Nonlocal planar Schrödinger-Poisson systems in the fractional Sobolev limiting case

Daniele Cassani [a,b,*], Zhisu Liu [c,a,1], Giulio Romani [a,b]

[a] *Dipartimento di Scienza e Alta Tecnologia, Università degli Studi dell'Insubria, Villa Toeplitz, Via G.B. Vico, 46, 21100 Varese, Italy*

[b] *RISM-Riemann International School of Mathematics, Villa Toeplitz, Via G.B. Vico, 46, 21100 Varese, Italy*

[c] *Center for Mathematical Sciences, School of Mathematics and Physics, China University of Geosciences, Wuhan, Hubei, 430074, PR China*

Abstract

We study the nonlinear Schrödinger equation for the s-fractional p-Laplacian strongly coupled with the Poisson equation in dimension two and with $p = \frac{2}{s}$, which is the limiting case for the embedding of the fractional Sobolev space $W^{s,p}(\mathbb{R}^2)$. We prove existence of solutions by means of a variational approximating procedure for an auxiliary Choquard equation in which the uniformly approximated sign-changing logarithmic kernel competes with the exponential nonlinearity. Qualitative properties of solutions such as symmetry and decay are also established by exploiting a suitable moving planes technique.
© 2023 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

* Corresponding author at: Dipartimento di Scienza e Alta Tecnologia, Università degli Studi dell'Insubria, Villa Toeplitz, Via G.B. Vico, 46, 21100 Varese, Italy.

E-mail addresses: daniele.cassani@uninsubria.it (D. Cassani), liuzhisu@cug.edu.cn (Z. Liu), giulio.romani@uninsubria.it (G. Romani).

1. Introduction

In this paper we investigate existence and symmetry of positive solutions for the following strong coupling of the $(s, \frac{2}{s})$-fractional Schrödinger equation with the Poisson equation in the whole plane

$$\begin{cases} (-\Delta)^s_{\frac{2}{s}} u + |u|^{\frac{2}{s}-2} u = \phi \, f(u) \\ -\Delta \phi = F(u) \end{cases} \tag{SP_s}$$

where $s \in (0, 1)$, f is a nonnegative real function and F its primitive vanishing at zero. The nonlocal operator in the Schrödinger equation is the s-fractional p-Laplacian with $p = \frac{2}{s}$, which is the so-called limiting case for the Sobolev embedding of the fractional Sobolev space (see Section 2)

$$W^{s,p}(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{2-sp}}(\mathbb{R}^2)$$

and it is well-known that this implies the nonlinearity may grow exponentially at infinity.

In the local case $s = 1$, (SP_s) reduces to the deeply studied Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u = \phi \, f(u) \\ -\Delta \phi = F(u) \end{cases} \tag{SP}$$

which emerges in several fields of Physics: in electrostatics, it models the interaction of two identically charged particles; in quantum mechanics yields a model for the self-interaction of the wave function with its own gravitational field; it is also related with the Hartree model for crystals, see [6] and references therein. In the higher-dimensional case $N \geq 3$, and when f has a polynomial growth, there is an extensive literature about (SP), see the survey [35] and references therein. A strategy to investigate (SP) is to move the attention to the corresponding Choquard equation

$$-\Delta u + u = (I_N * F(u)) \, f(u) \quad \text{in } \mathbb{R}^N, \tag{Ch}$$

obtained by solving the Poisson equation in (SP) by means of convolution with the (positive) Riesz kernel

$$I_N := \frac{C_N}{|x|^{N-2}}, \qquad N \geq 3$$

where C_N is an explicit positive constant depending on the dimension N, and then formally (see the discussion carried out in Section 3) inserting $\phi = \phi_u := I_N * F(u)$ in the Schrödinger equation. Besides the effect of variables reduction, the advantage of this approach is that the Choquard equation (Ch) can be studied by variational techniques. Moreover, new interesting phenomena arise, such as the appearance of a lower-critical exponent in addition to the usual upper-critical exponent, as in the Sobolev case, see [12,13] and references therein.

In dimension $N = 2$ just a few results are available. In this case the Riesz kernel is logarithmic

$$I_2 := \frac{1}{2\pi} \ln \frac{1}{|x|}, \tag{1.1}$$

and therefore unbounded both from below and from above, which together with the fact that it is sign changing introduces a major difficulty with respect to the higher dimensional case. In fact, the functional associated to the corresponding Choquard equation (Ch) may be not well-defined in the natural space $H^1(\mathbb{R}^2)$. When $f(u) = u$, the approach developed in [4,15,16,23], originating from the unpublished work of Stubbe [41], allows one to study the equation in a constrained subspace of $H^1(\mathbb{R}^2)$, in which the log-convolution term is well-defined. However, in dimension two it is well-known that $H^1(\mathbb{R}^2) \hookrightarrow L^q(\Omega)$ for all $q \geq 1$ but not in $L^\infty(\Omega)$ and that the maximal degree of summability for functions with membership in $H^1(\mathbb{R}^2)$ is exponential, namely $\int (e^{\alpha u^2} - 1)\, dx$ stays bounded for all $\alpha > 0$, see [38]. Several extensions and refinements to this result have been proposed, among which the extension to the higher-order Sobolev spaces $W^{m,p}$ with $m \in \mathbb{N}$, in the limiting case $N = mp$, see [39].

Exponential nonlinearities in (Ch) when $N = 2$, still maintaining the polynomial Riesz kernel and so loosing the connection to the Schrödinger-Poisson system, have been first considered in [1]. On the other hand, a special Schrödinger-Poisson system with logarithmic kernel and exponential nonlinearity, but not in gradient form as (SP), has been studied in [2]. The tuning of the two phenomena: logarithmic growth of the Riesz kernel and maximal exponential growth for system (SP) was first tackled in [11]. Here, the authors establish a proper functional setting by means of a log-weighted version of the Pohožaev-Trudinger inequality, so that the functional associated to (Ch) turns out to be well-defined. An extension of these techniques to quasilinear Schrödinger-Poisson systems was established in [8]. A different approach has been recently proposed in [10,32,33]. Instead of working directly on the logarithmic Choquard equation, the authors consider a family of approximating problems, each of them involving only a polynomial Riesz kernel, and prove that the limit of a sequence of approximating solutions converges to a solution to the original problem. This approach has the advantage of working in the $H^1(\mathbb{R}^2)$ space context and one finds a posteriori that the logarithmic convolution term turns out to be well-defined at least on the solution. Let us finally mention that this method has been recently used in [37] to cover also the zero-mass case.

As a consequence of [36,46], it is meaningful to study system (SP$_s$) in presence of exponential nonlinearities. Fractional Schrödinger-Poisson systems and fractional Choquard equations have been recently studied in the subcritical regime, precisely in higher dimension $N > sp$ and with polynomial nonlinearities, see [18,25,26,29]. To the very best of our knowledge, the only results for fractional Choquard equations with exponential nonlinearities are obtained in [17] in the one dimensional case, in [7] where the logarithmic kernel and the exponential nonlinearity are not combined, and in [45] where the Riesz kernel is polynomial; see also [44] for related results.

Here we address both the problem of proving existence and establishing qualitative properties of solutions to the fractional planar system (SP$_s$), in the limiting case of logarithmic kernel and exponential nonlinearity. The first nontrivial step is proving that solutions of the system can be obtained by solving the related Choquard equation

$$(-\Delta)^{\frac{s}{2}}_{\frac{s}{2}} u + |u|^{\frac{2}{s}-2} u = \frac{1}{2\pi} \left(\ln \frac{1}{|x|} * F(u) \right) f(u) \quad \text{in } \mathbb{R}^2. \tag{Ch$_s$}$$

Before stating our main results, let us briefly introduce the functional setting and make precise the definition of solution we deal with for (SP$_s$) and (Ch$_s$). The $(s, \frac{2}{s})$-fractional Laplace operator is pointwisely defined as

$$(-\Delta)^s_{\frac{2}{s}} u(x) := 2 \, PV \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^{\frac{2}{s}-2}(u(x) - u(y))}{|x - y|^4} \, dy, \quad x \in \mathbb{R}^2,$$

where PV stands for the Cauchy Principal Value which is well-defined for all $x \in \mathbb{R}^2$, for functions in $C^{1,1}_{loc}(\mathbb{R}^2)$ which enjoy suitable integrability conditions at infinity, see Section 3 and [14, Lemma 5.2].

The fractional Sobolev space $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ is defined by

$$W^{s,\frac{2}{s}}(\mathbb{R}^2) := \{u \in L^{\frac{2}{s}}(\mathbb{R}^2) : [u]_{s,\frac{2}{s}} < +\infty\},$$

where $[u]_{s,\frac{2}{s}}$ denotes the Gagliardo seminorm, which in this context reads as follows

$$[u]_{s,\frac{2}{s}} = \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy \right)^{\frac{s}{2}}.$$

The fractional Sobolev-Slobodeckij space $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ is a uniformly convex Banach space with norm

$$\|u\| := \left(\|u\|_{\frac{2}{s}}^{\frac{2}{s}} + [u]_{s,\frac{2}{s}}^{\frac{2}{s}} \right)^{\frac{s}{2}}.$$

For $\gamma > 0$ the weighted Lebesgue space $L_\gamma(\mathbb{R}^2)$ is defined as

$$L_\gamma(\mathbb{R}^2) := \left\{ u \in L^1_{loc}(\mathbb{R}^2) \, \Big| \int_{\mathbb{R}^2} \frac{|u(x)|}{1 + |x|^{2+2\gamma}} \, dx < +\infty \right\}.$$

Definition 1.1. For $f \in \mathcal{S}'(\mathbb{R}^2)$ we say that a function $\phi \in L_1(\mathbb{R}^2)$ is a solution of the linear Poisson equation $-\Delta \phi = f$ in $\mathbb{R}^2$ if

$$\int_{\mathbb{R}^2} \phi(-\Delta \varphi) = \langle f, \varphi \rangle \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Definition 1.2 (*Solution of* (SP$_s$)). We say that (u, ϕ) is a weak solution of the Schrödinger-Poisson system (SP$_s$) if

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^{\frac{2}{s}-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^4} \, dx \, dy + \int_{\mathbb{R}^2} |u|^{\frac{2}{s}-2} u\varphi \, dx = \int_{\mathbb{R}^2} \phi f(u)\varphi \, dx$$

for all $\varphi \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$, and ϕ solves $-\Delta\phi = F(u)$ in $\mathbb{R}^2$ in the sense of Definition 1.1.

Definition 1.3 (*Solution of* (Ch$_s$)). We say that $u \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ is a *weak solution* of (Ch$_s$) if

$$
\iint_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|u(x)-u(y)|^{\frac{2}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^4}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^2} |u|^{\frac{2}{s}-2}u\varphi\,\mathrm{d}x
$$

$$
= \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\ln\frac{1}{|x-y|}F(u(y))\,\mathrm{d}y\right)f(u(x))\varphi(x)\,\mathrm{d}x \tag{1.2}
$$

for all $\varphi \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$.

Assumptions. Throughout the paper, we consider the following assumptions on the nonlinearity f:

(f_1) $f \in C^1(\mathbb{R})$, $f \geq 0$, $f(t) = 0$ for $t \leq 0$, and $f(t) = o(t^{\frac{2}{s}-1})$ as $t \to 0^+$;
(f_2) there exist constants $b_1, b_2 > 0$ such that for any $t > 0$,

$$
0 < f(t) \leq b_1 + b_2\Phi_{2,s}(\alpha_*|t|^{\frac{2}{2-s}}),
$$

where $\Phi_{2,s}(\cdot)$ has exponential growth, see Proposition 2.1;
(f_3) there exists $\tau \in (0, s)$ such that

$$
1 - s + \tau \leq \frac{F(t)f'(t)}{f^2(t)} < 1 + \mu(s,\tau) \quad \text{for any } t > 0,
$$

where $\mu(s,\tau)$ is explicitly given in Lemma 5.8;
(f_4) $\displaystyle\lim_{t\to+\infty}\frac{F(t)f'(t)}{f^2(t)} = 1$ or equivalently $\displaystyle\lim_{t\to+\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{F(t)}{f(t)} = 0$;
(f_5) there exists $\beta_0 > 1$ depending on s such that

$$
f(t)F(t) \geq \beta\, t^{\frac{2}{s}} \quad \text{for all } t > T(s),
$$

for some $\beta > \beta_0$. The values of β_0 and $T(\cdot)$ are explicitly given in Lemma 5.5.

Main results. We are now in the position to state our main results. The first one, proved in Section 3, concerns the relationship between the solutions of the Choquard equation (Ch$_s$) and the corresponding Schrödinger-Poisson system (SP$_s$): this is the fractional counterpart of [8, Theorem 2.1]. In particular we make rigorous the fact that from a weak solution u of (Ch$_s$) one obtains a weak solution to (SP$_s$).

Theorem 1.4 ((Ch$_s$) $\implies$ (SP$_s$)). *Suppose the nonlinearity f satisfies (f_1)-(f_2). Let $u \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ be a positive weak solution of the Choquard equation* (Ch$_s$) *in the sense of Definition 1.3 and define $\phi_u := I_2 * F(u)$. Then $(u, \phi_u) \in W^{s,\frac{2}{s}}(\mathbb{R}^2) \times L_{\frac{1}{2}}(\mathbb{R}^2)$ is a solution of*

the Schrödinger-Poisson system (SP$_s$) in the sense of Definition 1.2. Moreover $u \in L^\infty(\mathbb{R}^2) \cap C_{\text{loc}}^{0,\nu}(\mathbb{R}^2)$ for some $\nu \in (0,1)$, and

$$\phi_u(x) = -\frac{\|F(u)\|_1}{2\pi} \ln|x| + o(1) \qquad as \ |x| \to +\infty. \tag{1.3}$$

Note that we do not prove that the two problems are equivalent, namely that they have the same set of solutions. Finding a proper functional setting in which this holds is still open, even for the local case (SP)–(Ch), see [8, Section 2].

The system (SP$_s$) is autonomous, thus we expect that positive solutions enjoy radial symmetry. In the next result, by exploiting the connection between (Ch$_s$) and (SP$_s$) established in Theorem 1.4, we show that this is indeed the case for solutions which are regular enough. The regularity will be needed to have a pointwisely defined $(s, \frac{2}{s})$-fractional Laplace operator and to be able to exploit a moving plane technique. This method has been firstly employed to (s, p)-fractional operators by Chen and Li in [14] for the single Schrödinger equation; here, we further develop it to cover the case of systems.

Theorem 1.5 (*Symmetry for* (Ch$_s$)). *Suppose* (f_1)-(f_2) *are satisfied, and let* $u \in C_{loc}^{1,1}(\mathbb{R}^2) \cap W^{s,\frac{2}{s}}(\mathbb{R}^2)$ *be a positive solution of* (Ch$_s$). *Then* u *is radially symmetric about the origin and monotone decreasing.*

Observe that in Theorems 1.4 and 1.5 we only need that f is subcritical or critical in the sense of Proposition 2.1. In order to establish the existence of solutions for (Ch$_s$) – and therefore for (SP$_s$) by Theorem 1.4 – we require the full set of assumptions (f_1)–(f_5).

Theorem 1.6 (*Existence for* (Ch$_s$)). *Suppose* f *satisfies* (f_1)-(f_5), *then* (Ch$_s$) *possesses a positive radially symmetric solution* $u \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ *such that*

$$\left| \int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|} * F(u) \right) F(u) \, \mathrm{d}x \right| < +\infty. \tag{1.4}$$

As a consequence of Theorems 1.4 and 1.6, we have

Corollary 1.7 (*Existence for* (SP$_s$)). *Suppose* f *satisfies* (f_1)-(f_5), *then* (SP$_s$) *possesses a solution* $(u, \phi) \in W^{s,\frac{2}{s}}(\mathbb{R}^2) \times L_{\frac{1}{2}}(\mathbb{R}^2)$ *such that:*

i) $u \in L^\infty(\mathbb{R}^2) \cap C_{\text{loc}}^{0,\nu}(\mathbb{R}^2)$ *for some* $\nu \in (0,1)$, *is positive, radially symmetric and* (1.4) *holds;*
ii) $\phi = \phi_u := I_2 * F(u)$ *and the asymptotic behaviour* (1.3) *is satisfied.*

Overview. In the next section we give motivations and discuss consequences of our assumptions together with some preliminaries. In Section 3 we study the relationship between solutions of the Choquard equation (Ch$_s$) and the related Schrödinger-Poisson system (SP$_s$) which is quite delicate as it depends on the notion of solution and regularity issues. Apparently there is no equivalence in general, even in the local case, see [8]; Theorem 1.4 is a step forward towards a complete understanding of this phenomenon. In Section 4 we prove the symmetry result of

Theorem 1.5 by extending the moving-plane technique of [14]. Finally, in Section 5 we exploit all previous results to prove Theorem 1.6 by means of a variational approach together with a uniform asymptotic approximation technique.

Notation. For $R > 0$ and $x_0 \in \mathbb{R}^2$ we denote by $B_R(x_0)$ the ball of radius R and centre x_0. Given a set $\Omega \subset \mathbb{R}^2$, we denote $\Omega^c := \mathbb{R}^2 \setminus \Omega$, and its characteristic function as χ_Ω. The space of the infinitely differentiable functions which are compactly supported is denoted by $C_0^\infty(\mathbb{R}^2)$, while $L^p(\mathbb{R}^2)$ with $p \in [1, +\infty]$ is the Lebesgue space of p-integrable functions. The norm of $L^p(\mathbb{R}^2)$ is denoted by $\|\cdot\|_p$. The spaces $C^{0,\nu}(\mathbb{R}^2)$ for $\nu \in (0, 1)$ are usual spaces of Hölder continuous functions. The space $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions and $\mathcal{S}'$ the dual space of tempered distributions. For $q > 0$ we define $q! := q(q-1)\cdots(q-\lfloor q \rfloor)$, where $\lfloor q \rfloor$ denotes the largest integer strictly less than q; if $q > 1$ its conjugate Hölder exponent is $q' := \frac{q}{q-1}$. Finally, $o_n(1)$ denotes a vanishing real sequence as $n \to +\infty$. Hereafter, the letter C will be used to denote positive constants which are independent of relevant quantities and whose value may change from line to line.

2. Preliminaries

The Pohožaev-Trudinger inequality has been extended to the Sobolev fractional setting by Parini and Ruf [36] for bounded domains and by Zhang [46] for the whole space, results which we resume below. Let

$$\Phi_{2,s}(t) := e^t - \sum_{j=0}^{j_{\frac{2}{s}}-2} \frac{t^j}{j!},$$

for $t \geq 0$, where $j_{\frac{2}{s}} := \min\{j \in \mathbb{N} : j \geq \frac{2}{s}\}$.

Proposition 2.1 ([36,46]). *Let $s \in (0, 1)$, then for all $\alpha > 0$ one has*

$$\int_{\mathbb{R}^2} \Phi_{2,s}(\alpha |u|^{\frac{2}{2-s}}) \, dx < +\infty. \tag{2.1}$$

Moreover, there exists $\alpha_* \in (0, +\infty)$ such that for $0 \leq \alpha < \alpha_*$,

$$\sup_{u \in W^{s,\frac{2}{s}}(\mathbb{R}^2), \, \|u\|_{W^{s,\frac{2}{s}}(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \Phi_{2,s}(\alpha |u|^{\frac{2}{2-s}}) \, dx < +\infty.$$

Hence

$$\alpha_* := \sup \left\{ \alpha : \sup_{u \in W^{s,\frac{2}{s}}(\mathbb{R}^2), \, \|u\|_{W^{s,\frac{2}{s}}(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \Phi_{2,s}(\alpha |u|^{\frac{2}{2-s}}) \, dx < +\infty \right\}.$$

Moreover, for $\alpha > \alpha_{s,2}^*$,

$$\sup_{u \in W^{s,\frac{2}{s}}(\mathbb{R}^2),\, \|u\|_{W^{s,\frac{2}{s}}(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \Phi_{2,s}(\alpha |u|^{\frac{2}{2-s}})\, dx = +\infty \,,$$

where

$$\alpha_{s,2}^* := 2^{\frac{4+s}{2-s}} \left(\pi^2 \Gamma(1 + \tfrac{2}{s}) \sum_{k=0}^{+\infty} \frac{1}{(1+k)^{2/s-1}} \right)^{\frac{s}{2-s}}.$$

Remark 2.2. As remarked in [36,46], the result of Proposition 2.1 is not sharp in the sense of Moser, since $\alpha_{s,2}^*$ is just an upper bound for the sharp exponent α_*, while obtaining the precise value of α_* is still an open problem.

Let us point out some immediate consequences of the assumptions (f_1)–(f_5):

Remark 2.3.

(i) From (f_1) and (f_2) it is easy to infer that there exists a constant C such that

$$F(t) \leq C \left(t^{\frac{2}{s}} + t^{\frac{2}{s}} \Phi_{2,s}(\alpha_* |t|^{\frac{2}{2-s}}) \right) \quad \text{for all } t > 0\,;$$

(ii) Assumption (f_3) implies that f is monotone increasing. Moreover,

$$\frac{d}{dt} \frac{F(t)}{f(t)} = \frac{f^2(t) - F(t) f'(t)}{f^2(t)} \leq s - \tau \,,$$

from which one infers

$$F(t) \leq (s - \tau) t f(t) \quad \text{for any } t \geq 0 \,,$$

a Ambrosetti-Rabinowitz type condition. The upper bound in (f_3) is fulfilled by nonlinearities of the kind $t \mapsto e^{t^\alpha}$ with $\alpha > 1$, in particular by the critical growth $t \mapsto e^{t^{\frac{2}{2-s}}}$;

(iii) Assumption (f_4) requires that the function f grows exponentially at infinity. Moreover, from (f_4) one may deduce that for any $\varepsilon > 0$ there exists M_ε such that for $t \geq M_\varepsilon$

$$F(t) \leq \varepsilon t f(t)\,. \tag{2.2}$$

Indeed, there exists $N_\varepsilon > 0$ such that $\frac{d}{ds} \frac{F(s)}{f(s)} < \frac{\varepsilon}{2}$ for all $s > N_\varepsilon$, and integrating over $[N_\varepsilon, t]$ one gets

$$\frac{F(t)}{f(t)} - \frac{F(N_\varepsilon)}{f(N_\varepsilon)} = \int_{N_\varepsilon}^{t} \frac{d}{ds} \frac{F(s)}{f(s)} \, ds < \frac{\varepsilon}{2}(t - N_\varepsilon)\,.$$

Then (2.2) follows, by restricting to $t > M_\varepsilon \geq N_\varepsilon$, where M_ε is chosen such that $\frac{F(N_\varepsilon)}{f(N_\varepsilon)} - \varepsilon N_\varepsilon < \frac{\varepsilon}{2} M_\varepsilon$;

(iv) (f_5) is a condition about middle-range values of $f(t)$, since by (f_4) it is automatically verified at infinity. It can be related to the De Figueiredo-Miyagaki-Ruf condition [22], and is crucial in order to estimate the mountain pass level and gain compactness, see Lemma 5.5. We point out that the fractional analogue of the Moser sequence, usually exploited to estimate the mountain pass level, is here spoiled by the fact that it is not known whether the value α_* coincides with $\alpha_{s,2}$ in Proposition 2.1. Assumption (f_5), combined with the choice of a simple fixed test function, which allows explicit computations of its seminorm, will enable us to give an explicit estimate of the mountain pass level.

(v) Finally let us give an example of nonlinearity satisfying (f_1)–(f_5). Let $A, B, t_* > 0$ and define

$$F(t) = \begin{cases} A\, t^{\frac{2}{s}+1} & \text{for } t \in (0, t_*), \\ B\, e^{t^{\frac{2}{2-s}}} & \text{for } t \geq t_*, \end{cases}$$

and extend it by 0 on $\mathbb{R}^-$. Note that F has critical growth at ∞, and that assumptions (f_1) and (f_2) are readily satisfied for any choice of the constants. Moreover, for $t \in (0, t_*)$ one has

$$\frac{F(t)f'(t)}{f^2(t)} = 1 - \frac{s}{2+s},$$

which is bigger than or equal to $1 - s + \tau$, e.g. with the choice $\tau = \tau(s) := \frac{s(1+s)}{2+s}$. Hence the lower bound in (f_3) is satisfied on $(0, t_*)$. Assumption (f_4) is also easily verified, since

$$\lim_{t \to +\infty} \frac{F(t)f'(t)}{f^2(t)} = \lim_{t \to +\infty} \left(1 + \frac{s}{2} t^{-\frac{2}{2-s}}\right) = 1.$$

By the same computation we also note that the lower bound in (f_3) is trivially satisfied on $(t_*, +\infty)$, whereas the upper bound in (f_3) holds, provided

$$\frac{s}{2} t^{-\frac{2}{2-s}} < \mu(s, \tau(s)),$$

with $\mu(s, \tau)$ given in Lemma 5.8, that is, for $t > t_*$ where

$$t_* := \left(\frac{s}{2\mu(s, \tau(s))}\right)^{\frac{2-s}{2}}.$$

In order for $f = F'$ to be continuous on $\mathbb{R}$, one needs to impose a linear dependence between the constants A and B. Finally, (f_5) is also satisfied, provided one chooses A and B large enough. This is a working example, however, let us point out that these restrictions on constants can be relaxed.

For $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^2$ we define

$$W_0^{s, \frac{2}{s}}(\Omega) := \left\{ u \in W^{s, \frac{2}{s}}(\mathbb{R}^2) \mid u|_{\mathbb{R}^2 \setminus \Omega} \equiv 0 \right\}.$$

The subspace of radial functions in $W^{s, \frac{2}{s}}(\mathbb{R}^2)$ is defined as

$$W_r^{s, \frac{2}{s}}(\mathbb{R}^2) := \{u \in W^{s, \frac{2}{s}}(\mathbb{R}^2) \mid u(x) = u(|x|)\}.$$

The following compactness result due to Lions [31] will be crucial in our analysis.

Lemma 2.4. *Let $s \in (0, 1]$. Then $W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ is compactly embedded in $L^q(\mathbb{R}^2)$ for any $\frac{2}{s} < q < \infty$.*

Let us also recall the well-known Hardy-Littlewood-Sobolev inequality, see [30, Theorem 4.3], which will be frequently used throughout the paper.

Lemma 2.5. *(Hardy-Littlewood-Sobolev inequality) Let $N \geq 1$, $q, r > 1$, and $\alpha \in (0, N)$ with $\frac{1}{q} + \frac{\alpha}{N} + \frac{1}{r} = 2$. There exists a constant $C = C(N, \alpha, q, r)$ such that for all $f \in L^q(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ one has*

$$\int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * f(x)\right) h(x) \, dx \leq C \|f\|_q \|h\|_r.$$

The following technical lemma will be useful in the sequel,

Lemma 2.6. *Assume $u \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$ and (f_1)-(f_2) hold. Then for any $0 < \kappa \leq 1$ we have*

$$\int\limits_{\mathbb{R}^2} \frac{F(u(y))}{|x - y|^\kappa} \, dy - \frac{1}{|x|^\kappa} \int\limits_{\mathbb{R}^2} F(u(y)) \, dy \to 0, \quad as \ |x| \to +\infty.$$

Proof. First, we have $F(u) \in L^1(\mathbb{R}^2)$ by Remark 2.3(i) and Proposition 2.1. Consider now $|x| \geq 1$ and note that

$$\int\limits_{\mathbb{R}^2} \frac{F(u(y))}{|x - y|^\kappa} \, dy - \frac{1}{|x|^\kappa} \int\limits_{\mathbb{R}^2} F(u(y)) \, dy = \int\limits_{\mathbb{R}^2} \left(\frac{1}{|x - y|^\kappa} - \frac{1}{|x|^\kappa}\right) F(u(y)) \, dy.$$

For $x, y \in \mathbb{R}^2$ define $g(x, y) := \frac{1}{|x-y|^\kappa} - \frac{1}{|x|^\kappa}$. Observe that $g(x, y) \to 0$ as $|x| \to +\infty$ for every $y \in \mathbb{R}^2$. Moreover,

$$-1 \leq g(x, y) \chi_{|x-y| \geq \frac{1}{2}}(y) \leq 2^\kappa \quad \text{for } x, y \in \mathbb{R}^2 \text{ with } |x| \geq 1.$$

Hence, Lebesgue's theorem implies that

$$\int\limits_{|y-x| \geq \frac{1}{2}} g(x, y) F(u(y)) \, dy \to 0 \quad \text{as } |x| \to +\infty.$$

Moreover, we have

$$0 \le \frac{1}{|x|^\kappa} \int\limits_{|x-y| \le \frac{1}{2}} F(u(y)) \, \mathrm{d}y \le \frac{1}{|x|^\kappa} \int\limits_{\mathbb{R}^2} F(u(y)) \, \mathrm{d}y \to 0 \quad \text{as } |x| \to +\infty \,,$$

and

$$0 \le \int\limits_{|x-y| \le \frac{1}{2}} \frac{1}{|x - y|^\kappa} F(u(y)) \, \mathrm{d}y$$

$$\le \left(\int\limits_{B_{\frac{1}{2}}(x)} \frac{1}{|y|^{\kappa\sigma}} \, \mathrm{d}y \right)^{\frac{1}{\sigma}} \left(\int\limits_{\mathbb{R}^2} F(u(y))^{\sigma'} \, \mathrm{d}y \right)^{\frac{1}{\sigma'}} \to 0 \quad \text{as } |x| \to +\infty \,, \tag{2.3}$$

by Theorem 2.1, for $\sigma > 1$ large enough and $\sigma' = \frac{\sigma}{\sigma-1}$. Based on the above estimates, we infer that

$$\int\limits_{\mathbb{R}^2} g(x, y) F(u(y)) \, \mathrm{d}y \to 0 \quad \text{as } |x| \to \infty \,. \quad \square$$

3. On the relationship between the nonlocal Choquard equation and the Schrödinger-Poisson system: proof of Theorem 1.4

It is commonly used in the literature [2,11,15,16,23] that solutions u of the Choquard equation (Ch$_s$) correspond to solutions (u, ϕ) of the Schrödinger-Poisson system (SP$_s$) by formally inverting the Laplace operator in the second equation and considering the only solution ϕ given by the convolution of the nonlinearity with the Riesz kernel, namely $\phi = \phi_u := I_N * F(u)$. However, the Sobolev limiting case $N = sp$ is affected by regularity issues which have roots in the lack of regularity of the Fourier transform, see [27]. This turns out to be a quite delicate situation which we redeem in Theorem 1.4 where we settle a suitable functional framework in order to obtain solutions for (SP$_s$) from solutions of the Choquard equation (Ch$_s$). The complete equivalence between (SP$_s$) and (Ch$_s$) is still open even in the non fractional case, see [8].

As a first step to prove Theorem 1.4, we show that solutions to (Ch$_s$) are bounded by means of a Nash-Moser iteration argument in the spirit of [3,34]. This fact, together with a suitable polynomial estimate of the decay at infinity, yields a log-weighted L^1-estimate for $F(u)$, which is the crucial ingredient to prove Theorem 1.4. Then, we end up with $C^{0,\nu}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\nu \in (0, 1)$, by standard regularity results. Throughout this section we assume (f_1)-(f_2).

Remark 3.1. Note that any nontrivial weak solution $u \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$ of (Ch$_s$) is positive. Indeed, take $u^- := \min\{u, 0\} \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$ as test function in (1.2), to get

$$0 = \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \frac{|u(x) - u(y)|^{\frac{2}{s}-2} (u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y + \int\limits_{\mathbb{R}^2} |u|^{\frac{2}{s}-2} u u^- \, \mathrm{d}x$$

$$- \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(\int\limits_{\mathbb{R}^2} \ln \frac{1}{|x - y|} F(u(y)) \, \mathrm{d}y \right) f(u(x)) u^-(x) \, \mathrm{d}x$$

$$\geq \iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|u^-(x) - u^-(y)|^{\frac{2}{s}}}{|x-y|^4} \, dx \, dy + \int_{\mathbb{R}^2} |u^-|^{\frac{2}{s}} \, dx \,,$$

which implies that $u^- \equiv 0$. Hence, using the strong maximum principle for the p-fractional Laplacian [19, Theorem 1.4] we can infer that $u > 0$ in $\mathbb{R}^2$.

Lemma 3.2. *Let $u \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ be a positive solution of the Choquard equation* (Ch$_s$)*, then $u \in L^\infty(\mathbb{R}^2)$.*

Proof. Since $f(u) \geq 0$, for a nonnegative test function $\varphi \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ we have

$$
\begin{aligned}
\iint_{\mathbb{R}^2 \mathbb{R}^2} &\frac{|u(x) - u(y)|^{\frac{2}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^4} \, dx \, dy + \int_{\mathbb{R}^2} |u|^{\frac{2}{s}-2} u\varphi \, dx \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(u(y)) \, dy \right) f(u(x))\varphi(x) \, dx \\
&\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{F(u(y))}{|x-y|} \, dy \right) f(u(x))\varphi(x) \, dx \\
&\leq C \int_{\mathbb{R}^2} f(u(x))\varphi(x) \, dx \,,
\end{aligned}
\tag{3.1}
$$

where in the last inequality we have used the fact that $\frac{1}{|\cdot|} * F(u)$ is bounded, which follows arguing as in [1, Lemma 4.1]. For any $L > 0$ and $\gamma > 1$ define

$$\psi(t) := t(\min\{t,L\})^{\frac{2}{s}(\gamma-1)}, \quad \Psi(t) := \int_0^t (\psi'(\tau))^{\frac{s}{2}} \, d\tau, \quad K(t) := \frac{s}{2}|t|^{\frac{2}{s}} \tag{3.2}$$

for $t > 0$. By Jensen's inequality it is possible to show that

$$|\Psi(a) - \Psi(b)|^{\frac{2}{s}} \leq K'(a-b)(\psi(a) - \psi(b)) \tag{3.3}$$

holds for any $a, b \in \mathbb{R}^+$. Therefore

$$
\begin{aligned}
|\Psi(u(x)) - \Psi(u(y))|^{\frac{2}{s}} \leq |u(x) - u(y)|^{\frac{2}{s}-2}(u(x)-u(y))[u(x)u_L(x)^{\frac{2}{s}(\gamma-1)} \\
- u(y)u_L(y)^{\frac{2}{s}(\gamma-1)}],
\end{aligned}
$$

where $u_L := \min\{u, L\}$. Taking $\psi(u) = u u_L^{\frac{2}{s}(\gamma-1)}$ as test function in (3.1), we obtain

$$[\Psi(u)]_{s,\frac{2}{s}}^{\frac{2}{s}} \le \iint\limits_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|u(x)-u(y)|^{\frac{2}{s}-2}(u(x)-u(y))[u(x)u_L(x)^{\frac{2}{s}(\gamma-1)}-u(y)u_L(y)^{\frac{2}{s}(\gamma-1)}]}{|x-y|^4}\,\mathrm{d}x\,\mathrm{d}y$$

$$\le C\int\limits_{\mathbb{R}^2} f(u)uu_L^{\frac{2}{s}(\gamma-1)}\,\mathrm{d}x - \int\limits_{\mathbb{R}^2} |u|^{\frac{2}{s}}u_L^{\frac{2}{s}(\gamma-1)}\,\mathrm{d}x\,.$$

$$(3.4)$$

Recall that from (f_1)-(f_2) we have that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$f(u) \le \varepsilon|u|^{\frac{2}{s}-1} + C_\varepsilon|u|^{\frac{2}{s}-1}\Phi_{2,s}(\alpha_*|u|^{\frac{2}{2-s}})\,.$$

Hence, from (3.4) we infer

$$[\Psi(u)]_{s,\frac{2}{s}}^{\frac{2}{s}} \le (C\varepsilon-1)\int\limits_{\mathbb{R}^2} |u|^{\frac{2}{s}}u_L^{\frac{2}{s}(\gamma-1)}\,\mathrm{d}x + C_\varepsilon\int\limits_{\mathbb{R}^2} \Phi_{2,s}(\alpha_*|u|^{\frac{2}{2-s}})|u|^{\frac{2}{s}}u_L^{\frac{2}{s}(\gamma-1)}\,\mathrm{d}x$$

$$\le (C\varepsilon-1)\int\limits_{\mathbb{R}^2} |u|^{\frac{2}{s}}u_L^{\frac{2}{s}(\gamma-1)}\,\mathrm{d}x$$

$$+ C_\varepsilon\left(\int\limits_{\mathbb{R}^2} |u|^{\frac{2p}{s}}u_L^{\frac{2p}{s}(\gamma-1)}\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int\limits_{\mathbb{R}^2} \Phi_{2,s}(\alpha_*p'|u|^{\frac{2}{2-s}})\,\mathrm{d}x\right)^{\frac{1}{p'}}$$

$$\le (C\varepsilon-1)\|uu_L^{\gamma-1}\|_{\frac{2}{s}}^{\frac{2}{s}} + C_\varepsilon\|uu_L^{\gamma-1}\|_{\frac{2p}{s}}^{\frac{2}{s}}\,,$$

$$(3.5)$$

for $p > 1$ and $p' = \frac{p}{p-1}$ by (f_2) and Theorem 2.1. By (3.3) one has $\Psi(u) \le uu_L^{\gamma-1}$. Moreover,

$$\Psi(u) \ge \frac{1}{\gamma}uu_L^{\gamma-1}\,,$$

due to the definition of Ψ. Indeed, on the one hand if $u(x) < L$ one has $\psi(u)(x) = u(x)^{\frac{2}{s}(\gamma-1)}$, thus

$$\Psi(u)(x) = \left(\frac{2}{s}(\gamma-1)+1\right)^{\frac{s}{2}}\int\limits_0^{u(x)} t^{\gamma-1}\,\mathrm{d}t = \frac{\left(\frac{2}{s}(\gamma-1)+1\right)^{\frac{s}{2}}}{\gamma}u(x)^\gamma \ge \frac{1}{\gamma}u(x)u_L(x)^{\gamma-1}.$$

On the other hand, if $u(x) > L$, one has

$$\Psi(u)(x) = \int\limits_0^L (\psi'(t))^{\frac{s}{2}}\,\mathrm{d}t + \int\limits_L^{u(x)} (\psi'(t))^{\frac{s}{2}}\,\mathrm{d}t = \frac{\left(\frac{2}{s}(\gamma-1)+1\right)^{\frac{s}{2}}}{\gamma}L^\gamma + L^{\gamma-1}(u(x)-L)$$

$$\ge \frac{L^\gamma}{\gamma} + \frac{L^{\gamma-1}}{\gamma}(u(x)-L) = \frac{1}{\gamma}u(x)u_L(x)^{\gamma-1}.$$

Hence, taking $\varepsilon = \frac{1}{2C}$ in (3.5) and using the Sobolev embedding $W^{s,\frac{2}{s}}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ with $q > \frac{2p}{s}$, one obtains

$$\frac{S_q}{\gamma^{\frac{2}{s}}} \|uu_L^{\gamma-1}\|_{\frac{2}{s}}^{\frac{2}{s}} \leq C \|uu_L^{\gamma-1}\|_{\frac{2p}{s}}^{\frac{2}{s}},$$

where S_q is the best constant of the embedding inequality (see [9] for related results). Letting $L \to +\infty$, we immediately deduce

$$\|u\|_{q\gamma}^{\frac{2}{s}} \leq C \frac{\gamma^{\frac{2}{s}}}{S_q} \|u\|_{\frac{2p\gamma}{s}}^{\frac{2}{s}}. \tag{3.6}$$

Finally, buying the line of [34, Lemma 21], we get $\|u\|_\infty \leq C$. □

Next, in order to prove a crucial decay estimate, let us recall a standard lemma from [14, Lemma 5.1].

Lemma 3.3. *For $G(t) = |t|^{\frac{2}{s}-2}t$, it is well-known that by the mean value theorem, we have*

$$G(t_2) - G(t_1) = G'(\xi)(t_2 - t_1)$$

for some $\xi \in [t_1, t_2]$. Then there exists a constant $c_0 > 0$ independent of t_1 and t_2 such that

$$|\xi| \geq c_0 \max\{|t_1|, |t_2|\}.$$

Lemma 3.4. *Let $u \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ be a positive solution of the Choquard equation (Ch_s). Then there exist $C, R > 0$ for which*

$$u(x) \leq \frac{C}{1 + |x|^{\frac{7s}{2(2-s)}}} \qquad \text{for all } |x| > R.$$

Proof. We first claim that

$$\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(u(y)) \, \mathrm{d}y \leq \ln \frac{2}{|x|} \int_{\left\{|y| \leq \frac{|x|}{2}\right\}} F(u(y)) \, \mathrm{d}y + C_0, \tag{3.7}$$

where C_0 is a constant independent of x. Indeed, denoting by

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 \ : \ |y| > \frac{|x|}{2}, \ |x - y| < 1\right\},$$

we have

$$\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(u(y)) \, dy \leq \int_{|y| \leq \frac{|x|}{2}} \ln \frac{2}{|x|} F(u(y)) \, dy + \int_{\Omega} \ln \frac{1}{|x-y|} F(u(y)) \, dy$$

$$\leq \int_{|y| \leq \frac{|x|}{2}} \ln \frac{2}{|x|} F(u(y)) \, dy + C \int_{|x-y| < 1} \ln \frac{1}{|x-y|} \, dy \,,$$

with C depending on $\|u\|_\infty$, from which (3.7) follows. Hence, there exists $R_1 > 0$ such that

$$\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(u(y)) \, dy < 0 \quad \text{for all } |x| \geq R_1 \,.$$

Hence, taking $\varphi \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ such that $\varphi \geq 0$ and $\operatorname{supp} \varphi \subset B_{R_1}(0)^c$, we deduce

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^{\frac{2}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^4} \, dx \, dy + \int_{\mathbb{R}^2} |u|^{\frac{2}{s}-2} u\varphi \, dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(u(y)) \, dy \right) f(u(x))\varphi(x) \, dx \leq 0. \tag{3.8}$$

Defining now $w(x) := \left(1 + |x|^{\frac{7s}{2(2-s)}}\right)^{-1}$, by [20, Lemma 7.1] there exists $R_2 > 0$ such that

$$\left|(-\Delta)^s_{\frac{2}{s}} w(x)\right| \leq \frac{c}{|x|^4} \quad \text{for} \quad |x| \geq R_2$$

and hence

$$(-\Delta)^s_{\frac{2}{s}} w + |w|^{\frac{2}{s}-2} w \geq 0, \quad \text{for} \quad |x| \geq \widetilde{R}_2 \tag{3.9}$$

for a suitable $\widetilde{R}_2 \geq R_2$. Note that $(-\Delta)^s_{\frac{2}{s}} w$ is pointwisely well-defined since $w \in C^{1,1}_{\text{loc}}(\mathbb{R}^2) \cap L_{s,\frac{2}{s}}(\mathbb{R}^2)$. Since $u \in L^\infty(\mathbb{R}^2)$ by Lemma 3.2, one may find $C_1 > 0$, depending on $\|u\|_\infty$, for which

$$\psi(x) := u(x) - C_1 w(x) \leq 0 \quad \text{for} \quad |x| = R_3 := \max\{R_1, \widetilde{R}_2\}.$$

Defining $\widetilde{\psi} := \psi \chi_{\mathbb{R}^2 \setminus B_{R_3}(0)}$ and taking $\widetilde{\psi}^+ = \max\{\widetilde{\psi}, 0\} \in W_0^{s,\frac{2}{s}}(B_{R_3}(0)^c)$ as a test function in (3.8), and noting that by homogeneity the inequality (3.9) holds also for $\bar{w} := C_1 w$, we deduce that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{h(x,y)(\widetilde{\psi}^+(x)-\widetilde{\psi}^+(y))}{|x-y|^4} \, dx \, dy + \int_{\mathbb{R}^2} \left(|u|^{\frac{2}{s}-2} u - |\bar{w}|^{\frac{2}{s}-2}\bar{w}\right) \widetilde{\psi}^+ \, dx \leq 0, \tag{3.10}$$

where

$$h(x, y) := |u(x) - u(y)|^{\frac{2}{s}-2}(u(x) - u(y)) - |\bar{w}(x) - \bar{w}(y)|^{\frac{2}{s}-2}(\bar{w}(x) - \bar{w}(y)).$$

Note that by Lemma 3.3 one has

$$\iint\limits_{\mathbb{R}^2\,\mathbb{R}^2} \frac{h(x, y)(\widetilde{\psi}^+(x) - \widetilde{\psi}^+(y))}{|x - y|^4}\, \mathrm{d}x\, \mathrm{d}y$$

$$\geq \iint\limits_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|\bar{h}(x, y)|^{\frac{2}{s}-2}(\widetilde{\psi}(x) - \widetilde{\psi}(y)))(\widetilde{\psi}^+(x) - \widetilde{\psi}^+(y))}{|x - y|^4}\, \mathrm{d}x\, \mathrm{d}y \geq 0,$$

where $\bar{h}(x, y)$ lies in the segment $[(u(x) - u(y)), (\bar{w}(x) - \bar{w}(y))]$. By (3.10) this implies

$$0 \leq \int\limits_{B_{R_3}(0)^c} \left(|u|^{\frac{2}{s}-2}u - |\bar{w}|^{\frac{2}{s}-2}\bar{w}\right)(u - \bar{w})^+ \mathrm{d}x \leq 0.$$

As a consequence we get

$$u(x) \leq C_1 w(x) \leq \frac{C_1}{1 + |x|^{\frac{7s}{2(2-s)}}} \quad \text{for} \quad |x| \geq R_3,$$

as desired. □

Lemma 3.5. *Let* $u \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$ *be a positive solution of* (Ch$_s$). *Then*

$$\int\limits_{\mathbb{R}^2} \ln(e + |x|) F(u)\, \mathrm{d}x < +\infty. \tag{3.11}$$

Proof. Lemma 3.4 and (f_2) imply that there exists $R > 0$ for which

$$F(u(x)) \leq C\left[1 + |x|^{\frac{7s}{2(2-s)}}\right]^{-\frac{2}{s}}$$

as $|x| > R$. Therefore, since $\|u\|_\infty \leq C$ by Lemma 3.2, one has

$$\int\limits_{\mathbb{R}^2} \ln(e + |x|) F(u)\, \mathrm{d}x = \int\limits_{|x|<R} \ln(e + |x|) F(u)\, \mathrm{d}x + \int\limits_{|x|\geq R} \ln(e + |x|) F(u)\, \mathrm{d}x$$

$$\leq C \int\limits_{|x|<R} \ln(e + |x|)\, \mathrm{d}x + C \int\limits_{|x|\geq R} \ln(e + |x|)\left[1 + |x|^{\frac{7s}{2(2-s)}}\right]^{-\frac{2}{s}}\, \mathrm{d}x$$

$$\leq C + C \int\limits_{|x|\geq R} |x| \cdot |x|^{-\frac{7}{2-s}}\, \mathrm{d}x < +\infty. □$$

We are now in the position to prove Theorem 1.4.

Proof of Theorem 1.4. We already proved that $u \in L^\infty(\mathbb{R}^2)$ in Lemma 3.2. To show that $\phi_u \in L_{\frac{1}{2}}(\mathbb{R}^2)$ we compute as follows:

$$2\pi \int_{\mathbb{R}^2} \frac{|\phi_u(x)|}{1+|x|^3} \, dx \leq \int_{\mathbb{R}^2} F(u(y)) \left(\int_{\mathbb{R}^2} \left| \ln \left(\frac{1}{|x-y|} \right) \right| \frac{1}{1+|x|^3} \, dx \right) dy$$

$$\leq \int_{\mathbb{R}^2} F(u(y)) \left(\int_{|x-y|>1} \frac{\ln|x-y|}{1+|x|^3} \, dx + \int_{|x-y|\leq 1} \ln \frac{1}{|x-y|} \, dx \right) dy$$

$$\leq \int_{\mathbb{R}^2} F(u(y)) \, dy \int_{\mathbb{R}^2} \frac{\ln(1+|x|)}{1+|x|^3} \, dx + \int_{\mathbb{R}^2} \ln(1+|y|) F(u(y)) \, dy \int_{\mathbb{R}^2} \frac{dx}{1+|x|^3}$$

$$+ \|F(u)\|_{L^1(\mathbb{R}^2)} \|\ln(\cdot)\|_{L^1(B_1(0))} < +\infty,$$

using the elementary estimate $\ln|x-y| \leq \ln(1+|x|) + \ln(1+|y|)$ for $|x-y| > 1$, and Lemma 3.5.

In order to show that (u, ϕ_u) solves (SP_s), we follow the approach of [8, Theorem 2.1]. By [27, Lemma 2.3] the function

$$\widetilde{v}_u(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \left(\frac{1+|y|}{|x-y|} \right) F(u(y)) \, dy$$

belongs to $H^1_{\text{loc}}(\mathbb{R}^2)$ and solves $-\Delta \widetilde{v}_u = F(u)$ in $\mathbb{R}^2$ in the sense of Definition 1.1. Moreover, by [27, Lemma 2.4] we know that all such solutions of $-\Delta \phi = F(u)$ in $\mathbb{R}^2$ are of the form $\phi = \widetilde{v}_u + p$ with p polynomial of degree at most one. We aim to prove that ϕ_u solves the Poisson equation in (SP_s) by showing that $\widetilde{v}_u - \phi_u$ is constant. Indeed,

$$\widetilde{v}_u(x) - \phi_u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\ln \left(\frac{1+|y|}{|x-y|} \right) - \ln \left(\frac{1}{|x-y|} \right) \right) F(u(y)) \, dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(1+|y|) F(u(y)) \, dy < +\infty$$

by Lemma 3.5. Finally, we prove the logarithmic behaviour (1.3) of ϕ_u at ∞ in the spirit of [15, Proposition 2.3 (ii)]. One has

$$2\pi \phi_u(x) + \ln|x| \int_{\mathbb{R}^2} F(u(y)) \, dy = \int_{\mathbb{R}^2} \ln \frac{|x|}{|x-y|} F(u(y)) \, dy =: - \int_{\mathbb{R}^2} l(x, y) F(u(y)) \, dy,$$

where $l(x, y) := \ln \frac{|x-y|}{|x|}$. Note that for $|x| \to +\infty$ one has $l(x, y) \to 0$ for any fixed $y \in \mathbb{R}^2$. Let now $|x| > 1$, then if $|x - y| \geq \frac{1}{2}$ one has

$$\ln \frac{1}{2} \le \ln \frac{1}{2} - \ln \frac{1}{|x|} \le l(x, y) \chi_{|x-y| \ge \frac{1}{2}}(y) \le \ln \left(\frac{|x| + |x||y|}{|x|} \right) \le \ln(1 + |y|).$$

This yields

$$\left| - \int_{|x-y| \ge \frac{1}{2}} l(x, y) F(u(y)) \, dy \right| \le \int_{\mathbb{R}^2} \max \{ \ln 2, \ln(1 + |y|) \} F(u(y)) \, dy$$

$$\le \| F(u) \|_{L^1(\mathbb{R}^2)} \ln 2 + \int_{\mathbb{R}^2} \ln(1 + |y|) F(u(y)) \, dy < +\infty,$$

by Lemma 3.5. By the dominated convergence theorem, we infer

$$\int_{|x-y| \ge \frac{1}{2}} l(x, y) F(u(y)) \, dy \to 0 \qquad \text{as } |x| \to +\infty. \tag{3.12}$$

On the other hand, recalling that $|x| > 1$, in the case $|x - y| \le \frac{1}{2}$, we have $|y| \ge \frac{|x|}{2}$, therefore

$$0 \le \int_{|x-y| \le \frac{1}{2}} \ln |x| F(u(y)) \, dy \le \int_{\mathbb{R}^2 \setminus B_{\frac{|x|}{2}}(0)} \ln(2(1 + |y|)) F(u(y)) \, dy \to 0$$

as $|x| \to +\infty$ for the same reason. Moreover, by (f_2) and Theorem 2.1 there exists $\beta > 1$ for which $F(u) \in L^\beta(\mathbb{R}^2)$, and we have that

$$0 \le - \int_{|x-y| \le \frac{1}{2}} \ln |x - y| F(u(y)) \, dy = \int_{B_{\frac{1}{2}}(0)} - \ln |z| F(u(x - z)) \, dz$$

$$\le \| \ln(\cdot) \|_{L^{\beta'}(B_{\frac{1}{2}}(0))} \| F(u) \|_{L^\beta(B_{\frac{1}{2}}(x))} \to 0$$

as $|x| \to +\infty$. Eventually we conclude that also

$$\int_{|x-y| \le \frac{1}{2}} l(x, y) F(u(y)) \, dy \to 0$$

as $|x| \to +\infty$, which together with (3.12) implies (1.3). $\square$

Finally, we prove that solutions of (Ch_s) are Hölder continuous. Following [21,28], for all measurable $u : \mathbb{R}^2 \to \mathbb{R}$, we define its $(s, \frac{2}{s})$-*non-local tail* centred at $x \in \mathbb{R}^2$ with radius $R > 0$ as

$$\text{Tail}(u; x, R) := R^2 \left(\int\limits_{\mathbb{R}^2 \setminus B_R(x)} \frac{|u(y)|^{\frac{2}{s}-1}}{|x-y|^4} \, dy \right)^{\frac{s}{2-s}}.$$

Lemma 3.6. *Let $u \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$ be a positive solution of (Ch$_s$), then $u \in C^{\nu}_{\text{loc}}(\mathbb{R}^2)$ for some $\nu \in (0, 1)$.*

Proof. First, as in the proof of Lemma 3.2, one has $(-\Delta)^s_{\frac{2}{s}} u \le \bar{C}$ weakly in $\mathbb{R}^2$ for some constant $\bar{C} > 0$. To prove the weak bound from below, take a test function $\varphi \ge 0$ and estimate

$$\int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \frac{|u(x) - u(y)|^{\frac{2}{s}-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^4} \, dx \, dy + \int\limits_{\mathbb{R}^2} |u|^{\frac{2}{s}-2} u \varphi \, dx$$

$$= \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(\int\limits_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(u(y)) \, dy \right) f(u(x)) \varphi(x) \, dx$$

$$\ge -\frac{1}{2\pi} \int\limits_{|x-y|\ge 1} \int\limits_{\mathbb{R}^2} \ln|x-y| F(u(y)) f(u(x)) \varphi(x) \, dx \, dy \tag{3.13}$$

$$\ge -\frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} (\ln(1 + |x|) + \ln(1 + |y|)) F(u(y)) f(u(x)) \varphi(x) \, dx \, dy.$$

Since $u \in L^{\infty}(\mathbb{R}^2)$ and by the decay estimate of Lemma 3.4 we have

$$\int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \ln(1 + |x|) F(u(y)) f(u(x)) \varphi(x) \, dx \, dy$$

$$\le C \int\limits_{\mathbb{R}^2} \frac{\ln(1 + |x|)\varphi(x)}{\left(1 + |x|^{\frac{7s}{2(2-s)}} \right)^{\frac{2}{s}-1}} \, dx \int\limits_{\mathbb{R}^2} \frac{dy}{\left(1 + |y|^{\frac{7s}{2(2-s)}} \right)^{\frac{2}{s}}}$$

$$\le C \int\limits_{\mathbb{R}^2} \frac{\ln(1 + |x|)\varphi(x)}{1 + |x|^{\frac{7}{2}}} \, dx \int\limits_{\mathbb{R}^2} \frac{dy}{1 + |y|^{\frac{7}{2-s}}} \le C \int\limits_{\mathbb{R}^2} \varphi,$$

and analogously

$$\int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \ln(1 + |y|) F(u(y)) f(u(x)) \varphi(x) \, dx \, dy \le C \int\limits_{\mathbb{R}^2} \frac{\varphi(x)}{1 + |x|^{\frac{7}{2}}} \, dx \int\limits_{\mathbb{R}^2} \frac{\ln(1 + |y|)}{1 + |y|^{\frac{7}{2-s}}} \, dx \le C \int\limits_{\mathbb{R}^2} \varphi.$$

Since $\int_{\mathbb{R}^2} |u|^{\frac{2}{s}-2} u \varphi \le C \int_{\mathbb{R}^2} \varphi$, this yields $(-\Delta)^s_{\frac{2}{s}} u \ge -C$ weakly. Finally, $\text{Tail}(u; x, R)$ is bounded uniformly in (x, R), since

$$\text{Tail}(u; x, R) \leq R^2 \|u\|_\infty \left(\int\limits_{\mathbb{R}^2 \setminus B_R(0)} \frac{dy}{|y|^4} \right)^{\frac{s}{2-s}} \leq C. \tag{3.14}$$

Then, by [28, Corollary 5.5] there exists a universal constant $\widetilde{C}$ and $\nu \in (0, 1)$ such that for all $x_0 \in \mathbb{R}^2$ one has

$$[u]_{C^\nu(B_R(x_0))} \leq \widetilde{C} R^{-\nu} \left((\bar{C} R^2)^{\frac{s}{2-s}} + \|u\|_{L^\infty(B_{2R}(x_0))} + \text{Tail}(u; x_0, 2R) \right) \leq C(R)$$

with $C(R)$ independent of the choice of x_0. This readily implies $u \in C^\nu_{\text{loc}}(\mathbb{R}^2)$. $\quad\square$

4. Symmetry of positive solutions: proof of Theorem 1.5

As a consequence of Theorem 1.4, we have that solutions u of (Ch$_s$) (which are positive, see Remark 3.1) correspond to solutions (u, ϕ) of (SP$_s$) which enjoy the following

$$u \in L^\infty(\mathbb{R}^2) \quad \text{and} \quad \phi(x) = \phi_u(x) = (I_2 * F(u))(x) \to -\infty \quad \text{as } |x| \to +\infty. \tag{4.1}$$

Next we are going to prove Theorem 1.5 and for this purpose we rely on the method of moving planes, which has been adapted to the p-fractional context by Chen and Li in [14], provided the operator $(-\Delta)^{\frac{s}{2}}_{\frac{2}{s}}$ is pointwisely defined. For this reason we require $u \in C^{1,1}_{\text{loc}}(\mathbb{R}^2) \cap W^{s, \frac{2}{s}}(\mathbb{R}^2)$, see [14, Lemma 5.2] and also [43].

For $\beta \in \mathbb{R}$, let us set

$$\Sigma_\beta := \{x \in \mathbb{R}^2 : x_1 < \beta\} \quad \text{and} \quad \partial\Sigma_\beta = \{x \in \mathbb{R}^2 : x_1 = \beta\}.$$

Moreover, for any $x \in \mathbb{R}^2$, denote by x^β the reflection of x with respect to $\partial\Sigma_\beta$, that is $x^\beta = (2\beta - x_1, x_2)$. Set also

$$u^\beta(x) := u(x^\beta) \quad \text{and} \quad \phi^\beta(x) := \phi(x^\beta) \quad \text{for } x \in \mathbb{R}^2.$$

Note that ϕ is a continuous function on $\mathbb{R}^2$ by (f_1) and Lemma 3.4. Defining

$$u_\beta := u^\beta - u \quad \text{and} \quad \phi_\beta := \phi^\beta - \phi,$$

the following holds

$$(-\Delta)^{\frac{s}{2}}_{\frac{2}{s}} u^\beta - (-\Delta)^{\frac{s}{2}}_{\frac{2}{s}} u + |u^\beta|^{\frac{2}{s}-2} u^\beta - |u|^{\frac{2}{s}-2} u = \phi_\beta f(u^\beta) + \phi h_\beta u_\beta, \quad x \in \Sigma_\beta, \tag{4.2}$$

where

$$h_\beta(x) := \begin{cases} \frac{f(u^\beta(x)) - f(u(x))}{u_\beta(x)}, & u_\beta(x) \neq 0, \\ f'(u(x)), & u_\beta(x) = 0. \end{cases}$$

Recalling that $f \in C^1(\mathbb{R})$, there exists $C = C(u) > 0$ such that $\|h_\beta\|_{L^\infty(\Sigma_\beta)} \le C$ for any $\beta \in \mathbb{R}$. Moreover,

$$-\Delta\phi_\beta = K_\beta(x)u_\beta, \quad x \in \Sigma_\beta, \tag{4.3}$$

where

$$K_\beta(x) := \begin{cases} \frac{F(u^\beta(x)) - F(u(x))}{u_\beta(x)}, & u_\beta(x) \ne 0, \\ f(u(x)), & u_\beta(x) = 0. \end{cases}$$

From the definition of ϕ, we deduce that

$$\phi_\beta(x) = \int_{\Sigma_\beta} \ln\frac{|x - y^\beta|}{|x - y|} K_\beta(y)u_\beta(y)\, dy. \tag{4.4}$$

Since $\frac{|x - y^\beta|}{|x - y|} > 1$ for any $x, y \in \Sigma_\beta$, we have $\phi_\beta \ge 0$ in Σ_β if $u_\beta \ge 0$ in Σ_β for every $\beta \in \mathbb{R}$. In what follows, for a function v we denote by $v^- := \min\{v, 0\}$ its negative part. Notice that ϕ^- is a nonpositive function with this convention. Actually we have the following lemma which is inspired by [14, Theorem 2.1].

Lemma 4.1. *If $\beta \in \mathbb{R}$ is such that $u_\beta \ge 0$ in Σ_β, then also $\phi_\beta \ge 0$ on Σ_β. Furthermore, either $u_\beta \equiv 0 \equiv \phi_\beta$ or $u_\beta > 0, \phi_\beta > 0$ on Σ_β.*

Proof. From the expression of ϕ_β in (4.4) we know that $u_\beta \ge 0$ in Σ_β implies $\phi_\beta \ge 0$ in Σ_β. In particular, $\phi_\beta > 0$ in Σ_β if $u_\beta \not\equiv 0$. Conversely, if $\phi_\beta \not\equiv 0$, then also $u_\beta \not\equiv 0$ again using (4.4). We *claim*:

$$u_\beta > 0 \qquad \text{for} \quad x \in \Sigma_\beta. \tag{4.5}$$

Indeed, if not, there exists $x_0 \in \Sigma_\beta$ such that $u_\beta(x_0) = \min_{\Sigma_\beta} u_\beta = 0$. Let $G(t) = |t|^{\frac{2}{s}-2}t$, then $G(t)$ is a strictly increasing function, and $G'(t) = (\frac{2}{s} - 1)t^{\frac{2}{s}-2} \ge 0$. On the one hand, a direct computation using the definition of the $(s, \frac{2}{s})$-fractional operator yields

$$(-\Delta)^s_{\frac{2}{s}} u^\beta(x_0) - (-\Delta)^s_{\frac{2}{s}} u(x_0) + |u^\beta(x_0)|^{\frac{2}{s}-2}u^\beta(x_0) - |u(x_0)|^{\frac{2}{s}-2}u(x_0)$$

$$= C \cdot PV \int_{\mathbb{R}^2} \frac{G(u^\beta(x_0) - u^\beta(y)) - G(u(x_0) - u(y))}{|x - y|^4}\, dy$$

$$= C \cdot PV \int_{\Sigma_\beta} \frac{G(u^\beta(x_0) - u^\beta(y)) - G(u(x_0) - u(y))}{|x - y|^4}\, dy$$

$$+ C \cdot PV \int_{\Sigma_\beta} \frac{G(u^\beta(x_0) - u(y)) - G(u(x_0) - u(y^\beta))}{|x - y^\beta|^4}\, dy \tag{4.6}$$

$$= C \cdot PV \int_{\Sigma_\beta} \left[\frac{1}{|x-y|^4} - \frac{1}{|x-y^\beta|^4} \right] \left(G(u^\beta(x_0) - u^\beta(y)) - G(u(x_0) - u(y)) \right) dy$$

$$+ C \cdot PV \int_{\Sigma_\beta} \frac{G'(\xi(y)) + G'(\eta(y))}{|x-y|^4} u_\beta(x_0) dy$$

$$= -C \cdot PV \int_{\Sigma_\beta} \left[\frac{1}{|x-y|^4} - \frac{1}{|x-y^\beta|^4} \right] G'(\theta(y)) u_\beta(y) dy \le 0,$$

where $\xi(y)$ lies between $u^\beta(x_0) - u^\beta(y)$ and $u(x_0) - u^\beta(y)$, $\eta(y)$ between $u^\beta(x_0) - u(y)$ and $u(x_0) - u(y)$, and $\theta(y)$ between $u^\beta(x_0) - u^\beta(y)$ and $u(x_0) - u(y)$. Here we have used the mean value theorem in the third and fourth identities, and the fact that

$$\frac{1}{|x-y|^4} > \frac{1}{|x-y^\beta|^4}, \quad \forall x, y \in \Sigma_\beta.$$

On the other hand, by assumption (f_1) and since $w_\beta(x_0) > 0$, observe that

$$\phi_\beta(x_0) f(u^\beta(x_0)) + \phi(x_0) h_\beta(x_0) u_\beta(x_0) = \phi_\beta(x_0) f(u^\beta(x_0)) > 0,$$

which implies by (4.2) that

$$(-\Delta)^s_{\frac{2}{s}} u^\beta(x_0) - (-\Delta)^s_{\frac{2}{s}} u(x_0) + |u^\beta(x_0)|^{\frac{2}{s}-2} u^\beta - |u(x_0)|^{\frac{2}{s}-2} u(x_0) > 0, \qquad (4.7)$$

which contradicts (4.6). □

Let us next recall a maximum principle for the p-fractional laplacian and a key boundary estimate lemma for anti-symmetric functions, both established in [14]. These results play an important role in carrying out the method of moving planes.

Lemma 4.2 (*A maximum principle for anti-symmetric functions*). *Let Ω be a bounded domain in Σ_β. Assume that $u \in C^{1,1}_{loc}(\mathbb{R}^2) \cap W^{s,\frac{2}{s}}(\mathbb{R}^2)$. If*

$$\begin{cases} (-\Delta)^s_{\frac{2}{s}} u^\beta(x) \ge (-\Delta)^s_{\frac{2}{s}} u(x) & x \in \Omega, \\ u^\beta(x) \ge u(x) & x \in \Sigma_\beta \setminus \Omega, \end{cases} \qquad (4.8)$$

then $u_\beta(x) := u^\beta(x) - u(x) \ge 0$ in Ω. If $u_\beta = 0$ at some point in Ω, then $u_\beta(x) = 0$ a.e. $x \in \mathbb{R}^2$. These conclusions hold for an unbounded region Ω if we further assume that

$$\lim_{|x| \to +\infty} u_\beta(x) \ge 0.$$

Lemma 4.3 (*A key boundary estimate*). *Assume that $u_{\beta_0} > 0$ for $x \in \Sigma_{\beta_0}$. Suppose $\beta_k \to \beta_0$, and $x^k \in \Sigma_{\beta_k}$, such that*

$$u_{\beta_k}(x^k) = \min_{\Sigma_{\beta_k}} u_{\beta_k} \leq 0 \qquad and \qquad x^k \to x_0 \in \partial \Sigma_{\beta_0}.$$

Let $\delta_k := \mathrm{dist}(x^k, \partial \Sigma_{\beta_k}) \equiv |\beta_k - x_1^k|$. Then

$$\limsup_{\delta_k \to 0} \frac{1}{\delta_k} \left\{ (-\Delta)_{\frac{s}{2}}^s u^\beta(x^k) - (-\Delta)_{\frac{s}{2}}^s u(x^k) \right\} < 0.$$

Lemma 4.4. *There exists $\beta < 0$ such that*

$$u_\beta(x) \geq 0, \quad for\ all\ x \in \Sigma_\beta. \tag{4.9}$$

Proof. Note that u satisfies (4.2). Suppose on the contrary that (4.9) is violated. Then, there exists $\beta_0 < 0$ such that for all $\beta < \beta_0$ one is always able to find $x_* \in \Sigma_\beta$ for which

$$u_\beta(x_*) = \min_{\Sigma_\beta} u_\beta < 0$$

holds First, observe that

$$0 \leq \ln \frac{|x_* - y^\beta|}{|x_* - y|} \leq \ln \left(1 + \frac{|y - y^\beta|}{|x_* - y|} \right) \leq \frac{|y - y^\beta|}{|x_* - y|} = \frac{2(\beta - y_1)}{|x_* - y|} \tag{4.10}$$

for any $x, y \in \Sigma_\beta$. Set

$$M_\beta := \{ u \in \Sigma_\beta : u_\beta < 0 \}.$$

Recalling that, by assumption (f_1), K_β is non-negative function. Based on the above facts, we deduce from the expression of ϕ_β in (4.4) that

$$\phi_\beta(x_*) := \int_{\Sigma_\beta} \ln \frac{|x_* - y^\beta|}{|x_* - y|} K_\beta(y) u_\beta(y) \, \mathrm{d}y$$

$$\geq u_\beta(x_*) \int_{M_\beta} \frac{2(\beta - y_1)}{|x_* - y|} K_\beta(y) \, \mathrm{d}y$$

$$\geq u_\beta(x_*) \int_{M_\beta} \frac{2(\beta - y_1)}{|x_* - y|} f(u(y)) \, \mathrm{d}y$$

$$\geq u_\beta(x_*) \int_{M_\beta \cap \{|x_* - y|\} > 1} \frac{2|y|}{|x_* - y|} f(u(y)) \, \mathrm{d}y$$

$$+ u_\beta(x_*) \int_{M_\beta \cap \{|x_* - y|\} \leq 1} \frac{2(\beta - y_1)}{|x_* - y|} f(u(y)) \, \mathrm{d}y \tag{4.11}$$

$$\geq u_\beta(x_*) \int\limits_{\Sigma_\beta} 2|y| f(u(y)) \mathrm{d}y$$

$$+ u_\beta(x_*) \left(\int\limits_{\{|x_*-y|\leq 1\}} \frac{\mathrm{d}y}{|x_* - y|^{\frac{3}{2}}} \right)^{\frac{2}{3}} \left(\int\limits_{\Sigma_\beta} |2(\beta - y_1) f(u(y))|^3 \, \mathrm{d}y \right)^{\frac{1}{3}}$$

$$=: u_\beta(x_*) c_\beta \,.$$

It is easy to check that c_β is finite and $c_\beta \to 0$ as $\beta \to -\infty$. Indeed $|y| f(u) \in L^1(\mathbb{R}^2)$ by (f_2) by Theorem 2.1, and the second integral is finite. Moreover, for $|\beta|$ large enough in $B_{|\beta|}(0)^c$ one has $|u(x)| \leq C \left(1 + |x|^{\frac{7s}{2(2-s)}} \right)$ by Lemma 3.4, and using (f_1) we obtain

$$\int\limits_{\Sigma_\beta} |2(\beta - y_1) f(u(y))|^3 \mathrm{d}y \leq C \int\limits_{B_{|\beta|}(0)^c} |\beta - y_1|^3 |y|^{\frac{21}{2}} \mathrm{d}y \leq C\beta^{-\frac{11}{2}} \to 0 \quad \text{as } \beta \to -\infty \,.$$

Moreover, by assumption (f_1), for any fixed $\varepsilon > 0$ and β sufficiently negative,

$$f(u^\beta(x_*)) \leq f(u(x_*)) \leq \varepsilon |u(x_*)|^{\frac{2}{s}-2} u(x_*) \,. \tag{4.12}$$

Thanks to (4.1) and the fact that h_β is a non-negative function, for sufficiently negative β,

$$\phi(x_*) h_\beta(x_*) u_\beta(x_*) \geq 0 \,. \tag{4.13}$$

Combining (4.11), (4.12), and (4.13) yields

$$\phi_\beta(x_*) f(u^\beta(x_*)) + \phi(x_*) h_\beta(x_*) u_\beta(x_*) \geq c_\beta |u(x_*)|^{\frac{2}{s}-2} u(x_*) u_\beta(x_*) \,. \tag{4.14}$$

It follows from Lemma 3.3 that there exists $c_o > 0$ such that

$$|u^\beta(x_*)|^{\frac{2}{s}-2} u^\beta(x_*) - |u(x_*)|^{\frac{2}{s}-2} u(x_*) \geq c_o |u(x_*)|^{\frac{2}{s}-2} u_\beta(x_*) \,. \tag{4.15}$$

By using (4.14) and (4.15) in (4.2) and β sufficiently negative so that $c_\beta u(x_*) < c_o$, we get

$$(-\Delta)_{\frac{2}{s}}^{\frac{s}{2}} u^\beta(x_*) - (-\Delta)_{\frac{2}{s}}^{\frac{s}{2}} u(x_*) > 0 \,. \tag{4.16}$$

Next, similar computations as in (4.6) yield

$$(-\Delta)^{\frac{s}{2}}_{\frac{s}{2}} u^\beta(x_*) - (-\Delta)^{\frac{s}{2}}_{\frac{s}{2}} u(x_*)$$

$$= C \cdot PV \int_{\Sigma_\beta} \left[\frac{1}{|x-y|^4} - \frac{1}{|x-y^\beta|^4} \right] \left(G(u^\beta(x_*) - u^\beta(y)) - G(u(x_*) - u(y)) \right) dy$$

$$+ C \cdot PV \int_{\Sigma_\beta} \frac{G'(\xi(y)) + G'(\eta(y))}{|x-y|^4} u_\beta(x_*) \, dy \tag{4.17}$$

$$\le C \cdot PV \int_{\Sigma_\beta} \left[\frac{1}{|x-y|^4} - \frac{1}{|x-y^\beta|^4} \right] \left(G(u^\beta(x_*) - u^\beta(y)) - G(u(x_*) - u(y)) \right) dy.$$

Since $G(t) = |t|^{\frac{2}{s}-2} t$ is a strictly increasing function, and

$$u^\beta(x_*) - u^\beta(y) - u(x_*) + u(y) = u_\beta(x_*) - u_\beta(y) \le 0,$$

we infer that

$$G(u^\beta(x_*) - u^\beta(y)) - G(u(x_*) - u(y)) \le 0.$$

This implies by (4.17) that

$$(-\Delta)^{\frac{s}{2}}_{\frac{s}{2}} u^\beta(x_0) - (-\Delta)^{\frac{s}{2}}_{\frac{s}{2}} u(x_*) \le 0,$$

which contradicts (4.16). □

Proof of Theorem 1.5. So far, Lemma 4.4 provides a starting point to move the plane $\partial \Sigma_\beta$. Now, let us move the plane to the right, as long as (4.9) holds, up to some limiting position. In particular, define

$$\beta_0 := \sup\{\beta \in \mathbb{R} \,|\, u_\mu(x) \ge 0, \, x \in \Sigma_\mu, \, \mu \le \beta\}. \tag{4.18}$$

Let us divide the proof into two steps:

Step 1. Let us prove that u is symmetric about the limiting plane Σ_{β_0}, that is,

$$u_{\beta_0} \equiv 0, \quad \text{for all } x \in \Sigma_{\beta_0}. \tag{4.19}$$

By contradiction, assume that (4.19) is false, that is to say, $u_{\beta_0}(x) \ge 0$ and $u_{\beta_0}(x) \ne 0$ for some $x \in \Sigma_{\beta_0}$. Moreover, it follows from (4.2) that

$$(-\Delta)^{\frac{s}{2}}_{\frac{s}{2}} u^{\beta_0} - (-\Delta)^{\frac{s}{2}}_{\frac{s}{2}} u + |u^{\beta_0}|^{\frac{2}{s}-2} u^{\beta_0} - |u|^{\frac{2}{s}-2} u - \phi h_{\beta_0} u_{\beta_0} \ge 0, \quad x \in \Sigma_{\beta_0}. \tag{4.20}$$

So, by the strong maximum principle (see Lemma 4.2) together with the fact that the map $t \mapsto |t|^{\frac{2}{s}-2} t$ is increasing and $t \mapsto \phi h_{\beta_0} t$ is linear, we have $u_{\beta_0} > 0$ and hence by (4.4) also

$$\phi_{\beta_0}(x) > 0, \quad \forall x \in \Sigma_{\beta_0}$$

by Lemma 4.1. Now, according to the definition of β_0, there exists a sequence $\beta_n \searrow \beta_0$, and $x^n \in \Sigma_{\beta_n}$ such that

$$u_{\beta_n}(x^n) = \min_{\Sigma_{\beta_n}} u_{\beta_n} < 0, \quad \text{and} \quad \nabla u_{\beta_n}(x^n) = 0. \tag{4.21}$$

Remind that $u \in C^1(\mathbb{R}^2)$. Up to subsequence, we *claim*:

$$x^n \to x^* \quad \text{as} \quad n \to +\infty. \tag{4.22}$$

Indeed, let $B_R := B_R(0)$ for $R > 0$, by (4.1), we can choose $R > 1$ large enough such that

$$\phi \le 0 \quad \text{in } \Sigma_{\beta_n} \setminus B_R \quad \text{for any } \beta_n \tag{4.23}$$

and

$$\left(\int_{|x^n - y| \le 1} \frac{dy}{|x^n - y|^{\frac{3}{2}}} \right)^{\frac{2}{3}} \left(\int_{\Sigma_{\beta_n} \setminus B_R} |2(\beta_n - y_1) f(u(y))|^3 dy \right)^{\frac{1}{3}} < \frac{c_o}{4},$$

$$\int_{\Sigma_{\beta_n} \setminus B_R} 2|y| f(u(y)) \, dy < \frac{c_o}{4} \quad \text{in } \Sigma_{\beta_n} \setminus B_R \quad \text{for any } \beta_n, \tag{4.24}$$

where c_o has appeared in (4.15). Assume by contradiction that there exists $N > 0$ such that $x^n \in \Sigma_{\beta_n} \setminus B_R$ for all $n > N$. Then by (4.23) and the definition of h_β, one has for $n > N$

$$\phi(x^n) h_{\beta_n}(x^n) u_{\beta_n}(x^n) \ge 0. \tag{4.25}$$

Moreover, by assumption (f_1), for any fixed $\varepsilon > 0$, taking N large enough,

$$f(u^{\beta_n}(x^n)) \le f(u(x^n)) \le \varepsilon |u(x^n)|^{\frac{2}{s}-2} u(x^n) \tag{4.26}$$

holds for $n > N$. Note that for $y \in B_R$ one has

$$\ln \frac{|x^n - y^{\beta_n}|}{|x^n - y|} \to 0 \quad \text{as } n \to \infty.$$

Set

$$M_{\beta_n} := \{u \in \Sigma_{\beta_n} : u_{\beta_n} < 0\}.$$

We deduce from (4.4), Lemma 3.4, (4.10), and (4.24) that for $n > N$ with N large enough

$$\phi_{\beta_n}(x^n) := \int\limits_{\Sigma_{\beta_n}} \ln\frac{|x^n - y^{\beta_n}|}{|x^n - y|} K_{\beta_n}(y)u_{\beta_n}(y)\,dy$$

$$\geq \int\limits_{M_{\beta_n}\cap B_R} \ln\frac{|x^n - y^{\beta_n}|}{|x^n - y|} K_{\beta_n}(y)u_{\beta_n}(y)\,dy$$

$$+ \int\limits_{M_{\beta_n}\setminus B_R} \frac{2(\beta_n - y_1)}{|x^n - y|} K_{\beta_n}(y)u_{\beta_n}(y)\,dy$$

$$\geq u_{\beta_n}(x^n) \int\limits_{M_{\beta_n}\setminus B_R} \frac{2(\beta_n - y_1)}{|x^n - y|} f(u(y))\,dy + o_n(1)\cdot u_{\beta_n}(x^n)$$

$$\geq u_{\beta_n}(x^n) \int\limits_{M_{\beta_n}\cap\{|x^n-y|\geq 1\}\setminus B_R} \frac{2|y|}{|x^n - y|} f(u(y))\,dy \qquad (4.27)$$

$$+ u_{\beta_n}(x^n) \int\limits_{M_{\beta_n}\cap\{|x^n-y|\leq 1\}\setminus B_R} \frac{2(\beta_n - y_1)}{|x^n - y|} f(u(y))\,dy + o_n(1)\cdot u_{\beta_n}(x^n)$$

$$\geq u_{\beta_n}(x^n)\left(\int\limits_{|x^n-y|\leq 1} \frac{dy}{|x^n - y|^{\frac{3}{2}}}\right)^{\frac{2}{3}}\left(\int\limits_{\Sigma_{\beta_n}\setminus B_R} |2(\beta_n - y_1)f(u(y))|^3 dy\right)^{\frac{1}{3}}$$

$$+ u_{\beta_n}(x^n) \int\limits_{\Sigma_{\beta_n}\setminus B_R} 2|y|f(u(y))\,dy + o_n(1)\cdot u_{\beta_n}(x^n)$$

$$\geq \frac{c_o}{2}u_{\beta_n}(x^n).$$

On the other hand, as for (4.15), it follows from Lemma 3.3 that for $n > N$ with N large enough

$$|u(x^n)|^{\frac{2}{s}-2}u(x^n) - |u^{\beta_n}(x^n)|^{\frac{2}{s}-2}u^{\beta_n}(x^n) \geq c_o|u(x^n)|^{\frac{2}{s}-2}u_{\beta_n}(x^n). \qquad (4.28)$$

Combining (4.25), (4.26) and (4.27), we have

$$\phi_{\beta_n}(x^n)f(u^{\beta_n}(x^n)) + \phi(x^n)h_{\beta_n}(x^n)u_{\beta_n}(x^n) \geq \varepsilon\frac{c_o}{2}|u(x^n)|^{\frac{2}{s}-2}u(x^n)u_{\beta_n}(x^n).$$

Take $\varepsilon > 0$ sufficiently small such that $\varepsilon u(x^n) < 2$, when n is large enough. Hence, from (4.28) and (4.2) we immediately deduce that for $n > N$ with N large enough

$$(-\Delta)^s_{\frac{2}{s}}u_{\beta_n}(x^n) - (-\Delta)^s_{\frac{2}{s}}u(x^n) > 0.$$

Moreover, arguing as in Lemma 4.4, one may deduce

$$(-\Delta)^s_{\frac{2}{s}}u_{\beta_n}(x^n) - (-\Delta)^s_{\frac{2}{s}}u(x^n) < 0.$$

This is a contradiction. Therefore, $(x^n)_n$ must be bounded. The claim (4.22) holds true. Thus, by (4.21) we have

$$u_{\beta_0}(x^*) = \min_{\Sigma_{\beta_0}} u_{\beta_0} \leq 0, \quad \text{and} \quad \nabla u_{\beta_0}(x^*) = 0. \tag{4.29}$$

By the definitions of δ_n, h_{β_n} and recalling that ϕ is a continuous function, it follows that

$$\liminf_{n\to\infty} \frac{\phi(x^n) h_{\beta_n}(x^n) u_{\beta_n}(x^n)}{\delta_n} = \phi(x^*) h_{\beta_0}(x^*) \liminf_{n\to\infty} \frac{u_\beta(x^n)}{\delta_n} = 0, \tag{4.30}$$

where $\delta_n := |\beta_n - x_1^n|$. By the definitions of K_β and M_β, Lemma 3.4, (4.10), and (f_1)-(f_2), there exists $C > 0$ such that for $R > 1$

$$
\begin{aligned}
\phi_{\beta_n}(x^n) &= \int_{\Sigma_{\beta_n}} \ln \frac{|x^n - y^{\beta_n}|}{|x^n - y|} K_{\beta_n}(y) u_{\beta_n}(y) \, dy \\
&\geq u_{\beta_n}(x^n) \left(\int_{M_{\beta_n} \cap B_R} \frac{2(\beta_n - y_1)}{|x^n - y|} f(u(y)) \, dy + \int_{M_{\beta_n} \setminus B_R} \frac{2(\beta_n - y_1)}{|x^n - y|} f(u(y)) \, dy \right) \\
&\geq u_{\beta_n}(x^n) \left(\int_{M_{\beta_n} \cap B_R} \frac{2|y|}{|x^n - y|} f(u(y)) \, dy + \int_{M_{\beta_n} \setminus B_R} 2|y| f(u(y)) \, dy \right) \\
&\geq C u_{\beta_n}(x^n).
\end{aligned}
\tag{4.31}
$$

Hence

$$\liminf_{n\to\infty} \frac{\phi_{\beta_n}(x^n) f(u^{\beta_n}(x_n))}{\delta_n} \geq \liminf_{n\to\infty} \frac{C u_{\beta_n}(x^n) f(u^{\beta_n}(x_n))}{\delta_n} = 0,$$

which implies, together with (4.31) and (4.2) for $\beta = \beta_n$ and $x = x^n$, that

$$\liminf_{\delta_n \to 0^+} \frac{1}{\delta_n} \left\{ (-\Delta)_{\frac{s}{s}}^{\frac{s}{2}} u^\beta(x^n) - (-\Delta)_{\frac{s}{s}}^{\frac{s}{2}} u(x^n) \right\} \geq 0.$$

This contradicts Lemma 4.3 and therefore (4.19) holds true.

Step 2. We next complete the proof showing that u is radially symmetric. Recalling Lemma 3.4 and the definition of β_0 in (4.18), we first have $\beta_0 < \infty$. It follows from Lemma 4.3 that $\beta_0 > -\infty$. According to Lemma 4.1 and Step 1, we get $u_{\beta_0} \equiv 0$ and $\phi_{\beta_0} \equiv 0$. By using the same argument for the second coordinate direction x_2, we can find $\beta_2 \in \mathbb{R}$ such that $u_{\beta_2} \equiv 0$ and $\phi_{\beta_2} \equiv 0$. Consider $\beta = (\beta_0, \beta_2)$, then $\tilde{u}(x) := u(x - \beta)$ and $\tilde{\phi}(x) := \phi(x - \beta)$ is a solution of equation (4.2). By invariance under translation, we may assume that $\tilde{u}(x) = \tilde{u}(-x)$ and $\tilde{\phi}(x) := \phi(-x)$ for $x \in \mathbb{R}^2$. So it is not hard to check that each symmetry hyperplane of $\tilde{u}(x)$ and $\tilde{\phi}(x)$ contains the origin. Thus, repeating the above arguments for an arbitrary direction replacing the x_1-coordinate direction, we deduce that $\tilde{u}(x)$ and $\tilde{\phi}(x)$ are symmetric with respect to any hyperplane containing the origin, thus radially symmetric. Moreover, as a byproduct of the method, $\tilde{u}(x)$ and $\tilde{\phi}(x)$ are also strictly decreasing in the distance from the symmetry centre. $\square$

5. Existence results for (Ch_s) by asymptotic approximation: proof of Theorem 1.6

As we mentioned in the Introduction, the applicability of variational methods to the planar Choquard equation (Ch_s) is not straightforward. Indeed (Ch_s) has, at least formally, a variational structure related to the energy functional

$$I(u) := \frac{s}{2} \iint_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|u(x) - u(y)|^{\frac{2}{s}}}{|x-y|^4} \, dx\, dy + \frac{s}{2} \int_{\mathbb{R}^2} |u|^{\frac{2}{s}} \, dx$$

$$-\frac{1}{4\pi} \iint_{\mathbb{R}^2\,\mathbb{R}^2} \ln\frac{1}{|x-y|} F(u(y)) F(u(x)) \, dx\, dy. \quad (5.1)$$

However, this energy functional is not well-defined on the natural Sobolev space $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ because of the presence of the convolution term and the fact that the logarithm is unbounded both from below and from above. To overcome this difficulty, inspired by [10,33] we will use an approximation technique as follows. Set

$$G_\alpha(x) := \frac{|x|^{-\alpha} - 1}{\alpha}, \qquad \alpha \in (0, 1],$$

$x \in \mathbb{R}^2$, and consider the modified approximating problem

$$(-\Delta)_{\frac{2}{s}}^{\frac{s}{2}} u + |u|^{\frac{2}{s}-2} u = (G_\alpha(\cdot) * F(u)) f(u) \quad \text{on } \mathbb{R}^2, \quad (5.2)$$

with corresponding functional

$$I_\alpha(u) := \frac{s}{2} \|u\|^{\frac{2}{s}} + \frac{1}{4\pi\alpha} \left[\int_{\mathbb{R}^2} F(u) \, dx \right]^2 - \frac{1}{4\pi\alpha} \iint_{\mathbb{R}^2\,\mathbb{R}^2} \frac{1}{|x-y|^\alpha} F(u(x)) F(u(y)) \, dx\, dy$$

$$= \frac{s}{2} \|u\|^{\frac{2}{s}} - \frac{1}{4\pi} \int_{\mathbb{R}^2} (G_\alpha(\cdot) * F(u)) F(u) \, dx.$$

Unlike the original functional I, the power-type singularity in G_α can be handled by the Hardy-Littlewood-Sobolev inequality (Lemma 2.5), and it is standard to prove that I_α is well-defined and C^1 on $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ with

$$I'_\alpha(u)v = \iint_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|u(x) - u(y)|^{\frac{2}{s}-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^4} \, dx\, dy + \int_{\mathbb{R}^2} |u|^{\frac{2}{s}-2} u v \, dx$$

$$-\frac{1}{2\pi} \int_{\mathbb{R}^2} (G_\alpha(\cdot) * F(u)) f(u) v \, dx$$

for $u, v \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$. In order to retrieve compactness in light of Lemma 2.4, we will restrict the functional I_α to the subspace $W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ of radially symmetric functions. Note that in the previous section we established symmetry of all strong positive solution for (Ch$_s$) belonging to $C_{\text{loc}}^{1,1}(\mathbb{R}^2)$. Because of the singular behaviour of the $(s, \frac{2}{s})$-fractional Laplacian, it seems difficult to prove regularity for weak solutions of (5.2) belonging to $W^{s, \frac{2}{s}}(\mathbb{R}^2)$, so that the assumptions for the symmetry result in Theorem 1.5 can be fulfilled (see [5]). However, since the problem is autonomous, it is natural to restrict I_α to the radial symmetric setting, by means of Palais' Principle of Symmetric Criticality (see [42]), for which critical points of I_α restricted to $W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ are still unconstrained critical points for I_α on $W^{s, \frac{2}{s}}(\mathbb{R}^2)$.

In the sequel we will use some elementary estimates which we collect in the next lemma.

Lemma 5.1. *Let $\alpha \in (0, 1]$. Then,*

$$\frac{t^{-\alpha} - 1}{\alpha} \geq \ln \frac{1}{t} \qquad \text{for all } t \in (0, 1].$$

Moreover, for all $\nu > \alpha$ there exists $C_\nu > 0$ such that

$$\frac{t^{-\alpha} - 1}{\alpha} \leq C_\nu t^{-\nu} \qquad \text{for all } t > 0.$$

Let us show that for all $\alpha \in (0, 1]$ the functional I_α enjoys a mountain-pass geometry.

Lemma 5.2. *Let $\alpha \in (0, 1]$ and assume (f_1)-(f_3). Then, there exist constants $\rho, \eta > 0$ and $e \in W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ such that:*

(i) $\|e\| > \rho$ and $I_\alpha(e) < 0$;

(ii) $I_\alpha|_{S_\rho} \geq \eta > 0$, where $S_\rho = \left\{ u \in W_r^{s, \frac{2}{s}}(\mathbb{R}^2) \mid \|u\| = \rho \right\}$.

Proof. (i) Take $e_0 \in W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ such that $e_0(x) = 1$ for $x \in B_{\frac{1}{8}}(0)$, $e_0(x) = 0$ for $x \in \mathbb{R}^2 \setminus B_{\frac{1}{4}}(0)$. For $t > 0$ set

$$\Psi(t) := \frac{1}{2} \left(\int_{\mathbb{R}^2} F(t e_0) \, dx \right)^2. \tag{5.3}$$

By Remark 2.3-(ii) we infer that

$$\frac{\Psi'(t)}{\Psi(t)} \geq \frac{2}{(s - \tau)t} \qquad \text{for all } t > 0$$

and integrating over $[1, t]$ we find

$$\Psi(t) \geq \Psi(1) t^{\frac{2}{s-\tau}} = \frac{1}{2} \left(\int_{\mathbb{R}^2} F(e_0) \, dx \right)^2 t^{\frac{2}{s-\tau}}. \tag{5.4}$$

It follows from Lemma 5.1, (5.3) and (5.4) that

$$
\begin{aligned}
I_\alpha(te_0) &\leq \frac{s}{2} t^{\frac{2}{s}} \|e_0\|^{\frac{2}{s}} - \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha} - 1}{\alpha} F\big(te_0(y)\big) F\big(te_0(x)\big) \, dx \, dy \\
&= \frac{s}{2} t^{\frac{2}{s}} \|e_0\|^{\frac{2}{s}} - \frac{1}{4\pi} \iint_{\{|x-y|\leq \frac{1}{2}\}} \frac{|x-y|^{-\alpha} - 1}{\alpha} F\big(te_0(y)\big) F\big(te_0(x)\big) \, dx \, dy \\
&\leq \frac{s}{2} t^{\frac{2}{s}} \|e_0\|^{\frac{2}{s}} - \frac{1}{4\pi} \iint_{\{|x-y|\leq \frac{1}{2}\}} \ln \frac{1}{|x-y|} F\big(te_0(y)\big) F\big(te_0(x)\big) \, dx \, dy \\
&\leq \frac{s}{2} t^{\frac{2}{s}} \|e_0\|^{\frac{2}{s}} - \frac{\ln 2}{4\pi} \left(\int_{\mathbb{R}^2} F(te_0) \, dx \right)^2 \\
&\leq \frac{s}{2} t^{\frac{2}{s}} \|e_0\|^{\frac{2}{s}} - \frac{\ln 2}{4\pi} \left(\int_{\mathbb{R}^2} F(e_0) \, dx \right)^2 t^{\frac{2}{s-\tau}}.
\end{aligned}
$$

Therefore, one can find $t_0 > 0$ large enough such that $I_\alpha(t_0 e_0) < 0$.

(ii) By Remark 2.3(i) there exists $C > 0$ such that

$$|F(u)|^{\frac{4}{3}} \leq C \left(|u|^{\frac{8}{3s}} + |u|^{\frac{8}{3s}} \Phi_{2,s}\big(\tfrac{4}{3}\alpha_* |u|^{\frac{2}{2-s}}\big) \right).$$

Hence, by Hardy-Littlewood-Sobolev's inequality and Theorem 2.1, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} F(u(x)) F(u(y)) \, dx \, dy &\leq C \left(\int_{\mathbb{R}^2} |F(u)|^{\frac{4}{3}} \, dx \right)^{\frac{3}{2}} \\
&\leq C \left(\|u\|_{\frac{8}{3s}}^{\frac{4}{s}} + \|u\|_{\frac{8}{3s} p'}^{\frac{4}{s}} \left(\int_{\mathbb{R}^2} \Phi_{2,s}\big(\tfrac{4}{3} p\alpha_* \|u\|^{\frac{2}{2-s}} \big(\tfrac{|u|}{\|u\|}\big)^{\frac{2}{2-s}}\big) \, dx \right)^{\frac{3}{2p}} \right)
\end{aligned} \tag{5.5}
$$

for some $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. By Theorem 2.1, we can find $p > 1$ such that the last factor is bounded independently of u provided $\|u\| < \left(\frac{3}{4}\right)^{\frac{2-s}{2}}$. From (5.5) we deduce that

$$I_\alpha(u) = \frac{s}{2} \|u\|^{\frac{2}{s}} - \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha} - 1}{\alpha} F(u(y)) F(u(x)) \, dx \, dy$$

$$\geq \frac{s}{2}\|u\|^{\frac{2}{s}} - \frac{1}{4\pi}\iint\limits_{\{|x-y|\leq 1\}} \frac{|x-y|^{-\alpha}-1}{\alpha}F(u(y))F(u(x))\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq \frac{s}{2}\|u\|^{\frac{2}{s}} - \frac{1}{4\pi}\iint\limits_{\{|x-y|\leq 1\}} \frac{F(u(y))F(u(x))}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq \frac{s}{2}\|u\|^{\frac{2}{s}} - C(\|u\|^{\frac{4}{s}}_{\frac{8}{3s}} + \|u\|^{\frac{4}{s}}_{\frac{8}{3s}p'}).$$

So, let $\|u\| = \rho > 0$ be sufficiently small, and recall the embedding $W_r^{s,\frac{2}{s}}(\mathbb{R}^2) \hookrightarrow L^t(\mathbb{R}^2)$ for all $t > \frac{2}{s}$, to obtain $\eta > 0$ such that $I_\alpha(u) \geq \eta$ for any $\alpha \in (0,1]$. $\quad\square$

As a consequence of Lemma 5.1, the mountain pass level

$$c_\alpha := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\alpha(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], W_r^{s,\frac{2}{s}}(\mathbb{R}^2)) : \gamma(0) = 0, \gamma(1) = e\},$$

turns out to be well defined. Moreover, from the Ekeland Variational Principle, the mountain pass geometry yields the existence of a Cerami sequence at level c_α for any fixed $\alpha \in (0,1]$, see e.g. [24]. Namely, there exists $(u_n^\alpha)_n \subset W_r^{s,\frac{2}{s}}(\mathbb{R}^2)$ such that

$$I_\alpha(u_n^\alpha) \to c_\alpha \quad \text{and} \quad (1+\|u_n^\alpha\|)I_\alpha'(u_n^\alpha) \to 0 \quad \text{in } (W^{s,\frac{2}{s}}(\mathbb{R}^2))'$$

as $n \to +\infty$. For the sake of a lighter notation, we will simply use $u_n := u_n^\alpha$.

Remark 5.3. Observe from Lemma 5.2 that there exist two constants $a, b > 0$ independent of α such that $a < c_\alpha < b$.

The next technical Lemma will be crucial in estimating the mountain pass level c_α. Let $R > 0$ and $\bar{w} \in C(\mathbb{R}^2)$ be such that

$$\bar{w}(x) := \begin{cases} 1 & \text{for } |x| \leq \frac{R}{2}, \\ 2 - \frac{2}{R}|x| & \text{for } |x| \in (\frac{R}{2}, R), \\ 0 & \text{for } |x| \geq R. \end{cases} \tag{5.6}$$

Lemma 5.4. *For all $R > 0$ and $s \in (0,1)$ we have $\bar{w} \in W_r^{s,\frac{2}{s}}(\mathbb{R}^2)$ and*

$$\|\bar{w}\|^{\frac{2}{s}} \leq \frac{\pi R^2}{4}\left(1 + \frac{s(2+3s)}{(2+s)(1+s)}\right) + \frac{4\pi^2}{2-s}\left(\frac{31}{18}s + \frac{10}{3}\right).$$

Proof. We have

$$
\|\bar{w}\|_{\frac{2}{s}}^{\frac{2}{s}} = \frac{\pi R^2}{4} + 2\pi \int_{\frac{R}{2}}^{R} \left|2 - \frac{2}{R}r\right|^{\frac{2}{s}} r \, dr = \frac{\pi R^2}{4} + 2\pi \left(\frac{2}{R}\right)^{\frac{2}{s}} \int_{0}^{\frac{R}{2}} y^{\frac{2}{s}} (R - y) \, dy
$$

$$
= \frac{\pi R^2}{4} + 2\pi \frac{R^2}{2} \left(\frac{s}{2+s} - \frac{s}{4+4s}\right) = \frac{\pi R^2}{4} \left(1 + \frac{s(2+3s)}{(2+s)(1+s)}\right).
$$
(5.7)

Let us now compute the seminorm of $\bar{w}$. Since $\bar{w}$ is radially symmetric, according to the equivalent formulations of the seminorm for radial functions established in [36, Proposition 4.3], we have

$$
[\bar{w}]_{s,\frac{2}{s}}^{\frac{2}{s}} = 4\pi^2 \int_{0}^{+\infty} \int_{0}^{+\infty} |\bar{w}(r) - \bar{w}(t)|^{\frac{2}{s}} rt \frac{r^2 + t^2}{|r^2 - t^2|^3} \, dr \, dt
$$

$$
= 8\pi^2 \left(\int_{0}^{\frac{R}{2}} \int_{\frac{R}{2}}^{R} + \int_{0}^{\frac{R}{2}} \int_{R}^{+\infty} + \int_{\frac{R}{2}}^{R} \int_{R}^{+\infty}\right) + 4\pi^2 \int_{\frac{R}{2}}^{R} \int_{\frac{R}{2}}^{R}
$$

$$
=: 8\pi^2 (I_1 + I_2 + I_3) + 4\pi^2 I_4
$$

and let us next estimate I_i, $i = 1, \ldots 4$, separately. Recalling that

$$
\frac{d}{dr} \left(\frac{1}{2} \frac{r^2}{(t^2 - r^2)^2}\right) = r \frac{r^2 + t^2}{(t^2 - r^2)^3},
$$
(5.8)

we have

$$
I_1 = \int_{\frac{R}{2}}^{R} \left|\frac{2}{R}t - 1\right|^{\frac{2}{s}} t \left(\int_{0}^{\frac{R}{2}} r \frac{r^2 + t^2}{(t^2 - r^2)^3} \, dr\right) dt = \frac{1}{2} \int_{\frac{R}{2}}^{R} \left|\frac{2}{R}t - 1\right|^{\frac{2}{s}} t \frac{R^2}{4} \frac{dt}{(t - \frac{R}{2})^2 (t + \frac{R}{2})^2}
$$

$$
\leq \frac{R}{8} \left(\frac{2}{R}\right)^{\frac{2}{s}} \int_{\frac{R}{2}}^{R} \left|t - \frac{R}{2}\right|^{\frac{2}{s} - 2} dt = \frac{s}{4(2 - s)}.
$$

Again using (5.8) we compute the second term I_2 as

$$
I_2 = \int_{0}^{\frac{R}{2}} r \left(\int_{R}^{+\infty} t \frac{r^2 + t^2}{(t^2 - r^2)^3} \, dt\right) dr = \frac{R^2}{2} \int_{0}^{\frac{R}{2}} \frac{r}{(R^2 - r^2)^2} \, dr = \frac{1}{3}.
$$

The third term I_3 can be estimated as

$$I_3 = \int_{\frac{R}{2}}^{R} \left| \frac{2}{R} r - 1 \right|^{\frac{2}{s}} r \left(\int_{R}^{+\infty} t \frac{r^2 + t^2}{(t^2 - r^2)^3} dt \right) dr$$

$$= \frac{1}{2} \left(\frac{2}{R} \right)^{\frac{2}{s}} \int_{\frac{R}{2}}^{R} (R - r)^{\frac{2}{s}} r \frac{R^2}{(R - r)^2 (R + r)^2} dr$$

$$\leq \frac{2}{9} R \left(\frac{2}{R} \right)^{\frac{2}{s}} \int_{\frac{R}{2}}^{R} (R - r)^{\frac{2}{s} - 2} dr = \frac{4}{9} \frac{s}{2 - s} .$$

Finally, integrating by parts we get

$$I_4 = \left(\frac{2}{R} \right)^{\frac{2}{s}} \int_{\frac{R}{2}}^{R} r \left(\int_{\frac{R}{2}}^{R} |r - t|^{\frac{2}{s}} t \frac{r^2 + t^2}{(t^2 - r^2)^3} dt \right) dr$$

$$= \left(\frac{2}{R} \right)^{\frac{2}{s}} \int_{\frac{R}{2}}^{R} r \left[|r - t|^{\frac{2}{s}} \cdot \frac{1}{2} \frac{t^2}{(t^2 - r^2)^2} \right]_{t = \frac{R}{2}}^{t = R} dr$$

$$+ \left(\frac{2}{R} \right)^{\frac{2}{s}} \int_{\frac{R}{2}}^{R} \frac{r}{s} \int_{\frac{R}{2}}^{r} \left[(r - t)^{\frac{2}{s} - 1} \cdot \frac{t^2}{(t^2 - r^2)^2} \right] dt \, dr$$

$$- \left(\frac{2}{R} \right)^{\frac{2}{s}} \int_{\frac{R}{2}}^{R} \frac{r}{s} \int_{r}^{R} \left[(t - r)^{\frac{2}{s} - 1} \cdot \frac{t^2}{(t^2 - r^2)^2} \right] dt \, dr$$

$$=: A_1 + A_2 - A_3 .$$

By inspection one has $A_3 \geq 0$. Moreover,

$$A_1 \leq \left(\frac{2}{R} \right)^{\frac{2}{s}} \left[\int_{\frac{R}{2}}^{R} \frac{r}{2} |r - R|^{\frac{2}{s} - 2} \frac{R^2}{(r + R)^2} dr - \int_{\frac{R}{2}}^{R} \frac{r}{2} |r - \frac{R}{2}|^{\frac{2}{s} - 2} \frac{\frac{1}{4} R^2}{(r + R/2)^2} dr \right]$$

$$\leq \left(\frac{2}{R} \right)^{\frac{2}{s}} \frac{R}{2} \int_{\frac{R}{2}}^{R} |r - R|^{\frac{2}{s} - 2} dr = \frac{s}{2 - s}$$

and

$$A_2 = \left(\frac{2}{R}\right)^{\frac{2}{s}} \int\limits_{\frac{R}{2}}^{R} \frac{r}{s} \left(\int\limits_{\frac{R}{2}}^{r} (r-t)^{\frac{2}{s}-3} \cdot \frac{t^2}{(t+r)^2} dt\right) dr$$

$$\leq \left(\frac{2}{R}\right)^{\frac{2}{s}} \frac{R}{s} \int\limits_{\frac{R}{2}}^{R} (r - \tfrac{R}{2})^{\frac{2}{s}-2} dr = \frac{2}{2-s}.$$

Eventually we get

$$[\bar{w}]_{s,\frac{2}{s}}^{\frac{2}{s}} = 4\pi^2 (2I_1 + 2I_2 + 2I_3 + I_4) \leq \frac{4\pi^2}{2-s}\left(\frac{31}{18}s + \frac{10}{3}\right). \tag{5.9}$$

The desired estimate for the norm of $\bar{w}$ is obtained combining (5.7) with (5.9). $\quad\square$

Lemma 5.5. *Suppose (f_1)–(f_5) hold. Then $c_\alpha < \frac{s}{4}$ for all $\alpha \in (0, 1]$.*

Proof. Let $R \in \left(0, \frac{1}{3}\right]$ to be fixed later, and $\bar{w} \in W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ be as in (5.6). Define $w := \|\bar{w}\|^{-1}\bar{w}$. As in the proof of Lemma 5.2, we deduce the existence of $T > 0$ such that

$$I_\alpha(Tw) := \max_{t \geq 0} I_\alpha(tw),$$

and therefore, since I_α is a functional of class C^1, one has $\frac{d}{dt}\big|_{t=T} I_\alpha(tw) = I'_\alpha(Tw)w = 0$. Since by definition $c_\alpha \leq I_\alpha(Tw)$, we aim at proving that $I_\alpha(Tw) < \frac{s}{4}$. Suppose by contradiction that for all $\varepsilon > 0$ there exists $\alpha \in (0, 1]$ such that

$$I_\alpha(Tw) \geq \frac{s}{4},$$

then

$$\frac{s}{2}T^{\frac{2}{s}} \geq \frac{s}{4} + \frac{1}{4\pi}\int\limits_{\mathbb{R}^2} (G_\alpha(x) * F(Tw)) F(Tw)\, dx. \tag{5.10}$$

Observe that if $x, y \in B_R(0)$, with $R \leq \frac{1}{3}$ then $G_\alpha(x-y) \geq \ln\frac{1}{|x-y|} \geq \ln\frac{3}{2} > 0$. Thus, since w supported in $B_R(0)$, it follows from (5.10) that

$$T \geq \left(\frac{1}{2}\right)^{\frac{s}{2}}. \tag{5.11}$$

Moreover, $I'_\alpha(Tw)Tw = 0$ yields

$$T^{\frac{2}{s}} = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} (G_\alpha(x) * F(Tw)) f(Tw)Tw\, dx. \tag{5.12}$$

Let us estimate from below the right-hand side. We have

$$
\begin{aligned}
\int_{\mathbb{R}^2} & (G_\alpha(x) * F(Tw)) f(Tw) Tw \, dx \\
&\geq \int_{B_{\frac{R}{2}} \times B_{\frac{R}{2}}} G_\alpha(x - y) F(Tw(y)) f(Tw(x)) Tw(x) \, dx \, dy \\
&\geq \int_{B_{\frac{R}{2}} \times B_{\frac{R}{2}}} \ln \frac{1}{|x - y|} F(Tw(y)) f(Tw(x)) Tw(x) \, dx \, dy \\
&\geq \ln 3 \int_{|y| \leq \frac{R}{2}} F(Tw(y)) \, dy \int_{|x| \leq \frac{R}{2}} f(Tw(x)) Tw(x) \, dx \\
&\geq \ln 3 \left(\int_{|x| \leq \frac{R}{2}} \sqrt{F(Tw(x)) f(Tw(x)) Tw(x)} \, dx \right)^2
\end{aligned}
\tag{5.13}
$$

by the Hölder inequality. Note that in $B_{\frac{R}{2}}(0)$ one has

$$
Tw = T \|\bar{w}\|^{-1} \geq \frac{\|\bar{w}\|^{-1}}{2^{\frac{s}{2}}}
$$

by (5.11). By the computations of the norm of $\bar{w}$ carried out in Lemma 5.4, one has $\|\bar{w}\| < \frac{1}{2^{\frac{s}{2}} T(s)}$ by taking $R = \frac{1}{3}$ and

$$
T(s) := \left(\frac{\pi}{18} \frac{4s^2 + 5s + 2}{(2 + s)(1 + s)} + \frac{4\pi^2}{9} \frac{31s + 60}{2 - s} \right)^{-\frac{s}{2}}.
\tag{5.14}
$$

Hence, by (f_5) we deduce from (5.12) and (5.13) that

$$
\begin{aligned}
T^{\frac{2}{s}} &\geq \frac{\beta}{2\pi} \ln 3 \left(\int_{|x| \leq \frac{R}{2}} (Tw(x))^{\frac{2+s}{2s}} \, dx \right)^2 = \frac{\beta}{2\pi} \ln 3 |B_{\frac{R}{2}}(0)|^2 \left(\frac{T}{\|\bar{w}\|} \right)^{\frac{2}{s}+1} \\
&= \beta \frac{\pi R^4 \ln 3}{32} \frac{T^{\frac{2}{s}+1}}{\|\bar{w}\|^{\frac{2}{s}+1}},
\end{aligned}
$$

which implies

$$
T < \beta T \leq \frac{32 \|\bar{w}\|^{\frac{2}{s}+1}}{\pi R^4 \ln 3} \leq \frac{32}{\pi R^4 \ln 3} \left(\frac{1}{2^{\frac{s}{2}} T(s)} \right)^{\frac{2}{s}+1} = \frac{2^{4-\frac{s}{2}} 3^4}{\pi \ln 3} \left(\frac{1}{T(s)} \right)^{\frac{2}{s}+1},
\tag{5.15}
$$

since $\beta > 1$ and taking $R = \frac{1}{3}$. Hence, combining (5.15) and (5.11), we infer

$$\frac{2^{4-\frac{s}{2}}3^4}{\pi \ln 3}\left(\frac{1}{T(s)}\right)^{\frac{2}{s}+1} \geq \beta T \geq \beta \left(\frac{1}{2}\right)^{\frac{s}{2}}.$$

This is in contradiction with (f_5) provided one takes

$$\beta > \beta_0 := \frac{2^4 \, 3^4}{\pi \ln 3}\left(\frac{1}{T(s)}\right)^{\frac{2}{s}+1}. \quad \square \tag{5.16}$$

The estimate of the mountain-pass level obtained in Lemma 5.5 enables us to obtain the following compactness results

Lemma 5.6. *Assume that (f_1)–(f_5) hold. Let $(u_n)_n \subset W^{s,\frac{2}{s}}(\mathbb{R}^2)$ be a Cerami sequence of I_α at level c_α, then $(u_n)_n$ is bounded in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ with*

$$\|u_n\|^{\frac{2}{s}} < \frac{s}{\tau}, \tag{5.17}$$

as well as

$$\left|\int_{\mathbb{R}^2} [G_\alpha(x) * F(u_n)] F(u_n) \, dx\right| < C, \qquad \left|\int_{\mathbb{R}^2} [G_\alpha(x) * F(u_n)] f(u_n) u_n \, dx\right| < C. \tag{5.18}$$

Proof. Since $(u_n)_n \subset W^{s,\frac{2}{s}}(\mathbb{R}^2)$ is a Cerami sequence, one has

$$\frac{s}{2}\|u_n\|^{\frac{2}{s}} - \frac{1}{4\pi}\int_{\mathbb{R}^2} [G_\alpha(x) * F(u_n)] F(u_n) \, dx \to c_\alpha \tag{5.19}$$

and for all $v \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$

$$\iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2}(u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^4} \, dx \, dy$$

$$+ \int_{\mathbb{R}^2} |u_n|^{\frac{2}{s}-2} u_n v \, dx - \frac{1}{2\pi}\int_{\mathbb{R}^2} [G_\alpha(x) * F(u_n)] f(u_n) v \, dx = o_n(1)\|v\|. \tag{5.20}$$

Taking $v = u_n$, we get

$$\iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x-y|^4} \, dx \, dy + \int_{\mathbb{R}^2} |u_n|^{\frac{2}{s}} \, dx - \frac{1}{2\pi}\int_{\mathbb{R}^2} [G_\alpha(x) * F(u_n)] f(u_n) u_n \, dx = o_n(1). \tag{5.21}$$

In order to prove the boundedness of $(u_n)_n$, we introduce a suitable test function as follows

$$v_n := \begin{cases} \frac{F(u_n)}{f(u_n)}, & u_n > 0, \\ (s - \tau)u_n, & u_n \leq 0, \end{cases}$$

where τ is the positive constant appearing in (f_3). It is easy to check that $|v_n| \leq C|u_n|$ since $F(t) \leq (s - \tau)f(t)t$ by Remark 2.3(ii) and $f(t) = 0$ if and only if $t \leq 0$. Furthermore,

$$
\begin{aligned}
\iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|v_n(x) - v_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y &= \iint_{u_n(x)>0; u_n(y)>0} \frac{\left| \frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))} \right|^{\frac{2}{s}}}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + 2 \iint_{u_n(x)\leq 0; u_n(y)>0} \frac{\left| (s - \tau)u_n(x) - \frac{F(u_n(y))}{f(u_n(y))} \right|^{\frac{2}{s}}}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + (s - \tau)^{\frac{2}{s}} \iint_{u_n(x)\leq 0; u_n(y)\leq 0} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{\left(1 - \frac{F(\xi_n(x,y))f'(\xi_n(x,y))}{f^2(\xi_n(x,y))}\right)^{\frac{2}{s}} |u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + 3(s - \tau)^{\frac{2}{s}} \iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq C \iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y,
\end{aligned}
\tag{5.22}
$$

by (f_4), where $\xi_n(x, y) \in \mathbb{R}$. Thus v_n is well defined in $W^{s, \frac{2}{s}}(\mathbb{R}^2)$. Taking $v = v_n$ in (5.20) and recalling that $f(t) = 0$ and $F(t) = 0$ for any $t \leq 0$, we infer

$$
\iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2}(u_n(x) - u_n(y))(v_n(x) - v_n(y))}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y + (s - \tau) \int_{\{u_n<0\}} |u_n|^{\frac{2}{s}} \, \mathrm{d}x
$$

$$
+ \int_{\mathbb{R}^2} |u_n|^{\frac{2}{s}-2} u_n \frac{F(u_n)}{f(u_n)} \, \mathrm{d}x - \frac{1}{2\pi} \int_{\mathbb{R}^2} [G_\alpha(x) * F(u_n)]F(u_n) \, \mathrm{d}x = o_n(1)\|u_n\|.
$$

Recalling (5.19), this yields

$$
\iint_{\mathbb{R}^2 \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2}(u_n(x) - u_n(y))(v_n(x) - v_n(y))}{|x - y|^4} \, \mathrm{d}x \, \mathrm{d}y
$$

$$+ \int_{\mathbb{R}^2} (s - \tau)|u_n|^{\frac{2}{s}} \, dx + 2c_\alpha - s\|u_n\|^{\frac{2}{s}} \geq o_n(1)\|u_n\|. \tag{5.23}$$

Similarly to (5.22), we also obtain

$$\iint_{\mathbb{R}^2 \, \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2} (u_n(x) - u_n(y)) (v_n(x) - v_n(y))}{|x - y|^4} \, dx \, dy$$

$$\leq \iint_{u_n(x)>0; u_n(y)>0} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2} (u_n(x) - u_n(y)) \left(\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))}\right)}{|x - y|^4} \, dx \, dy$$

$$+ 2 \iint_{u_n(x)\leq 0; u_n(y)>0} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2} (u_n(x) - u_n(y)) \left((s - \tau)u_n(x) - \frac{F(u_n(y))}{f(u_n(y))}\right)}{|x - y|^4} \, dx \, dy$$

$$+ \iint_{u_n(x)\leq 0; u_n(y)\leq 0} \frac{(s - \tau) |u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy$$

$$\leq (s - \tau) \iint_{\mathbb{R}^2 \, \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy.$$

Combining this and (5.23) we get

$$\tau \iint_{\mathbb{R}^2 \, \mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy + \tau \int_{\mathbb{R}^2} |u_n|^{\frac{2}{s}} \, dx \leq o_n(1)\|u_n\| + 2c_\alpha.$$

As a consequence, we have

$$\|u_n\|^{\frac{2}{s}} \leq \frac{2}{\tau} c_\alpha < \frac{s}{\tau},$$

by (5.5), where we note that the constant on the right is independent of n and α. Finally, from (5.19) and (5.21) we immediately obtain (5.18). $\square$

Remark 5.7. Thanks to the uniform boundedness of Cerami sequences of Lemma 5.6, from now on we can always suppose that Cerami sequences at level c_α are nonnegative. Indeed, $u_n^- := \min\{u_n, 0\} \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$ and thus, following the same argument used in Remark 3.1, one has

$$\iint_{\mathbb{R}^2 \, \mathbb{R}^2} \frac{|u_n^-(x) - u_n^-(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy + \int_{\mathbb{R}^2} |u_n^-|^{\frac{2}{s}} \leq I_\alpha'(u_n)u_n^- \leq \|I_\alpha'(u_n)\|\|u_n\| = o_n(1),$$

since $\|u_n\| \leq C$ by Lemma 5.6. This implies that $u_n^- \to 0$ in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ as $n \to +\infty$ and therefore, setting $u_n^+ := \max\{u_n, 0\}$, $(u_n^+)_n$ is a Cerami sequence of I_α at level c_α, which we will denote simply by $(u_n)_n$.

Lemma 5.8. *Suppose (f_1)–(f_5) hold and let $(u_n)_n$ be a Cerami sequence of I_α at level c_α. Then there exists $C > 0$ independent of n and α such that*

$$\int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq C \quad \text{and} \quad \int_{\mathbb{R}^2} F(u_n)^\kappa \, dx \leq C$$

for any $\kappa \in \left[1, \frac{1}{\gamma_{s,\tau}}\right)$, where $\gamma_{s,\tau} \in (0,1)$ is a constant depending just on s and τ.

In the proof we will use the following

Lemma 5.9. *Let $W \in [0, 1)$ and $q > 1$. Then*

$$(1+W)^q \leq 1 + qW \max\{1, (1+W)^{q-1}\} \tag{5.24}$$

and

$$(1+W)^q - W - 1 \leq \frac{q!}{q - \lfloor q \rfloor} \frac{W}{1-W}. \tag{5.25}$$

Proof. We have

$$(1+W)^q = 1 + q \int_1^{1+W} t^{q-1} dt \leq 1 + qW \max\{1, (1+W)^{q-1}\}.$$

Next, let $N = \lfloor q \rfloor$. Hence, applying (5.24) iteratively, and from $\max\{1, (1+W)^{q-(N+1)}\} = 1$, we get

$$
\begin{aligned}
(1+W)^q - W - 1 &\leq 1 + qW(1+W)^{q-1} - W - 1 \\
&\leq 1 + qW + q(q-1)W^2 + \cdots + q(q-1)\cdots(q-N)W^{N+1} - W - 1 \\
&\leq (q-1)W + q(q-1)W^2 + \cdots + q(q-1)\cdots(q-N)W^{N+1} \\
&\leq q(q-1)\cdots(q-N+1)W(1+W+\cdots+W^N) \\
&= \frac{q!}{q-\lfloor q \rfloor} W \frac{1-W^{N+1}}{1-W} \\
&\leq \frac{q!}{q-\lfloor q \rfloor} \frac{W}{1-W}. \quad \square
\end{aligned}
$$

Proof of Lemma 5.8. Since $(u_n)_n$ is a Cerami sequence of I_α at level c_α, by Lemma 5.5 we have

$$\|u_n\|^{\frac{2}{s}} < \frac{s}{\tau}. \tag{5.26}$$

Let us introduce the following auxiliary function

$$G(t) := t - \frac{1}{s}\frac{F(t)}{f(t)} \quad \text{for } t \geq 0,$$

and define $v_n := G(u_n)$. Similarly to the proof of Lemma 5.6, one has $v_n \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$. We first show that there exists $\gamma_{s,\tau} \in (0,1)$ depending just on s and τ, such that

$$\|v_n\|^{\frac{2}{s}} \leq \gamma_{s,\tau} < 1 \tag{5.27}$$

for n large enough. Indeed, recall by (5.19)-(5.20) that

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} [G_\alpha(x) * F(u_n)] F(u_n)\,\mathrm{d}x = s\|u_n\|^{\frac{2}{s}} - 2c_\alpha + o_n(1)$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2}(u_n(x) - u_n(y))}{|x - y|^4}\left(\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))}\right)\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \int_{\mathbb{R}^2} |u_n|^{\frac{2}{s}-2}u_n\frac{F(u_n)}{f(u_n)}\,\mathrm{d}x - \frac{1}{2\pi}\int_{\mathbb{R}^2}[G_\alpha(x) * F(u_n)]F(u_n)\,\mathrm{d}x = o_n(1),$$

which together imply

$$\frac{1}{s}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}-2}(u_n(x) - u_n(y))}{|x - y|^4}\left(\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))}\right)\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \frac{1}{s}\int_{\mathbb{R}^2} |u_n|^{\frac{2}{s}-2}u_n\frac{F(u_n)}{f(u_n)}\,\mathrm{d}x - \|u_n\|^{\frac{2}{s}} - \frac{2}{s}c_\alpha = o_n(1). \tag{5.28}$$

Hence, one has

$$\|v_n\|^{\frac{2}{s}} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|G(u_n(x)) - G(u_n(y))|^{\frac{2}{s}}}{|x - y|^4}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^2}|G(u_n)|^{\frac{2}{s}}\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|G(u_n(x)) - G(u_n(y))|^{\frac{2}{s}} + \frac{1}{s}|u_n(x) - u_n(y)|^{\frac{2}{s}-2}(u_n(x) - u_n(y))\left(\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))}\right)}{|x - y|^4}\,\mathrm{d}y\,\mathrm{d}x$$

$$+ \int_{\mathbb{R}^2}|G(u_n)|^{\frac{2}{s}}\,\mathrm{d}x + \frac{1}{s}\int_{\mathbb{R}^2}|u_n|^{\frac{2}{s}-2}u_n\frac{F(u_n)}{f(u_n)}\,\mathrm{d}x + \frac{2}{s}c_\alpha - [u_n]_{s,\frac{2}{s}}^{\frac{2}{s}} - \|u_n\|_{\frac{2}{s}}^{\frac{2}{s}} + o_n(1). \tag{5.29}$$

Observe that for all $x, y \in \mathbb{R}^2$ by the mean value theorem one has

$$\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))} = \left(1 - \frac{F(\theta(x,y))f'(\theta(x,y))}{f^2(\theta(x,y))}\right)(u_n(x) - u_n(y)),$$

for some $\theta(x, y) \in (\min\{u_n(x), u_n(y)\}, \max\{u_n(x), u_n(y)\})$. Therefore, by the definition of G we also get

$$G(u_n(x)) - G(u_n(y)) = \left(1 - \frac{1}{s} + \frac{1}{s}\frac{F(\theta(x,y))f'(\theta(x,y))}{f^2(\theta(x,y))}\right)(u_n(x) - u_n(y)).$$

Hence, we can estimate as follows:

$$Z(u_n) :=$$

$$\iint_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|G(u_n(x)) - G(u_n(y))|^{\frac{2}{s}} + \frac{1}{s}|u_n(x) - u_n(y)|^{\frac{2}{s}-2}(u_n(x) - u_n(y))\left(\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))}\right)}{|x - y|^4}\,dy\,dx$$

$$= \iint_{\mathbb{R}^2\,\mathbb{R}^2} \left[\left|1 - \frac{1}{s}\left(1 - \frac{F(\theta(x,y))f'(\theta(x,y))}{f^2(\theta(x,y))}\right)\right|^{\frac{2}{s}} + \frac{1}{s}\left(1 - \frac{F(\theta(x,y))f'(\theta(x,y))}{f^2(\theta(x,y))}\right)\right] \cdot$$

$$\cdot \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4}\,dx\,dy. \quad (5.30)$$

Let

$$A := \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \frac{Ff'}{f^2}(\theta(x,y)) > 1\right\}$$

and split $\mathbb{R}^4 = A \cup (\mathbb{R}^4 \setminus A)$ and $Z(u_n) = Z_A(u_n) + Z_{\mathbb{R}^4\setminus A}(u_n)$ accordingly. On $\mathbb{R}^4 \setminus A$ we have

$$\left|1 - \frac{1}{s}\left(1 - \frac{F(\theta(x,y))f'(\theta(x,y))}{f^2(\theta(x,y))}\right)\right|^{\frac{2}{s}} \leq 1 - \frac{1}{s}\left(1 - \frac{F(\theta(x,y))f'(\theta(x,y))}{f^2(\theta(x,y))}\right),$$

and therefore

$$Z_{\mathbb{R}^4\setminus A}(u_n) \leq \int_{\mathbb{R}^4\setminus A} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4}\,dx\,dy. \quad (5.31)$$

On the other hand, invoking the estimate (5.25) with $q = \frac{2}{s}$ and $W = W(x, y) = \frac{1}{s}\left(\frac{F(\theta(x,y))f'(\theta(x,y))}{f^2(\theta(x,y))} - 1\right) > 0$, we obtain

$$Z_A(u_n) - \int\limits_A \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy = \int\limits_A \left((1 + W)^{\frac{2}{s}} - W - 1 \right) \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy$$

$$\leq \frac{\frac{2}{s}!}{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor} \int\limits_A \frac{W}{1 - W} \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy \, .$$

Requiring

$$\frac{F(t) f'(t)}{f^2(t)} < 1 + \frac{s}{2} \quad \text{for all } t > 0 \, , \tag{5.32}$$

one has $\|W\|_\infty < \frac{1}{2}$. Hence,

$$Z_A(u_n) - \int\limits_A \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy \leq \frac{2^{\frac{2}{s}}!}{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor} \|W\|_\infty \int\limits_A \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy \, . \tag{5.33}$$

Finally, from (5.31), (5.33) and (5.17), we obtain

$$Z(u_n) - [u_n]_{s, \frac{2}{s}}^{\frac{2}{s}} \leq \frac{2^{\frac{2}{s}}!}{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor} \|W\|_\infty \int\limits_A \frac{|u_n(x) - u_n(y)|^{\frac{2}{s}}}{|x - y|^4} \, dx \, dy$$

$$\leq \frac{2^{\frac{2}{s}}!}{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor} \|W\|_\infty \frac{s}{\tau} \, .$$

We can now estimate the norm of v_n. From (5.29) we get

$$\|v_n\|^{\frac{2}{s}} = Z(u_n) - [u_n]_{s, \frac{2}{s}}^{\frac{2}{s}} + \int\limits_{\mathbb{R}^2} |G(u_n)|^{\frac{2}{s}} \, dx + \frac{1}{s} \int\limits_{\mathbb{R}^2} |u_n|^{\frac{2}{s} - 2} u_n \frac{F(u_n)}{f(u_n)} \, dx$$

$$+ \frac{2}{s} c_\alpha - \|u_n\|_{\frac{2}{s}}^{\frac{2}{s}} + o_n(1)$$

$$\leq \frac{2^{\frac{2}{s}}!}{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor} \|W\|_\infty \frac{s}{\tau} + \int\limits_{\mathbb{R}^2} |u_n|^{\frac{2}{s}} \left| 1 - \frac{F(u_n)}{s f(u_n) u_n} \right|^{\frac{2}{s}} \, dx + \frac{1}{s} \int\limits_{\mathbb{R}^2} |u_n|^{\frac{2}{s} - 2} u_n \frac{F(u_n)}{f(u_n)} \, dx$$

$$+ \frac{2}{s} c_\alpha - \|u_n\|_{\frac{2}{s}}^{\frac{2}{s}} + o_n(1)$$

$$\leq \frac{2^{\frac{2}{s}}!}{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor} \|W\|_\infty \frac{s}{\tau} + \int\limits_{\mathbb{R}^2} |u_n|^{\frac{2}{s}} \left(1 - \frac{F(u_n)}{s f(u_n) u_n} \right) \, dx + \frac{1}{s} \int\limits_{\mathbb{R}^2} |u_n|^{\frac{2}{s} - 2} u_n \frac{F(u_n)}{f(u_n)} \, dx$$

$$+ \frac{2}{s} c_\alpha - \|u_n\|_{\frac{2}{s}}^{\frac{2}{s}} + o_n(1)$$

$$\leq \frac{2^{\frac{2}{s} \frac{2}{s}}!}{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor} \frac{s}{\tau} \left(\left\| \frac{F f'}{f^2} \right\|_\infty - 1 \right) + \frac{1}{2} + o_n(1) \, ,$$

having used Remark 2.3(ii) and the estimate of c_α in Lemma 5.5. In order to prove the claim (5.27), we need to check that the constant on the right-hand side is strictly less than 1. This is equivalent to the following

$$\left\| \frac{F f'}{f^2} \right\|_\infty < 1 + \frac{1}{2} \frac{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor}{\frac{2}{s} \frac{2}{s}!} \frac{\tau}{s}. \tag{5.34}$$

Defining now

$$\mu(s, \tau) := \min \left\{ \frac{s}{2}, \frac{1}{2} \frac{\frac{2}{s} - \lfloor \frac{2}{s} \rfloor}{\frac{2}{s} \frac{2}{s}!} \frac{\tau}{s} \right\}, \tag{5.35}$$

assumption (f_3) guarantees that both conditions (5.32) and (5.34) are satisfied.

Next we aim at estimating the L^1 norm of a suitable power of $(F(u_n))_n$ by using (5.27). By (f_4), for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\frac{F(t)}{f(t)} \leq \varepsilon(t - t_\varepsilon) + \frac{F(t_\varepsilon)}{f(t_\varepsilon)} \quad \text{for all } t \geq t_\varepsilon.$$

Hence, by (f_3) one has

$$v_n = G(u_n) = \left(1 - \frac{\varepsilon}{s}\right)(u_n - t_\varepsilon) + t_\varepsilon - \frac{1}{s} \frac{F(t_\varepsilon)}{f(t_\varepsilon)} \geq \left(1 - \frac{\varepsilon}{s}\right)(u_n - t_\varepsilon), \tag{5.36}$$

which implies that for all $x \in \mathbb{R}^2$

$$u_n(x) \leq t_\varepsilon + \frac{v_n(x)}{1 - \bar{\varepsilon}}, \tag{5.37}$$

where $\bar{\varepsilon} := \frac{\varepsilon}{s}$. Hence, by (f_1)-(f_2) we have that for any given $\varepsilon > 0$, there exists C_ε such that

$$\int_{\mathbb{R}^2} F(u_n)^\kappa \, dx \leq \int_{|u_n| < t_\varepsilon} F(u_n)^\kappa \, dx + \int_{|u_n| \geq t_\varepsilon} F(u_n)^\kappa \, dx$$

$$\leq C_\varepsilon \|u_n\|^{\frac{2}{s}\kappa} + C_\varepsilon \int_{|u_n| \geq t_\varepsilon} \Phi_{2,s}\left(\alpha_* \kappa \left(t_\varepsilon + \frac{v_n}{1 - \bar{\varepsilon}}\right)^{\frac{2}{2-s}}\right) dx. \tag{5.38}$$

Moreover,

$$\Phi_{2,s}\left(\alpha_* \kappa \left(t_\varepsilon + \frac{v_n}{1 - \bar{\varepsilon}}\right)^{\frac{2}{2-s}}\right) \leq C_\varepsilon \Phi_{2,s}\left(\alpha_* \kappa (1 + \bar{\varepsilon}) \left(\frac{v_n}{1 - \bar{\varepsilon}}\right)^{\frac{2}{2-s}}\right).$$

In view of (5.36) and Remark 2.3(ii), $v_n \geq \frac{\tau}{s} t_\varepsilon$ if $u_n \geq t_\varepsilon$, and then it follows from (5.38) that

$$\int\limits_{\mathbb{R}^2} F(u_n)^\kappa \, dx \le C_\varepsilon \|u_n\|^{\frac{2}{s}\kappa} + C_\varepsilon \int\limits_{|u_n| \ge t_\varepsilon} \Phi_{2,s}\left(\alpha_* \kappa (1+\bar{\varepsilon})\left(\frac{v_n}{1-\bar{\varepsilon}}\right)^{\frac{2}{2-s}}\right) dx. \tag{5.39}$$

Since $\|v_n\|^{\frac{2}{s}} \le \gamma_{s,\tau} + \sigma < 1$ for n large enough and $\sigma > 0$ small enough by (5.27), the following holds

$$\frac{\kappa(1+\bar{\varepsilon})}{(1-\bar{\varepsilon})^{\frac{2}{2-s}}} \|v_n\|^{\frac{2}{2-s}} \le \frac{\kappa(1+\bar{\varepsilon})}{(1-\bar{\varepsilon})^{\frac{2}{2-s}}}\left(\gamma_{s,\tau} + \sigma\right) < 1 \tag{5.40}$$

for $\varepsilon > 0$ small enough and $\kappa \in \left[1, \frac{1}{\gamma_{s,\tau}}\right)$. As a consequence, from (5.39) we obtain

$$\int\limits_{\mathbb{R}^2} F(u_n)^\kappa \, dx \le C$$

for some C independent of n and α. Similarly, one can also prove that

$$\int\limits_{\mathbb{R}^2} f(u_n) u_n \, dx \le C$$

for some C independent of n and α. $\quad\square$

Now we are in the position to prove the existence of a nontrivial critical point for the approximating functional I_α, for α sufficiently small.

Proposition 5.10. *Assume that (f_1)–(f_5) hold. For all $\alpha \in (0,1)$ sufficiently small there exists a positive $u_\alpha \in W_r^{s,\frac{2}{s}}(\mathbb{R}^2)$ such that $I_\alpha'(u_\alpha) = 0$.*

Proof. By Lemma 5.2 and Remark 5.7, for all $\alpha \in (0,1)$ there exists a nonnegative Cerami sequence $(u_n^\alpha)_n \subset W_r^{s,\frac{2}{s}}(\mathbb{R}^2)$ of I_α at level c_α, which is bounded in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ by Lemma 5.6. Hence there exists a nonnegative $u_\alpha \in W_r^{s,\frac{2}{s}}(\mathbb{R}^2)$ such that, up to a subsequence,

$$\begin{aligned}
u_n^\alpha &\rightharpoonup u_\alpha \quad \text{in } W^{s,\frac{2}{s}}(\mathbb{R}^2), \\
u_n^\alpha &\to u_\alpha \quad \text{a.e. in } \mathbb{R}^2, \\
u_n^\alpha &\to u_\alpha \quad \text{in } L^p(\mathbb{R}^2), \quad \text{for all } p \in \left(\tfrac{2}{s}, +\infty\right),
\end{aligned} \tag{5.41}$$

as $n \to +\infty$. Note that the compactness in the Lebesgue spaces is retrieved because we are restricting to the space of radially symmetric functions in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$, see Lemma 2.4. By the definition of $(u_n^\alpha)_n$ one has

$$\|u_n^\alpha\|^2 + \frac{1}{2\pi\alpha} \int\limits_{\mathbb{R}^2} F(u_n^\alpha) \, dx \int\limits_{\mathbb{R}^2} f(u_n^\alpha) u_n^\alpha \, dx = 2c_\alpha + \frac{1}{2\pi\alpha} \int\limits_{\mathbb{R}^2}\left(\frac{1}{|x|^\alpha} * F(u_n^\alpha)\right) f(u_n^\alpha) u_n^\alpha \, dx + o_n(1).$$

Lemmas 5.6 and 5.8 yield $C > 0$ independent of n and α such that

$$\int_{\mathbb{R}^2} \left(\frac{1}{|x|^\alpha} * F(u_n^\alpha) \right) f(u_n^\alpha(x)) u_n^\alpha(x) \, dx \leq C \, . \tag{5.42}$$

Next we *claim* that for any $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \left(\frac{1}{|x|^\alpha} * F(u_n^\alpha) \right) f(u_n^\alpha(x)) \varphi(x) \, dx \to \int_{\mathbb{R}^2} \left(\frac{1}{|x|^\alpha} * F(u_\alpha) \right) f(u_\alpha(x)) \varphi(x) \, dx \, , \tag{5.43}$$

as $n \to \infty$. Indeed, first by Lemma 5.8, for $x \in \mathbb{R}^2$ one has

$$\int_{\mathbb{R}^2} \frac{F(u_n^\alpha(y))}{|x-y|^\alpha} \, dy \leq \int_{|x-y| \leq 1} \frac{F(u_n^\alpha(y))}{|x-y|^\alpha} \, dy + \int_{|x-y| > 1} \frac{F(u_n^\alpha(y))}{|x-y|^\alpha} \, dy$$

$$\leq \left(\int_{|x-y| \leq 1} \frac{dy}{|x-y|^{\frac{3}{2}}} \right)^{\frac{2\alpha}{3}} \left(\int_{\mathbb{R}^2} F(u_n^\alpha(y))^{\frac{3}{3-2\alpha}} \, dy \right)^{\frac{3-2\alpha}{3}} + \int_{\mathbb{R}^2} F(u_n^\alpha(y)) \, dy \leq C, \tag{5.44}$$

provided $\alpha > 0$ is small enough. Hence, in order to have (5.43), it is sufficient to apply [22, Lemma 2.1] to the sequence

$$g(x, u_n^\alpha(x)) := \int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} F(u_n^\alpha(y)) \, dy \, f(u_n^\alpha(x))$$

restricted to any bounded domain Ω. Indeed, $g(\cdot, u_n^\alpha) \in L^1(\Omega)$ by (5.44) and (f_2). Similarly, we can also prove

$$f(u_n^\alpha) \to f(u_\alpha) \quad \text{in } L_{loc}^1(\mathbb{R}^2) \, . \tag{5.45}$$

Moreover, by (5.26), (5.37), (5.27), and using the mean value theorem, we deduce

$$\int_{\mathbb{R}^2} |F(u_n^\alpha) - F(u_\alpha)| \, dx = \int_{\mathbb{R}^2} |f(u_\alpha + \tau_n(x)(u_n^\alpha - u_\alpha))(u_n^\alpha - u_\alpha)| \, dx$$

$$\leq \varepsilon \int_{\mathbb{R}^2} (|u_n^\alpha| + |u_\alpha|)^{\frac{2}{s}-1} |u_n^\alpha - u_\alpha| \, dx$$

$$+ C_\varepsilon \int_{\mathbb{R}^2} \Phi_{2,s} \left(\alpha_* \left(u_\alpha + \tau_n(x)(u_n^\alpha - u_\alpha) \right)^{\frac{2}{2-s}} \right) |(u_n^\alpha + u_\alpha)(u_n^\alpha - u_\alpha)| \, dx$$

$$\leq \varepsilon \left(\|u_n^\alpha\|_{\frac{2}{s}} + \|u_\alpha\|_{\frac{2}{s}} \right) \|u_n^\alpha - u_\alpha\|_{\frac{2}{s}}$$

$$+ C_\varepsilon \left(\int\limits_{\{u_n^\alpha > u_\alpha\}} \Phi_{2,s} \left(\alpha_* |u_n^\alpha|^{\frac{2}{2-s}} \right) |(u_n^\alpha + u_\alpha)(u_n^\alpha - u_\alpha)| \, dx \right.$$

$$\left. + \int\limits_{\{u_n^\alpha \le u_\alpha\}} \Phi_{2,s} \left(\alpha_* |u_\alpha|^{\frac{2}{2-s}} \right) |(u_n^\alpha + u_\alpha)(u_n^\alpha - u_\alpha)| \, dx \right) \tag{5.46}$$

$$\le C\varepsilon + C_\varepsilon \left(\int\limits_{\{u_n^\alpha > u_\alpha\}} \Phi_{2,s} \left(\alpha_* \left| t_\varepsilon + \frac{v_n}{1-\bar\varepsilon} \right|^{\frac{2}{2-s}} \right) |(u_n^\alpha + u_\alpha)(u_n^\alpha - u_\alpha)| \, dx \right)$$

$$\le C\varepsilon + C\|u_n^\alpha\|_{\theta'\eta} \left(\int\limits_{\mathbb{R}^2} \Phi_{2,s} \left(\alpha_* \theta' \eta \left| t_\varepsilon + \frac{v_n}{1-\bar\varepsilon} \right|^{\frac{2}{2-s}} \right) \right)^{\frac{1}{\theta'\eta'}} \left(\int\limits_{\mathbb{R}^2} |u_n^\alpha - u_\alpha|^\theta \, dx \right)^{\frac{1}{\theta}}$$

$$\le C\varepsilon + C_\varepsilon \left(\int\limits_{\mathbb{R}^2} |u_n^\alpha - u_\alpha|^\theta \, dx \right)^{\frac{1}{\theta}} = C\varepsilon + o_n(1),$$

where $\tau_n(x) \in (0,1)$, $\bar\varepsilon = \frac{\varepsilon}{s}$, and for ε small enough. Note that in the second-to-last inequality we have used Hölder's inequality with $\theta > \frac{2}{s}$ and $\theta'\eta > \frac{2}{s}$. Similarly, we can also infer

$$\int\limits_{\mathbb{R}^2} |f(u_n^\alpha)(u_n^\alpha - u_\alpha)| \, dx \to 0, \quad \text{as } n \to \infty. \tag{5.47}$$

Consequently, combining (5.41), (5.46), (5.43) and (5.45) we get $I_\alpha'(u_\alpha) = 0$. Finally, we need to show that u_α is not trivial, by proving that $I_\alpha(u_\alpha) \ne 0$. First, (5.44) and (5.47) imply

$$\left| \int\limits_{\mathbb{R}^2} \left[\left(\frac{1}{|x|^\alpha} * F(u_n^\alpha) \right) f(u_n^\alpha(x)) - \left(\frac{1}{|x|^\alpha} * F(u_\alpha) \right) f(u_\alpha(x)) \right] (u_n^\alpha(x) - u_\alpha(x)) \, dx \right|$$

$$\le \left| \int\limits_{\mathbb{R}^2} \left[\left(\frac{1}{|x|^\alpha} * F(u_n^\alpha) \right) - \left(\frac{1}{|x|^\alpha} * F(u_\alpha) \right) \right] f(u_n^\alpha(x))(u_n^\alpha(x) - u_\alpha(x)) \, dx \right|$$

$$+ \left| \int\limits_{\mathbb{R}^2} \left(\frac{1}{|x|^\alpha} * F(u_\alpha) \right) \left[f(u_n^\alpha(x)) - f(u_\alpha(x)) \right] (u_n^\alpha(x) - u_\alpha(x)) \, dx \right|$$

$$= o_n(1).$$

This, together with (5.46) and (5.47), implies that

$$o_n(1) = (I_\alpha'(u_n^\alpha) - I_\alpha'(u_\alpha))(u_n^\alpha - u_\alpha)$$

$$\ge C\|u_n^\alpha - u_\alpha\|^{\frac{2}{s}} + \frac{1}{\alpha} \left[\int\limits_{\mathbb{R}^2} F(u_n^\alpha) dx \int\limits_{\mathbb{R}^2} f(u_n^\alpha) u_n^\alpha \, dx - \int\limits_{\mathbb{R}^2} F(u_\alpha) dx \int\limits_{\mathbb{R}^2} f(u_\alpha) u_\alpha \, dx \right]$$

$$+ \frac{1}{\alpha} \left| \int\limits_{\mathbb{R}^2} \left[\left(\frac{1}{|x|^\alpha} * F(u_n^\alpha) \right) f(u_n^\alpha(x)) \right. \right.$$

$$\left. \left. - \left(\frac{1}{|x|^\alpha} * F(u_\alpha) \right) f(u_\alpha(x)) \right] (u_n^\alpha(x) - u_\alpha(x)) \, \mathrm{d}x \right|$$

$$= C \|u_n^\alpha - u_\alpha\|^{\frac{2}{s}} + o_n(1) \, .$$

Note that we used the monotonicity of the $(s, \frac{2}{s})$-Laplacian, which relies on the well-known inequality

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq C_p |a - b|^p \, ,$$

which holds for all $p \geq 2$ and $a, b \in \mathbb{R}^N$, see e.g. [40]. As a consequence, $u_n^\alpha \to u_\alpha$ in $W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ as $n \to +\infty$, and therefore $I(u_\alpha) = c_\alpha \neq 0$, which implies that $u_\alpha \neq 0$. Hence, using the strong maximum principle for the p-fractional Laplacian [19, Theorem 1.4] we eventually obtain that $u_\alpha > 0$ in $\mathbb{R}^2$. $\square$

So far, we have proved that there exists $\bar{\alpha} \in (0, 1)$ such that, for all $\alpha \in (0, \bar{\alpha})$, the approximating functional I_α has a positive critical point in $W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$. The last step consists in showing that actually the sequence $(u_\alpha)_\alpha$ converges, up to a subsequence, to a nontrivial critical point of the original functional I.

Thanks to the a-priori bound (5.17), which is independent of α, we have that $(u_\alpha)_\alpha$ is uniformly bounded in $W^{s, \frac{2}{s}}(\mathbb{R}^2)$ for $\alpha \in (0, \bar{\alpha})$ with

$$\|u_\alpha\|^{\frac{2}{s}} < \frac{s}{\tau} \, . \tag{5.48}$$

Hence, up to a subsequence, there exists $u_0 \in W_r^{s, \frac{2}{s}}(\mathbb{R}^2)$ such that

$$\begin{aligned}
u_\alpha &\rightharpoonup u_0 \quad \text{in } W^{s, \frac{2}{s}}(\mathbb{R}^2), \\
u_\alpha &\to u_0 \quad \text{a.e. in } \mathbb{R}^2, \\
u_\alpha &\to u_0 \quad \text{in } L^p(\mathbb{R}^2), \quad \text{for all } p \in \left(\frac{2}{s}, +\infty \right),
\end{aligned} \tag{5.49}$$

as $\alpha \to 0^+$ by Lemma 2.4.

Let us first prove some regularity results for u_α. The strategy is similar to the one employed in Section 3 and thus here we just highlight the differences.

Lemma 5.11. *For any* $\omega \in \left[1, \frac{1}{\gamma_{s,\tau}} \right)$ *there exist* $\bar{\alpha} \in (0, 1)$ *and a constant* $C > 0$, *independent of* α, *such that*

$$\int\limits_{|x-y| \leq 1} \frac{F(u_\alpha(y))}{|x - y|^{\frac{4(\omega-1)}{3\omega}}} \, \mathrm{d}y \leq C \, ,$$

and

$$\int_{|x-y|\leq 1} \frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \, dy \to 0 \quad as \;\; |x| \to +\infty, \tag{5.50}$$

uniformly for $\alpha \in (0, \bar{\alpha})$.

Proof. Let us define $v_\alpha := G(u_\alpha)$, as in Lemma 5.8. Using a similar argument, we deduce that $\sup_{\alpha \in (0,\bar\alpha)} \|v_\alpha\| < 1$, and that for all $\omega \in \left[1, \frac{1}{\gamma_{s,\tau}}\right)$ there exists $C > 0$ independent of α such that

$$\int_{\mathbb{R}^2} F(u_\alpha)^\omega \, dx < C.$$

Hence, by Hölder's inequality,

$$\int_{\{|x-y|\leq 1\}} \frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \, dy \leq \left(\int_{\{|x-y|\leq 1\}} \frac{dy}{|x-y|^{\frac{4}{3}}} \right)^{\frac{\omega-1}{\omega}} \left(\int_{\{|x-y|\leq 1\}} F(u_\alpha)^\omega \, dy \right)^{\frac{1}{\omega}} \leq C, \tag{5.51}$$

where C is independent of α. Moreover, again as in Lemma 5.8, for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$u_\alpha(x) \leq t_\varepsilon + \frac{v_\alpha(x)}{1 - \bar\varepsilon} \quad \text{for any } x \in \mathbb{R}^2, \tag{5.52}$$

with $\bar\varepsilon := \frac{\varepsilon}{s}$, which implies that there exists $C_\varepsilon > 0$ such that

$$u_\alpha^{\frac{2}{2-s}} \leq C_\varepsilon t_\varepsilon^{\frac{2}{2-s}} + (1 + \bar\varepsilon)^{\frac{2}{2-s}} \frac{v_\alpha^{\frac{2}{2-s}}}{(1 - \bar\varepsilon)^{\frac{2}{2-s}}}.$$

Therefore, in order to show (5.50), we can combine (5.51) and (5.52) and get

$$\int_{|x-y|\leq 1} \frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \, dy$$

$$\leq C_\varepsilon \left(\int_{|x-y|\leq 1} |u_\alpha|^{\frac{2}{s}\omega} + |u_\alpha|^{\frac{2}{s}\omega} \Phi_{2,s}\left(\alpha_*\omega(1+\bar\varepsilon)^{\frac{2}{2-s}} \frac{v_\alpha^{\frac{2}{2-s}}}{(1-\bar\varepsilon)^{\frac{2}{2-s}}} \right) dy \right)^{\frac{1}{\omega}}$$

$$\leq C_\varepsilon \left(\int_{B_1(x)} |u_\alpha|^{\frac{2}{s}\omega} dy + \left(\int_{B_1(x)} |u_\alpha|^{\frac{2}{s}\omega\zeta} dy \right)^{\frac{1}{\zeta}} \left(\int_{\mathbb{R}^2} \Phi_{2,s}\left(\alpha_*\omega\zeta'(1+\bar\varepsilon)^{\frac{2}{2-s}} \frac{v_\alpha^{\frac{2}{2-s}}}{(1-\bar\varepsilon)^{\frac{2}{2-s}}} \right) dy \right)^{\frac{1}{\zeta'}} \right)^{\frac{1}{\omega}}$$

$$\leq C_\varepsilon \left(\int_{B_1(x)} |u_\alpha|^{\frac{2}{s}\omega} dy + C_\varepsilon \left(\int_{B_1(x)} |u_\alpha|^{\frac{2}{s}\omega\zeta} dy \right)^{\frac{1}{\zeta}} \right)^{\frac{1}{\omega}},$$

where $\zeta' = \frac{\zeta}{\zeta-1}$, and in the last inequality we choose $\omega\zeta'$ sufficiently close to 1 and we take advantage of the fact that $\|v_\alpha\| < 1$. Thanks to (5.49) we obtain (5.50) and conclude the proof. $\quad\square$

Lemma 5.12. *Let $\alpha \in (0, \bar{\alpha})$ and $u_\alpha \in W^{s, \frac{2}{s}}(\mathbb{R}^2)$ be a weak solution of* (5.2), *then there exists $C > 0$ independent of α such that $\|u_\alpha\|_\infty \leq C$. Furthermore, $u_\alpha \in C^v_{\mathrm{loc}}(\mathbb{R}^2)$ for some $v \in (0, 1)$ and there exists $R > 0$ such that for $|x| \geq R$,*

$$|u_\alpha(x)| \leq \frac{C}{1 + |x|^{\frac{7s}{2(2-s)}}}$$

for some $C > 0$ uniformly for $\alpha \in (0, \frac{4(\omega - 1)}{3\omega})$.

Proof. Since u_α is a nonnegative function, by (5.2) and Lemma 5.1 we obtain

$$
\begin{aligned}
(-\Delta)^{\frac{s}{2}}_{\frac{2}{s}} u_\alpha + |u_\alpha|^{\frac{2}{s}-2} u_\alpha &\leq \left(\int\limits_{|x-y|\leq 1} \frac{|x-y|^{-\alpha} - 1}{\alpha} F(u_\alpha(y)) \, \mathrm{d}y \right) f(u_\alpha(x)) \\
&\leq \int\limits_{|x-y|\leq 1} \frac{F(u_\alpha(y)) \, \mathrm{d}y}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} f(u_\alpha(x))
\end{aligned}
\tag{5.53}
$$

weakly in $W^{s, \frac{2}{s}}(\mathbb{R}^2)$. For any $L > 0$ and $\gamma > 1$ consider the functions ψ and Ψ defined in (3.2). The proof follows the same line as in Lemma 3.2. Taking $\psi(u_\alpha) = u_\alpha u_{L,\alpha}^{\frac{2}{s}(\gamma-1)}$ as a test function in (5.53), we obtain

$$[\Psi(u_\alpha)]^{\frac{2}{s}}_{s, \frac{2}{s}}$$

$$
\leq \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{|u_\alpha(x) - u_\alpha(y)|^{\frac{2}{s}-2}(u_\alpha(x) - u_\alpha(y))[u_\alpha(x)u_{L,\alpha}^{\frac{2(\gamma-1)}{s}}(x) - u_\alpha(y)u_{L,\alpha}^{\frac{2(\gamma-1)}{s}}(y)]}{|x-y|^4} \, \mathrm{d}y \, \mathrm{d}x
$$

$$
\leq \iint\limits_{|x-y|\leq 1} \frac{F(u_\alpha(y)) \mathrm{d}y}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} f(u_\alpha(x)) u_\alpha u_{L,\alpha}^{\frac{2}{s}(\gamma-1)} \, \mathrm{d}x - \int\limits_{\mathbb{R}^2} |u_\alpha|^{\frac{2}{s}} u_{L,\alpha}^{\frac{2}{s}(\gamma-1)} \, \mathrm{d}x . \tag{5.54}
$$

By (f_1)-(f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$f(u_\alpha) \leq \varepsilon |u_\alpha|^{\frac{2}{s}-1} + C_\varepsilon |u_\alpha|^{\frac{2}{s}-1} \Phi_{2,s}(\alpha_* |u_\alpha|^{\frac{2}{2-s}}).$$

So, for $\varepsilon > 0$ sufficiently small, using Young's inequality, Proposition 2.1, Lemma 5.11, and (5.54), we get

$$[\Psi(u_\alpha)]^{\frac{2}{s}}_{s, \frac{2}{s}} \leq (C\varepsilon - 1)\|u_\alpha u_{L,\alpha}^{\gamma-1}\|^{\frac{2}{s}}_{\frac{2}{s}} + C_\varepsilon \|u_\alpha u_{L,\alpha}^{\gamma-1}\|^{\frac{2}{s}}_{\frac{2p}{s}}, \tag{5.55}$$

where p is sufficiently large. Noting that all the constants are independent of α, we can follow the argument of Lemma 3.2 and obtain $\|u_\alpha\|_\infty \leq C$, with C independent of α. Finally, the decay estimate and Hölder's regularity can be obtained following step by step the proofs of Lemmas 3.4 and 3.6, paying only attention to the fact that G_α substitutes the logarithmic Riesz kernel. □

We are finally in a position to prove Theorem 1.6, namely the existence of a weak solution of (Ch_s). Next we prove that the sequence of solutions $(u_\alpha)_\alpha$ for the approximated problems (5.2) has a nontrivial accumulation point $u_0 \in W^{s,\frac{2}{s}}(\mathbb{R}^2)$ satisfying $I'(u_0) = 0$.

Proof of Theorem 1.6. Let us divide the proof into two steps.

Step 1. We show that $u_0 \in W_r^{s,\frac{2}{s}}(\mathbb{R}^2)$ satisfies $I'(u_0) = 0$.

For any $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have

$$
I'_\alpha(u_\alpha)\varphi = \iint\limits_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|u_\alpha(x) - u_\alpha(y)|^{\frac{2}{s}-2}(u_\alpha(x) - u_\alpha(y))(\varphi(x) - \varphi(y))}{|x - y|^4} \, \mathrm{d}y \, \mathrm{d}x
$$

$$
+ \int\limits_{\mathbb{R}^2} |u_\alpha|^{\frac{2}{s}-2} u_\alpha \varphi \, \mathrm{d}x - \frac{1}{2\pi} \iint\limits_{\mathbb{R}^2\,\mathbb{R}^2} \frac{|x - y|^{-\alpha} - 1}{\alpha} F(u_\alpha(y)) \, \mathrm{d}y f(u_\alpha(x)) \varphi \, \mathrm{d}x . \quad (5.56)
$$

The first two terms are easy to handle thanks to the convergence in (5.49). Thus we focus on the third term and let us split $\mathbb{R}^2 = \{|x - y| \leq 1\} \cup \{|x - y| > 1\}$. On the one hand, let $\omega \in \left(1, \frac{1}{\gamma_{s,\tau}}\right)$ be fixed, then by Lemmas 5.1 and 5.12, together with the continuity of f, for all $x, y \in \mathbb{R}^2$ such that $|x - y| \leq 1$ we get

$$
\left| \frac{|x - y|^{-\alpha} - 1}{\alpha} F(u_\alpha(y)) f(u_\alpha(x)) \varphi(x) \right| \leq \frac{C_\omega}{|x - y|^{\frac{4(\omega-1)}{3\omega}}} F(u_\alpha(y)) f(u_\alpha(x)) |\varphi(x)|
$$

$$
\leq \frac{C}{|x - y|^{\frac{4(\omega-1)}{3\omega}}} |\varphi(x)|
$$

$$(5.57)$$

for $\alpha < \frac{4(\omega-1)}{3\omega}$, which is readily integrable in $\{(x, y) \in \mathbb{R}^2 \mid |x - y| \leq 1\}$. Thus, recalling (5.49), the Lebesgue dominated convergence theorem implies

$$
\iint\limits_{|x-y|\leq 1} \frac{|x - y|^{-\alpha} - 1}{\alpha} F(u_\alpha(y)) \, \mathrm{d}y f(u_\alpha(x)) \varphi(x) \, \mathrm{d}x
$$

$$
\to \iint\limits_{|x-y|\leq 1} \ln \frac{1}{|x - y|} F(u_0(y)) \, \mathrm{d}y f(u_0(x)) \varphi(x) \, \mathrm{d}x
$$

$$(5.58)$$

as $\alpha \to 0^+$. On the other hand, when $|x - y| \geq 1$, there exists $\tau = \tau(|x - y|) \in (0, 1)$ such that

$$
0 \geq G_\alpha(x - y) = \frac{|x - y|^{-\alpha} - 1}{\alpha} = -|x - y|^{-\tau\alpha} \ln |x - y|, \quad (5.59)
$$

applying the mean value theorem to the function $h_b(\alpha) = b^{-\alpha}$ with $b = |x - y|$. Since φ has a compact support, it follows from Lemma 5.12, (f_1), and the monotonicity of F, that

$$\left| \frac{|x-y|^{-\alpha}-1}{\alpha} F(u_\alpha(y)) f(u_\alpha(x)) \varphi(x) \right|$$

$$= |x-y|^{-\tau\alpha} \ln|x-y| F(u_\alpha(y)) f(u_\alpha(x)) |\varphi(x)|$$

$$\leq (|x|+|y|) F(u_\alpha(y)) f(u_\alpha(x)) |\varphi(x)| \tag{5.60}$$

$$\leq (1+|y|) \frac{C}{1+|y|^{\frac{7}{2-s}}} |\varphi(x)|$$

$$\leq \frac{C}{1+|y|^{\frac{5}{2}}} |\varphi(x)|$$

which is readily integrable on $\{(x,y) \in \mathbb{R}^2 \,|\, |x-y| \geq 1\}$. Hence, again by the Lebesgue dominated convergence theorem, this implies

$$\iint\limits_{|x-y|\geq 1} \frac{|x-y|^{-\alpha}-1}{\alpha} F(u_\alpha(y)) \, dy f(u_\alpha(x)) \varphi(x) \, dx$$

$$\to - \iint\limits_{|x-y|\geq 1} \ln|x-y| F(u_0(y)) \, dy f(u_0(x)) \varphi(x) \, dx. \tag{5.61}$$

Finally, we need to prove the finiteness of

$$\int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \ln|x-y| F(u_0(y)) \, dy F(u_0(x)) \, dx.$$

By Fatou's lemma, we have

$$\left| \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \ln|x-y| F(u_0(y)) \, dy \, F(u_0(x)) \, dx \right|$$

$$\leq \liminf_{\alpha\to 0} \left(\iint\limits_{|x-y|\leq 1} G_\alpha(x-y) F(u_\alpha(y)) \, dy \, F(u_\alpha(x)) \, dx \right. \tag{5.62}$$

$$\left. - \iint\limits_{|x-y|\geq 1} G_\alpha(x-y) F(u_\alpha(y)) \, dy \, F(u_\alpha(x)) \, dx \right).$$

Using Lemma 5.1 with $\nu = \frac{4(\omega-1)}{3\omega}$, Lemma 5.8 with $\kappa = 1$, and Lemma 5.11, we have

$$\iint\limits_{|x-y|\leq 1} G_\alpha(x-y) F(u_\alpha(y)) \, dy \, F(u_\alpha(x)) \, dx$$

$$\leq C \int\limits_{\mathbb{R}^2} F(u_\alpha(x)) \left(\int\limits_{|x-y|\leq 1} \frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \, dy \right) dx \leq C \tag{5.63}$$

uniformly for $\alpha \in \left(0, \frac{4(\omega-1)}{3\omega}\right)$. So, by Remark 5.3 and (5.63) we deduce

$$
\begin{aligned}
&-\frac{1}{4\pi} \iint_{|x-y|\geq 1} G_\alpha(x-y)F(u_\alpha(y))\,\mathrm{d}y\,F(u_\alpha(x))\,\mathrm{d}x \\
&\leq I_\alpha(u_\alpha) + \frac{1}{4\pi} \iint_{|x-y|\leq 1} G_\alpha(x-y)F(u_\alpha(y))\,\mathrm{d}y\,F(u_\alpha(x))\,\mathrm{d}x - \frac{s}{2}\|u_\alpha\|^{\frac{2}{s}} \leq C
\end{aligned}
\tag{5.64}
$$

uniformly for α sufficiently small. Combining (5.62)-(5.64), we have

$$
\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| F(u_0(y))\,\mathrm{d}y\,F(u_0(x))\,\mathrm{d}x \right| < +\infty.
\tag{5.65}
$$

Based on (5.58), (5.61) and (5.65), by taking the limit as $\alpha \to 0^+$ in (5.56), we have $I'(u_0) = 0$ with $I(u_0) < +\infty$, that is, $u_0 \in W_r^{s,\frac{2}{s}}(\mathbb{R}^2)$ solves equation (Ch$_s$).

Step 2. To conclude the proof, we need to show that $u_0 \neq 0$ and that $u_\alpha \to u_0$ in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$. Assume on the contrary that $u_\alpha \rightharpoonup 0$ in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$, and so $u_\alpha \to 0$ in $L^t(\mathbb{R}^2)$ for $t \in (\frac{2}{s}, +\infty)$. Similarly to (5.47), we obtain $\int_{\mathbb{R}^2} f(u_\alpha)u_\alpha\,\mathrm{d}x = o_\alpha(1)$. Hence, by Lemma 5.1 and Lemma 5.11, we have

$$
\begin{aligned}
0 = I'_\alpha(u_\alpha)u_\alpha &= \|u_\alpha\|^{\frac{2}{s}} - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\alpha(x-y)F(u_\alpha(y))f(u_\alpha(x))u_\alpha(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \|u_\alpha\|^{\frac{2}{s}} - \frac{1}{2\pi}\iint_{\{|x-y|\leq 1\}} G_\alpha(x-y)F(u_\alpha(y))f(u_\alpha(x))u_\alpha(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \|u_\alpha\|^{\frac{2}{s}} - \frac{1}{2\pi}\iint_{\{|x-y|\leq 1\}} \frac{1}{|x-y|^{\frac{4(\omega-1)}{\omega}}} F(u_\alpha(y))f(u_\alpha(x))u_\alpha(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \|u_\alpha\|^{\frac{2}{s}} - C\int_{\mathbb{R}^2} f(u_\alpha(x))u_\alpha(x)\,\mathrm{d}x \\
&= \|u_\alpha\|^{\frac{2}{s}} + o_\alpha(1),
\end{aligned}
$$

which yields $u_\alpha \to 0$ in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ as $\alpha \to 0^+$. Then, using Remark 5.3 and (5.59), we have

$$
\begin{aligned}
a \leq c_\alpha + o_\alpha(1) &= I_\alpha(u_\alpha) \\
&= \frac{s}{2}\|u_\alpha\|^{\frac{2}{s}} - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\alpha(x-y)F(u_\alpha(y))F(u_\alpha(x))\,\mathrm{d}x\,\mathrm{d}y \\
&\leq -\frac{1}{4\pi}\iint_{\{|x-y|\geq 1\}} G_\alpha(x-y)F(u_\alpha(y))F(u_\alpha(x))\,\mathrm{d}x\,\mathrm{d}y + o_\alpha(1)
\end{aligned}
$$

$$= -\frac{1}{4\pi} \iint_{\{|x-y|\geq 1\}} |x-y|^{-\tau(x,y)\alpha} \ln|x-y| F(u_\alpha(y)) F(u_\alpha(x)) \,\mathrm{d}x\,\mathrm{d}y + o_\alpha(1)$$

$$\leq \frac{1}{4\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| F(u_\alpha(y)) F(u_\alpha(x)) \,\mathrm{d}x\,\mathrm{d}y + o_\alpha(1)$$

$$\leq \frac{\|F(u_\alpha)\|_1}{2\pi} \int_{\mathbb{R}^2} |x| F(u_\alpha(x)) \,\mathrm{d}x + o_\alpha(1)$$

$$\leq C\left(R \int_{|x|\leq R} F(u_\alpha(x)) \,\mathrm{d}x + \int_{|x|\geq R} |x| F(u_\alpha(x)) \,\mathrm{d}x \right) + o_\alpha(1)$$

$$\leq C\left(\int_{|x|\leq R} |u_\alpha|^{\frac{2}{s}} \,\mathrm{d}x + \int_{|x|\geq R} \frac{C|x|}{1+|x|^{\frac{7}{2-s}}} \,\mathrm{d}x \right) + o_\alpha(1)$$

$$= o_\alpha(1),$$

which yields a contradiction. Note that in the last inequality we have used Remark 2.3(i) with (5.48) to estimate the first term, and (f_1) together with the monotonicity of F and the decay given by Lemma 5.12, to estimate the second term. Finally, similarly to (5.58) and (5.61), by Lemma 5.12 and the Lebesgue dominated convergence theorem, we have

$$\iint_{\mathbb{R}^2}\iint_{\mathbb{R}^2} \frac{|x-y|^{-\alpha}-1}{\alpha} F(u_\alpha(y)) f(u_\alpha(x)) u_\alpha(x) \,\mathrm{d}x\,\mathrm{d}y$$

$$\to -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| F(u_0(y)) f(u_0(x)) u_0(x) \,\mathrm{d}x\,\mathrm{d}y,$$

(5.66)

from which we conclude that $u_\alpha \to u_0$ in $W^{s,\frac{2}{s}}(\mathbb{R}^2)$ as $\alpha \to 0^+$. $\square$

Data availability

No data was used for the research described in the article.

References

[1] C.O. Alves, D. Cassani, C. Tarsi, M. Yang, Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in $\mathbb{R}^2$, J. Differ. Equ. 261 (3) (2016) 1933–1972.

[2] C.O. Alves, G. Figueiredo, Existence of positive solution for a planar Schrödinger-Poisson system with exponential growth, J. Math. Phys. 60 (1) (2019) 011503.

[3] V. Ambrosio, T. Isernia, Multiplicity and concentration results for some nonlinear Schrödinger equations with the fractional p-Laplacian, Discrete Contin. Dyn. Syst. 38 (11) (2018) 5835–5881.

[4] D. Bonheure, S. Cingolani, J. Van Schaftingen, The logarithmic Choquard equation: sharp asymptotics and nonde-generacy of the groundstate, J. Funct. Anal. 272 (12) (2017) 5255–5281.

[5] L. Brasco, E. Lindgren, A. Schikorra, Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case, Adv. Math. 338 (2018) 782–846.

[6] V. Benci, D. Fortunato, Variational Methods in Nonlinear Field Equations, Springer, 2014.

[7] E. de S. Böer, O.H. Miyagaki, Existence and multiplicity of solutions for the fractional p-Laplacian Choquard logarithmic equation involving a nonlinearity with exponential critical and subcritical growth, J. Math. Phys. 62 (5) (2021) 051507.

[8] C.D. Bucur, D. Cassani, C. Tarsi, Quasilinear logarithmic Choquard equations with exponential growth in $\mathbb{R}^N$, J. Differ. Equ. 328 (2022) 261–294.

[9] D. Cassani, L. Du, Fine bounds for best constants of fractional subcritical Sobolev embeddings and applications to nonlocal PDEs, Adv. Nonlinear Anal. 12 (2023), 28 pp.

[10] D. Cassani, L. Du, Z. Liu, Positive solutions to the planar Choquard equation via asymptotic approximation, Preprint, https://arxiv.org/abs/2305.10905, 2023.

[11] D. Cassani, C. Tarsi, Schrödinger-Newton equations in dimension two via a Pohozaev-Trudinger log-weighted inequality, Calc. Var. Partial Differ. Equ. 60 (5) (2021) 197.

[12] D. Cassani, J. Van Schaftingen, J. Zhang, Groundstates for Choquard type equations with Hardy-Littlewood-Sobolev lower critical exponent, Proc. R. Soc. Edinb., Sect. A 150 (2020) 1377–1400.

[13] D. Cassani, J. Zhang, Choquard-type equations with Hardy-Littlewood-Sobolev upper-critical growth, Adv. Nonlinear Anal. 8 (2019) 1184–1212.

[14] W. Chen, C. Li, Maximum principle for the fractional p-Laplacian and symmetry of solutions, Adv. Math. 335 (2018) 735–758.

[15] S. Cingolani, T. Weth, On the planar Schrödinger-Poisson system, Ann. Inst. Henri Poincaré C, Anal. Non Linéaire 33 (1) (2016) 169–197.

[16] S. Cingolani, T. Weth, Trudinger-Moser-type inequality with logarithmic convolution potentials, J. Lond. Math. Soc. (2) 105 (3) (2022) 1897–1935.

[17] R. Clemente, J.C. de Albuquerque, E. Barboza, Existence of solutions for a fractional Choquard-type equation in $\mathbb{R}$ with critical exponential growth, Z. Angew. Math. Phys. 72 (1) (2021) 16.

[18] P. d'Avenia, G. Siciliano, M. Squassina, On fractional Choquard equations, Math. Models Methods Appl. Sci. 25 (2015) 1447–1476.

[19] L. Del Pezzo, A. Quaas, A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian, J. Differ. Equ. 263 (2017) 765–778.

[20] L. Del Pezzo, A. Quaas, Spectrum of the fractional p-Laplacian in $\mathbb{R}^N$ and decay estimate for positive solutions of a Schrödinger equation, Nonlinear Anal. 193 (2020) 111479.

[21] A. Di Castro, T. Kuusi, G. Palatucci, Local behavior of fractional p-minimizers, Ann. Inst. Henri Poincaré C, Anal. Non Linéaire 33 (5) (2016) 1279–1299.

[22] D. de Figueiredo, O.H. Miyagaki, B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, Calc. Var. Partial Differ. Equ. 3 (1995) 139–153.

[23] M. Du, T. Weth, Ground states and high energy solutions of the planar Schrödinger-Poisson system, Nonlinearity 30 (9) (2017) 3492–3515.

[24] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer, 1990.

[25] W. Guan, V.D. Rădulescu, D.-B. Wang, Bound states of fractional Choquard equations with Hardy-Littlewood-Sobolev critical exponent, J. Differ. Equ. 355 (2023) 219–247.

[26] X. He, D.-B. Wang, Multiple bound state solutions for fractional Schrödinger-Poisson systems with critical nonlocal terms, J. Geom. Anal. 33 (2023) 194.

[27] A. Hyder, Structure of conformal metrics on $\mathbb{R}^n$ with constant Q-curvature, Differ. Integral Equ. 32 (7–8) (2019) 423–454.

[28] A. Iannizzotto, S. Mosconi, M. Squassina, Global Hölder regularity for the fractional p-Laplacian, Rev. Mat. Iberoam. 32 (2016) 1353–1392.

[29] C.T. Ledesma, O.H. Miyagaki, Positive solutions for a class of fractional Choquard equations in exterior domains, Milan J. Math. 90 (2022) 519–554.

[30] E.H. Lieb, M. Loss, Analysis, second edition, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[31] P. Lions, Symétrie et compacité dans les espaces de Sobolev, J. Funct. Anal. 49 (1982) 315–334.

[32] Z. Liu, V.D. Rădulescu, J. Zhang, A planar Schrödinger-Newton system with Trudienger-Moser critical growth, Calc. Var. Partial Differ. Equ. 62 (2023), 31 pp.

[33] Z. Liu, V.D. Rădulescu, C. Tang, J. Zhang, Another look at planar Schrödinger-Newton systems, J. Differ. Equ. 328 (2022) 65–104.

[34] G. Molica Bisci, N. Thin, L. Vilasi, On a class of nonlocal Schrödinger equations with exponential growth, Adv. Differ. Equ. 27 (2022) 571–610.

[35] V. Moroz, J. Van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl. 19 (2017) 773–813.

[36] E. Parini, B. Ruf, On the Moser-Trudinger inequality in fractional Sobolev-Slobodeckij spaces, J. Anal. Math. 138 (2019) 281–300.

[37] G. Romani, Schrödinger-Poisson systems with zero mass in the Sobolev limiting case, Preprint, https://arxiv.org/pdf/2310.08460.pdf, 2023.

[38] B. Ruf, A sharp Trudinger-Moser type inequality for unbounded domains in R^2, J. Funct. Anal. 219 (2005) 340–367.

[39] B. Ruf, F. Sani, Sharp Adams-type inequalities in R^n, Trans. Am. Math. Soc. 365 (2) (2013) 645–670.

[40] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, Lectures Notes in Math., vol. 665, Springer, Berlin, 1978.

[41] J. Stubbe, Bound states of two-dimensional Schrödinger-Newton equations, arXiv:0807.4059v1, 2008.

[42] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

[43] L. Wu, W. Chen, The sliding methods for the fractional p-Laplacian, Adv. Math. 361 (2020) 106933, 26 pp.

[44] S. Yuan, V.D. Rădulescu, S. Chen, L. Wen, Fractional Choquard logarithmic equations with Stein-Weiss potential, J. Math. Anal. Appl. 526 (2023) 127214.

[45] S. Yuan, V.D. Rădulescu, X. Tang, L. Zhang, Concentrating solutions for singularly perturbed fractional (N/s)-Laplacian equations with nonlocal reaction, Forum Math. (2023).

[46] C. Zhang, Trudinger-Moser inequalities in fractional Sobolev-Slobodeckij spaces and multiplicity of weak solutions to the fractional-Laplacian equation, Adv. Nonlinear Stud. 19 (2019) 197–217.