ON THE CONSTRUCTION OF NON-SIMPLE BLOW-UP SOLUTIONS FOR THE SINGULAR LIOUVILLE EQUATION WITH A POTENTIAL

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Abstract. We are concerned with the existence of blowing-up solutions to the following boundary value problem

$$-\Delta u = \lambda V(x)e^u - 4\pi N\boldsymbol{\delta}_0 \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1,$$

where B_1 is the unit ball in $\mathbb{R}^2$ centered at the origin, $V(x)$ is a positive smooth potential, N is a positive integer ($N \geq 1$). Here $\boldsymbol{\delta}_0$ defines the Dirac measure with pole at 0, and $\lambda > 0$ is a small parameter. We assume that $N = 1$ and, under some suitable assumptions on the derivatives of the potential V at 0, we find a solution which exhibits a non-simple blow-up profile as $\lambda \to 0^+$.

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1. Introduction

Given Ω a smooth and bounded domain in $\mathbb{R}^2$ containing the origin, consider the following Liouville equation with Dirac mass measure

$$\begin{cases} -\Delta u = \lambda V(x)e^u - 4\pi N\boldsymbol{\delta}_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here λ is a positive small parameter, the potential V is a positive and smooth function, $\boldsymbol{\delta}_0$ denotes Dirac mass supported at 0 and N is a positive integer.

Problem (1.1) is motivated by its applications in conformal geometry and several fields of physics, where quite a few semilinear elliptic equations defined in two dimensional spaces with an exponential nonlinear term are very commonly observed and studied. The well known prescribing Gauss curvature equation, mean field equation, Liouville type equations from the Chern-Simons self-dual theory, and systems of equations of the Toda system are a few examples of this family. The analysis of these equations is usually challenging as the interesting exponential nonlinear term is always related to the lack of compactness in the variational approach. One important feature of these equations is the blow-up phenomenon, the understanding of which is closely related to results on existence, compactness, a-priori estimates, etc.

The asymptotic behaviour of family of blowing up solutions u_k can be referred to the papers [6], [8], [19], [20], [22], [24] for the regular problem, i.e. when $N = 0$. An extension to the singular case $N > 0$ is contained in [3]-[5]. If a blow-up point p is either a regular point or a "non-quantized" singular source, the asymptotic behavior of u_k around p is well understood (see [3, 5, 7, 8, 15, 18, 31, 32]). As a matter of fact, u_k satisfies the spherical Harnack inequality around 0, which implies that, after scaling, the sequence u_k behaves as a single bubble around the maximum point. However, if p happens to be a quantized singular source, the so-called "non-simple" blow-up phenomenon does happen (see [17, 28, 29, 30]), which is equivalent to stating that u_k violates the

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spherical Harnack inequality around p. The study of non-simple blow-up solutions, whether or not the blow-up point has to be a critical point of coefficient functions, has been a major challenge for Liouville equations and its research has intrigued people for years. Recently significant progress has been made by Kuo-Lin, Bartolucci-Tarantello and other authors ([5, 10, 17, 28, 29, 30]. In particular it is established in [4] and [17] that there are $N+1$ local maximum points and they are evenly distributed on $\mathbb{S}^1$ after scaling according to their magnitude. In [28] and [29] Harnack inequalities and second order vanishing conditions for non-simple blow-ups are obtained.

The case $N \in \mathbb{N}$ is more difficult to treat, and at the same time the most relevant to physical applications. Indeed, in vortex theory the number N represents vortex multiplicity, so that in that context the most interesting case is precisely when it is a positive integer. The difference between the case $N \in \mathbb{N}$ and $N \notin \mathbb{N}$ is also analytically essential. Indeed, as usual in problems involving concentration phenomena like (1.1), after suitable rescaling of the blowing-up around a concentration point one sees a limiting equation which, in this case, takes the form of the planar singular Liouville equation:

$$-\Delta U = e^U - 4\pi N \delta_0, \quad \int_{\mathbb{R}^2} e^U dx < \infty;$$

only if $N \in \mathbb{N}$ the above limiting equation admits non-radial solutions around 0 since the family of all solutions extends to one carrying an extra parameter (see [23]). This suggests that if $N \in \mathbb{N}$ and the blow-up point happens to be the singular source, then solutions of (1.1) may exhibit non-simple blow-up phenomenon.

So, from analytical viewpoints the study of non-simple blow-up solutions is far more challenging than simple blow-up solutions, but the impact of this study may be even more significant because they represent certain situations in the blow-up analysis of systems of Liouville equations. Indeed, if local maxima of blow-up solutions in a system tend to one point, the profile of solutions can be described by a Liouville equation with quantized singular source. For all this reasons, it is desirable to know exactly when non-simple blow-up phenomenon happens.

However, the question on the existence of non-simple blowing-up solutions to (1.1) concentrating at 0 is far from being completely settled. A first definite answer is provided by [11] which rules out the non-simple blow-up phenomenon for (1.1) if the potential V is constant: more precisely it is established that there is no non-simple blow-up sequence for (1.1) with $V = const.$, even if we are in the presence of multiples singularities $\sum_i N_i \boldsymbol{\delta}_{p_i}$. Apart from this, only partial results are known: in [10] the construction of solutions exhibiting a non simple blow-up profile at 0 is carried out for equation (1.1) with $V \equiv 1$ provided that Ω is the unit ball and the weight of the source is a positive number $N = N_\lambda$ close an integer N from the right side. On the other hand, in[12], for any fixed positive integer N, it is proved the existence of a solution to (1.1) with $V \equiv 1$, where $\boldsymbol{\delta}_0$ is replaced by $\boldsymbol{\delta}_{p_\lambda}$ for a suitable $p_\lambda \in \Omega$, with $N+1$ blowing up points at the vertices of a sufficiently tiny regular polygon centered in p_λ; moreover the location of p_λ is determined by the geometry of the domain in a $\lambda-$dependent way and does not seem possible to be prescribed arbitrarily. To our knowledge, the existence of non-simple blow-up phenomenon for (1.1) for a fixed V and a fixed N independent of λ is still open, even in the case of the ball: the only example is constructed in [9] for a special class of potentials of the form $V(|x|^{N+1})$.

In this paper we investigate the existence of non-simple blow-up solutions when Ω is the unit ball B_1 centered at the origin, the potential $V_\lambda = V$ is fixed and $N = 1$:

$$\begin{cases} -\Delta u = \lambda V(x) e^u - 4\pi\delta_0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{1.2}$$

Let us pass to enumerate the hypotheses on the potential V that will be steadily used throughout the paper.

(H1) $\inf_{B_1} V(x) > c > 0$ for a positive constant c independent of λ and, without loss of generality, we may assume $V(0) = 1$;

(H2) $V(x)$ is even, i.e.
$$V(x) = V(-x) \quad \forall x \in B_1.$$

Furthermore we will require sufficient regularity of V at 0 together with crucial conditions on the derivatives of V at 0:

(H3) $V(x)$ is of class C^1 in the closed unit ball $\overline{B}_1$ and the following holds:
$$\begin{aligned} V(x) = 1 &+ A_0(x_1^4 + x_2^4) + A_1(x_1^3 x_2 - x_1 x_2^3) + A_2 x_1^2 x_2^2 \\ &+ D_0(x_1^6 - x_2^6) + D_1(x_1^5 x_2 + x_1 x_2^5) + D_2(x_1^4 x_2^2 - x_1^2 x_2^4) + D_3 x_1^3 x_2^3 + O(|x|^7) \end{aligned} \tag{1.3}$$
for some constants $A_0, A_1, A_2, D_0, D_1, D_2, D_3 \in \mathbb{R}$.

Let us comment on assumption (H3): in [28], [29], [30] the second and the third authors proved that if non simple blow up scenarios occur for equation (1.2), then the first derivatives as well as the Laplacian of coefficient functions must tend to zero at the singular source; so the vanishing of the second order terms in the expansion (1.3) is not surprising. Moreover, the analysis reveals that the relation between the forth derivatives and between the sixth derivatives plays a crucial role since it guarantees that the non simple blow-up solutions can be accurately approximated by global solutions by allowing an a priori estimate for the error which turns out to be sufficiently small (see Remark 4.5 and Remark 6.1).

In order to provide the exact formulation of the result let us fix some notation: in the following $G(x, y)$ is the Green's function of $-\Delta$ over Ω under Dirichlet boundary conditions and $H(x, y)$ denotes its regular part:
$$H(x, y) := G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$
In the case of the unit ball we have the explicit formula for the regular part of the Green function in B_1 which is given by
$$H(x, y) = \frac{1}{2\pi} \log \left(|x| \left| y - \frac{x}{|x|^2} \right| \right), \quad x, y \in B_1. \tag{1.4}$$

Then the main result of this paper provides a sufficient condition on the potential V, in addition to the assumptions (H1) – (H2) – (H3), which implies that (1.2) admits a family of non-simple blowing-up solutions. Such a sufficient condition is expressed in terms of the concept of stable zeroes for a suitable vector field.

Theorem 1.1. *Assume that hypotheses* (H1) – (H2) – (H3) *hold and, in addition,*
$$A_0 = 2, \;\; A_1 = 0 \;\; A_2 = 4. \tag{1.5}$$
Let $\xi \in \mathbb{R}^2$, $\xi \neq 0$, be a zero for the following vector field which is stable under uniform perturbations[1]
$$F : (\xi_1, \xi_2) \longmapsto \begin{pmatrix} 3D_0 \xi_1^2 + D_1 \xi_1 \xi_2 + \dfrac{3D_0 - D_2}{4} \xi_2^2 + \dfrac{15D_0 - D_2}{4} \\ \dfrac{D_1}{2} \xi_1^2 + \dfrac{3D_0 - D_2}{2} \xi_1 \xi_2 + 3\dfrac{2D_1 + D_3}{8} \xi_2^2 + \dfrac{10D_1 + 3D_3}{8} \end{pmatrix}. \tag{1.6}$$

[1] Given $F : \mathbb{R}^2 \to \mathbb{R}^2$ a continuous vector filed, we say that ξ is a zero for F which is stable with respect to uniform perturbations if $F(\xi) = 0$ and for any neighborhood U of ξ and $\epsilon > 0$ there exists $\eta > 0$ such that if $\Psi : U \to \mathbb{R}^2$ is continuous and $\|\Psi - F\|_\infty \leq \eta$, then Ψ has a zero in U. A sufficient condition which implies that 0 is a stable zero of a vector field F is $deg(F, U, 0) \neq 0$ for some neighborhood U of ξ, where deg denotes the standard Brower degree.

Then, for λ sufficiently small the problem (1.2) *has a family of solutions u_λ satisfying $u_\lambda(x) = u_\lambda(-x)$ and blowing up at the origin as $\lambda \to 0^+$:*

$$\lambda e^{u_\lambda} \to 16\pi\boldsymbol{\delta}_0 \quad \textit{in the measure sense.}$$

More precisely there exist $\delta = \delta(\lambda) > 0$ and $b = b(\lambda) \in B_1$ in a neighborhood of 0 *such that u_λ satisfies*

$$u_\lambda + 4\pi G(x,0) = -2\log\left(\mu^4 + |x^2 - b|^2\right) + 8\pi H(x^2, b) + o(1)$$

in H^1-sense, where

$$b(\lambda) = \frac{\xi_0}{4\sqrt{2}}\sqrt{\lambda\log\frac{1}{\lambda}}\big(1 + o(1)\big), \qquad \mu^2(\lambda) = \frac{\sqrt{\lambda}}{4\sqrt{2}}(1 + o(1)). \tag{1.7}$$

In particular, $\mu^2 = o(|b|)$.

The solution constructed in Theorem 1.1 reveals a non-simple blow-up profile: indeed, denoting by $\pm\beta$ the square complex roots of b, since the rate of convergence $\beta \to 0$ is lower than the speed of the concentration parameter $\mu \to 0$ (see estimate (1.7)), then u_λ develops 2 local maximum points concentrating at 0 which are arranged close to two opposite vertices. The analysis shows that the configuration of the limiting local maxima is determined by the interaction of two crucial aspects: the effect of the potential V, which tends to shrink the bubble to 0, and the boundary effect, represented by the Robin function $H(\xi,\xi)$, which tends to repel the bubble from 0. On the other hand, the existence of this kind of non-simple blow-up is still open for more general potential V. Indeed, as we will observe in Remark 6.1, if we apply our method for generic values A_0, A_1, A_2 not satisfying (1.5), then we find out that the forces exerted between the potential and the boundary may not balance and we are unable to catch a solution different from the radially symmetric one.

Remark 1.2. *Let us observe that F actually corresponds to a gradient field, precisely $F(\xi) = \nabla J(\xi)$, where the potential J is given by*

$$J(\xi) = D_0\xi_1^3 + \frac{D_1}{2}\xi_1^2\xi_2 + \frac{3D_0 - D_2}{4}\xi_1\xi_2^2 + \frac{2D_1 + D_3}{8}\xi_2^3 + \frac{15D_0 - D_2}{4}\xi_1 + \frac{10D_1 + 3D_3}{8}\xi_2.$$

Example 1.3. *Let us provide explicit examples of coefficients D_0, D_1, D_2 for which 0 is a stable zero for F, so that, according to Theorem 1.1 the corresponding V will produce a non simple blowing up solution for eqation* (1.2)*. Indeed, if we take $D_0 = 0$, $D_1 = 2$, $D_2 = -4$, $D_3 = -4$, then the potential J defined in Remark 1.2 becomes*

$$J(\xi) = \xi_1^2\xi_2 + \xi_1\xi_2^2 + \xi_1 + \xi_2.$$

It is immediate to check that $(1,-1)$ (respectively $(-1,1)$) is a critical points for J, so

$$F(1,-1) = \nabla J(1,-1) = 0$$

Moreover, the Hessian matrix of J at $(1,-1)$ is given by

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $(1,-1)$ (respectively $(-1,1)$) turns out to be a nondegenerate critical point for J of saddle type, so deg$(F,U,0) \neq 0$ where deg denotes the standard Brower degree and U is a sufficiently small neighbourhood of $(1,-1)$. Consequently, $(1,-1)$ is a stable (with respect to uniform perturbations) zero for F. Then, according to Theorem 1.1, if V satisfies (H1) – (H2) *and*

$$V(x) = 1 + 2x_1^4 + 4x_1^2x_2^2 + 2x_2^4 + 2(x_1^5x_2 + x_1x_2^5) + 4(x_1^4x_2^2 - x_1^2x_2^4) - 4x_1^3x_2^3 + O(|x|^7)$$

then $\xi_0 := (1,-1)$ (respectively $\xi_0 := (-1,1)$) gives rise to a non-simple blowing up family of solutions to (1.2)*.*

The phenomena of non-simple bubbling solutions not only occur in single equations, but also in systems. In a recent work of the third author and Gu ([16]) the non-simple blow-up behaviours are studied for singular Liouville systems. In another work of the second, third authors and Wu [27] non-simple blowup is ruled out for Toda systems. Examples of non-simple blow-up solutions are available for other models: we recall, for instance, the Liouville equation with anisotropic coefficients in [26] and the Toda system in [1].

The proofs use singular perturbation methods which combine the variational approach with a Lyapunov-Schmidt type procedure. Roughly speaking, the first step consists in the construction of an approximate solution, which should turn out to be precise enough. In view of the expected asymptotic behaviour, the shape of such approximate solution will resemble, after the change of variables $x \mapsto x^{1/2}$, a *bubble* of the form (2.6) with a suitable choice of the parameter $\delta = \delta(\lambda, b)$. We point out that in the new variables the potential $V(x^{1/2})$ would not be regular at the origin, in general; however a careful evaluation carried out in Lemma 4.2 shows that the delicate balance in the coefficients of the Taylor expansion given by hypothesis (H3) guarantees, among other things, that it is three times differentiable at the origin (see also Remark 4.3). Then we look for a solution to (1.2) in a small neighborhood of the first approximation. As quite standard in singular perturbation theory, a crucial ingredient is nondegeneracy of the explicit family of solutions of the limiting Liouville problem (2.5), as established in [2]. This allows us to study the invertibility of the linearized operator associated to the problem (1.2) under suitable orthogonality conditions. Next we introduce an intermediate problem and a fixed point argument will provide a solution for an auxiliary equation, which turns out to be solvable for any choice of b. Finally we test the auxiliary equation on the elements of the kernel of the linearized operator and we find out that, in order to find an *exact* solution of (1.2), the location of the maximum points, which is detected by the parameter b, should be a zero for a reduced finite dimensional map. The main technical difficulty in the proof is that we need to expand the reduced map up to higher orders to catch a nontrivial zero b, which will give rise to a non-simple blow-up solution. Moreover the method fails for $N \geq 2$: indeed if we try to apply our technique to $N \geq 2$, then the analogous of assumption (H3) would give that the potential has vanishing derivatives up to the order $N+1$ at 0, and this implies that the approximation rate for the reduced finite dimensional map is unfortunately not sufficiently small to carry out the argument.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results, notation, and the definition of the approximating solution. Moreover, a more general version of Theorems 1.1 is stated there (see Theorem 2.1). In Section 3 we sketch the solvability of the linearized problem by referring to [13] and [14] for the proof. The error up to which the approximating solution solves problem (1.2) is estimated in Section 4. Section 5 considers the solvability of an auxiliary problem by a contraction argument. In Section 6 we complete the proof of Theorem 1.1. In Appendix A we collect some results, most of them well-known, which are usually referred to throughout the paper.

NOTATION: In our estimates throughout the paper, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variables under consideration.

2. Preliminaries and statement of the main results

We are going to provide an equivalent formulation of problem (1.2) and Theorem 1.1. Indeed, setting v the regular part of u, namely

$$v = u + 4\pi G(x,0) = u + 2\log\frac{1}{|x|}, \tag{2.1}$$

problem (1.2) is then equivalent to solving the following boundary value problem

$$\begin{cases} -\Delta v = \lambda |x|^2 V(x) e^v & \text{in } B_1 \\ v = 0 & \text{on } \partial B_1 \end{cases}. \tag{2.2}$$

Here G and H are the Green's function and its regular part as defined in the introduction.

In what follows, we identify $x=(x_1,x_2)\in\mathbb{R}^2$ with $x_1+\mathrm{i}x_2\in\mathbb{C}$ and we denote by $x\,y$ the multiplication of the complex numbers x,y and, analogously, by x^2 the square of the complex number x.

Since V and the solutions considered in the paper are even, we can rewrite problem (2.2) as a regular Liouville problem: more precisely, denoting by $x^{\frac12}$ the complex 2-roots of x, the change of variables

$$w(x) = v\big(x^{\frac12}\big) \tag{2.3}$$

transforms problem (2.2) into a (regular) Liouville problem of the form

$$\begin{cases} -\Delta w = \dfrac{\lambda}{4} V\big(x^{\frac12}\big) e^w & \text{in } B_1 \\ w = 0 & \text{on } \partial B_1 \end{cases}. \tag{2.4}$$

Theorem 1.1 will be a consequence of a more general result concerning Liouville-type problems. In order to provide such a result, we now give a construction of a suitable approximate solution for (2.4). We can associate to (2.4) a limiting problem of Liouville type which will play a crucial role in the construction of blowing up solutions as $\lambda\to 0^+$:

$$-\Delta W = e^W \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{W(x)}dx < +\infty. \tag{2.5}$$

All solutions of this problem are given, in complex notation, by the three-parameter family of functions

$$W_{\delta,b}(x) := \log\frac{8\delta^2}{(\delta^2+|x-b|^2)^2} \quad \delta>0,\ b\in\mathbb{C}. \tag{2.6}$$

The following quantization property holds:

$$\int_{\mathbb{R}^2} e^{W_{\delta,b}(x)}dx = 8\pi. \tag{2.7}$$

In the following we agree that

$$W_\lambda(x) = W_{\delta,b}(x), \quad \delta>0,\ b\in\mathbb{C},$$

where the value $\delta=\delta(\lambda,b)$ is defined by

$$\delta^2 := \frac{\lambda}{32} V(b^{\frac12}) e^{8\pi H(b,b)} = \frac{\lambda}{32} V(b^{\frac12})(1-|b|^2)^4. \tag{2.8}$$

We point out that the diagonal $H(b,b)$ appearing in (2.8) is called the Robin function of the domain and in the case of the ball it takes the form

$$H(x,x) = \frac{1}{2\pi}\log(1-|x|^2), \quad x\in B_1$$

according to (1.4). To obtain a better first approximation, we need to modify the function W_λ in order to satisfy the zero boundary condition. Precisely, we consider the projection PW_λ onto the space $H_0^1(B_1)$, where the projection $P : H^1(\mathbb{R}^N) \to H_0^1(B_1)$ is defined as the unique solution of the problem

$$\Delta Pv = \Delta v \quad \text{in } B_1, \qquad Pv = 0 \quad \text{on } \partial B_1.$$

We recall that the regular part $H(x,b)$ of the Green function, defined in (1.4), is harmonic in B_1 and satisfies $H(x,b) = \frac{1}{2\pi}\log|x-b|$ for $x \in \partial B_1$; a straightforward computation gives that for any $x \in \partial B_1$

$$\begin{aligned} PW_\lambda - W_\lambda + \log\left(8\delta^2\right) - 8\pi H(x,b) &= -W_\lambda + \log\left(8\delta^2\right) - 4\log|x-b| = 2\log\Big(1 + \frac{\delta^2}{|x-b|^2}\Big) \\ &= 2\frac{\delta^2}{|x-b|^2} + O(\delta^4) = 2\frac{\delta^2}{1+O(|b|)} + O(\delta^4) \\ &= 2\delta^2 + O(\delta^2|b|) + O(\delta^4) \end{aligned}$$

with uniform estimate for $x \in \partial B_1$ and b in a small neighborhood of 0. Since the expressions $PW_\lambda - W_\lambda + \log\left(8\delta^2\right) - 8\pi H(x,b)$ and $2\delta^2$ are harmonic in B_1, then the maximum principle applies and implies the following asymptotic expansion

$$\begin{aligned} PW_\lambda =& W_\lambda - \log\left(8\delta^2\right) + 8\pi H(x,b) + 2\delta^2 + O(\delta^2|b|) + O(\delta^4) \\ =& -2\log\left(\delta^2 + |x-b|^2\right) + 8\pi H(x,b) + 2\delta^2 + O(\delta^2|b|) + O(\delta^4) \end{aligned} \tag{2.9}$$

uniformly for $x \in \overline{B}_1$ and b in a small neighborhood of 0.

We point out that, in order to simplify the notation, in our estimates throughout the paper we will describe the asymptotic behaviors of quantities under considerations in terms of $\delta = \delta(\lambda, b)$ instead of the parameter λ of the equation. Clearly according to (2.8) δ has the same rate as $\lambda^{\frac{1}{2}}$, so at each step we can easily pass to the analogous asymptotic in terms of λ: for instance, in (2.9) the error term "$O(\delta^4)$" can be equivalently replaced by "$O(\lambda^2)$".

We shall look for a solution to (2.4) in a small neighborhood of the first approximation, namely a solution of the form

$$w_\lambda = PW_\lambda + \phi_\lambda,$$

where the rest term ϕ_λ is small in $H_0^1(B_1)$-norm.

Let us reformulate the main theorem for problem (2.4), which prove that a non symmetric blow-up occurs for problem (2.4). More precisely, we provide a solution which develops a bubble centered at a point b; and since the rate of convergence $b \to 0^+$ is lower than the speed of the concentration parameter $\delta \to 0^+$ (see estimate (2.10)), then the blowing up turns out to be non symmetric in the first approximation.

Theorem 2.1. *Assume that hypotheses* (H1) – (H3) *and* (1.5) *hold. Let $b \in \mathbb{R}^2$ be a zero for the vector field* (1.6) *which is stable under uniform perturbations. Then, for λ sufficiently small the problem* (2.4) *has a family of solutions w_λ satisfying*

$$w_\lambda = -2\log\left(\delta^2 + |x-b_\lambda|^2\right) + 8\pi H(x, b_\lambda) + o(1)$$

in H^1-sense, where

$$b_\lambda = \frac{\xi_0}{4\sqrt{2}}\sqrt{\lambda\log\frac{1}{\lambda}}\big(1+o(1)\big). \tag{2.10}$$

In particular, by (2.8), $\delta^2 = o(|b_\lambda|)$.

In the remaining part of this paper we will prove Theorems 2.1 and at the end of Section 6 we shall see how Theorems 1.1 follows quite directly as a corollary.

We end this section by setting notation and basic well-known facts which will be of use in the rest of the paper. Given Ω a bounded domain, we denote by $\|\cdot\|$ and $\|\cdot\|_p$ the norms in the space $H^1_0(\Omega)$ and $L^p(\Omega)$, respectively, namely

$$\|u\| := \|u\|_{H^1_0(\Omega)}, \qquad \|u\|_p := \|u\|_{L^p(\Omega)} \quad \forall u \in H^1_0(\Omega).$$

In next lemma we recall the well-known Moser-Trudinger inequality ([21, 25]).

Lemma 2.2. *There exists $C > 0$ such that for any bounded domain Ω in $\mathbb{R}^2$*

$$\int_\Omega e^{\frac{4\pi u^2}{\|u\|^2}} dy \le C|\Omega| \quad \forall u \in H^1_0(\Omega),$$

where $|\Omega|$ stands for the measure of the domain Ω. In particular, for any $q \ge 1$

$$\|e^u\|_q \le C^{\frac{1}{q}} |\Omega|^{\frac{1}{q}} e^{\frac{q}{16\pi}\|u\|^2} \quad \forall u \in H^1_0(\Omega).$$

As commented in the introduction, our proof uses the singular perturbation methods. For that, the nondegeneracy of the functions that we use to build our approximating solution is essential. Next proposition is devoted to the nondegeneracy of the finite mass solutions of the Liouville equation (see [2] for the proof).

Proposition 2.3. *Assume that $\xi \in \mathbb{R}^2$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ solves the problem*

$$-\Delta\phi = \frac{8}{(1+|z-\xi|^2)^2}\phi \ \text{ in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |\nabla\phi(z)|^2 dz < +\infty. \tag{2.11}$$

Then there exist c_0, c_1, $c_2 \in \mathbb{R}$ such that

$$\phi(z) = c_0 Z_0 + c_1 Z_1 + c_2 Z_2,$$

$$Z_0(z) := \frac{1-|z-\xi|^2}{1+|z-\xi|^2}, \quad Z_1(z) := \frac{z_1-\xi_1}{1+|z-\xi|^2}, \quad Z_2(z) := \frac{z_2-\xi_2}{1+|z-\xi|^2}.$$

3. Analysis of the linearized operator

According to Proposition 2.3, by the change of variable $x = \delta z$, we immediately get that all solutions ψ of

$$-\Delta\psi = \frac{8\delta^2}{(\delta^2+|x-b|^2)^2}\psi = e^{W_\lambda}\psi \quad \text{in} \quad \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |\nabla\phi(x)|^2 dx < +\infty.$$

are linear combinations of the functions

$$Z^0_{\delta,b}(x) = \frac{\delta^2-|x-b|^2}{\delta^2+|x-b|^2}, \ Z^1_{\delta,b}(x) = \frac{\delta(x_1-b_1)}{\delta^2+|x-b|^2}, \ Z^2_{\delta,b}(x) = \frac{\delta(x_2-b_2)}{\delta^2+|x-b|^2}.$$

We introduce the projections $PZ^j_{\delta,b}$ onto $H^1_0(B_1)$. It is immediate that

$$PZ^0_{\delta,b}(x) = Z^0_{\delta,b}(x) + 1 + O\left(\delta^2\right) = \frac{2\delta^2}{\delta^2+|x-b|^2} + O(\delta^2) \tag{3.1}$$

and

$$PZ^j_{\delta,b}(x) = Z^j_{\delta,b}(x) + O(\delta) \text{ for } j = 1,2 \tag{3.2}$$

uniformly with respect to $x \in \overline{B}_1$ and $b > 0$ in a small neighborhood of 0.

We agree that $Z^j_\lambda := Z^j_{\delta,b}$ for any $j = 0, 1, 2$, where δ is defined in terms of λ and b according to (2.8). Let us consider the following linear problem: given $h \in H^1_0(B_1)$, find a function $\phi \in H^1_0(B_1)$ and constant $c_1, c_2 \in \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\phi - \dfrac{\lambda}{4} V\big(x^{\frac{1}{2}}\big) e^{PW_\lambda}\phi = \Delta h + \displaystyle\sum_{j=1,2} c_j Z^j_\lambda e^{W_\lambda} \\ \displaystyle\int_{B_1} \nabla\phi\nabla PZ^j_\lambda = 0 \;\; j = 1, 2 \end{cases} . \tag{3.3}$$

In order to solve problem (3.3), we need to establish an a priori estimate. For the proof we refer to [13] (Proposition 3.1) or [14] (Proposition 3.1).

Proposition 3.1. *There exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$, any b in a small neighborhood of 0 and any $h \in H^1_0(B_1)$, if $(\phi, c_1, c_2) \in H^1_0(B_1) \times \mathbb{R}^2$ solves (3.3), then the following holds*

$$\|\phi\| \le C|\log\delta|\|h\|.$$

For any $p > 1$, let

$$i^*_p : L^p(B_1) \to H^1_0(B_1) \tag{3.4}$$

be the adjoint operator of the embedding $i_p : H^1_0(B_1) \hookrightarrow L^{\frac{p}{p-1}}(B_1)$, i.e. $u = i^*_p(v)$ if and only if $-\Delta u = v$ in B_1, $u = 0$ on ∂B_1. We point out that i^*_p is a continuous mapping, namely

$$\|i^*_p(v)\| \le c_p\|v\|_p, \text{ for any } v \in L^p(B_1), \tag{3.5}$$

for some constant c_p which depends on p. Next let us set

$$K := \operatorname{span}\left\{PZ^1_\lambda,\, PZ^2_\lambda\right\}$$

and

$$K^\perp := \left\{\phi \in H^1_0(B_1) \;:\; \int_{B_1} \nabla\phi\nabla PZ^j_\lambda dx = 0 \;\; j = 1, 2\right\}$$

and denote by

$$\Pi : H^1_0(B_1) \to K, \qquad \Pi^\perp : H^1_0(B_1) \to K^\perp$$

the corresponding projections. Let $L : K^\perp \to K^\perp$ be the linear operator defined by

$$L(\phi) := \frac{1}{4}\Pi^\perp\Big(i^*_p\big(\lambda V\big(x^{\frac{1}{2}}\big)e^{PW_\lambda}\phi\big)\Big) - \phi. \tag{3.6}$$

Notice that problem (3.3) reduces to

$$L(\phi) = \Pi^\perp h, \quad \phi \in K^\perp.$$

As a consequence of Proposition 3.1 we derive the invertibility of L.

Proposition 3.2. *For any $p > 1$ there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$, any b in a small neighborhood of 0 and any $h \in K^\perp$ there is a unique solution $\phi \in K^\perp$ to the problem*

$$L(\phi) = h.$$

In particular, L is invertible; moreover,

$$\|L^{-1}\| \le C|\log\delta|.$$

Proof. Observe that the operator $\phi \mapsto \Pi^\perp\big(i^*_p(\lambda V\big(x^{\frac{1}{2}}\big)e^{PW_\lambda}\phi)\big)$ is a compact operator in $K^\perp$. Let us consider the case $h = 0$, and take $\phi \in K^\perp$ with $L(\phi) = 0$. In other words, ϕ solves the system (3.3) with $h = 0$ for some $c_1, c_2 \in \mathbb{R}$. Proposition 3.1 implies $\phi \equiv 0$. Then, Fredholm's alternative implies the existence and uniqueness result.

Once we have existence, the norm estimate follows directly from Proposition 3.1. □

4. Estimate of the error term

The goal of this section is to provide an estimate of the error up to which the approximate solution PW_λ solves problem (2.4). First of all, we perform the following estimates.

Lemma 4.1. *Let $\gamma = 0, 1, 2$ and $p > 1$ be fixed. The following holds:*

$$\||x-b|^\gamma e^{W_\lambda}\|_p \le C\delta^\gamma \delta^{-2\frac{p-1}{p}}, \quad \||x-b|^\gamma \lambda e^{PW_\lambda}\|_p \le C\delta^\gamma \delta^{-2\frac{p-1}{p}} \tag{4.1}$$

uniformly for b in a small neighborhood of 0.

Proof. We compute

$$\||x-b|^\gamma e^{W_\lambda}\|_p^p = 8^p\delta^{2p}\int_{B_1}\frac{|x-b|^{\gamma p}}{(\delta^2+|x-b|^2)^{2p}}dx \le 8^p\delta^{\gamma p-2(p-1)}\int_{\mathbb{R}^2}\frac{|z|^{\gamma p}}{(1+|z|^2)^{2p}}dz.$$

Taking into account that the last integral is finite for $\gamma = 0, 1, 2$ and $p > 1$ we deduce the first part of (4.1). To prove the second part it is sufficient to observe that by (2.9) and by the choice of δ in (2.8) we derive

$$\lambda e^{PW_\lambda} = \frac{\lambda}{8\delta^2}e^{W_\lambda+O(1)} = e^{W_\lambda}(1+O(1)). \tag{4.2}$$

□

Lemma 4.2. *Assume that hypotheses* (H1) − (H3) *hold. There exists $P : \mathbb{R}^2 \to \mathbb{R}$ a homogeneous polynomial of degree 2 such that*

$$\begin{aligned}
\frac{V(x^{\frac12})}{V(b^{\frac12})} &= 1 + 2A_0b_1(x_1-b_1) + \frac{A_1}{2}\Big(b_1(x_2-b_2)+b_2(x_1-b_1)\Big) + \Big(A_0+\frac{A_2}{2}\Big)b_2(x_2-b_2)\\
&+ D_0\Big((x_1-b_1)^3+3b_1^2(x_1-b_1)\Big) + \Big(\frac{D_1}{4}+\frac{D_3}{8}\Big)\Big((x_2-b_2)^3+3b_2^2(x_2-b_2)\Big)\\
&+ \frac{D_1}{2}\Big((x_1-b_1)^2(x_2-b_2)+b_1^2(x_2-b_2)+2b_1b_2(x_1-b_1)\Big)\\
&+ \frac14\Big(3D_0-D_2\Big)\Big((x_1-b_1)(x_2-b_2)^2+b_2^2(x_1-b_1)+2b_1b_2(x_2-b_2)\Big)\\
&+ P(x-b) + O(|b||x-b|^2) + O(|b|^3|x-b|) + O(|x-b|^{\frac72}) + O(|b|^{\frac72})
\end{aligned}$$

uniformly for b in a small neighborhood of 0.

Proof. Let us first consider a more general potential V of the form

$$V(x) = 1 + \sum_{j=0}^4 A_jx_1^{4-j}x_2^j + \sum_{j=0}^6 D_jx_1^{6-j}x_2^j + O(|x|^7), \quad A_j, D_j \in \mathbb{R},$$

and, using the polar coodinates $x = \rho e^{\mathrm{i}\theta} = (\rho\cos\theta, \rho\sin\theta)$, we have

$$V(x^{\frac12}) = 1 + \rho^2\sum_{j=0}^4 A_j\cos^{4-j}\frac{\theta}{2}\sin^j\frac{\theta}{2} + \rho^3\sum_{j=0}^6 D_j\cos^{6-j}\frac{\theta}{2}\sin^j\frac{\theta}{2} + O(|x|^{\frac72}).$$

Now we use standard trigonometric identities to obtain:

$$\cos^4\frac{\theta}{2} = \frac{\sin^2\theta+2\cos\theta+2\cos^2\theta}{4}, \qquad \sin^4\frac{\theta}{2} = \frac{\sin^2\theta-2\cos\theta+2\cos^2\theta}{4}$$

$$\cos^6\frac{\theta}{2} = \frac{1+4\cos^3\theta+3\cos\theta\sin^2\theta+3\cos^2\theta}{8}, \quad \sin^6\frac{\theta}{2} = \frac{1-4\cos^3\theta-3\sin^2\theta\cos\theta+3\cos^2\theta}{8}$$

$$\cos\frac{\theta}{2}\sin^3\frac{\theta}{2} = \sin\theta\frac{1-\cos\theta}{4}, \qquad \sin\frac{\theta}{2}\cos^3\frac{\theta}{2} = \sin\theta\frac{1+\cos\theta}{4}, \qquad \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac14\sin^2\theta,$$

$$\cos^5\frac{\theta}{2}\sin\frac{\theta}{2}=\sin\theta\frac{2\cos^2\theta+\sin^2\theta+2\cos\theta}{8},\qquad \sin^5\frac{\theta}{2}\cos\frac{\theta}{2}=\sin\theta\frac{2\cos^2\theta+\sin^2\theta-2\cos\theta}{8},$$
$$\sin^3\frac{\theta}{2}\sin^3\frac{\theta}{2}=\frac{1}{8}\sin^3\theta,\qquad \cos^2\frac{\theta}{2}\sin^4\frac{\theta}{2}=\sin^2\theta\frac{1-\cos\theta}{8},\qquad \sin^2\frac{\theta}{2}\cos^4\frac{\theta}{2}=\sin^2\theta\frac{1+\cos\theta}{8}.$$

According to (H3) we get

$$A_0=A_4,\ \ A_1=-A_3,\ \ D_0=-D_6,\ \ D_1=D_5,\ \ D_4=-D_2, \tag{4.3}$$

so we derive

$$\begin{aligned} V(x^{\frac{1}{2}}) &= 1+A_0\Big(x_1^2+\frac{x_2^2}{2}\Big)+\frac{A_1}{2}x_1x_2+\frac{A_2}{4}x_2^2\\ &\quad+D_0\Big(x_1^3+\frac{3}{4}x_1x_2^2\Big)+D_1\Big(\frac{x_1^2x_2}{2}+\frac{x_2^3}{4}\Big)-\frac{D_2}{4}x_1x_2^2+\frac{D_3}{8}x_2^3+O(|x|^{3+\frac{1}{2}})\\ &= 1+A_0x_1^2+\frac{A_1}{2}x_1x_2+\Big(\frac{A_0}{2}+\frac{A_2}{4}\Big)x_2^2\\ &\quad+D_0x_1^3+\frac{D_1}{2}x_1^2x_2+\frac{1}{4}\Big(3D_0-D_2\Big)x_1x_2^2+\Big(\frac{D_1}{4}+\frac{D_3}{8}\Big)x_2^3+O(|x|^{3+\frac{1}{2}}). \end{aligned} \tag{4.4}$$

Next observe that, setting $x_1=(x_1-b_1)+b_1$ and $x_2=(x_2-b_2)+b_2$ and making trivial computations we get

$$\begin{aligned} &A_0x_1^2+\frac{A_1}{2}x_1x_2+\Big(\frac{A_0}{2}+\frac{A_2}{4}\Big)x_2^2\\ &\quad+D_0x_1^3+\frac{D_1}{2}x_1^2x_2+\frac{1}{4}\Big(3D_0-D_2\Big)x_1x_2^2+\Big(\frac{D_1}{4}+\frac{D_3}{8}\Big)x_2^3\\ &=2A_0b_1(x_1-b_1)+\frac{A_1}{2}\Big(b_1(x_2-b_2)+b_2(x_1-b_1)\Big)+\Big(A_0+\frac{A_2}{2}\Big)b_2(x_2-b_2)\\ &\quad+D_0\Big((x_1-b_1)^3+3b_1^2(x_1-b_1)\Big)+\frac{D_1}{2}\Big((x_1-b_1)^2(x_2-b_2)+b_1^2(x_2-b_2)+2b_1b_2(x_1-b_1)\Big)\\ &\quad+\frac{1}{4}\Big(3D_0-D_2\Big)\Big((x_1-b_1)(x_2-b_2)^2+b_2^2(x_1-b_1)+2b_1b_2(x_2-b_2)\Big)\\ &\quad+\Big(\frac{D_1}{4}+\frac{D_3}{8}\Big)\Big((x_2-b_2)^3+3b_2^2(x_2-b_2)\Big)\\ &\quad+A_0b_1^2+\frac{A_1}{2}b_1b_2+\Big(\frac{A_0}{2}+\frac{A_2}{4}\Big)b_2^2+D_0b_1^3+\frac{D_1}{2}b_1^2b_2+\frac{1}{4}\Big(3D_0-D_2\Big)b_1b_2^2+\Big(\frac{D_1}{4}+\frac{D_3}{8}\Big)b_2^3\\ &\quad+P(x-b)+O(|b||x-b|^2). \end{aligned}$$

Finally, since by (4.4)

$$\begin{aligned} V(b^{\frac{1}{2}}) =&\ 1+A_0b_1^2+\frac{A_1}{2}b_1b_2+\Big(\frac{A_0}{2}+\frac{A_2}{4}\Big)b_2^2\\ &+D_0b_1^3+\frac{D_1}{2}b_1^2b_2+\frac{1}{4}\Big(3D_0-D_2\Big)b_1b_2^2+\Big(\frac{D_1}{4}+\frac{D_3}{8}\Big)b_2^3+O(|b|^{3+\frac{1}{2}}) \end{aligned}$$

and, consequently, $\frac{1}{V(b^{\frac{1}{2}})}=1+O(|b|^2)$, substituting into (4.4) we obtain the thesis.

□

Remark 4.3. *Let us observe that thanks to the symmetry of the coefficients* (4.3) *we obtain that* $V(x^{\frac{1}{2}})$ *turns out to be three times differentiable at* 0*: indeed the choice of coefficients implies that the two sums* $\sum_{j=0}^{4}A_j\cos^{4-j}\frac{\theta}{2}\sin^j\frac{\theta}{2}$ *and* $\sum_{j=0}^{6}D_j\cos^{6-j}\frac{\theta}{2}\sin^j\frac{\theta}{2}$ *turn out to be polynomials in the variables* $\cos\theta$, $\sin\theta$ *of degree 2 and 3 respectively.*

Now we are in the position to provide the error estimate.

Proposition 4.4. *Assume that hypotheses* (H1) − (H2) − (H3) *and* (1.5) *hold and define*

$$R_\lambda := \frac{\lambda}{4} V\big(x^{\frac{1}{2}}\big) e^{PW_\lambda} + \Delta PW_\lambda = \frac{\lambda}{4} V\big(x^{\frac{1}{2}}\big) e^{PW_\lambda} - e^{W_\lambda}.$$

Then the following holds

$$\begin{aligned} R_\lambda &= 2\delta^2 e^{W_\lambda} + D_0 e^{W_\lambda}\Big((x_1-b_1)^3 + 3b_1^2(x_1-b_1)\Big) \\ &+ \frac{D_1}{2} e^{W_\lambda}\Big((x_1-b_1)^2(x_2-b_2) + b_1^2(x_2-b_2) + 2b_1b_2(x_1-b_1)\Big) \\ &+ \frac{1}{4}\Big(3D_0 - D_2\Big)e^{W_\lambda}\Big((x_1-b_1)(x_2-b_2)^2 + b_2^2(x_1-b_1) + 2b_1b_2(x_2-b_2)\Big) \\ &+ \Big(\frac{B_1}{4} + \frac{D_3}{8}\Big)e^{W_\lambda}\Big((x_2-b_2)^3 + 3b_2^2(x_2-b_2)\Big) + P(x-b)e^{W_\lambda} \\ &+ O(\delta^2|x-b|)e^{W_\lambda} + O(|b||x-b|^2)e^{W_\lambda} + O(|b|^3|x-b|)e^{W_\lambda} + O(|x-b|^{\frac{7}{2}})e^{W_\lambda} \\ &+ O(|b|^{\frac{7}{2}})e^{W_\lambda} + O(\delta^2|b|)e^{W_\lambda} + O(\delta^4)e^{W_\lambda} \end{aligned} \tag{4.5}$$

uniformly for b in a small neighborhood of 0*. Moreover for any $p > 1$*

$$\|R_\lambda\|_p \le C(\delta^2 + |b|^3)\delta^{-2\frac{p-1}{p}}$$

uniformly for b in a small neighborhood of 0*.*

Proof. By (2.9) and the choice of δ in (2.8) we derive

$$\begin{aligned} \frac{\lambda}{4} V(x^{\frac{1}{2}})e^{PW_\lambda} &= \frac{\lambda}{32\delta^2} V(x^{\frac{1}{2}}) e^{W_\lambda + 8\pi H(x,b) + 2\delta^2 + O(\delta^2|b|) + O(\delta^4)} \\ &= \frac{V(x^{\frac{1}{2}})}{V(b^{\frac{1}{2}})} e^{W_\lambda} e^{8\pi(H(x,b) - H(b,b)) + 2\delta^2 + O(\delta^2|b|) + O(\delta^4)} \\ &= \frac{V(x^{\frac{1}{2}})}{V(b^{\frac{1}{2}})} e^{W_\lambda} e^{8\pi(H(x,b) - H(b,b))}\big(1 + 2\delta^2 + O(\delta^2|b|) + O(\delta^4)\big). \end{aligned} \tag{4.6}$$

Using the expression of H given in (1.4) we compute

$$\begin{aligned} H(x,b) &= \frac{1}{4\pi}\log\Big(1 + |x|^2|b|^2 - 2b_1x_1 - 2b_2x_2\Big) \\ &= \frac{1}{4\pi}\log\Big(1 + |x-b|^2|b|^2 + |b|^4 - 2|b|^2 - 2b_1(x_1-b_1) - 2b_2(x_2-b_2) + O(|b|^3|x-b|)\Big) \end{aligned}$$

by which

$$\begin{aligned} &e^{8\pi(H(x,b)-H(b,b))} \\ &= \frac{(1 + |x|^2|b|^2 - 2b_1x_1 - 2b_2x_2)^2}{(1-|b|^2)^4} \\ &= \frac{\big(1 + |x-b|^2|b|^2 + |b|^4 - 2b_1(x_1-b_1) - 2b_2(x_2-b_2) - 2|b|^2 + O(|b|^3|x-b|)\big)^2}{(1-|b|^2)^4} \\ &= \left(1 + \frac{|x-b|^2|b|^2 - 2b_1(x_1-b_1) - 2b_2(x_2-b_2) + O(|b|^3|x-b|)}{(1-|b|^2)^2}\right)^2 \\ &= \Big(1 - 2b_1(x_1-b_1) - 2b_2(x_2-b_2) + O(|b|^3|x-b|) + O(|b|^2|x-b|^2)\Big)^2 \\ &= 1 - 4b_1(x_1-b_1) - 4b_2(x_2-b_2) + O(|b|^2|x-b|^2) + O(|b|^3|x-b|). \end{aligned}$$

Then (4.6) becomes

$$\begin{aligned}\frac{\lambda}{4}V(x^{\frac{1}{2}})e^{PW_\lambda} = (1+2\delta^2)\frac{V(x^{\frac{1}{2}})}{V(b^{\frac{1}{2}})}e^{W_\lambda} - 4\frac{V(x^{\frac{1}{2}})}{V(b^{\frac{1}{2}})}e^{W_\lambda}(b_1(x_1-b_1)+b_2(x_2-b_2) \\ + e^{W_\lambda}\Big(O(|b|^2|x-b|^2)+O(|b|^3|x-b|)+O(\delta^2|b|)+O(\delta^4)\Big).\end{aligned} \tag{4.7}$$

Using the expansion provided by Lemma 4.2 into (4.7), and the crucial assumption (1.5), we get the estimate (4.5). Observe that (4.5) can be written in more approximate way as

$$R_\lambda = e^{W_\lambda}\Big(O(\delta^2)+O(|b|^2|x-b|)+O(|x-b|^2)+O(|b|^{\frac{7}{2}}\Big).$$

So, by applying Lemma 4.4 we obtain the L^p estimate. □

Remark 4.5. *We observe that for general coefficients A_0, A_1, A_2, after substituting the expansion of Lemma 4.2 into* (4.7) *we obtain that the following term*

$$e^{W_\lambda}(2A_0-4)b_1(x_1-b_1)+e^{W_\lambda}\Big(A_0+\frac{A_2}{2}-4\Big)b_2(x_2-b_2)+e^{W_\lambda}\frac{A_1}{2}\Big(b_1(x_2-b_2)+b_2(x_1-b_1)\Big)$$

does not vanish and actually shall represent the leading term in the estimate of the error R_λ. This will explain later in Remark 6.1 why the result of Theorem 1.1 fails in general without the assumption (1.5)*.*

5. The nonlinear problem: a contraction argument

In order to solve (2.4), let us consider the following intermediate problem:

$$\begin{cases} -\Delta(PW_\lambda+\phi)-\dfrac{\lambda}{4}V(x^{\frac{1}{2}})e^{PW_\lambda+\phi} = \displaystyle\sum_{j=1,2} c_j Z_\lambda^j e^{W_\lambda}, \\ \phi\in H_0^1(B_1), \quad \displaystyle\int_{B_1}\nabla\phi\nabla PZ_\lambda^j dx = 0, \quad j=1,2. \end{cases} \tag{5.1}$$

Then it is convenient to solve as a first step the problem for ϕ as a function of b.

Let us rewrite problem (5.1) in a more convenient way. In what follows we denote by $N : H_0^1(B_1)\to K^\perp$ the nonlinear operator

$$N(\phi) = \Pi^\perp\left(i_p^*\left(\frac{\lambda}{4}V(x^{\frac{1}{2}})e^{PW_\lambda}(e^\phi-1-\phi)\right)\right).$$

Therefore problem (5.1) turns out to be equivalent to the problem

$$L(\phi)+N(\phi)=\tilde{R}, \quad \phi\in K^\perp \tag{5.2}$$

where, recalling Lemma 4.1,

$$\tilde{R} = \Pi^\perp\left(i_p^*(R_\lambda)\right) = \Pi^\perp\left(PW_\lambda - i_p^*\left(\frac{\lambda}{4}V(x^{\frac{1}{2}})e^{PW_\lambda}\right)\right).$$

We need the following auxiliary lemma.

Lemma 5.1. *For any $p>1$ and any $\phi_1,\phi_2\in H_0^1(B_1)$ with $\|\phi\|_1$, $\|\phi_2\|<1$ the following holds*

$$\|e^{\phi_1}-\phi_1-e^{\phi_2}+\phi_2\|_p \le C(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\|, \tag{5.3}$$

$$\|N(\phi_1)-N(\phi_2)\| \le C\delta^{-2\frac{p^2-1}{p^2}}(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\| \tag{5.4}$$

uniformly for b in a small neighborhood of 0.

Proof. A straightforward computation gives that the inequality $|e^a - a - e^b + b| \le e^{|a|+|b|}(|a|+|b|)|a-b|$ holds for all $a, b \in \mathbb{R}$. Then, by applying Hölder's inequality with $\frac{1}{q} + \frac{1}{r} + \frac{1}{t} = 1$, we derive

$$\|e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2\|_p \le C\|e^{|\phi_1|+|\phi_2|}\|_{pq}(\|\phi_1\|_{pr} + \|\phi_2\|_{pr})\|\phi_1 - \phi_2\|_{pt}$$

and (5.3) follows by using Lemma 2.2 and the continuity of the embeddings $H_0^1(B_1) \subset L^{pr}(B_1)$ and $H_0^1(B_1) \subset L^{pt}(B_1)$. Let us prove (5.4). According to (3.5) we get

$$\|N(\phi_1) - N(\phi_2)\| \le C\|\lambda V(x^{\frac{1}{2}})e^{PW_\lambda}(e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2)\|_p,$$

and by Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we derive

$$\begin{aligned}\|N(\phi_1) - N(\phi_2)\| &\le C\|\lambda V(x^{\frac{1}{2}})e^{PW_\lambda}\|_{p^2}\|e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2|\|_{pq} \\ &\le C\|\lambda V(x^{\frac{1}{2}})e^{PW_\lambda}\|_{p^2}(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|\end{aligned}$$

by (5.3), and the conclusion follows by Lemma 4.1. □

Problem (5.1) or, equivalently, problem (5.2) turns out to be solvable for any choice of point b in a small neighbourhood of 0, provided that λ is sufficiently small. Indeed we have the following result.

Proposition 5.2. *Assume* (H1) – (H2) – (H3) *and* (1.5) *hold and let* $\varepsilon > 0$ *be a fixed small number. Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda \in (0, \lambda_0)$ *and any* $b \in \mathbb{R}^2$ *with* $|b| \le \delta^{\frac{2}{3}}$ *there is a unique* $\phi_\lambda = \phi_{\lambda,b} \in K^\perp$ *satisfying* (5.1) *for some* $c_1, c_2 \in \mathbb{R}$ *and*

$$\|\phi_\lambda\| \le \delta^{2-\varepsilon}. \tag{5.5}$$

Moreover the map $b \mapsto \phi_{\lambda,b} \in H_0^1(B_1)$ *is continuous.*

Proof. Since problem (5.2) is equivalent to problem (5.1), we will show that problem (5.2) can be solved via a contraction mapping argument. Indeed, in virtue of Proposition 3.2, let us introduce the map

$$T := L^{-1}(\tilde{R} - N(\phi)), \quad \phi \in K^\perp.$$

Let us fix $p > 1$ sufficiently close to 1. By (3.5) and Proposition 4.4, if $|b| \le \delta^{\frac{2}{3}}$ we get

$$\|\tilde{R}\| \le C\delta^{2-\frac{\varepsilon}{2}}. \tag{5.6}$$

Next, by (5.4),

$$\|N(\phi_1) - N(\phi_2)\| \le C\delta^{-\frac{\varepsilon}{2}}(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \quad \forall \phi_1, \phi_2 \in H_0^1(B_1), \|\phi_1\|, \|\phi_2\| < 1. \tag{5.7}$$

In particular, by taking $\phi_2 = 0$,

$$\|N(\phi)\| \le C\delta^{-\frac{\varepsilon}{2}}\|\phi\|^2 \quad \forall \phi \in H_0^1(B_1), \|\phi\| < 1. \tag{5.8}$$

We claim that T is a contraction map over the ball

$$\mathcal{B} := \left\{\phi \in K^\perp \,\middle|\, \|\phi\| \le \delta^{2-\varepsilon}\right\}$$

provided that λ is small enough. Indeed, combining Proposition 3.2, (5.6), (5.7), (5.8), for any $\phi \in \mathcal{B}$ we have

$$\|T(\phi)\| \le C|\log\delta|\big(\|\tilde{R}\| + \|N(\phi)\|\big) \le C|\log\delta|\delta^{2-\frac{\varepsilon}{2}} < \delta^{2-\varepsilon}.$$

Similarly, for any $\phi_1, \phi_2 \in \mathcal{B}$

$$\|T(\phi_1) - T(\phi_2)\| \le C|\log\delta|\|N(\phi_1) - N(\phi_2)\| \le C\delta^{-\frac{\varepsilon}{2}}|\log\delta|(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \le \frac{1}{2}\|\phi_1 - \phi_2\|.$$

Uniqueness of solutions implies continuous dependence of $\phi_\lambda = \phi_{\lambda,b}$ on b. □

6. Proof of Theorems 1.1 and Theorem 2.1

During this section we assume that the crucial assumption (H1) – (H2) – (A3) and (1.5) of Theorem 1.1 hold.

After problem (5.1) has been solved according to Proposition 5.2, then we find a solution to the original problem (2.4) if $b \in \mathbb{R}^2$ is such that $|b| \le \delta^{\frac{2}{3}}$ and

$$c_1 = c_2 = 0.$$

Let us find the condition satisfied by b in order to get c_1, c_2 equal to zero.

Proof of Theorem 2.1. We multiply the equation in (5.1) by PZ^i_λ and integrate over B_1:

$$\begin{aligned} \int_{B_1} \nabla(PW_\lambda + \phi_\lambda)\nabla PZ^i_\lambda dx - \frac{\lambda}{4}\int_{B_1} V\big(x^{\frac{1}{2}}\big)e^{PW_\lambda+\phi_\lambda} PZ^i_\lambda dx \\ = \sum_{h=1,2} c_h \int_{B_1} Z^h_\lambda e^{W_\lambda} PZ^i_\lambda dx. \end{aligned} \tag{6.1}$$

The object is now to expand each integral of the above identity and analyze the leading term. In the remaining part of the section all the estimates hold uniformly for $|b| \le \delta^{\frac{2}{3}}$, without further notice.

Let us begin by observing that the orthogonality in (5.1) gives

$$\int_{B_1} \nabla\phi_\lambda \nabla PZ^i_\lambda dx = \int_{B_1} e^{W_\lambda}\phi_\lambda Z^i_\lambda dx = 0 \tag{6.2}$$

and, by (3.2),

$$\int_{B_1} Z^h_\lambda e^{W_\lambda} PZ^i_\lambda dx = \int_{\mathbb{R}^2} \frac{8 z_i z_h}{(1+|z|^2)^4} dz + o(1) = \begin{cases} \dfrac{2}{3}\pi + o(1) & \text{if } h = i \\ o(1) & \text{if } h \ne i \end{cases} \tag{6.3}$$

where we have used that $\int_{\mathbb{R}^2} \frac{z_i^2}{(1+|z|^2)^4} dz = \frac{2}{3}\pi$ and $\int_{\mathbb{R}^2} \frac{z_1 z_2}{(1+|z|^2)^4} dz = 0$. Using the definition of R_λ in Lemma 4.4, (6.2) and (6.3), then (6.1) becomes

$$\int_{B_1} R_\lambda PZ^i_\lambda dx + \frac{\lambda}{4}\int_{B_1} V(x^{\frac{1}{2}}) e^{PW_\lambda}(e^{\phi_\lambda} - 1) PZ^i_\lambda dx = \begin{cases} -\dfrac{2}{3}\pi + o(1) & \text{if } h = i \\ o(1) & \text{if } h \ne i \end{cases}. \tag{6.4}$$

Let us first estimate the term containing the function ϕ_λ: recalling (6.2)

$$\begin{aligned} \frac{\lambda}{4}\int_{B_1} V(x^{\frac{1}{2}}) e^{PW_\lambda}(e^{\phi_\lambda} - 1) PZ^i_\lambda dx = \int_{B_1} R_\lambda (e^{\phi_\lambda} - 1) PZ^i_\lambda dx \\ + \int_{B_1} e^{W_\lambda}(e^{\phi_\lambda} - 1 - \phi_\lambda) PZ^i_\lambda dx \\ + \int_{B_1} e^{W_\lambda}\phi_\lambda (PZ^i_\lambda - Z^i_\lambda) dx. \end{aligned} \tag{6.5}$$

Now, let us fix $\varepsilon > 0$ sufficiently small and $p > 1$ sufficiently close to 1. Next let $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, (5.3) with $\phi_2 = 0$ and Proposition 5.2 give

$$\|e^{\phi_\lambda} - 1 - \phi_\lambda\|_q \le C\|\phi_\lambda\|^2 \le C\delta^{4-2\varepsilon}$$

and, consequently,

$$\|e^{\phi_\lambda} - 1\|_q \le C\|\phi_\lambda\| \le C\delta^{2-\varepsilon}. \tag{6.6}$$

Therefore, Lemma 4.1 implies

$$\begin{aligned}\int_{B_1} e^{W_\lambda}(e^{\phi_\lambda}-1-\phi_\lambda)PZ^i_\lambda dx &= O(\|e^{W_\lambda}(e^{\phi_\lambda}-1-\phi_\lambda)\|_1) = O(\|e^{W_\lambda}\|_p\|e^{\phi_\lambda}-1-\phi_\lambda\|_q)\\ &= O\Big(\delta^{4-2\frac{p-1}{p}-2\varepsilon}\Big).\end{aligned} \tag{6.7}$$

Now, by Lemma 4.4

$$\begin{aligned}\int_{B_1} R_\lambda(e^{\phi_\lambda}-1)PZ^i_\lambda dx &= O\big(\big\|R_\lambda(e^{\phi_\lambda}-1)\big\|_1\big) = O\big(\|R_\lambda\|_p\|e^{\phi_\lambda}-1\|_q\big)\\ &= O\Big(\delta^{4-2\frac{p-1}{p}-\varepsilon}\Big).\end{aligned} \tag{6.8}$$

Finally by Lemma A.3 and Lemma 4.1, using that $|b|\le \delta^{\frac{2}{3}}$,

$$\begin{aligned}\int_{B_1} e^{W_\lambda}\phi_\lambda(PZ^i_\lambda - Z^i_\lambda)dx &= -\delta\int_{B_1} e^{W_\lambda}\phi_\lambda(x_1-b_1)dx + O\Big(\delta^{\frac{5}{3}}\int_{B_1} e^{W_\lambda}|\phi_\lambda|dx\Big)\\ &= O(\delta\||x-b|e^{W_\lambda}\|_p\|\phi_\lambda\|) + O(\delta^{\frac{5}{3}}\|e^{W_\lambda}\|_p\|\phi_\lambda\|)\\ &= O(\delta^{4-\varepsilon-2\frac{p-1}{p}}) + O(\delta^{\frac{11}{3}-2\frac{p-1}{p}-\varepsilon}).\end{aligned} \tag{6.9}$$

By inserting (6.7)-(6.8)-(6.9) into (6.5), we obtain

$$\lambda\int_{B_1} V\big(x^{\frac{1}{\alpha}}\big)e^{PW_\lambda}(e^{\phi_\lambda}-1)PZ^i_\lambda dx = O(\delta^3) \tag{6.10}$$

provided that that ε is chosen sufficiently close to 0 and p sufficiently close to 1. Next, by (4.5), using Lemma A.2 and Lemma A.4, we get

$$\begin{aligned}\int_{B_1} R_\lambda PZ^1_\lambda dx = 2\pi\delta\Big(&3b_1^2D_0 + D_1b_1b_2 + \frac{15D_0-D_2}{4}\delta^2\log\frac{1}{\delta} + \frac{3D_0-D_2}{4}b_2^2\Big)\\ &+O(\delta^3)+O(\delta^2|b|)+O(|b|^{\frac{7}{2}})+O(|b|^3\delta).\end{aligned}$$

$$\begin{aligned}\int_{B_1} R_\lambda PZ^2_\lambda dx = 2\pi\delta\Big(&\frac{D_1}{2}b_1^2 + \frac{3D_0-D_2}{2}b_1b_2 + \frac{10D_1+3D_3}{8}\delta^2\log\frac{1}{\delta} + 3\frac{2D_1+D_3}{8}b_2^2\Big)\\ &+O(\delta^3)+O(\delta^2|b|)+O(|b|^{\frac{7}{2}})+O(|b|^3\delta).\end{aligned}$$

By inserting the above identity and (6.10) into (6.4) we deduce

$$2\pi\delta^2\log\frac{1}{\delta}F\Bigg(\frac{b}{\delta\sqrt{\log\frac{1}{\delta}}}\Bigg) + O(\delta^3)+O(\delta^2|b|)+O(|b|^{\frac{7}{2}})+O(|b|^3\delta) = -\frac{2}{3}\pi c + o(|c|), \tag{6.11}$$

where $c=(c_1,c_2)$ and $F:\mathbb{R}^2\to\mathbb{R}^2$ denotes the vector field defined in (1.6).

Now let $\xi_0\neq 0$ be a zero for F which is stable under uniform perturbations according to Theorem 1.1; then (6.11) gives that the following holds

$$\delta^2\log\frac{1}{\delta}F\Bigg(\frac{b}{\delta\sqrt{\log\frac{1}{\delta}}}\Bigg) + o\Big(\delta^2\log\frac{1}{\delta}\Big) = -\frac{c}{3} + o(|c|) \;\text{ unif. for } |b|\le 2|\xi_0|\delta\sqrt{\log\frac{1}{\delta}}. \tag{6.12}$$

Now, setting

$$\tilde b = \frac{b}{\delta\sqrt{\log\frac{1}{\delta}}},$$

we rewrite (6.12) as

$$\delta^2 \log \frac{1}{\delta}\Big(F(\tilde{b}) + o(1)\Big) = -\frac{c}{3} + o(|c|) \text{ unif. for } |\tilde{b}| \le 2|\xi_0|. \tag{6.13}$$

The continuity of the map $b \mapsto \phi_\lambda = \phi_{\lambda,b}$ guaranteed by Proposition 5.2 implies that the left hand side of (6.13) is continuous too. So, the uniform stability gives that, if $\eta > 0$ is sufficiently small, then for λ small enough the left hand side of (6.13) has a zero $\tilde{b}_\lambda$ with $|\tilde{b}_\lambda - \xi_0| \le \eta$ or, equivalently, the left hand side of (6.12) has a zero b_λ with $\big|\frac{b_\lambda}{\delta\sqrt{\log\frac{1}{\delta}}} - \xi_0\big| \le \eta$. The arbitrariness of η implies

$$b_\lambda = \xi_0 \delta \sqrt{\log \frac{1}{\delta}}(1 + o(1)).$$

Remark 6.1. *We point put that, for general coefficients A_0, A_1, A_2, then the error term R_λ reduces to the expression in Remark 4.5 at the leading part and, thanks to Lemma A.2, when we multiply it against PZ_i^λ we actually obtain*

$$2\pi\delta(2A_0 - 4)b_1 + \pi A_1 \delta b_2 + h.o.t, \quad 2\pi\delta\Big(A_0 + \frac{A_2}{2} - 4\Big)b_2 + \pi A_1 \delta b_1 + h.o.t.$$

which in general admits the only trivial zero $b = 0$ for the leading term, so we are unable to catch a non-simple blow-up solution without the assumption (1.5).

6.1. **Proof of Theorems 1.1.** Theorem 2.1 provides a solution to the problem (2.4) of the form

$$w_\lambda = PW_\lambda + \phi_\lambda$$

where $\phi_\lambda = \phi_{\lambda,b_\lambda} \in H_0^1(B_1)$ satisfies (5.5) and $b = b_\lambda$ satisfies (1.7).

Moreover, using (6.6) and Lemma 4.1, by Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ we get

$$\begin{aligned}
\lambda \|V(y^{\frac{1}{2}})(e^{w_\lambda} - e^{PW_\lambda})\|_1 &= \lambda \|V(y^{\frac{1}{2}}) e^{PW_\lambda}(e^{\phi_\lambda} - 1)\|_1 \\
&\le \lambda \|e^{PW_\lambda}\|_p \|e^{\phi_\lambda} - 1\|_q \\
&= O(\delta^{2-2\frac{p-1}{p}-\varepsilon}) = o(1),
\end{aligned}$$

if p is chosen sufficiently close to 1 and ε sufficiently close to 0. Similarly, by Proposition 4.4,

$$\Big\| \frac{\lambda}{4} V(y^{\frac{1}{2}}) e^{PW_\lambda} - e^{W_\lambda} \Big\|_1 = \|R_\lambda\|_1 = O(\delta^{2-2\frac{p-1}{p}}) = o(1).$$

Therefore

$$\Big\| \frac{\lambda}{4} V(y^{\frac{1}{2}}) e^{w_\lambda} - e^{W_\lambda} \Big\|_1 = o(1).$$

Clearly, by (2.1) and (2.3),

$$u_\lambda(x) = w_\lambda(x^2) - 4\pi G(x, 0) = w_\lambda(x^2) - 2\log \frac{1}{|x|}$$

solves equation (1.1) and

$$\begin{aligned}
\|\lambda V(x) e^{u_\lambda(x)} - 4|x|^2 e^{W_\lambda(x^2)}\|_1 &= 4 \Big\| \frac{\lambda}{4} |x|^2 V(x) e^{w_\lambda(x^2)} - |x|^2 e^{W_\lambda(x^2)} \Big\|_1 \\
&= 2 \Big\| \frac{\lambda}{4} V(y^{\frac{1}{2}}) e^{w_\lambda(y)} - e^{W_\lambda(y)} \Big\|_1 = o(1)
\end{aligned}$$

by Lemma A.5. Hence, recalling (2.7) and Lemma A.5,

$$\lambda \int_{B_1} V(x) e^{u_\lambda} dx = 4 \int_{\mathbb{R}^2} |x|^2 V(x) e^{W_\lambda(x^2)} dx + o(1)$$
$$= 2 \int_{\mathbb{R}^2} V(y^{\frac{1}{2}}) e^{W_\lambda(y)} dy + o(1) = 16\pi + o(1).$$

Similarly for every neighborhood U of 0

$$\lambda \int_U V(x) e^{u_\lambda} dx \to 16\pi.$$

Theorem 1.1 is thus completely proved by setting $\mu^2 = \delta$.

Appendix A

In this appendix we derive some crucial integral estimates which arise in the asymptotic expansion of the energy of approximate solution PW_λ.

Lemma A.1. *The following holds:*

$$\int_{B_1} e^{W_\lambda} |x-b| dx = O(\delta), \quad \int_{B_1} e^{W_\lambda} |x-b|^2 dx = 16\pi\delta^2 |\log\delta| + O(\delta^2), \quad \int_{B_1} e^{W_\lambda} |x-b|^3 dx = O(\delta^2)$$

uniformly for b *in a small neighborhood of* 0.

Proof. We compute

$$\int_{B_1} e^{W_\lambda} |x-b| dx \le 8\delta \int_{\mathbb{R}^2} \frac{1}{(1+|z-\delta^{-1}b|^2)^2} |z-\delta^{-1}b| dz = 8\delta \int_{\mathbb{R}^2} \frac{|z|}{(1+|z|^2)^2} dz$$

and the first estimate follows. In order to show the second estimate let us observe that $B(b, 1-|b|) \subset B(0,1) \subset B(b, 1+|b|)$, so we compute

$$\begin{aligned}
\int_{B_1} e^{W_\lambda} |x-b|^2 dx &= 8 \int_{B_1} \frac{\delta^2 |x-b|^2}{(\delta^2 + |x-b|^2)^2} dx \\
&= 8 \int_{B(b,1-|b|)} \frac{\delta^2 |x-b|^2}{(\delta^2 + |x-b|^2)^2} dx + O\Big(\int_{B(b,1+|b|)\setminus B(b,1-|b|)} \frac{\delta^2 |x-b|^2}{(\delta^2 + |x-b|^2)^2} dx \Big) \\
&= 8 \int_{B(0,1-|b|)} \frac{\delta^2 |x|^2}{(\delta^2 + |x|^2)^3} dx + O\Big(\int_{B(0,1+|b|)\setminus B(0,1-|b|)} \frac{\delta^2 |x|^2}{(\delta^2 + |x|^2)^2} dx \Big) \\
&= 8\delta^2 \int_{|z| \le \frac{1-|b|}{\delta}} \frac{|z|^2}{(1+|z|^2)^2} dz + O\Big(\delta^2 \int_{\frac{1-|b|}{\delta} \le |z| \le \frac{1+|b|}{\delta}} \frac{1}{|z|^2} dz \Big) \\
&= 8\delta^2 \int_{|z| \le \frac{1-|b|}{\delta}} \frac{1}{1+|z|^2} dz + O(\delta^2) \\
&= 16\pi\delta^2 |\log\delta| + O(\delta^2).
\end{aligned}$$

In order to prove the third estimate, let $R > 1$ so that $B(0,1) \subset B(b,R)$ if b lies in a small neighborhood of 0. Then,

$$\int_{B_1} e^{W_\lambda} |x-b|^3 dx = 8\delta^3 \int_{|z| \le \frac{1}{\delta}} \frac{1}{(1+|z-\delta^{-1}b|^2)^2} |z-\delta^{-1}b|^3 dz$$
$$\le 8\delta^3 \int_{B(0,\frac{R}{\delta})} \frac{|z|^3}{(1+|z|^2)^2} dz \le C\delta^2.$$

□

Since the key part in the proof of Theorem 2.1 relies in testing the equation (5.1) with PZ^i_λ in order to catch the leading terms, a crucial step consists in the evaluation of some integral estimates, as provided by the following lemma.

Lemma A.2. *The following holds for $i, j = 1, 2$:*

$$\int_{B_1} e^{W_\lambda} PZ^i_\lambda dx = O(\delta),$$

$$\int_{B_1} e^{W_\lambda} PZ^i_\lambda (x_i - b_i) dx = 2\pi\delta + O(\delta^2), \quad \int_{B_1} e^{W_\lambda} PZ^i_\lambda (x_j - b_j) dx = O(\delta^2) \quad i \neq j,$$

$$\int_{B_1} e^{W_\lambda} |PZ^i_\lambda| |x - b|^2 dx = O(\delta^2),$$

$$\int_{B_1} e^{W_\lambda} PZ^i_\lambda (x_i - b_i)^3 dx = 6\pi\delta^3 \log\frac{1}{\delta} + O(\delta^3) \quad \int_{B_1} e^{W_\lambda} PZ^i_\lambda (x_j - b_j)^3 dx = O(\delta^3) \quad i \neq j,$$

$$\int_{B_1} e^{W_\lambda} PZ^i_\lambda (x_j - b_j)^2 (x_i - b_i) dx = 2\pi\delta^3 \log\frac{1}{\delta} + O(\delta^3) \quad i \neq j,$$

$$\int_{B_1} e^{W_\lambda} PZ^i_\lambda (x_i - b_i)^2 (x_j - b_j) dx = O(\delta^3) \quad i \neq j,$$

$$\int_{B_1} e^{W_\lambda} |PZ^i_\lambda| |x - b|^{\frac{7}{2}} dx = O(\delta^3)$$

uniformly for b in a small neighborhood of 0.

Proof. We compute

$$\begin{aligned}\int_{B_1} e^{W_\lambda} Z^i_\lambda dx &= 8\int_{|z|\leq\frac{1}{\delta}} \frac{1}{(1 + |z - \delta^{-1}b|^2)^3}(z_i - \delta^{-1}b_i) dz \\ &= 8\int_{\mathbb{R}^2} \frac{1}{(1 + |z - \delta^{-1}b|^2)^3}(z_i - \delta^{-1}b_i) dz + O(\delta^3) \\ &= 8\int_{\mathbb{R}^2} \frac{z_i}{(1 + |z|^2)^3} dz + O(\delta^3) = O(\delta^3),\end{aligned}$$

since $\int_{\mathbb{R}^2} \frac{z_i}{(1+|z||^2)^3} dz = 0$ by oddness. Next

$$\begin{aligned}\int_{B_1} e^{W_\lambda} Z^i_\lambda (x_i - b_i) dx &= 8\delta \int_{|z|\leq\frac{1}{\delta}} \frac{1}{(1 + |z - \delta^{-1}b|^2)^3}(z_i - \delta^{-1}b_i)^2 dz \\ &= 8\delta \int_{\mathbb{R}^2} \frac{1}{(1 + |z - \delta^{-1}b|^2)^3}(z_i - \delta^{-1}b_i)^2 dz + O(\delta^3) \\ &= 8\delta \int_{\mathbb{R}^2} \frac{z_i^2}{(1 + |z|^2)^3} dz + O(\delta^3) \\ &= 2\pi\delta + O(\delta^3)\end{aligned}$$

where we have used the identity $\int_{\mathbb{R}^2} \frac{(z_i)^2}{(1+|z|^2)^3} = \frac{1}{2}\int_{\mathbb{R}^2} \frac{|z|^2}{(1+|z|^2)^3} = \frac{\pi}{4}$. Similarly for $i \neq j$

$$\int_{B_1} e^{W_\lambda} Z^i_\lambda (x_j - b_j) dx = 8\delta \int_{\mathbb{R}^2} \frac{z_i z_j}{(1 + |z|^2)^3} dz + O(\delta^3) = O(\delta^3)$$

since $\int_{\mathbb{R}^2} \frac{z_i z_j}{(1+|z|^2)^3} dz = 0$. Next,

$$\int_{B_1} e^{W_\lambda} |Z^i_\lambda| |x - b|^2 dx \leq 8\delta^2 \int_{\mathbb{R}^2} \frac{|x - b|^3}{(\delta^2 + |x - b|^2)^3} dx = 8\delta^2 \int_{\mathbb{R}^2} \frac{|z|^3}{(1 + |z|^2)^3} dz \leq C\delta^2.$$

Using that $B(b,1-|b|)\subset B(0,1)\subset B(b,1+|b|)$, we compute

$$\begin{aligned}
&\int_{B_1} e^{W_\lambda} Z^i_\lambda (x_i-b_i)^3 dx\\
&=8\delta^3\int_{B_1}\frac{(x_i-b_i)^4}{(\delta^2+|x-b|^2)^3}dx\\
&=8\delta^3\int_{B(b,1-|b|)}\frac{(x_i-b_i)^4}{(\delta^2+|x-b|^2)^3}dx+O\Big(\delta^3\int_{B(b,1+|b|)\setminus B(b,1-|b|)}\frac{|x-b|^4}{(\delta^2+|x-b|^2)^3}dx\Big)\\
&=8\delta^3\int_{B(0,1-|b|)}\frac{(x_i)^4}{(\delta^2+|x|^2)^3}dx+O\Big(\delta^3\int_{B(0,1+|b|)\setminus B(0,1-|b|)}\frac{|x|^4}{(\delta^2+|x|^2)^3}dx\Big)\\
&=8\delta^3\int_{|z|\le\frac{1-|b|}{\delta}}\frac{(z_i)^4}{(1+|z|^2)^3}dz+O(\delta^3)\\
&=6\pi\delta^3|\log\delta|+O(\delta^3)
\end{aligned}$$

where we have used the identity $\int_{|z|\le r}\frac{(z_i)^4}{(1+|z|^2)^3}dz=\frac{3}{8}\pi\log(1+r^2)+\frac34\frac{\pi}{1+r^2}-\frac{3}{16}\frac{\pi}{(1+r^2)^2}-\frac{9}{16}\pi$.

Similarly, for $i\neq j$

$$\int_{B_1} e^{W_\lambda} Z^i_\lambda (x_j-b_j)^3 dx=8\delta^3\int_{|z|\le\frac{1-|b|}{\delta}}\frac{z_i(z_j)^3}{(1+|z|^2)^3}dz+O(\delta^3)=O(\delta^3)$$

since $\int_{|z|\le r}\frac{z_i(z_j)^3}{(1+|z|^2)^3}dz=0$.

Next, for $i\neq j$

$$\begin{aligned}
\int_{B_1} e^{W_\lambda} Z^i_\lambda (x_j-b_j)^2(x_i-b_i) dx&=8\delta^3\int_{|z|\le\frac{1-|b|}{\delta}}\frac{(z_i)^2(z_j)^2}{(1+|z|^2)^3}dz+O(\delta^3)\\
&=2\pi\delta^3|\log\delta|+O(\delta^3)
\end{aligned}$$

where the last equality follows by $\int_{|z|\le r}\frac{(z_i)^2(z_j)^2}{(1+|z|^2)^3}dz=\frac{\pi}{8}\log(1+r^2)+\frac14\frac{\pi}{1+r^2}-\frac{1}{16}\frac{\pi}{(1+r^2)^2}-\frac{3}{16}\pi$. Similarly for $i\neq j$

$$\int_{B_1} e^{W_\lambda} Z^i_\lambda (x_i-b_i)^2(x_j-b_j) dx=8\delta^3\int_{|z|\le\frac{1-b}{\delta}}\frac{(z_i)^3z_j}{(1+|z|^2)^3}dz+O(\delta^3)=O(\delta^3)$$

by $\int_{|z|\le r}\frac{(z_i)^3z_j}{(1+|z|^2)^3}dz=0$. Finally

$$\int_{B_1} e^{W_\lambda}|Z^i_\lambda||x-b|^{\frac72}dx\le 8\delta^2\int_{B(b,1+|b|)}\frac{\delta|x-b|^{\frac92}}{(\delta^2+|x-b|^2)^3}dx=8\delta^{\frac72}\int_{|z|\le\frac{1+|b|}{\delta}}\frac{|z|^{\frac92}}{(1+|z|^2)^3}dz\le C\delta^3.$$

Taking into account that $PZ^i_\lambda=Z^i_\lambda+O(\delta)$ by (3.2), and recalling Lemma A.1, the above integral estimates give the thesis. □

In order to derive next integral estimate we need to expand the projections PZ^i_λ to a higher order with respect to (3.2).

Lemma A.3. *For $i=1,2$ the following holds:*

$$PZ^i_\lambda=Z^i_\lambda(x)-\delta(x_i-b_i)+O(\delta^3)+O(\delta|b|)\ \text{in}\ B_1$$

uniformly for b in a small neighborhood of 0.

Proof. Let us consider $i = 1$. Observe that

$$\begin{aligned}\text{if } |x| = 1: \quad Z^1_\lambda(x) &= \frac{\delta(x_1 - b_1)}{\delta^2 + |x-b|^2} = \frac{\delta(x_1 - b_1)}{1 + \delta^2 + O(|b|)} \\ &= \delta(x_1 - b_1)\Big(1 + O(|b|) + O(\delta^2)\Big) \\ &= \delta(x_1 - b_1) + O(|b|\delta) + O(\delta^3).\end{aligned}$$

Therefore, if we set

$$\hat{Z}^1_\lambda := Z^1_\lambda(x) - \delta(x_1 - b_1),$$

we get

$$\hat{Z}^1_\lambda(x) = O(\delta^3) + O(\delta|b|) \text{ if } |x| = 1$$

and

$$-\Delta \hat{Z}^1_\lambda(x) = -\Delta Z^1_\lambda(x) = \Delta P Z^1_\lambda(x) \text{ in } B_1.$$

Hence, since by construction $PZ^1_\lambda = 0$ for $|x| = 1$, the maximum principle applies and gives

$$PZ^1_\lambda = \hat{Z}^1_\lambda + O(\delta^3) + O(\delta|b|) = Z^1_\lambda(x) - \delta(x_1 - b_1) + O(\delta^3) + O(\delta|b|) \text{ in } B_1.$$

□

Lemma A.4. *Let P be a homogeneous polynomial of degree 2. Then the following holds:*

$$\int_{B_1} e^{W_\lambda} PZ^i_\lambda P(x-b)dx = O(\delta^3) + O(\delta^2|b|) \quad i = 1, 2$$

uniformly for b in a small neighborhood of 0.

Proof. We compute

$$\begin{aligned}\int_{B_1} e^{W_\lambda} Z^i_\lambda P(x-b)dx &= 8\delta^2 \int_{|z|\le\frac{1}{\delta}} \frac{P(z - \delta^{-1}b)}{(1 + |z - \delta^{-1}b|^2)^3}(z_i - \delta^{-1}b_i)dz \\ &= 8\delta^2 \int_{\mathbb{R}^2} \frac{P(z)}{(1+|z|^2)^3} z_i dz + O(\delta^3) = O(\delta^3)\end{aligned} \tag{A.1}$$

where we have used that $\int_{\mathbb{R}^2} \frac{P(z)}{(1+|z|^2)^3} z_i dz = 0$ by oddness. Taking into account of Lemma A.3 we get

$$\begin{aligned}\int_{B_1} e^{W_\lambda} PZ^i_\lambda P(x-b)dx &= \int_{B_1} e^{W_\lambda} Z^i_\lambda P(x-b)dx - \delta \int_{B_1} e^{W_\lambda}(x_i - b_i)P(x-b)dx \\ &\quad + \Big(O(\delta^3) + O(\delta|b|)\Big) \int_{B_1} e^{W_\lambda}|x-b|^2 dx \\ &= \int_{B_1} e^{W_\lambda} Z^i_\lambda P(x-b)dx + O(\delta) \int_{B_1} e^{W_\lambda}|x-b|^3 dx \\ &\quad + \Big(O(\delta^3) + O(\delta|b|)\Big) \int_{B_1} e^{W_\lambda}|x-b|^2 dx\end{aligned}$$

and the thesis follows by (A.1), and recalling Lemma A.1. □

Finally we deduce some integral identities associated to the change of variable $x \mapsto x^\alpha$ which appears frequently when dealing with α-symmetric functions.

Lemma A.5. *Let $\alpha \in \mathbb{N}$, $\alpha \ge 2$, and let $f \in L^1(B_1)$. Then we have that $|x|^{2(\alpha-1)} f(x^\alpha) \in L^1(B_1)$ and*

$$\int_{B_1} |x|^{2(\alpha-1)} f(x^\alpha)dx = \frac{1}{\alpha} \int_{B_1} f(y)dy.$$

Proof. It is sufficient to prove the thesis for a smooth function f. Using the polar coordinates (ρ,θ) and then applying the change of variables $(\rho',\theta')=(\rho^\alpha,\alpha\theta)$

$$\begin{aligned}\int_{B_1}|x|^{2(\alpha-1)}f(x^\alpha)dx &= \int_0^{+\infty}d\rho\int_0^{2\pi}\rho^{2\alpha-1}f(\rho^\alpha e^{\mathrm{i}\alpha\theta})d\theta\\ &= \frac{1}{\alpha^2}\int_0^{+\infty}d\rho'\int_0^{2\alpha\pi}\rho' f(\rho' e^{\mathrm{i}\theta'})d\theta'\\ &= \frac{1}{\alpha}\int_0^{+\infty}d\rho'\int_0^{2\pi}\rho'|f(\rho' e^{\mathrm{i}\theta'})|^2 d\theta'\\ &= \frac{1}{\alpha}\int_{B_1}f(y)dy.\end{aligned}$$

□

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