting it at the point where the try has been scored (see Figure 1-left). In this context, an optimization problem arises: which is the distance from the try line that maximizes the angle under which the kicker sees the posts? To make students identifying and solving such a problem, the following tasks were revised starting from a previous version of the activity designed by the authors.

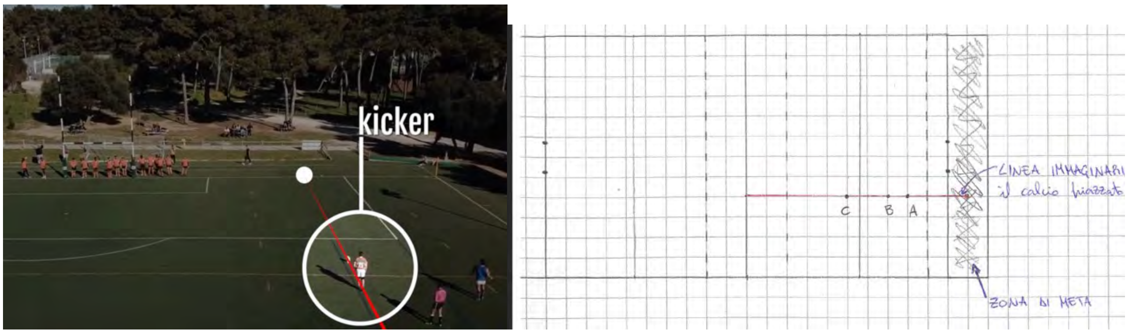

Figure 1: Left: The conversion kick; the white dot and the red line correspond to the try point and the imaginary line, respectively. Right: an example of student's drawing for Task 1. The "try zone" and the "imaginary line on which it is possible to shoot the conversion kick" are also reported

In the first meeting, Veronica started with a brief introduction of the rules of rugby as well as the yard sizes and the purpose of the game, supported by slides (using non-mathematical representation). Afterwards, the students were introduced to the rule of the conversion kick supported by the text: "The conversion kick must be shoot from any point belonging to an imaginary line which is perpendicular to the try line intersecting at the point where the try has been scored". Then, the students were asked to draw the rugby yard and to mark a point from which it is possible to kick the conversion kick (**Task 1)**. Task 1 was meant to guide students in understanding the conversion kick context and in starting to build the mathematical representation of it (see Figure 1-right). In **Task 2**, with the support of a large-size paper sheet (A0) on which a half of the yard was schematically represented, students were asked to work in groups to make some conversion kick tests, using a small paper ball. Task 2 was followed by a discussion, which will be analysed in detail in the next section, prompted by the following questions: **1) Does the kick position have an impact on the result? 2) Does the try point have an impact on the result? 3) Once the try point is fixed, which geometric element has an impact on the result?** Task 2 was proposed to support students in understanding and structuring the real situation to get to the formulation of the optimization problem mentioned above. During the second meeting, students worked in groups to introduce a coordinate system (**Task 3**), thus pursuing the mathematization of the problem emerged at the end of Task 2. Then, in **Task 4**, they were prompted to explore the variation of the angle under which the posts are seen by the kicker and the variation of both the distance from the try line and the try point, using GeoGebra. Finally, **Task 5** was meant to guide students in finding the solution of the problem: the optimal locus of points for the conversion kick is a hyperbole with vertices at the posts. Tasks 4 and 5 were designed to help students in mathematizing, working mathematically, interpreting and validating.

Data analysis

The data consist of the video recording and drawings produced by the students. In this work, the episode under consideration is the class discussion that occurred at the end of Task 2. The episode is analysed according to the diachronic and synchronic analyses of the semiotic bundle (Arzarello et al., 2009). The synchronic analysis allows to consider the relationships among different resources simultaneously activated by the students at a certain moment, while diachronic analysis focuses on the resources activated by students in successive moments. In terms of synchronic analysis, we describe the relationship between words, expressions and inscriptions activated at the same time to identify one or more steps of the modelling cycle; while, in terms of diachronic analysis, we describe the evolution of those resources during the discussion that lead to the formulation of the mathematical modelling problem. For instance, at a certain moment a student (Luca) uses the following words: "I would have defined the angle in a different way. The point in which we kick is the angle's vertex and the two half-lines which pass from the posts delimit the angle". At the same time, he describes the angle with gestures and, afterwards, he draws the angle at the blackboard (inscriptions). This synchronic analysis allows us to identify that Luca is performing the "mathematizing "step of the modelling cycle. From the diachronic point of view, we link this moment to a previous one where another student (Robert) had referred to the angle in a different way. We recall that the questions that prompted the discussion regard the interplay between the kick position and the difficulty of the conversion kick. The episode under analysis lasted 30 min and can be divided into six moments, corresponding to the interventions of six students, fictionally named respectively: Roberto, Gianni, Luca, Mario, Gabriele and Paolo.

Moment 1 starts with Veronica inviting Roberto, a student who could realize a particularly difficult "finger kick" at the beginning of task 2, at the blackboard. The teacher prompts Roberto to represent that particular kick (point R1 in Figure 2-left) and an easier one. Roberto first draws the point R1 and the segment R1-O, called by Roberto as "the trajectory" (see Figure 2-left). Afterwards he draws also points R2 and R3, with the respective "trajectories", affirming that the kick is easier. However, while he strongly states that the kick in R1 is more difficult, he appears to be hesitant when saying that R2 and R3 are easier positions, and he does not reply to the teacher who asks for the reason why. Then, the teacher prompts Roberto to take an example closer to the posts and he chooses to represent a point in the middle of the posts themselves, marking first the try point R4 and, afterwards, the kick points R5 and R6 (Figure 2-left), stating that the trajectory is the same for the two points. The teacher, then, asks Roberto to represent an intermediate case. He draws the green points and segments (Figure 2-left) stating that the difficulty of the kicks follows the same pattern as for the points R1, R2 and R3. From this first excerpt, we observe that Figure 1-right is mathematically enriched by Roberto by adding two not real-world elements: the centre of the posts, O, and the segment which links the kick point to the centre of the posts (the "trajectory" in Roberto's words). In particular, the second one shows that Roberto has already introduced a simplification by representing the trajectory as a straight line. Moreover, Roberto seems to relate the difficulty of the kick with the trajectory. We notice that, for the points R5 and R6, Roberto states that "the trajectory is the same" even if its length is changing between the two points. We can then conclude that the property of the trajectory on which Roberto focuses is not its length. We can guess that Roberto thinks to the direction as the trajectory's property that influence the difficulty of a kick, but, since he does not answer when the teacher asks the reason why points R2 and R3 are easier than point R1, we think that he cannot make this fact explicit.

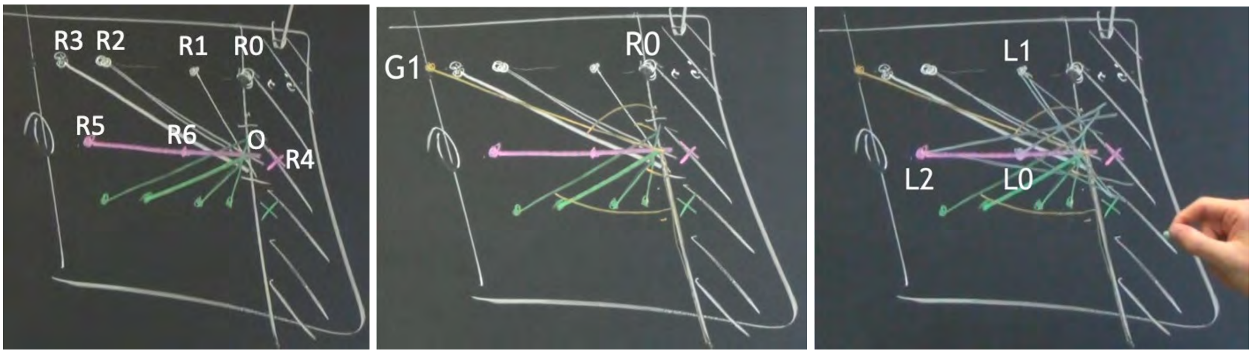

Figure 2: Left: Roberto's drawing. Center: Gianni's drawing with the point G1 and the angles represented in yellow. Right: Luca's drawing. Points L0 and L1 correspond to the vertexes of the two angles (in grey) for two different kicking positions.

At moment 2, a novel element to Roberto's drawing is introduced by Gianni, who notices that the try line and the trajectory identified by Roberto can be considered as the sides of an angle. Gianni's sketch of this angle for different kick points is depicted in yellow in Figure 2-center. Gianni establishes and makes explicit a relationship between the difficulty of a kick and the width of the angle, saying: "the smaller this angle, the more difficult it is to kick". Veronica, then, puts the attention on the pink elements in Figure 2-center: Gianni explains that this is the easiest case because "the angle reaches the maximum width". The discussion then shifts on the try point R0: Gianni indicates and draws point G1, identifying it as the one for which the maximum width is reached. We notice that, in doing this, Gianni's tone of voice is uncertain. Afterwards, Gianni shifts the attention once more to the the pink elements in Figure 2-center, repeating that it is the simplest case. We notice that Gianni goes ahead on the same path as Roberto, by adding a new mathematical element, the angle, and, thus, making explicit the vague idea introduced by Roberto. Thus, Gianni further structures the situation model and starts to mathematize it. The fact that Gianni tries to re-shift the attention on the central case (represented in pink in Figure 2) makes us think that he is trying to put the attention on a case which can be well described by the angle he has introduced: indeed, the angle explains well the fact that it is more difficult to kick from a lateral position than from a central one. On the other hand, the hesitation shown by Gianni indicates that he probably starts to feel uncomfortable about the fact that the angle he has proposed suggests that, regardless from where the try point is, the most advantageous kicking position is always on the half-way line.

At moment 3, Luca takes the floor, saying “I would have defined the angle in a different way: the point in which we kick is the angle’s vertex and the two half-lines which pass from the posts delimit the angle”. While saying this, he also draws the angle in the space in front of him using gestures. Afterwards, when prompted by the teacher, he adds this new element to the previous drawing for two different kicking positions (see angles with vertexes in points L0 and L1 in Figure 2-right. Luca further adds: “depending on the angle’s width we have a more or less open view on the posts”. He then specifies, pointing the segment which connects the posts, that “it’s a better view on the space between the posts”. He further comments that, with the angle introduced by Roberto and Gianni before, the focus is put on the midpoint of the posts, which is not always visible from the kicker perspective. He adds also that “from our perspective the posts have to be as far as possible”, accompanying his words with the gesture of moving his indexes away. Luca, then supports his angle’s proposal by focusing the attention on points L0 and L2 (Figure 2-right) and saying that his angle is “more accurate” because the angle varies with the distance from the try line (contrarily to the angle proposed by Gianni), explaining that “here [pointing at L0] it’s easier from closer because the angle is wider, while, going further, it is more difficult”. We notice that, while Gianni follows the same path as Roberto, Luca goes ahead in partial opposition to them: he bases his strategy on the drawing made by Roberto and Gianni, but he proposes a different angle. We want to point out that, despite Luca being a rugby player, he is not the first student taking the floor in the discussion. To propose his angle, which will turn out to be a key element for building an accurate drawing of the modelling problem, he exploited the inaccurate drawing proposed by Roberto and Gianni. The previous discussion, summarized in the drawing by Roberto and Gianni, is the stimulus for Luca to find a mathematical element (the angle) to represent an element belonging to the reality (“the view on the posts”, in Luca’s words) which, as a rugby player, he identifies as a fundamental one. The use of shared drawings is an important support to the exchanges among students: as a consequence, Luca activated and shared a validation of the proposal of his classmates. Validation is evident both when Luca criticizes the focus put on the midpoint of the posts by Gianni’s proposal and when he affirms that his model is more accurate in explaining the central case.

At the beginning of moment 4, various elements are added to the drawing by Mario, who starts his intervention by saying that he would define, in his words, “a piecewise line” to discriminate the case “inside and outside the posts”. Inside the posts, he proposes to shoot from as close as possible to the try line because “the kick is sure”. In the case outside the posts, Mario explains his strategy using the angle proposed by Luca: “when we are far from the posts, to increase the possibilities to score [he draws angle M0 in Figure 4-left], we must move backwards because the angle increases slightly [he makes a gesture with the hand in the direction of the arrow in Figure 5-left]. If we go closer, we just have to be here [he draws point M1 in Figure 4-center] to obtain the same result [he draws the angle M2 in Figure 4-center]".

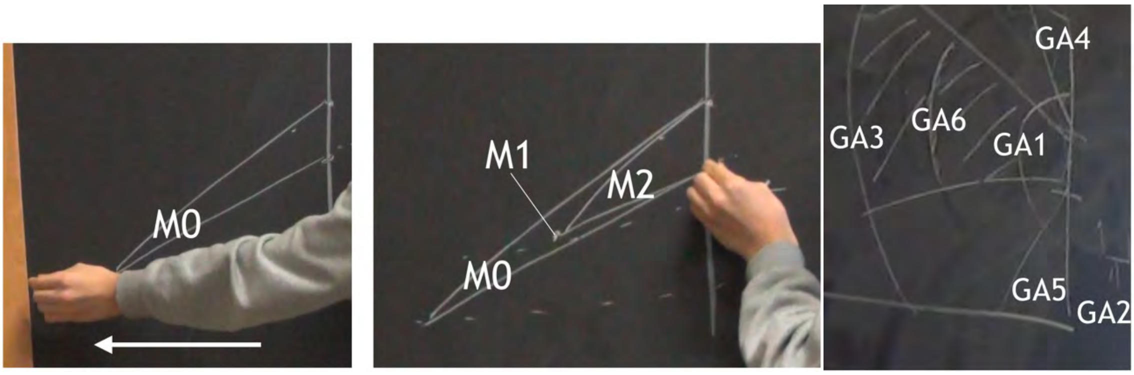

Figure 3: Left and Center: Mario's drawing. Right: Gabriele's drawing

Mario, using Luca's angle, is introducing a new element: the locus of points which allow to have the same view on the posts, thus adding an important element toward the definition of the problem. A second important element introduced by Mario is that, far from the try line, the posts can be approximated as a single point and, according to him, this fact can support the proposal of Roberto and Gianni, for which the midpoint between the posts is an important element. By doing this, Mario introduces for the first time the edge case of what happens when the kicker is very far from the try line and carries out a validation of Gianni and Roberto's proposal. Mario's drawing and his remark about the edge case far from the try line, stimulates a new intervention by Luca. In opposition to Mario's statement that the posts tend to seem closer when moving far from the try line, Luca points the attention toward what happens when we start moving away from the try line: "if I am on the try line, I can see a single post. If I make a step to the side, I see the posts as if they were joint, the more I go backwards, the more the posts appear to be separated. The more I am toward the out line [toward the sides of the yard] the more I have to go far so that I can see a wider space". The last remark by Mario is a trigger for Luca to describe what happens in the opposite edge case, on the try line, even if Luca makes his remark in contrast to Mario's one, not recognizing that both are true.

Afterwards, the teacher focuses the attention on the fact that an argument in favour to the angle proposed by Luca is that, having the vertex on the kicker, it can be more functional in representing his point of view. She then prompts the students suggesting them to think about what happens to the two proposed angles very far from the try line. Gianni answers that the angle he has proposed gets wider and wider but without reaching 90 degrees. The teacher then asks: "In Luca's case that angle is zero on the try line, then it gets wider and, afterwards, does it grow indefinitely?" Luca answers: "No, at a certain point it will decrease again. It will reach a point of maximum amplitude, which is the best point to kick and then it starts decreasing". We notice that the last discussion allows Luca to introduce a new element: the existence of an optimal position for the kick, mathematically described by the point in which the angle is maximum. Despite the fundamental elements introduced by the students in the discussion, we remark that, at this point, no problem has been formulated yet. In this sense, the intervention of Gabriele at moment 5 is fundamental. He proposes a completely different perspective and drawing with respect to his classmates, represented in Figure 3-right. Drawing the half circle GA1, he says that "its radius depends both on the power of the kick and on the success rate". He further states that the area within that half-circle must be eliminated because the kick would be too difficult given the structure of the posts, which he represents as GA2 in Figure 3-right. Gabriele then draws a second half circle (identified by GA3 in Figure 3-right and he states that the points outside GA3 must be eliminated "because I have to use too much power and, above all, because there is a quite low success rate, since the ball has to run a longer trajectory". He then eliminates the lateral zones GA4 and GA5 and he further identifies a central zone, GA6, by referring in an unclear way to the strategies proposed by his classmates, naming them "Gianni's idea" and "Luca's idea". He proposes to carry out a "probability calculation using a gaussian (distribution, ed.) in this zone [GA6] to further identify the optimal points." We point out that, even if Gabriele's drawing is inaccurate and, in some ways, confusing, it is the first one in which the problem of finding the optimal points for shooting the kick clearly emerges. Moreover, Gabriele also introduces in his proposal many real-world elements (the power of the kick and the three-dimensional structure of the posts) which gives the teacher the opportunity to discuss about the fact that it is possible to build different mathematical models of different complexity for the same situation.

The problem is more accurately expressed, at moment 6, by Paolo who suggests to "find which is the best distance to shoot the kick" and he further says: "if we score a very lateral try because the defender leads us laterally, we have to go backwards to have a better view for shooting the kick". We remark that Paolo uses Gabriele's proposal to more accurately define the problem: while Gabriele a-priori eliminates the very lateral zones, Paolo considers, also the possibility of scoring very lateral tries. Luca supports Paolo's proposal, saying that "the central zone identified by Gabriele [see zone GA6 in Figure 3-right] is the one from which shooting the kick is easier but, from a statistical perspective, in matches lateral tries are much more likely". Both Paolo and Luca activate a validation of the problem and the solution delineated by Gabriele and they propose a problem which, from their perspective, is more useful as a support for decision making in rugby.

Discussions and conclusions

We recall that in this work the research question was: Could accurate and inaccurate drawings help students in going throughout the modelling cycle when dealing with complex MM activities and, if yes, where and how? To address this question, a classroom discussion leading to the identification of a mathematical modelling problem has been analysed, using diachronic and synchronic analysis (Arzarello et al., 2009) and the mathematical modelling cycle of Blum (2015). The analysis suggests that drawings (both accurate and inaccurate) help students' exchanges when they are dealing with complex MM activities. But the pivotal aspect in this context is that drawings have to be shared among students. Indeed, results show how students use their classmates' drawings either refining them or as a stimulus for proposing alternative drawings. While in Rellensmann et al. (2022, p. 413) it is said that "inaccurate drawings can hinder the student from constructing a correct mathematical model because the drawing illustrates incorrect relationships between the objects which the students translate into an incorrect mathematical model", referring to drawings which are not shared among students, we argue that, when resources are shared between students, also inaccurate drawings can help students in progressing throughout the modelling process. Moreover, we noticed in which steps of the modelling cycle students were helped during the MM activities. For instance, Luca performed a validation step when the inaccurate drawings provided by his classmates prompted him to propose the correct angle because the midpoint of the posts is not always visible from the kicker perspective. Somehow this is in contrast with previous findings: "Students normally do not validate their solutions, it seems to be part of the "contract didatique": checking the correctness and suitability of a solution is exclusively the teacher's responsibility" (Blum, 2015, p. 79). Finally, we can notice how the synchronic and diachronic analysis of the semiotic bundle shed light on students' modelling processes, confirming once more that students (and more in general modellers) don't proceed throughout the modelling cycle in a sequential way: as an example, students propose solutions without a proper mathematization and mathematical work or they activate validation processes in the first phases of the modelling cycle.

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