Walter Rudin meets Elias M. Stein

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Walter Rudin and Elias M. Stein were giants in the world of mathematics. They were loved and admired from students and researchers to teachers and academics, both young and old. They touched many of us through their inspiring books at the undergraduate and postgraduate level. Although they were leading researchers in both harmonic analysis and several complex variables, we are not aware whether they interacted and discussed mathematics. In this article, Rudin and Stein meet mathematically through a reformulation of the beautiful theory of Fourier series with gaps that Rudin developed in the 1950s as an equivalent Fourier restriction problem from the 1970s, a problem Stein proposed and which remains a fundamental, central problem in Euclidean harmonic analysis today.

Walter Rudin was born in Vienna on 2 May, 1921 and emigrated to the US in 1945, completing his PhD at Duke University in 1949. While a C. L. E. Moore Instructor at MIT in the early 1950s, Walter was asked to teach a real analysis course but he could not find a textbook that he liked so he decided to write *Principles of Mathematical Analysis* which despite its age, has remained the paragon of high quality. After a stint of teaching at the University of Rochester, he took up a position at the University of Wisconsin, Madison in 1959 where he remained until his retirement as Vilas Professor in 1991. He died at his home in Madison on 20 May, 2010.

Elias M. Stein (known to friends and colleagues as Eli) was born in Antwerp on 13 January, 1931 and emigrated with his family to the US in 1941, settling in New York where Eli attended high school. He went to the University of Chicago, received his PhD in 1955, and then went to MIT as a C.L.E. Moore Instructor before Antoni Zygmund told Eli "it's time to return to Chicago." In 1963, Stein moved to Princeton University as a full professor where he remained until he died on 23 December, 2018.

Between 2003 and 2011, Eli expanded the presentation of Walter's *Principles* and published a series of four books aimed at advanced undergraduates. This series is quickly becoming an important part of any young analyst's education.

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However the majority of books written by Rudin and Stein are postgraduate textbooks and research monographs (too many to list here), mainly in the areas of harmonic analysis and several complex variables where both men were central figures.

In this article, these two luminaries meet in the world of mathematical analysis. We look back at some important work Rudin did in the 1950s and recast it in terms of a far-reaching problem from the 1970s that Stein gave us.

1950s: a golden age for Fourier analysis

At a 1946 conference in Princeton,[1] Zygmund gave a scathing review of the post world war state of harmonic/Fourier analysis, describing the area as fettered with unsolved problems with no guiding theme or programme. This all changed in the 1950s in two profound ways. First, the 1950s represented a convergence of the point of view that the most appropriate setting for Fourier analysis is furnished by the class of locally compact abelian groups. This abstraction is not done for the sake of mere generalisation. It not only gives conceptual clarification to classical problems, it also leads to the introduction of new, interesting analytical problems. A beautiful example is the theory Rudin developed in his paper *Trigonometric series with gaps* [14] which we will turn to momentarily.

The second profound change from the 1950s is the advent of the *real variable method* which emerged from the seminal paper of Calderón and Zygmund [6], freeing us from the complex method which tied us to one dimension in the study of Fourier series and the Fourier transform. This led Stein to propose a series of fundamental problems addressing basic properties of Fourier series and the Fourier transform in higher dimensions. These problems are interconnected and the core conjectures underpinning each problem still remain unsolved today despite the heroic efforts of many eminent mathematicians. One of these problems is the *Fourier restriction problem* which Stein introduced in the mid to late 1960s and which we will discuss in more detail below.

Trigonometric series with gaps

This paper [14] of Rudin introduced several kinds of sparse sets of integers with interesting properties. In the 1920s Sidon showed that a continuous function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, with a lacunary[2] Fourier series $f \sim \sum c_k e^{2\pi i 2^k \theta}$, automatically has an absolutely convergent Fourier series; i.e. $\sum_k |c_k| < \infty$. Giving us a glimpse of how he thinks, Rudin took this isolated result and realised that there is a rich theory of sets $\Lambda \subset \mathbb{Z}$ with the property that $C_\Lambda(\mathbb{T}) \subset A(\mathbb{T})$. Here $A(\mathbb{T})$ is the space of absolutely convergent Fourier series and $C_\Lambda(\mathbb{T})$ is the closed subspace of continuous functions on the circle $\mathbb{T}$ which are Fourier supported in Λ; that is, the Fourier coefficients $\widehat{f}(n) = 0$ for all $n \notin \Lambda$.

[1] Problems of mathematics, Princeton University bicentennial conferences, series 2, conference 2, Princeton, New Jersey, 1947.

[2] More generally, the sequence $\{2^k\}$ can be replaced by any sequence of positive integers $\{b_k\}$ satisfying $\inf_k b_{k+1}/b_k > 1$.

Rudin called such spectral sets *Sidon sets* and observed they have interesting arithmetic properties. He first developed the theory on the circle $\mathbb{T}$ and then extended it to any compact, abelian group G, detailing a programme to characterise Sidon sets in terms of their arithmetic properties. He intimated that the key to unlock this arithmetic characterisation is the following improving bound: for $F = \exp(L^2)$, there is a constant $C > 0$ such that

$$\|f\|_F \;\le\; C\,\|f\|_{L^2} \quad \text{for all} \;\; f \in C_\Lambda. \tag{F}$$

The function space $\exp(L^2)$ lies near L^∞; in fact, $L^\infty \subset \exp(L^2) \subset L^p$ for all finite $p < \infty$. In the 1930s, Zygmund established (F) with $F = \exp(L^2)$ for lacunary sequences Λ and Rudin extended this to any Sidon set on any compact abelian group G. Pisier established the reverse implication, showing that (F) with $F = \exp(L^2)$ characterises when Λ is a Sidon set and he did this on any compact abelian group; see [10, 11]. This led Pisier to his arithmetic characterisation of Sidon sets, the definitive result in the theory of Sidon sets; see [12].

There are many sparse families of spectral sets Λ which are defined by or characterised by (F) for some endpoint function space F near L^∞. In his *gaps* paper, Rudin introduced and developed the theory of $\Lambda(p)$ sets which are defined by (F) for $F = L^p$ when $p > 2$. Rudin conjectured that the squares $\Lambda = \{n^2\}$ is a $\Lambda(p)$ set for all $p < 4$ but this seems difficult and remains unsolved (see [3]). A deep result of Bourgain [4] established that for every $p > 2$, there is a $\Lambda(p)$ set which is not a $\Lambda(p + \epsilon)$ set for any $\epsilon > 0$.

In a different paper [13] from the 1950s, Rudin introduced *Paley* sets $\Lambda \subset \mathbb{Z}$ and showed that the bound (F) holds for $F = BMO(\mathbb{T})$ if and only if $\sup_{I\in\mathcal{D}} \#[\Lambda \cap I] < \infty$ where $\mathcal{D}$ is the set of dyadic intervals $\{\pm[2^k, 2^{k+1}] : k \in \mathbb{N}\}$; in other words, Λ is a finite union of lacunary sequences. Similar to $\exp(L^2)$, the function space BMO of bounded mean oscillation lies near L^∞, again $L^\infty \subset BMO \subset L^p$ for all finite $p < \infty$.

The Fourier restriction problem

The Fourier transform

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi\cdot x}\, dx,$$

defined initially for Lebesgue integrable functions, is a fundamental object in many different areas. In the 1960s, Stein introduced the Fourier restriction problem which seeks to understand the singularities that arise when one computes the Fourier transform of $L^p(\mathbb{R}^n)$ functions. When $p = 1$, the Fourier transform is well behaved. A basic fact is that the Fourier transform of $f \in L^1(\mathbb{R}^n)$ is a continuous, bounded function and so restricting $\widehat{f}$ to *any* set $S \subset \mathbb{R}^n$ defines a function on S, continuous with respect to the induced topology. On the other hand, the Fourier transform defines a unitary operator from L^2 onto L^2 and so the singularities of $\widehat{f}$ for $f \in L^2(\mathbb{R}^n)$ that arise are those of an arbitrary L^2 function. Hence it can be identically ∞ on any set S of measure zero.

By interpolation, one can define the Fourier transform for $f \in L^p(\mathbb{R}^n)$ when $1 \le p \le 2$ and $\widehat{f} \in L^{p'}$ where p' is the conjugate exponent to p, satisfying $1/p + 1/p' = 1$. However when $1 \le p < 2$, the mapping $f \to \widehat{f}$ is *not* onto $L^{p'}$ (we have already seen this when $p = 1$; the function $\widehat{f}$ is not a general L^∞ function, it is also continuous).

Stein observed that there is a range $1 \le p < p_0$ for some $p_0(n) > 1$ so that for any $f \in L^p(\mathbb{R}^n)$, one can make sense of $\widehat{f}$ as an L^2 density on the unit sphere $\mathbb{S}^{n-1} = \{|x| = 1\}$. More precisely, he showed the Fourier restriction operator $\mathcal{R}f = \widehat{f}|_{\mathbb{S}^{n-1}}$, defined initially on test functions, extends to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^2(\mathbb{S}^{n-1})$ when $1 \le p < p_0$ and so the singularities of $\widehat{f}$ on $\mathbb{S}^{n-1}$ can be no worse than an L^2 function on the sphere. The fact that the unit sphere $\mathbb{S}^{n-1}$ has curvature is crucial for Stein's observation; if S is a compact piece of a hyperplane, then for any $p > 1$, there is an $f \in L^p(\mathbb{R}^n)$ such that $\widehat{f}|_S \equiv \infty$. See [15] for this example and a general discussion of the Fourier restriction phenomenon.

The Fourier restriction problem is to determine the complete $L^p \to L^q$ mapping properties of $\mathcal{R}$ and the conjecture is that

$$\|\mathcal{R}f\|_{L^q(\mathbb{S}^{n-1})} \;\le\; C\, \|f\|_{L^p(\mathbb{R}^n)} \tag{1}$$

holds if and only if $1 \le p < 2n/(n+1)$ and $(n+1)q \le (n-1)p'$. Here the space $L^q(\mathbb{S}^{n-1})$ is defined with respect to surface measure $d\sigma$. In two dimensions $n = 2$, the conjecture was solved by Fefferman and Stein [8] in 1970 and independently by Zygmund [18] in 1974 but it remains open in dimensions $n \ge 3$. One can formulate conjectures for the Fourier restriction problem associated to other varieties of varying dimension with nonvanishing curvature.

Remarkably the Fourier restriction problem has profound applications in disparate areas of mathematics, from PDEs in the form of fundamental Strichartz estimates to the recent solution of *decoupling conjectures* which led to Bourgain, Demeter and Guth's [5] complete resolution of the Vinogradov Mean Value Theorem, a central problem in analytic number theory from the 1930s.

Duality

The golden age of the 1950s brought to the fore the central role that the principle of duality plays in Fourier analysis. By duality, we can give an equivalent formulation of the improving bound (F): if E is a Banach space of functions near L^1 such that $E^* = F$ (the Banach space dual of E is F with norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively), then

$$\Big(\sum_{n \in \Lambda} |\widehat{f}(n)|^2 \Big)^{1/2} \;\le\; C\, \|f\|_E \tag{E}$$

with the same constant C appearing in (F). The bound in (E) holds for all functions $f \in E$ and we are *restricting* the Fourier coefficients $\{\widehat{f}(n)\}$ to the given sequence Λ (for general compact abelian groups G, the spectral set Λ lies in the discrete Fourier dual group $\widehat{G}$). Here we see a connection between trigonometric series with gaps and certain discrete variants of the Fourier restriction problem.

Our discussion of Rudin's 1950s theory can be rephrased as follows: Pisier's deep result says that Λ is a Sidon set if and only if (E) holds with $E = L\sqrt{\log L}$ and this holds on any compact abelian group. A spectral set is a $\Lambda(p)$ set for $p > 2$ if and only if (E) holds with $E = L^{p'}$. Furthermore a set $\Lambda \subset \mathbb{Z}$ is a finite union of lacunary sequences if and only if (E) holds for $E = H^1(\mathbb{T})$, the Hardy space on the circle whose Banach space dual is $BMO(\mathbb{T})$.

Also by duality, we can reformulate the Fourier restriction problem in terms of the extension operator

$$\mathcal{E}g(\xi) \ := \ \int_{\mathbb{S}^{n-1}} g(\omega) e^{-2\pi i \xi\cdot\omega} d\sigma(\omega);$$

the Fourier restriction conjecture (1) is equivalent to

$$\|\mathcal{E}g\|_{L^{p'}(\mathbb{R}^n)} \ \le \ C\, \|g\|_{L^{q'}(\mathbb{S}^{n-1})}$$

holding if and only if $1 \le p < 2n/(n+1)$ and $(n+1)q \le (n-1)p'$, Here p' and q' are the conjugate exponents of p and q, respectively. At the endpoint, the exponents $p = q = 2n/(n+1)$ agree but the bound (1) is known to fail (in fact it suffices to take $g \equiv 1$ and check that $\mathcal{E}1 \in L^{p'}$ precisely when $p' > 2n/(n-1)$).

In [16] Tomas proved a local Fourier restriction estimate in two dimensions (the one dimensional sphere $\mathbb{S}^1 = \mathbb{T}$ is the circle) at the endpoint $p' = q' = 4$;

$$\|\mathcal{E}g\|_{L^4(B_R)} \ \le \ C[\log R]^{1/4}\, \|g\|_{L^4(\mathbb{T})}$$

where $B_R = \{|x| \le R\}$. Nowadays it is known that this local estimate implies the Fefferman-Stein/Zygmund result that (1) holds when $n = 2$.

In Zygmund's paper [18], a connection was *almost* made between bounds (E) or (F) for trigonometric series with gaps and Euclidean Fourier restriction bounds. Zygmund proved two theorems in [18]. His second result established (1) for $n = 2$ but his first result, answering a question by Fefferman, showed

$$\Big(\sum_{|\underline{n}|=R} |\widehat{f}(\underline{n})|^2\Big)^{1/2} \ \le \ 5^{1/4}\, \|f\|_{L^{4/3}(\mathbb{T}^2)}, \tag{2}$$

the sum being taken over all lattice points in $S_R = \{\underline{n} = (m,n) \in \mathbb{Z}^2 : m^2 + n^2 = R^2\}$. He remarks that the bounds (1) and (2) are analogous and his proof of (1) for $n = 2$ is modelled on the proof he gave for (2). Furthermore he comments that his proof of (2) is quite general and works for any spectral set S which has the property that $\#\{(r,s) \in S^2 : r \pm s = t\}$ is bounded for all t.

The bound (2) simply says S_R is a $\Lambda(4)$ set. It is the bound (E) for the compact group $G = \mathbb{T}^2$, the Banach space $E = L^{4/3}(\mathbb{T}^2)$ and the spectral set S_R. In Rudin's gaps paper the connection between $\Lambda(4)$ sets and the number of representations as the sum of two elements from the spectral set is made explicit (see Theorem 4.5 in [14]). In what follows, we will show that (2) is equivalent to a Fourier restriction bound for the two dimensional torus $\mathbb{T}^2$.

W. Rudin meets E. M. Stein

Here Rudin's theory of trigonometric series with gaps meets Stein's theory of Fourier restriction. We will work on the compact group $\mathbb{T}^n$ instead of the circle. Continuous functions $f \in C(\mathbb{T}^n)$ can be identified with (multiply) periodic functions on $\mathbb{R}^n$; they have a Fourier series

$$\sum_{\underline{n}\in\mathbb{Z}^n} \widehat{f}(\underline{n}) e^{2\pi i \underline{n}\cdot\underline{\theta}} \quad \text{where} \quad \widehat{f}(\underline{n}) = \int_{\mathbb{T}^n} f(\underline{\theta}) e^{-2\pi i \underline{\theta}\cdot\underline{n}}\, d\underline{\theta}.$$

The result is a general statement about spectral sets Λ that lie in the n-dimensional lattice $\mathbb{Z}^n$ and Banach function spaces $E = E(\mathbb{T}^n)$ with norm $\|\cdot\|_E$ which are continuously embedded in $L^1(\mathbb{T}^n)$.

We will also need to assume E continuously includes $L^p(\mathbb{T}^n)$ for some $1 < p \le 4/3$; thus we will assume there are two continuous embeddings

$$L^p(\mathbb{T}^n) \hookrightarrow E \hookrightarrow L^1(\mathbb{T}^n) \quad \text{for some } p \le 4/3.$$

This implies that there is a constant C such that the following norm inequalities (for some $p \le 4/3$) hold:

$$C^{-1}\,\|f\|_{L^1(\mathbb{T}^n)} \;\le\; \|f\|_E \;\le\; C\,\|f\|_{L^p(\mathbb{T}^n)}. \tag{3}$$

Therefore if $F = E^*$ is the Banach space dual of E, then $L^\infty(\mathbb{T}^n) \hookrightarrow F \hookrightarrow L^{p'}(\mathbb{T}^n)$; or equivalently,

$$C^{-1}\,\|f\|_{L^{p'}(\mathbb{T}^n)} \;\le\; \|f\|_F \;\le\; C\,\|f\|_{L^\infty(\mathbb{T}^n)}. \tag{4}$$

We will consider the extension operator for $\mathbb{T}^n$: let $\underline{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \simeq \mathbb{R}^{2n}$ and define

$$\mathcal{E}g(\underline{z}) \;:=\; \int_{\mathbb{T}^n} g(\underline{\theta})\, e^{-2\pi i \langle \underline{z}, \underline{\theta}\rangle}\, d\underline{\theta}.$$

Finally we set $B_R = \{\underline{z} \in \mathbb{C}^n : |z_j| \le R, \forall j\}$, a polydisc of radius R sitting in $\mathbb{R}^{2n}$.

Theorem 1. *Let Λ and E be as above. Then the following two statements are equivalent:*

(W. Rudin) *There is a constant A_Λ such that*

$$\|f\|_{F(\mathbb{T}^n)} \;\le\; A_\Lambda\,\|f\|_{L^2(\mathbb{T}^n)}, \quad \forall f \in C_\Lambda(\mathbb{T}^n).$$

(E. M. Stein) *There is a constant B_Λ such that*

$$\|\mathcal{E}g\|_{L^4(B_R)} \;\le\; B_\Lambda\,(\log R)^{n/4}\,\|g\|_{E(\mathbb{T}^n)},$$

for all $R \ge 2$ and for all $g \in C_\Lambda(\mathbb{T}^n)$.

The (W. Rudin) statement is simply the bound (F) (or equivalently (E)) in the setting of the compact group $\mathbb{T}^n$. The theorem does not recover the local Fourier restriction endpoint bound of Tomas since E cannot be taken to be L^4 as we are assuming that E continuously includes L^p for some $1 < p \leq 4/3$ (see (3)). The (E. M. Stein) statement is an extension of the Tomas local endpoint Fourier restriction bound to functions in some function space $E(\mathbb{T}^n)$ with Fourier support in $\Lambda \subset \mathbb{Z}^n$.

Applying the theorem to $E = L^{4/3}(\mathbb{T}^2)$ and $\Lambda = S_R$, the lattice points on a circle of radius R, we see that Zygmund's result (2) is in fact equivalent to an endpoint Euclidean Fourier restriction bound for $\mathbb{T}^2$.

Again if $\Lambda \subset \mathbb{Z}^n$ is a Sidon set (that is, if $C_\Lambda(\mathbb{T}^n) \subset A(\mathbb{T}^n)$, the space of absolutely convergent Fourier series), then we can take $E = L\sqrt{\log L}(\mathbb{T}^n)$ so that $E^* = F = \exp(L^2)(\mathbb{T}^n)$. Interestingly, the product of Sidon sets is not a Sidon set (for example, $\{2^k\} \times \{2^\ell\} \subset \mathbb{Z}^2$ is not a Sidon set). It was shown in [1] that if $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n \subseteq \mathbb{Z}^n$ is an n-fold product of (countably infinite) spectral sets $\Lambda_j \in \mathbb{Z}$, then each $\Lambda_j \subset \mathbb{Z}$ is a Sidon set if and only if (E) holds for $E = L(\log L)^{n/2}(\mathbb{T}^n)$. We have $E^* = F = \exp(L^{2/n})(\mathbb{T}^n)$. See also [11].

Rudin's theorem for Paley sets in $\mathbb{Z}$ was extended by Oberlin [9] to the n-dimensional torus $\mathbb{T}^n$. Specifically, the bound (E) holds for $E = H^1(\mathbb{T}^n)$ if and only if $\sup_R \#[\Lambda \cap R] < \infty$ where the supremum is taken over all dyadic rectangles $R \subset \mathbb{Z}^n$. In this case, $E^* = F = BMO(\mathbb{T}^n)$.

The proof

First we introduce some notation. We write points $\underline{z} \in \mathbb{C}^n \simeq \mathbb{R}^{2n}$ in polar form $z_j = r_j e^{i\theta_j}$ for each component of $\underline{z}$. We think of $\underline{r} = (r_1, \ldots, r_n)$ as the n radii for $\underline{z}$ and $\underline{\theta} = (\theta_1, \ldots, \theta_n)$ as parametrising points on $\mathbb{T}^n$ so that points in $\mathbb{R}^{2n}$ can be expressed as

$$\underline{r}e^{i\underline{\theta}} = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \in \mathbb{R}^{2n}$$

and functions on $\mathbb{R}^{2n}$ can be formally represented as

$$F(\underline{r}e^{i\underline{\theta}}) \ = \ \sum_{\underline{k} \in \mathbb{Z}^n} f_{\underline{k}}(\underline{r}) e^{i\underline{k}\cdot\underline{\theta}}.$$

We define the following mixed norms for functions $F : \mathbb{R}^{2n} \to \mathbb{C}$,

$$\|F\|^p_{L^p_+ L^2_{\mathbb{T}^n}} := \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{T}^n} |F(\underline{r}e^{i\underline{\theta}})|^2 \, d\underline{\theta} \Big)^{p/2} r_1 \cdots r_n d\underline{r}.$$

With this notation the polydisc in the theorem can be expressed as $B_R = \{\underline{r}e^{i\underline{\theta}} : \forall j,\ 0 \leq r_j \leq R\}$.

The proof factors through a variant of a result of L. Vega [17]. Namely,

$$\|\mathcal{E}g\|_{L^4_+ L^2_{\mathbb{T}^n}(B_R)} \ \leq \ C\,(\log R)^{n/4} \|g\|_{L^2(\mathbb{T}^n)}. \tag{5}$$

The theorem of L. Vega is a global mixed norm $L^p_{\text{rad}} L^2_{\text{ang}}$ estimate for the Fourier extension operator on the sphere $\mathbb{S}^{d-1}$. See also [7]. The proof of (5) follows from classical bounds for Bessel functions which give the $(\log R)^{n/4}$ bound, see e.g. [2].

If the (W. Rudin) statement holds (that is, (F) and hence (E) holds for Λ on $\mathbb{T}^n$), then our spectral set is a $\Lambda(4)$ set. To see this, we use (4) with $p' \geq 4$, together with (F) for Λ, to conclude

$$\|f\|_{L^4(\mathbb{T}^n)} \;\leq\; \|f\|_{L^{p'}(\mathbb{T}^n)} \;\leq\; C\,\|f\|_F \;\leq\; A_\Lambda\,\|f\|_{L^2(\mathbb{T}^n)}$$

for every $f \in C_\Lambda(\mathbb{T}^n)$. Hence, for every $\underline{r} \in \mathbb{R}^n_+$,

$$\int_{\mathbb{T}^n} |\mathcal{E}g(\underline{r}e^{i\underline{\theta}})|^4\, d\underline{\theta} \leq A_\Lambda^4 \Big(\int_{\mathbb{T}^n} |\mathcal{E}g(\underline{r}e^{i\underline{\theta}})|^2\, d\underline{\theta}\Big)^{4/2}$$

whenever $g \in C_\Lambda(\mathbb{T}^n)$.[3] Integrating over $\mathcal{B}_R := \{\underline{r} : \forall j,\, 0 \leq r_j \leq R\}$ shows that

$$\big\|\mathcal{E}g\big\|_{L^4(B_R)} \;\leq\; A_\Lambda\,\big\|\mathcal{E}g\big\|_{L^4_+ L^2_{\mathbb{T}^n}(B_R)}$$

for $g \in C_\Lambda(\mathbb{T}^n)$.

Therefore by (5),

$$\big\|\mathcal{E}g\big\|_{L^4(B_R)} \;\leq\; B_\Lambda\,(\log R)^{n/4}\|g\|_{L^2(\mathbb{T}^n)}$$

and

$$\|g\|_{L^2(\mathbb{T}^n)} = \Big(\sum_{\underline{n}\in\Lambda} |\widehat{g}(\underline{n})|^2\Big)^{1/2} \leq C\|g\|_{E(\mathbb{T}^n)}$$

for $g \in C_\Lambda(\mathbb{T}^n)$ by (E). This shows that the (E. M. Stein) statement holds.

The reverse implication is more involved and so, for clarity, we only give the proof when $n = 1$. The extension to general $n \geq 1$ is straightforward. Suppose that the (E. M. Stein) statement holds for $n = 1$ and fix a sequence $\{a_k\}_{k\in\Lambda}$. Our goal is to show

$$\Big\| \sum_{k\in\Lambda_N} a_k e^{ik(\cdot)}\Big\|_F \;\leq\; A_\Lambda \Big(\sum_{k\in\Lambda_N} |a_k|^2\Big)^{1/2} \tag{6}$$

with a constant A_Λ which is independent of N. Here $\Lambda_N = \{k \in \Lambda : |k| \leq N\}$. This will establish (F), the (W. Rudin) statement.

From $\{a_k\}$, we will construct a function

$$H(re^{i\theta}) := \sum_{k\in\Lambda_N} \mathfrak{h}_{k,N}(r) e^{ik\theta} \tag{7}$$

with certain properties, including that each $\mathfrak{h}_{k,N}$ is supported in $\{r \leq N^2\}$ and so H is supported in the disc B_{N^2}. To construct $\mathfrak{h}_{k,N}$, we first note that

$$\widehat{H}(e^{i\theta}) \;=\; \sum_{k\in\Lambda_N} \Big[\int_{\mathbb{R}_+} \mathfrak{h}_{k,N}(r) J_k(2\pi r) r\, dr\Big] e^{ik\theta}$$

[3] Note that $g \in C_\Lambda(\mathbb{T}^n)$ implies that $\mathcal{E}g(\underline{r}\,\cdot) \in C_\Lambda(\mathbb{T}^n)$.

where J_k is the classical Bessel function of order k. We use the asymptotic formula

$$J_k(2\pi r) = \sqrt{\frac{2}{\pi r}}\,\cos(2\pi r - k\pi/2 - \pi/4) + O(r^{-3/2})$$

where, importantly, the implicit constant in $O(r^{-3/2})$ is uniform in k for $r \geq 5k$. See [2].

We now construct a function $h_{k,N}(r)$ which depends on N and k. For every $m \in \mathbb{N}$ satisfying

$$5N \leq m \leq N^2, \text{ if } k \equiv j \bmod 4,\ j = 0, 1, 2, 3,$$

we set $h_{k,N}(r) = \sqrt{2}\, r^{-3/2}$ for

$$m + (2j+1)/8 \ \leq\ r\ \leq m + (2j+1)/8 + 10^{-10}$$

and we set $h_{k,N}(r) = 0$ otherwise. Hence for $5k \leq N$,

$$\int_0^{N^2} h_{k,N}(r) J_k(2\pi r) r dr \ = \tag{8}$$

$$\sum_{m=5N}^{N^2} \frac{2}{\sqrt{\pi}} \int_{m+(2j+1)/8}^{m+(2j+1)/8+10^{-10}} \frac{\cos(2\pi r - j\pi/2 - \pi/4))}{r}\, dr$$

$$+ \quad O(N^{-1}) \quad = \quad c \log N \ + \ O(N^{-1})$$

for some $c = c_{k,N} > 0$ with $c \sim 1$.

Let us denote the integral in (8) by $A_{k,N}$. Hence $|A_{k,N}| \sim \log N$. With $\{a_k\}_{k\in\Lambda_N}$ in (6), we define $\{g_k\}_{k\in\Lambda_N}$ by the relation

$$g_k\, A_{k,N} \ = \ c \log N\, a_k \quad \text{so that} \quad |g_k| \lesssim |a_k|.$$

We finally arrive at our $\mathfrak{h}_{k,N}(r) := g_k h_{k,N}(r)$ which defines H in (7). Note that $\mathfrak{h}_{k,N}$ is supported in $\{r \leq N^2\}$ since the same is true for $h_{k,N}$.

The dual formulation of the (E. M. Stein) statement with $R = N^2$ implies (since $4/3 \leq 2$)

$$\big\|\widehat{H}|_{\mathbb{T}}\|_F \ \leq \ B_\Lambda (\log N)^{1/4} \|H\|_{L_+^{4/3} L_{\mathbb{T}}^2} \tag{9}$$

where

$$\|H\|_{L_+^{4/3} L_{\mathbb{T}}^2} \ = \ \Big(\int_0^\infty \Big(\sum_{k\in\Lambda_N} |\mathfrak{h}_{k,N}(r)|^2\Big)^{\frac{1}{2}\frac{4}{3}} r\, dr\Big)^{3/4}.$$

Since each $\mathfrak{h}_{k,N}$ is supported in $[N, N^2]$ and $|\mathfrak{h}_{k,N}(r)| \lesssim r^{-3/2}|a_k|$, we see that

$$\|H\|_{L_+^{4/3} L_{\mathbb{T}}^2} \leq C \Big(\int_N^{N^2} \Big[\frac{1}{r^{3/2}}\Big]^{4/3} r dr\Big)^{3/4} \Big(\sum_{k\in\Lambda_N} |a_k|^2\Big)^{1/2}$$

and so

$$\|H\|_{L^{4/3}_+ L^2_{\mathbb{T}}} \;\le\; C\,(\log N)^{3/4} \Big(\sum_{k\in\Lambda_N} |a_k|^2 \Big)^{1/2}.$$

Also

$$\widehat{H}(e^{i\theta}) \;=\; [c\log N] \sum_{k\in\Lambda_N} a_k e^{ik\theta},$$

implying

$$\|\widehat{H}|_{\mathbb{T}}\|_F \;=\; [c\log N] \big\| \sum_{k\in\Lambda_N} a_k e^{ik(\cdot)} \big\|_F.$$

Therefore (9) implies

$$[c\log N]\; \big\| \sum_{k\in\Lambda_N} a_k e^{ik(\cdot)} \big\|_F \;\le\; A_\Lambda\; \log N \Big(\sum_{k\in\Lambda_N} |a_k|^2 \Big)^{1/2},$$

showing that (6) holds, establishing the (W. Rudin) statement.

Conclusion

Here we tried to draw some parallels between two great men; similar life stories, similar research interests and similar influence through their books and monographs. And although they may not have interacted mathematically as working researchers, their mathematics is nonetheless intimately connected. This is the beauty of mathematics.

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