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On the stabilization of forking and cyclic trajectories for nonlinear systems ⋆

Nicolas Kessler * **Lorenzo Fagiano** *

* *Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy (e-mail: nicolasmatthias.kessler@polimi.it, lorenzo.fagiano@polimi.it).*

Abstract: Stabilizing a reference trajectory for a nonlinear system is a common, non-trivial task in control theory. An approach to solve this problem is to approximate the nonlinear system along the trajectory as an uncertain linear time-varying one, and to solve an optimization problem featuring Linear Matrix Inequality (LMI) constraints to derive a stabilizing, smooth, gain-scheduled control law. Such an approach is extended here by considering a set of reference trajectories instead of a single one, such that switching among them is permitted. These switching events are commonly encountered in industrial plants, such as energy generation systems, and are of high relevance in practice. The approach allows one to derive a gain-scheduled control law guaranteeing asymptotic stability also during the switching and accounting for the linearization errors. Simulation results on a chemical system highlight the effectiveness of the method.

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1. INTRODUCTION

A common approach to control of industrial systems is to generate a set of admissible open-loop trajectories which are to be stabilized by corresponding feedback laws. Generating admissible open-loop trajectories can be done efficiently via open-loop optimal control Lee and Markus (1967), flatness Sun et al. (2022), iterative learning control Cobb et al. (2017), physical insight or possibly trial and error. However, the design of a stabilizing feedback still remains a challenge, especially when the trajectory is allowed to be cyclic or have forks due to decisions in a higher level planner. Indeed, industrial systems are often operated along cyclic trajectories, obtained by switching between different operating modes. High-level decision layers may command a switch between predefined reference trajectories and possibly also between different control algorithms. Examples are systems for manufacturing Konter and Thumann (2001) or power generation, such as airborne wind energy systems Fagiano et al. (2021). While a substantial amount of work has been carried out on stabilizing a single reference trajectory, the same does not hold when considering such switching behaviors. The current approaches to evaluate close-loop stability in these cases include brute-force analysis via Monte Carlo simulations, the study of the overall system as a hybrid one where the discrete state represents a selection of the mode of operation, or nonlinear model predictive control. All these solutions, however, appear to be not tractable either in the off-line phase or in real-time operation, especially for relatively fast dynamics.

In this paper, we propose an approach that guarantees asymptotic stability of the trajectories of interest also across the reference switching events, and it is computationally tractable off-line and very efficient on-line since it yields a smooth, gain-scheduled feedback law suitable for implementation on embedded systems with fast sampling rates. Further, we propose a novel method to estimate the basin of attraction for the stabilized trajectories.

In the remainder, we understand a forking trajectory as a reference trajectory along which a given state at a specific time may have two successors, depending on a decision, of which the outcome is not known a-priori during the design process of the feedback. To address the resulting stabilization problem, we extend a recently proposed approach based on LMIs Kessler and Fagiano (2023). The main idea is to consider the points (or regions) in the state space where switches can occur as nodes, and each trajectory as a set of directed edges connecting two such nodes. Then, we extend the result in Kessler and Fagiano (2023) by considering all possible successors for each node, instead of one distinct successor as done for single-trajectory problems. After presenting the method and its theoretical properties, we showcase its behavior on a chemical plant model.

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2. PROBLEM FORMULATION

We consider a nonlinear, time-invariant system in discrete time k:

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$$x(k+1) = f(x(k), u(k)), \tag{1}$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u \in \mathbb{U} \subseteq \mathbb{R}^m$ the input vector. Associated with this nonlinear system we consider a digraph $\mathbf{G}$ consisting of a finite set of nodes $\mathbf{V} \subset \mathbb{X}$, representing discrete reachable states of interest, and edges $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$ connecting these nodes; each edge thus represents the input value that steers the system from a node to another one in one time step. For the sake of compactness, we refer to the elements in $\mathbf{V}$ with their index i and denote its cardinality as $M = |\mathbf{V}|$. Correspondingly, we denote the edges as $\mathbf{E} = \{V_i, V_j\} \hat{=} \{i, j\}$. We denote with $\mathcal{F}$ the map associating an edge to a specific input, implicitly assuming it is unique

$$\mathcal{F} : \mathbf{E} \to \mathbb{U}$$
$$\mathcal{F}(E = \{i, j\}) \in \{u \in \mathbb{U} | x_j = f(x_i, u)\}.$$

For each node $V_i \in \mathbf{V}$ we denote by $\mathcal{N}^{\text{out}}(i)$ the set of out-neighbors of the node. For each edge $E \in \mathbf{E}$, we denote with E^- the node in $\mathbf{V}$ which is left by the edge and by E^+ the node in $\mathbf{V}$ which is entered by the edge. We denote by $\mathbf{W}$ a set of walks in $\mathbf{G}$. Each walk $W_r \in \mathbf{W}$ represents a reference trajectory of interest for the nonlinear system (1):

$$W_r = \{(x_{\text{ref}}^{W_r,\ell}, u_{\text{ref}}^{W_r,\ell})\},\ \ell = 0, \ldots, N_{W_r}$$
$$\mathbf{W} = \{W_r\},\ r = 1, \ldots, N_r$$

Let $\bar{r}(k)$ and $\bar{\ell}(k)$ be the indices denoting the active reference trajectory and the active node in that trajectory at time k, respectively. Then, we can specify $x_{\text{ref}}(k)$ and $u_{\text{ref}}(k)$ as follows:

$$x_{\text{ref}}(k) = x_{\text{ref}}^{W_{\bar{r}(k)},\bar{\ell}(k)}$$
$$u_{\text{ref}}(k) = u_{\text{ref}}^{W_{\bar{r}(k)},\bar{\ell}(k)}.$$

Though this definition masks the dependency on $\bar{r}(k)$ and $\bar{\ell}(k)$, for the sake of brevity, this dependency is assumed to be implicitly known. For a given state-input pair $(x(k), u(k))$, $\Delta x(k) := x(k) - x_{\text{ref}}(k)$ denotes the tracking error and $\Delta u(k) := u(k) - u_{\text{ref}}(k)$ denotes the input deviation from the reference. We consider the presence of desired sets where these deviations shall lie during the whole considered time interval, given by $\Delta x(k) \in \Delta\mathbb{X}$, $\Delta u(k) \in \Delta\mathbb{U}$, $\forall W_r \in \mathbf{W}$, $\forall k = 0, \ldots N_{W_r}$, containing the origin in their interior.

We consider the following assumptions, where $\mathbb{A} \oplus \mathbb{B}$ is the Minkowski sum of the two generic sets $\mathbb{A}$ and $\mathbb{B}$:

$$\mathbb{A} \oplus \mathbb{B} \doteq \{a + b : a \in \mathbb{A},\ b \in \mathbb{B}\}.$$

Assumption 1. The sets $\Delta\mathbb{X}$ and $\Delta\mathbb{U}$ are compact, nonempty and contain the origin in their interior.

Assumption 2. $\forall W_r \in \mathbf{W}$ and each $\ell = 0, \ldots, N_{W_r}$, function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in (1) is differentiable in the set

$$\left(\{x_{\text{ref}}^{W_r,\ell}\} \oplus \Delta\mathbb{X}\right) \times \left(\{u_{\text{ref}}^{W_r,\ell}\} \oplus \Delta\mathbb{U}\right).$$

For simplicity, in the remainder we consider a hypercuboid for the inputs $\Delta u_{\min} \leq \Delta u(k) \leq \Delta u_{\max}$, which is also a very common case in practice.

As mentioned, we want to consider trajectories that may have cycles, merge with other trajectories, or forks. Considering the graph formalism, wherever forks or merges occur, there will be nodes with multiple successors or predecessors respectively. The precise task is to design a (generally) time-varying, feedback controller $\mathcal{C}(\Delta x, k)$ such that the origin of the closed-loop system describing the error dynamics

$$\Delta x(k+1) = f(\Delta x(k) + x_{\text{ref}}(k), u_{\text{ref}}(k) + \mathcal{C}(\Delta x(k))) - f(x_{\text{ref}}(k), u_{\text{ref}}(k)) \tag{2}$$

is asymptotically stable for each trajectory

$$(x_{\text{ref}}(k), u_{\text{ref}}(k)),\ k = 0, \ldots, N_{W_r}, W_r \in \mathbf{W} \tag{3}$$

as well as for the overall trajectory obtained by any possible switching events allowed by the graph $\mathbf{G}$.

3. PROPOSED APPROACH AND MAIN RESULTS

We propose a partitioning followed by the construction of an uncertain system and propose a gain-scheduled, linear feedback. Around each node $x_i \in \mathbf{V}$ and edge $E \in \mathbf{E}$ we extend the discrete value x_i and $u_i = \mathcal{F}(E)$ to a set by adding the allowed deviation employing the Minkowski sum. Based on these sets, we define the uncertain error dynamics

$$\forall E \in \mathbf{E} :$$
$$\mathbb{A}_E = \left\{ A = \left.\frac{\partial f}{\partial x}\right|_{(x,u)},\ (x, u) \in E^- \oplus \Delta\mathbb{X} \times \mathcal{F}(E) \oplus \Delta\mathbb{U} \right\}$$
$$\mathbb{B}_E = \left\{ B = \left.\frac{\partial f}{\partial u}\right|_{(x,u)},\ (x, u) \in E^- \oplus \Delta\mathbb{X} \times \mathcal{F}(E) \oplus \Delta\mathbb{U} \right\}$$

To each edge $E \in \mathbf{E}$, we associate an uncertain linear system

$$\forall E \in \mathbf{E} : \mathbb{S}_E = \ (\mathbb{A}_E \times \mathbb{B}_E).$$

We call the graph of uncertain linear systems associated with $\mathbf{G}$ the uncertain linear time varying (LTV) system

$$\mathbb{S} = (\mathbf{G}, \mathcal{S}) \qquad \mathcal{S}(E) = \mathbb{S}_E. \tag{4}$$

Let us denote $E_{W_r,0}$ to $E_{W_r,N_{W_r}-1}$ the edges of a walk W_r. The uncertain LTV system obtained along a walk $W_r \in \mathbf{W}$ is denoted

$$\mathbb{S}_{W_r} = \{\mathbb{S}_{E_{W_r,0}}, \ldots, \mathbb{S}_{E_{W_r,N_{W_r}-1}}\}.$$

For the sake of brevity, given a node i that is part of a trajectory, we let $\bar{\mathbf{V}}_i$ be the union of i itself and its out-neighbors, i.e.

$$\bar{\mathbf{V}}_i = \{i\} \cup \mathcal{N}^{\text{out}}(i).$$

For each walk W_r, let us further introduce the set of admissible scheduling sequences, Π_r:

$$\Pi_r \doteq \left\{ \begin{array}{l|l} \boldsymbol{\pi}_r = & \pi_r(\ell) \in [0,1]^{N_{W_r}} \\ \{\pi_r(0), & \sum_{i=1}^{N_{W_r}} \pi_i(\ell) = 1 \\ \ldots, & \pi_{r,i}(\ell) \in [0,1], \forall i \in \{\ell, \ell+1\} \\ \pi_r(N_{W_r})\} & \pi_{r,i}(\ell) = 0,\ \forall i \notin \{\ell, \ell+1\} \end{array} \right\} \tag{5}$$

where $\pi_i(\ell)$ denotes the i^{th} entry of vector $\pi(\ell)$. We associate to each node a feedback gain K_i, which is to be designed. Correspondingly, each reference state $x_{\text{ref}}^{W_r,\ell}$ at step ℓ of the trajectory W_r will have an associated feedback gain $K^{W_r,\ell}$. Then, for chosen scheduling sequences $\boldsymbol{\pi}_r \in \Pi_r$, $r = 1, \ldots, N_r$, we consider a controller with the following structure:

$$\mathcal{C}(\Delta x, \bar{r}(k), \bar{\ell}(k)) = \underbrace{-\sum_{j=1}^{N_{W_{\bar{r}}}} K^{W_{\bar{r}},j} \pi_{\bar{r},j}(\bar{\ell}(k))}_{K(\bar{r}(k),\bar{\ell}(k))} \Delta x(k). \quad (6)$$

In this last paragraph we address the extension to robust control. The model we employ is a generalization of Takagi-Sugeno Fuzzy models, which consider a linear parameter varying plant and an additive uncertainty as a polytopic set of matrices López-Estrada et al. (2019); the method can be applied to a graph of systems, where a Takagi-Sugeno Fuzzy model is associated with each edge. The type of stability we establish is polyquadratic Lyapunov stability, cf. Daafouz and Bernussou (2001). To examine the effect of additive disturbance, well-known methods that exploit Lyapunov stability can be used, for example the methods described in Khalil and Grizzle (2002) in the chapter "Stability of Perturbed Systems".

3.1 Computing the Feedback Gains

In this section, we provide a result that allows one to compute K_i such that the uncertain LTV system (4) is stabilized and therefore each trajectory (3) is stabilized with a gain-scheduled control law. Then, in Section 3.2 we derive sufficient conditions on the initial perturbation $\Delta x(0)$ guaranteeing that the actual trajectories $(\Delta x(k), \Delta u(k))$ belong to $\Delta\mathbb{X} \times \Delta\mathbb{U}$, $\forall k = 0, \ldots, N_{W_r}, W_r \in \mathbf{W}$, thus achieving consistency of the obtained trajectories with our standing assumptions.

Let us first assume that for any walk $W_r \tilde{=} \{w_{r,i} = w_{r,1}, \ldots, w_{r,N_{W_r}}\} \in \mathbf{W}$ the following hard-switching scheduling sequence $\bar{\boldsymbol{\pi}}_r \in \Pi_r$ is used:

$$\bar{\boldsymbol{\pi}}_r \in \Pi_r : \pi_r(\ell) \in \{0,1\}^{N_{W_r}} \quad (7)$$

i.e. where all except one of the elements in $\bar{\pi}$ are 0. The element with value 1 pertains to the node representing $\Delta x_{\text{ref}}(k)$.

Lemma 3. (Gain-scheduled Stabilizability of the uncertain LTV system). The uncertain LTV system (4) can be asymptotically stabilized at the origin along a reference walk $W_r \in \mathbf{W}$ by a controller of the form (6) if there exist matrices $G_i \in \mathbb{R}^{n\times n}$, $R_i \in \mathbb{R}^{m\times n}$ and $Q_i \in \mathbb{S}_{++}^{n\times n}$ such that

$$0 \prec \begin{bmatrix} G_i + G_i^T - Q_i & (\cdot) \\ AG_i - BR_i & Q_j \end{bmatrix} \quad \begin{array}{l} \forall i = 1, \ldots, M \\ \forall j \in \{i, \mathcal{N}^{\text{out}}(i)\} \\ \forall (A, B) \in \mathbb{S}_E \end{array} \quad (8)$$

In this case, a stabilizing controller is the feedback law (6) with the gains $K_i = R_i G_i^{-1}$ and the hard-switching scheduling sequence $\bar{\boldsymbol{\pi}}_r$ (7).

Proof. By considering each edge $E \in \mathbf{E}$ a reference trajectory, stability under Feedback (6) with the gains $K_i = R_i G_i^{-1}$ and the hard-switching scheduling sequence $\bar{\boldsymbol{\pi}}$ (7) along each edge is proven by Lemma 3.1 in Kessler and Fagiano (2023). In what follows, we prove stability of any uncertain LTV system $\mathbb{S}$ under Feedback (6). Consider the node first node of the reference trajectory's (3) walk W_r, denoted $w_{r,1} = i$. If the trajectory ends in i, this concludes the proof, because each node is stabilized. Else, there will be a successor $j \in \mathcal{N}^{\text{out}}(i)$. In virtue of LMI (8) the uncertain linear systems $\forall j \in \mathcal{N}^{\text{out}}(i) : \{\mathbb{S}_{\{i;j\}}\}$ are stable under the proposed feedback law. From node j onward, it is equivalent to consider the tail of W_r, $W_{r,+} = W_r \backslash \{w_{r,1}\}$. Observe that by construction $W_{r,+} \in \mathbf{W}$, thus, $W_{r,+}$ is stabilized and the proof applies recursively, until $W_{r,+}$ consists of only one node and the proof is concluded.

We now state our main result, which generalizes Lemma 3 to any gain scheduling sequence $\boldsymbol{\pi}_r \in \Pi_r$ and to deal with asymptotic stability of the nonlinear system (2) (instead of its uncertain LTV approximation) under the standing assumptions along any walk $W_r \in \mathbf{W}$. Asymptotic stability of the nonlinear trajectory can be guaranteed by a control law with associated Lyapunov function for the closed loop uncertain LTV system (4), depending polytopically on $\boldsymbol{\pi}$, according to the following Theorem.

Theorem 4. (Gain-scheduled Stabilizability of the nonlinear system along compatible reference trajectories). Consider the model (1) and a reference trajectory compatible with $(\mathbf{V}, \mathbf{E})$ $(x_{\text{ref}}(k), u_{\text{ref}}(k))$, $k = 0, \ldots, N_{W_r}, W_r \in \mathbf{W}$. Let Assumptions 1 and 2 hold. Consider a feedback controller of the form (6) with any scheduling sequence $\boldsymbol{\pi}_r \in \Pi_r$. Further assume that the close loop state-input perturbation trajectories, $(\Delta x(k), \Delta u(k))$, belong to $\Delta\mathbb{X}\times\Delta\mathbb{U}$ for all $k = 0, \ldots, N_{W_r}$. Then, the model (2) is asymptotically stable at the origin if there exist matrices $G_i \in \mathbb{R}^{n\times n}$, $R_i \in \mathbb{R}^{m\times n}$ $Q_i \in \mathbb{S}_{++}^{n\times n}$, $i = 1, \ldots, M$, such that

$$0 \prec \begin{bmatrix} G_i + G_i^T - Q_i & (\cdot) \\ AG_i - BR_i & Q_j \end{bmatrix} \quad \begin{array}{l} \forall i = 1, \ldots, M \\ \forall j, \hat{j} \in \{i, \mathcal{N}^{\text{out}}(i)\} \\ \forall (A, B) \in \mathbb{S}_{\hat{j}} \end{array} \quad (9)$$

and in between any two subsequent partitions $\{i, j\} \in \mathbf{E}$ a set of stabilizing feedback gains K_i for control Law (6) is given as

$$K_i \in \text{Co}\{R_i G_i^{-1}, R_j G_j^{-1}\}, \quad (10)$$

where Co denotes the convex hull.

Proof. We first establish stability of the system (1) under feedback law (6) with any scheduling sequence $\boldsymbol{\pi}_r \in \Pi_r$ and gains K_i along each individual edge by applying Theorem 3.1 in Kessler and Fagiano (2023). In virtue of Proposition 3.1 in Kessler and Fagiano (2023) the nonlinear system's dynamics are covered by the uncertain LTV system (4). This holds true for any trajectory compatible with $(\mathbf{V}, \mathbf{E})$, because $\forall E \in \mathbf{E}$ the uncertain LTV system $\mathbb{S}_E$ is stabilized. The proof can be extended to not only consider individual edges, but the whole graph and and uncertain LTV system (4) as done in the proof of Lemma 3 concluding the proof.

To obtain the Feedback (6) from suitable matrices K_i each $K^{W_r,j}$ is assigned to the K_i with i corresponding to the index of the node in the trajectory.

Concluding this Section, we define the closed-loop uncertain LTV system and solutions of uncertain LTV systems in form (4). Let each node $V_i \in \mathbf{V}$ be associated with a set of linear feedback gains $\mathbf{K}_i$. For each edge $E \in \mathbb{E}$ consider the linear construction

$$\forall E \in \mathbf{E} :$$
$$\mathbf{A}_{E,\kappa} = \{A - BK : A \in \mathbf{A}_E,\, B \in \mathbf{B}_E,\, K \in \mathbf{K}_{E^-}\}$$

the closed loop system. We call the graph of uncertain linear closed-loop systems associated with $\mathbb{G}$ the uncertain LTV system

$$\begin{aligned} \mathbf{S}_\kappa &= (\mathbb{G}, \mathcal{S}_\kappa) \\ \mathcal{S}_\kappa(E) &= \mathbf{A}_{E,\kappa}. \end{aligned} \tag{11}$$

Next, we shall define solutions to the autonomous uncertain LTV systems (11). We denote by

$$\Delta \mathbf{X}_{W_r}(\Delta x_0) = \{\Delta x(0) = x_0, \Delta x(1), \ldots, \Delta x(N_{W_r})\}$$

a trajectory of $\Delta x(k)$ starting in Δx_0 associated with a walk $W_r \in \mathbf{W}$.

Definition 5. (Solutions of the autonomous System (11)).A trajectory $\Delta \mathbf{X}_{W_r}(\Delta x_0)$ s called a solution to a system in form (11) along a walk $W_r = w_1, \ldots, w_{N_{W_r}} \in \mathbf{W}$ starting in Δx_0 if and only if

$$\begin{aligned} \Delta x(0) &= x_0 \\ \forall k = 0, \ldots, N_{W_r} - 1 : \Delta x(k+1) &\in \\ \{x \in \mathbb{R}^n : \exists A \in \mathbf{A}_{\{w_k, w_{k+1}\},\kappa} : x &= Ax(k)\}. \end{aligned} \tag{12}$$

3.2 Basin of attraction

Analogously to our prior work, we continue by establishing a basin of attraction for any compatible trajectory such that the bounds

$$\forall W_r \in \mathbf{W} : \Delta x(k) \in \Delta \mathbb{X}, \quad \Delta u(k) \in \Delta \mathbb{U} \quad \forall k = 0, \ldots, N_{W_r}$$

are satisfied. We start by defining invariant ellipsoids, then consider the strongly connected components in $\mathbf{G}$ first, then show, that it is sufficient to consider the condensation digraph C$\{\mathbf{G}\}$ and conclude by establishing a basin of attraction for a set of trajectories. A definition of the condensation digraph can be found in Bullo (2020) along with further references. A box-constraint $\Delta u(k) \in \Delta \mathbb{U}$ $\forall k = 1, \ldots, N_{W_r}$ can be enforced analogously to Kothare et al. (1996); Cuzzola et al. (2002) and extending the semidefinite programs accordingly is straight forward. For the sake of compactness and brevity, we focus on enforcing constraints on Δx.

Let us define an invariant ellipsoid for $\mathbb{S}$ associated with a graph $(\mathbf{V}, \mathbf{E})$. We call an ellipsoid

$$\mathcal{E}(Q) = \{x \in \mathbb{R}^n : x^T Q^{-1} x \le 1\}, Q \in \mathbb{S}^{n \times n}_{++},$$

where $\mathbb{S}^{n\times n}_{++}$ denotes the set of positive-definite, symmetric $n \times n$ matrices, invariant, if the following holds:

$$\forall k \in \mathbb{N} : x(0) \in \mathcal{E}(Q) \implies x(k) \in \mathcal{E}(Q).$$

In the trivial case, that there is a constant Lyapunov function for $\mathbb{S}$, i.e. $Q_1 = Q_2 = \ldots = Q_M$, $\mathcal{E}(Q_1)$ is an invariant ellipsoid, see Boyd et al. (1994).

Let us now consider a strongly connected graph $\mathbf{G}$. For all possible walks with associated trajectories we aim to find the worst-case expansion of $\|\Delta x(k)\|$ under feedback (6), which can be understood as an $\mathcal{H}_\infty$-norm of $\mathbb{S}$ under the proposed feedback.

Definition 6. ($\mathcal{H}_\infty$-norm of a walk $\|W_r\|_\infty$).Given a walk $W_r \in \mathbf{W}$ and a trajectory $(x_{\text{ref}}(k), u_{\text{ref}}(k))$, $k = 0, \ldots, N_{W_r}$ compatible with the graph $\mathbf{V} = W_r, \mathbf{E} = \{\{\mathbf{V}_1, \mathbf{V}_2\}, \ldots, \{\mathbf{V}_{M-1}, \mathbf{V}_M\}\}$ and the uncertain LTV system $\mathbb{S}$ induced by W_r under feedback (6) with solutions $\{\Delta x(0), \Delta x(1), \ldots, \Delta x(N_{W_r})\} = \Delta \mathbf{X}_W(\Delta x_0)$, $||W_r||_\infty$ is defined as follows:

$$||W_r||_\infty = \underset{\Delta x_0 \in \Delta \mathbb{X}, k=1,\ldots,N_{W_r}, \Delta x(k) \in \Delta \mathbf{X}_{W_r}(\Delta x_0)}{\text{maximize}} \quad \frac{||\Delta x(k)||}{||\Delta x_0||}$$

The next result provides an upper bound to $\|W_r\|_\infty$

Lemma 7. (Upper bound on the $\mathcal{H}_\infty$-norm of a walk $||W_r||_\infty$). Given a walk $W_r \in \mathbf{W}, |W_r| = N_{W_r}$ and the uncertain LTV system $\mathbb{S}$ induced by W_r under feedback ((6)) with LMIs (9) being feasible, then an upper bound on $||W_r||_\infty$ is given by $\gamma^* \lambda_{\max}(Q_1)/\lambda_{\min}(Q_1)$ with γ^* being the minimum of

$$\begin{aligned} \underset{\gamma_i \in \mathbb{R}_+, \gamma \in \mathbb{R}_+}{\text{minimize}} \quad & \gamma \\ \gamma_1 &= 1 \\ \gamma_{N_{W_r}} Q_{N_{W_r}} &\preceq \gamma Q_1 \\ \forall i = 1, \ldots, N_{W_r} - 1 &: \\ \gamma_i Q_i &\preceq \gamma_{i+1} Q_{i+1}. \end{aligned} \tag{13}$$

Proof. Recall, that along each trajectory, the Lyapunov function of the closed-loop uncertain LTV system is negative decrescendandt along each edge. Thus, for each edge $\{i; j\} \in \mathbf{E}$, it holds that

$$x \in \mathcal{E}(Q_i) \implies x_+ \in \mathcal{E}(Q_i),$$

therefore from any node i the ellipsoid Q_i will be mapped into Q_i while transitioning along the edge $\{i; j\}$. Further, for all except for the last node in $\mathbf{W}$ each node's invariant ellipsoid contains the previous node's invariant ellipsoids in virtue of LMIs (13). Thus, $||W_r||_\infty$ is given by the minimum, γ^*, of the semidefinite program

$$\begin{aligned} \underset{\gamma}{\text{minimize}} \quad & \gamma \\ \gamma_M Q_M &\preceq \gamma Q_1, \end{aligned}$$

searching the minimum scaling of Q_1 until it contains $\gamma_M Q_M$. Finally the term $\lambda_{\max}(Q_1)/\lambda_{\min}(Q_1)$ accounts for the shape of the initial ellipsoid.

Next, we establish an upper-bound for the $\mathcal{H}_\infty$-norm of cycles in a graph.

Lemma 8. (Upper bound on the $\mathcal{H}_\infty$-norm of a cycle $||C||_\infty$). Given a cycle $C \in \mathbf{W}$, consider the walk $W_C = C, |W_C| = N_C$, then $||C||_\infty = ||W_C||_\infty$.

Proof. Due to monotonic descent of the Lyapunov function, $\Delta x(0) \in \mathcal{E}(Q_1) \implies \Delta x(N_C) \in \mathcal{E}(Q_{N_C}) = \mathcal{E}(Q_1)$, traversing the cycles at least once is a contracting operation and it is sufficient to consider the norm of the walk of a single traversal of the cycle.

Remark 9. A survey on methods to find all circuits, therefore all cycles, in a directed graph is given in Mateti and Deo (1976). An efficient algorithm is proposed in Johnson (1975).

Next, we aim to bound the $\mathcal{H}_\infty$-norm of a connected component $\mathbf{G}_C$. Let $\forall V_i \in \mathbf{V} : \mathbf{C}(V_i) = \{C \in \mathbf{C} : V_i \in C\}$ be the set of all cycles containing a node V_i. We define the $\mathcal{H}_\infty$-norm of a node V_i as

$$||V_i||_\infty = \underset{C \in \mathbf{C}(V_i)}{\text{maximize}} \quad ||C||_\infty.$$

Next, we associate with each edge $E = \{i, j\}$ an $\mathcal{H}_\infty$-norm

$$||E = \{i, j\}||_\infty = ||W = \{i, j\}||_\infty.$$

Let the graph $\bar{\mathbf{G}} = (\bar{\mathbf{V}}, \bar{\mathbf{E}})$ be obtained by performing a nodes splitting on $\mathbf{G}$ with weights $||V_i||_\infty$ on the nodes and $||E||_\infty$ on the edges and denote by $\bar{\mathbf{P}}$ be the set of paths in $\bar{\mathbf{G}}$. Note, that without a node splitting, we would associate a weight on the nodes which poses an issue with most algorithms on digraphs, since they consider weighted digraphs with weights solely on the edges. The node splitting associates with each node a pair of nodes connected by an edge with the weight of the node, thus yielding a suitable, weighted digraph. A precise definition of node splitting is given in Trevisan (2011). We propose the following bound on the $\mathcal{H}_\infty$-norm of all walks inside a connected component:

Lemma 10. (Upper bound on the $\mathcal{H}_\infty$-norm of all walks inside a connected component $\mathbf{G}_C$).The $\mathcal{H}_\infty$-norm of $\mathbf{G}_C$ is given by

$$||\mathbf{G}_C||_\infty = \underset{P\in\bar{\mathbf{P}}}{\text{maximize}} \quad ||P||_\infty. \tag{14}$$

Proof. By construction, any walk $W_r \in \mathbf{W}$ can be considered a path $P \in \mathbf{P}$, with $\mathbf{P}$ being the set of paths in $\mathbf{G}$ after replacing the cycles it contains by their first node. For each cycle replaced, the increase of the error bound from visiting a node can be at most $||V_i||_\infty$. Further, along the path P, the increase of the error is bounded by $||P||_\infty$. It remains to determine the path with the maximum product of its own norm and those of the nodes. This exact operation is implemented in Equation (14) by searching for the path with the maximum $\mathcal{H}_\infty$-norm in $\bar{\mathbf{G}}$.

Remark 11. A method to compute all paths in a directed graph is laid out in Thorelli (1966).

The last step is to establish a $\mathcal{H}_\infty$-norm of the closed-loop uncertain LTV system (11). Consider the condensation graph C$\{\mathbf{G}\}$. Analogously as in the last step, for each node in it, associate with the node the $\mathcal{H}_\infty$-norm of the represented connected component, for each edge the $\mathcal{H}_\infty$-norm of an edge and perform a node splitting to obtain the weighted digraph $\tilde{\mathbf{G}}$. Because the condensation digraph always is a tree, the set of walks is equal to the set of paths denoted $\tilde{\mathbf{W}}$. We propose the following upper bound

Lemma 12. (Upper bound on the $\mathcal{H}_\infty$-norm of all walks inside a graph $\mathbf{G}$). Let $\tilde{\mathbf{G}}$ be obtained by constructing C$\{\mathbf{G}\}$, upper-bounding the $\mathcal{H}_\infty$-norm of all nodes and edges in C$\{\mathbf{G}\}$ and performing a node splitting. Let $\tilde{\mathbf{W}}$ be the set of all walks in C$\{\mathbf{G}\}$. The $\mathcal{H}_\infty$-norm of $\mathbf{G}$ is given by

$$||\mathbf{G}||_\infty = \underset{W\in\tilde{\mathbf{W}}}{\text{maximize}} \quad ||W||_\infty. \tag{15}$$

Proof. For each connected component $\mathbf{G}_C$ visited, the error can increase at most by $||\mathbf{G}||_\infty$. Further, along any walk $\tilde{W} \in \tilde{\mathbf{W}}$, the increase of the error is bounded by $||\tilde{W}||_\infty$ and node splitting accounts for visiting connected components. Thus, the maximum increase of the error in $\mathbf{G}$ is given by Equation (15).

To satisfy $x(k) \in \Delta\mathbf{X}$, we propose the following, where

$$\mathcal{B}(\Delta\mathbb{X}) = \underset{l\in\{\ell\in\mathbb{R}^+:\{x\in\mathbb{R}^n:||x||_2\le\ell\}\subseteq\Delta\mathbb{X}\}}{\text{maximize}} \quad \{l\}$$

denotes the largest euclidean norm-ball fitting inside $\Delta\mathbb{X}$.

Theorem 13. (Minimal basin of attraction of $\mathbf{S}$ with bound $\forall W \in \mathbf{W} : \forall k = 1, \ldots, N_W : \Delta x(k) \in \Delta\mathbb{X}$).Let $\rho = ||\mathbf{G}||_\infty$ be an upper bound ρ on $\mathcal{H}_\infty$-norm of the set of uncertain LTV system $\mathbb{S}$ under feedback (6). A minimal basin of attraction is given by

$$\Delta x(0) \in \{x : \rho||x||_2 \le \mathcal{B}(\Delta\mathbb{X})\}. \tag{16}$$

Proof. In virtue of Lemma 12

$$\begin{aligned} &\forall W_r \in \mathbf{W} : \forall k \in 0, \ldots, N_{W_r} : \\ &\Delta x(0) \in \{x : \rho||x||_2 \le \mathcal{B}(\Delta\mathbb{X})\} \\ &\implies \Delta x(k) \in \mathcal{B}(\Delta\mathbb{X}) \\ &\implies \Delta x(k) \in \Delta\mathbb{X}. \end{aligned}$$

Remark 14. The provided estimates are conservative, because they hold for the entire uncertain system (4). Further, the methods proposed here do not account for a minimum decrease of the Lyapunov function, which can further improve this result. In case that there are multiple sources in C$\{\mathbf{G}\}$, a better result is obtained by computing each source's basin of attraction individually.

4. SIMULATION RESULT

We demonstrate our result on a continuously stirred tank reactor (CSTR) inspired by Müller, Matthias A and Angeli, David and Allgöwer, Frank and Amrit, Rishi and Rawlings, James B modeled in continuous time as

$$r_1(t) = 10^3 x_1(t)^2 e^{-1/x_3(t)} + 400 x_1(t) e^{-0.55/x_3(t)}$$
$$r_2(t) = 10^3 x_1(t)^2 e^{-1/x_3(t)}$$
$$\dot{x}(t) = f(x(t), u(t)) = \begin{bmatrix} 1 - r_1(t) - x_1(t) \\ r_2(t) \\ u(t) - x_3(t) \end{bmatrix}$$

where $\dot{x} \doteq \dfrac{x(t)}{dt}$ and t is the continuous time variable. The first two states are the concentration of the waste and the harvested products, respectively. The third state represents the temperature which is controlled by the heating power u provided to the system. The constraints are given by x_1, x_2, $x_3 \ge 0$, $0.0 \le u \le 0.499$. All variables in the model are normalized.

As laid out in Bailey et al. (1971); Müller, Matthias A and Angeli, David and Allgöwer, Frank and Amrit, Rishi and Rawlings, James B the CSTR is operated most profitably in a periodic operation. Another mode of operation is to maintain the optimal steady state. We aim to design a gain-scheduled controller and feed forward trajectories such that the CSTR can be driven from steady-state operation to periodic operation and back to steady-state operation. This control problem features cyclic operation of the plant and a forking set of reference trajectories. For the simulation, we assumed a sinusoidal disturbance on the temperature influx and used a fourth-order Runge-Kutta integration with sampling rate $h = 0.02$. Our simulation result showing two cycles in shown in Figure 1. As expected, the system is stabilized and deviates less from the reference trajectory under our proposed feedback. Further, rigorous stability guarantees are provided. To upper-bound the $\mathcal{H}_\infty$-norm of the closed loop system, the cyclic operation is analyzed first. Then, the condensation digraph is formed, resulting in a path-graph, whose $\mathcal{H}_\infty$-norm upper-bounds the closed-loop system as in Lemma 12. Regarding the theoretical guarantees, employing invariant ellipsoids, we can show that the region of attraction for a 6.4% deviation in any coordinate is bounded by the box $\{\Delta x \in \mathbb{R}^3 \mid |\Delta x_1| \le 4.8\%, |\Delta x_2| \le 0.8\%, |\Delta x_3| \le 2.5\%\}$.

5. CONCLUSION

We proposed an approach to stabilization of trajectories containing cycles and possibly merging points or forks. By

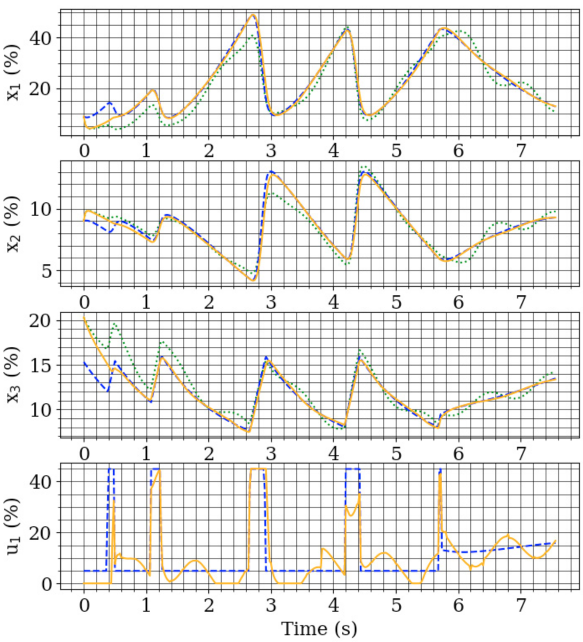

Fig. 1. Stabilizing the CSTR from steady state to a cyclic trajectory and back to steady state. Dashed: reference trajectory, continuous: closed-loop trajectory, dotted: open-loop trajectory.

considering the set of trajectories as a graph, where the states are the nodes and the edges represent control inputs steering the system from one node to the next, LMI conditions for stability of a single trajectory have be extended to cover any transition in the resulting graph. We then presented a result on finding the basin of attraction for the whole graph of trajectories. The results are demonstrated on a CSTR operated in the optimal steady state or cyclic operation.

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