Algebraic Geometry 10 (3) (2023) 348–393
doi:10.14231/AG-2023-012

Irreducible symplectic varieties from moduli spaces of sheaves on K3 and Abelian surfaces

Arvid Perego and Antonio Rapagnetta

Abstract

We show that the moduli spaces of sheaves on a projective K3 surface are irreducible symplectic varieties, and that the same holds for the fibers of the Albanese map of moduli spaces of sheaves on an Abelian surface.

1. Introduction and main results

A *holomorphic symplectic form* on a complex manifold X is an everywhere nondegenerate, closed, holomorphic 2-form on X. A complex manifold admitting a holomorphic symplectic form is called a *holomorphic symplectic manifold*. We let $h^{p,0}(X)$ be the dimension of the vector space $H^0(X, \Omega^p_X)$.

A connected compact Kähler manifold X is an *irreducible symplectic manifold* if it is holomorphic symplectic, simply connected and $h^{2,0}(X) = 1$. In particular, an irreducible symplectic manifold has even complex dimension and trivial canonical bundle.

By the Bogomolov decomposition theorem, irreducible symplectic manifolds are one of the three types of manifolds which are building blocks for compact Kähler manifolds with numerically trivial canonical bundle. There are very few known deformation classes of irreducible symplectic manifolds, namely:

(1) a compact, connected smooth complex surface is an irreducible symplectic manifold if and only if it is a K3 surface;

(2) if S is a K3 surface and $n \in \mathbb{N}$ with $n \geqslant 2$, the Hilbert scheme $\mathrm{Hilb}^n(S)$ of n points on S is an irreducible symplectic manifold of dimension $2n$ (see [Bea83, Théorème 3 and Proposition 6]);

(3) if T is a 2-dimensional complex torus and $n \in \mathbb{N}$ with $n \geqslant 2$, the generalized Kummer variety $\mathrm{Kum}^n(T)$ is an irreducible symplectic manifold of dimension $2n$ (see [Bea83, Théorème 4 and Proposition 8]);

(4) there are two more known deformation classes: OG_6, in dimension 6, and OG_{10}, in dimension 10 (see [O'G99, O'G03]).

A possible way to obtain new examples of varieties behaving like irreducible symplectic manifolds is to enlarge the family of varieties we are considering by including singular varieties. This is a very natural step, in particular in view of the minimal model program (MMP).

Received 14 May 2018, accepted in final form 10 October 2022.
2020 Mathematics Subject Classification 14D20, 14J60, 14J28, 14J42.
Keywords: moduli spaces of sheaves, K3 surfaces, irreducible symplectic varieties.

Indeed, if X is a connected complex projective manifold with $\kappa(X) = 0$, if the MMP works for X, then it produces a birational map $X \dashrightarrow Y$, where Y has terminal singularities and nef canonical divisor. Assuming the abundance conjecture, we get that a multiple of K_Y is trivial. So, for the classification of projective varieties whose Kodaira dimension is 0, it is central to extend the Bogomolov decomposition to normal projective varieties having terminal singularities and torsion (that is, numerically trivial by [Kaw85, Theorem 8.2]) canonical divisor.

A singular version of the Bogomolov decomposition theorem has recently been obtained. For singular projective varieties with Kawamata log terminal (klt) singularities, this is [HP19, Theorem 1.5] (whose proof is the combination of several results contained in [GGK19, Dru18, GKP16, DG18]), extended to compact Kähler spaces with log terminal singularities and numerically trivial canonical bundle by [BGL22, Theorem A].

The role played by irreducible symplectic manifolds in the Bogomolov decomposition theorem is played in these generalizations by *irreducible symplectic varieties*, whose definition was first given in [GKP16]. We will present the definition of an irreducible symplectic variety only in the projective setting since this is the one we will need in the present paper (for a more general definition, see [BGL22]).

We need the following notation: if X is a normal algebraic variety and X_{reg} is the smooth locus of X whose open embedding in X is $j : X_{\mathrm{reg}} \to X$, for every $p \in \mathbb{N}$ such that $0 \leqslant p \leqslant \dim(X)$, we let

$$\Omega_X^{[p]} := j_*\Omega^p_{X_{\mathrm{reg}}} = \left(\wedge^p \Omega_X\right)^{**},$$

whose global sections are called *reflexive p-forms* on X. A reflexive p-form on X is then a holomorphic p-form on X_{reg}.

If $f\colon Y \to X$ is a finite, dominant morphism between irreducible normal algebraic varieties, there is a morphism of reflexive sheaves $f^*\Omega_X^{[p]} \to \Omega_Y^{[p]}$, induced by the usual pull-back morphism of forms on the smooth loci, giving a morphism $f^{[*]}\colon H^0\big(X, \Omega_X^{[p]}\big) \to H^0\big(Y, \Omega_Y^{[p]}\big)$, called the *reflexive pull-back morphism.*

We first recall the definitions of a symplectic form and a symplectic variety (see [Bea00]).

DEFINITION 1.1. Let X be a normal algebraic variety.

(1) A *symplectic form* on X is a closed reflexive 2-form σ on X which is nondegenerate at each point of X_{reg}.
(2) If σ is a symplectic form on X, the pair (X, σ) is a *symplectic variety* if for every resolution $f\colon \widetilde{X} \to X$ of the singularities of X, the holomorphic symplectic form $\sigma_{\mathrm{reg}} := \sigma_{|X_{\mathrm{reg}}}$ extends to a holomorphic 2-form on $\widetilde{X}$.
(3) If (X, σ) is a symplectic variety and $f\colon \widetilde{X} \to X$ is a resolution of the singularities over which σ_{reg} extends to a holomorphic symplectic form, we say that f is a *symplectic resolution.*

We now define irreducible symplectic varieties following [GKP16]. Recall that if X and Y are two irreducible normal projective varieties, a *finite quasi-étale morphism* $f\colon Y \to X$ is a finite morphism which is étale in codimension 1.

DEFINITION 1.2. An *irreducible symplectic variety* is a normal projective variety X with canonical singularities that has a symplectic form $\sigma \in H^0\big(X, \Omega_X^{[2]}\big)$ such that for every finite quasi-étale morphism $f\colon Y \to X$, the exterior algebra of reflexive forms on Y is spanned by $f^{[*]}\sigma$.

Remark 1.3. If X is a normal projective variety, by the definition of $\Omega_X^{[p]}$, we have $H^0(X,\Omega_X^{[p]}) = H^0(X_{\mathrm{reg}},\Omega^p_{X_{\mathrm{reg}}})$. Theorem 1.4 of [GKKP11] implies that if X is quasi-projective with klt singularities and $\pi\colon \widetilde{X}\to X$ is a log-resolution, then for every $p\in\mathbb{N}$ such that $0\leqslant p\leqslant \dim(X)$, the sheaf $\pi_*\Omega^p_{\widetilde{X}}$ is reflexive. This implies in particular (see [GKKP11, Observation 1.3]) that $H^0(X,\Omega_X^{[p]})\simeq H^0(\widetilde{X},\Omega^p_{\widetilde{X}})$.

The definition of irreducible symplectic variety is motivated by the description of the algebra of holomorphic forms of an irreducible symplectic manifold, which is spanned by a holomorphic symplectic form (see [Bea83, Proposition 3]).

By [GGK19, Corollary 13.3], an irreducible symplectic variety X is simply connected, so the $\mathbb{Z}$-module $H^2(X,\mathbb{Z})$ is free. Moreover, the fact that irreducible symplectic varieties are simply connected together with the Bogomolov decomposition theorem imply that smooth irreducible symplectic varieties are irreducible symplectic manifolds.

Remark 1.4. A symplectic resolution Y of an irreducible symplectic variety X is an irreducible symplectic manifold. Indeed, X is klt and simply connected, so Y is simply connected as well (see [Tak03]), and by [GKKP11, Theorem 1.4], we have $h^0(Y,\Omega^2_Y) = h^0(X,\Omega_X^{[2]}) = 1$ since X is irreducible symplectic. Anyway, there are symplectic varieties having an irreducible symplectic manifold as a symplectic resolution but which are not irreducible symplectic varieties, as in the following.

Example 1.5. If X is a K3 surface and $m\geqslant 2$, then $Y=\mathrm{Sym}^m(X)$ has $\mathrm{Hilb}^m(X)$ as symplectic resolution, but it is not an irreducible symplectic variety since X^m is a finite quasi-étale cover of Y.

Examples of irreducible symplectic varieties in dimension 4 are known (see [Per20] for an overview). Among them, we cite the partial resolution of the quotient of $\mathrm{Hilb}^2(S)$ for S a K3 surface (respectively, $\mathrm{Kum}^2(T)$ for T an Abelian surface) by the action of a symplectic involution [Men15, KM18], and the quotients of $\mathrm{Hilb}^2(S)$ by the action of a symplectic automorphism of order 3, 5, 7 or 11 (see [Men18, Men22]).

The aim of this work is to provide a wide family of examples of irreducible symplectic varieties in higher dimension.

1.1 Notation and main results of the paper

The aim of the present paper is to provide families of irreducible symplectic varieties using moduli spaces of sheaves on K3 or Abelian surfaces.

In the following, S will be a projective K3 surface or an Abelian surface, and we let $\widetilde{H}(S,\mathbb{Z}) := H^{2*}(S,\mathbb{Z})$. An element $v\in\widetilde{H}(S,\mathbb{Z})$ will be written $v=(v_0,v_1,v_2)$, where $v_i\in H^{2i}(S,\mathbb{Z})$ and $v_0,v_2\in\mathbb{Z}$. It will be called *Mukai vector* if $v_0\geqslant 0$, $v_1\in\mathrm{NS}(S)$ and if when $v_0=0$, either v_1 is the first Chern class of an effective divisor, or $v_1=0$ and $v_2>0$. Moreover, we let $\rho(S)$ be the rank of the Néron–Severi group of S.

Recall that $\widetilde{H}(S,\mathbb{Z})$ has a pure weight 2 Hodge structure and a compatible lattice structure given by the Mukai pairing $(\cdot,\cdot)$ (see [HL97, Definitions 6.1.5 and 6.1.11]). We let $v^2 := (v,v)$ for every Mukai vector v, and we call $\widetilde{H}(S,\mathbb{Z})$ the *Mukai lattice* of S.

If $\mathscr{F}$ is a coherent sheaf on S, we define its *Mukai vector* as

$$v(\mathscr{F}) := \mathrm{ch}(\mathscr{F})\sqrt{\mathrm{td}(S)} = (\mathrm{rk}(\mathscr{F}), c_1(\mathscr{F}), \mathrm{ch}_2(\mathscr{F})+\epsilon(S)\mathrm{rk}(\mathscr{F}))\,,$$

where $\epsilon(S):=1$ if S is K3 and $\epsilon(S):=0$ if S is Abelian.

Now let v be a Mukai vector on S, and suppose that H is a polarization which is general with respect to v (see Definition 2.8). We write $M_v(S,H)$ (respectively, $M_v^s(S,H)$) for the moduli space of Gieseker H-semistable (respectively, H-stable) sheaves on S with Mukai vector v.

If S is Abelian and $v^2>0$, we have a dominant isotrivial fibration $a_v\colon M_v(S,H)\to S\times\widehat{S}$ (see [Yos01, Section 4.1]), where $\widehat{S}$ is the dual of S. We let $K_v(S,H):=a_v^{-1}(0_S,\mathscr{O}_S)$ and $K_v^s(S,H):=K_v(S,H)\cap M_v^s(S,H)$. The morphism a_v is known to be the Albanese morphism of $M_v(S,H)$ if v is primitive (see [Yos01, Theorem 0.1]); we will show that this holds in all the cases we will consider (see Corollary 3.7).

If no confusion on S and H is possible, we drop them from the notation. Moreover, we will always write $v=mw$, where $m\in\mathbb{N}$ and w is a primitive Mukai vector on S.

If $M_v^s\neq\emptyset$, then it is a holomorphic symplectic quasi-projective manifold of dimension v^2+2 (see [Muk84]).

If $m=1$ and S is K3, then $M_v^s\neq\emptyset$ if and only if $v^2\geqslant -2$ (see [Yos99a, Theorem 0.1]). If S is Abelian, then $M_v^s\neq\emptyset$ if and only if $v^2\geqslant 0$ (see [Yos01, Theorem 0.1], and compare with [KLS06, Section 2.4]). If $w^2>0$, then M_v and K_v are normal, irreducible projective varieties (see [KLS06, Theorem 4.4] and [PR14, Remark A.1]).

If $v^2\leqslant 0$, we have a precise description of M_v and K_v:

(1) If $v^2<0$ and S is K3, then M_v is either empty or a point (see [Muk87]). If S is Abelian, then $M_v=\emptyset$ (see [Yos01]).
(2) If $v^2=0$ and S is K3, then M_v is either a K3 surface (if $m=1$, see [Muk87]) or a symmetric product of a K3 surface (see [KLS06, Section 1]), in which case M_v is not irreducible symplectic (see Example 1.5).
(3) If $v^2=0$ and S is Abelian, then M_v is either an Abelian surface A (if $m=1$, see [Muk87]) or a symmetric product of an Abelian surface A (see [KLS06, Section 1]). Then M_v is not simply connected, and the fiber of the sum morphism $M_v\to A$ is either a point (if $m=1$) or a symplectic variety which is not irreducible symplectic (the proof of this is similar to Example 1.5).

Because of this, we will only be interested in Mukai vectors v such that $v^2>0$. We will need the following.

Definition 1.6. Let $m,k\in\mathbb{N}$ with $m,k>0$. A Mukai vector v on S will be said of *type* (m,k) if $v=mw$ for a primitive Mukai vector $w\in\widetilde{H}(S,\mathbb{Z})$ such that $w^2=2k$.

If S a projective K3 surface or an Abelian surface, v is a Mukai vector of type (m,k) on S and H is a polarization on S that is general with respect to v, then M_v is a nonempty, irreducible, normal projective variety of dimension $2m^2k+2$ (see [KLS06, Theorem 4.4]), which is symplectic and whose smooth locus is M_v^s. If S is Abelian and $(m,k)\neq(1,1)$, then K_v is a nonempty, irreducible, normal projective variety of dimension $2m^2k-2$, which is symplectic and whose regular locus is K_v^s. If $(m,k)=(1,1)$, then M_v is isomorphic to $S\times\widehat{S}$ and K_v is a point.

The first result we will prove is the following.

Theorem 1.7. *Let $m,k\in\mathbb{N}$ with $m,k>0$, and for $i=1,2$, let S_i be a projective K3 or Abelian surface, v_i a Mukai vector on S_i of type (m,k) and H_i a polarization on S_i which is general with respect to v_i.*

(1) *If S_1 and S_2 are both K3 surfaces or both Abelian surfaces, then $M_{v_1}(S_1,H_1)$ and $M_{v_2}(S_2,H_2)$ are deformation equivalent, and the deformation is locally trivial.*

(2) *If S_1 and S_2 are two Abelian surfaces, then $K_{v_1}(S_1, H_1)$ and $K_{v_2}(S_2, H_2)$ are deformation equivalent, and the deformation is locally trivial.*

Remark 1.8. In Theorem 1.7, and all along the paper, we will say that a morphism $f\colon \mathscr{X} \to T$ of complex varieties is a *locally trivial deformation* of a complex variety X if T is connected and f is a proper flat morphism verifying the following two conditions (see [FK87]):

(1) There is a point $t_0 \in T$ such that $X = f^{-1}(t_0)$.
(2) For every $x \in \mathscr{X}$, the deformation germ $(\mathscr{X}, x) \to (T, f(x))$ is isomorphic to the trivial deformation $\big(f^{-1}(f(x)), x\big) \times (T, f(x))$ of the germ $\big(f^{-1}(f(x)), x\big)$.

Remark 1.9. The deformation relating $M_{v_1}(S_1, H_1)$ and $M_{v_2}(S_2, H_2)$ in Theorem 1.7 is obtained using only deformations of the moduli spaces induced by deformations of the base surfaces S_i together with the Mukai vectors and the polarizations (see Section 2.3 for the definition) and isomorphisms between moduli spaces induced by Fourier–Mukai transforms. In particular, $M_{v_1}(S_1, H_1)$ and $M_{v_2}(S_2, H_2)$ are locally trivially deformation equivalent in the algebraic category.

As a consequence of Theorem 1.7, starting from K3 surfaces (respectively, Abelian surfaces), we get a single locally trivial deformation class for every pair (m, k) of strictly positive integers.

The proof of Theorem 1.7 is the content of Section 2 of the present paper. For $m = 1$, it is due to several authors (see [Muk84, Bea83, O'G97, Yos99a, Yos01]). For $(m, k) = (2, 1)$, the proof of Theorem 1.7 is in [PR13]. The deformation equivalence in Theorem 1.7(1) is basically due to Yoshioka: for Mukai vectors with positive rank, it is [Yos03, Proposition 3.6]; the rank 0 case is not explicitly stated but can be obtained as in [Yos09, Corollary 3.5]. As it is an important result which plays a central role in our paper, we decided to include a complete proof. The local triviality of the deformation follows from [Nam06, Main Theorem].

Yoshioka's original proof of the deformation equivalence involves two main technical tools: deformations of K3 or Abelian surfaces inducing deformations of the moduli spaces, and Fourier–Mukai transforms. As a by-product of it, one gets the nonemptyness and normality of the moduli spaces. Based on his proof of the deformation equivalence of the moduli spaces, and using a different argument to deal with particular cases, Yoshioka also proves their irreducibility (see [Yos03, Theorem 3.18]).

The proof of Theorem 1.7(1) we propose is a re-elaborated version of Yoshioka's proof that we tried to keep as elementary as possible. It uses the same tools together with [KLS06, Theorem 4.4], which proves the irreducibility and the normality of the moduli spaces independently of [Yos03] and [Yos09] (and which implies the irreducibility and the normality of the K_v, as shown in [PR14, Remark A.1]).

We observe that the proof of [KLS06, Theorem 4.4] uses the fact that if v is primitive and $v^2 \geqslant 0$, then $M_v \neq \emptyset$, which was proved by Yoshioka in [Yos99a] and [Yos01], independently of [Yos03] and [Yos09] (compare with [KLS06, Section 2.4]).

The use of [KLS06, Theorem 4.4] allows us to minimize the technicalities involved in the proof of the deformation equivalence of moduli spaces. The only Fourier–Mukai transforms we will use are those corresponding to tensorization with line bundles: the one whose kernel is the ideal of the diagonal (for K3 surfaces), and the one whose kernel is the Poincaré bundle (for Abelian surfaces). We only need to check the preservation of the semistability under these functors in the most natural direction (see Section 2.4).

The aim of Section 3 is to show the following, which is the main result of this paper.

THEOREM 1.10. *Let $m, k \in \mathbb{N}$ with $m, k > 0$, and let S be a projective K3 or Abelian surface, v a Mukai vector on S of type (m, k) and H a polarization on S which is general with respect to v.*

(1) *If S is K3, then $M_v(S, H)$ is an irreducible symplectic variety.*
(2) *If S is Abelian and $(m, k) \neq (1, 1)$, then $K_v(S, H)$ is an irreducible symplectic variety.*

Theorem 1.10 provides an answer to [GGK19, Question 14.10]: it implies that, for general polarizations, all moduli spaces of sheaves on a projective K3 surface (and all the fibers of the Albanese morphism of moduli spaces of sheaves on Abelian surfaces) are irreducible symplectic varieties, with the only exception of symmetric products.

The examples of irreducible symplectic varieties given in Theorem 1.10 naturally fall into three main cases:

(1) If S is K3, then M_v is smooth if and only if $m = 1$ (and M_v is deformation equivalent to $\mathrm{Hilb}^{k+1}(S)$). If S is Abelian, then K_v is smooth if and only if $m = 1$ (it is a point if $k = 1$, a K3 surface if $k = 2$ and deformation equivalent to $\mathrm{Kum}^{k-1}(S)$ if $k \geqslant 3$).
(2) If S is K3, then M_v has a symplectic resolution if and only if $(m, k) = (2, 1)$ (which is in the deformation class OG_{10}). If S is Abelian, then K_v has a symplectic resolution if and only if $(m, k) = (2, 1)$ (which is in the deformation class OG_6).
(3) In all other cases, M_v and K_v have terminal singularities and no symplectic resolutions. Indeed, both M_v and K_v are normal, irreducible projective varieties having a symplectic form on their smooth loci, and their singular loci have codimension at least equal to 4 (see for example [KLS06]). The main result of [Fle88] implies that they are symplectic varieties, and by [Nam01, Corollary 1], they have terminal singularities. As shown in [KLS06], the moduli spaces M_v and K_v are locally factorial and do not admit any symplectic resolution.

The proof of Theorem 1.10 uses Theorem 1.7 to reduce to a surface S such that $\mathrm{NS}(S) = \mathbb{Z} \cdot h$, where h is the first Chern class of an ample divisor H with $H^2 = 2k$. Taking $v = m(0, h, 0)$, if S is a K3 surface, we show that M_v and M_v^s are simply connected; if S is Abelian, we show that K_v and K_v^s are simply connected (if $(m, k) \neq (2, 1)$). We then calculate the numbers $h^0\big(M_v, \Omega_{M_v}^{[p]}\big)$ and $h^0\big(K_v, \Omega_{K_v}^{[p]}\big)$ by comparing them with $h^0\big(M_{v'}, \Omega_{M_{v'}}^{p}\big)$ and $h^0\big(K_{v'}, \Omega_{K_{v'}}^{p}\big)$, where v' is the primitive Mukai vector $v' = \big(0, mh, 1 - m^2 k\big)$.

Remark 1.11. The irreducible symplectic varieties we get with Theorem 1.10 all have simply connected smooth locus, up to one exception, namely when S is an Abelian surface and $(m, k) = (2, 1)$; in this case, the fundamental group of K_v^s is $\mathbb{Z}/2\mathbb{Z}$ (see [MRS18, Section 4] or Theorem 3.6). In any case, all the irreducible symplectic varieties we obtain have smooth locus with finite fundamental group.

Remark 1.12. A natural open question concerns the computation of the fundamental invariants, that is, the Beauville–Bogomolov form and the Fujiki constant for M_v and K_v in the case where they do not admit a symplectic resolution. We treat this problem in [PR20].

2. Deformations of moduli spaces

In this section, we study how moduli spaces vary under deformations and isomorphisms. In Section 2.1, we recall the notion of polarization which is general with respect to a Mukai vector v and some notions related to that: v-generic polarizations, v-walls and v-chambers. Section 2.2

is devoted to the morphism a_v in the case of Abelian surfaces. In Section 2.3, we introduce deformations of moduli spaces induced by deformations of the datum of a surface S, a Mukai vector v on S and a polarization H along smooth, connected varieties. In Section 2.4, we study isomorphisms between moduli spaces coming from Fourier–Mukai transforms.

Section 2.5 is devoted to the proof of Theorem 1.7, which is the main result of Section 2. Our proof of Theorem 1.7 is heavily based on [KLS06, Theorem 4.4], which asserts that if v is a Mukai vector on a projective K3 or Abelian surface S and H is a v-generic polarization, then $M_v(S,H)$ is a normal, irreducible projective variety of the expected dimension. Theorem 4.4 of [KLS06] is based on the nonemptyness of moduli spaces of sheaves for generic polarizations and primitive Mukai vectors of positive square [Yos01, Theorems 0.1 and 8.1] (compare with [KLS06, Section 2.4]).

These assumptions could be avoided (following Yoshioka) using [Muk84, Theorem 1.17] and stronger versions of Propositions 2.29 and 2.33.

2.1 General and generic polarizations

We recall the definition of a v-generic polarization introduced in [HL97] and [Yos01], that we will use all along the paper, and the notion of a polarization which is general with respect to v, introduced in [Yos09].

2.1.1 *Generic polarizations.* In what follows, S will always denote a projective K3 or Abelian surface and $v=(v_0,v_1,v_2)$ a Mukai vector on S. We will furthermore suppose that when $\rho(S)>1$, if $v_0=0$, then $v_2\neq 0$ (the case $v=(0,v_1,0)$ will be briefly discussed at the beginning of Section 2.1.2; see Example 2.7).

We associate with each Mukai vector v of this form a set W_v of divisors on S, whose definition depends on v_0: the case $v_0>0$ will be different from the case $v_0=0$.

If $v_0>0$, we let

$$|v|=\tfrac{1}{4}v_0^2(v,v)+\tfrac{1}{2}v_0^{2\epsilon(S)+2},$$

where we recall that $\epsilon(S)=1$ if S is K3 and $\epsilon(S)=0$ if S is Abelian. The rational number $|v|$ only depends on v_0 and v^2. If $|v|>0$, we define

$$W_v:=\left\{D\in \mathrm{NS}(S)\,|\,-|v|\leqslant D^2<0\right\},$$

and we let $W_v:=\emptyset$ if $|v|=0$. We notice that if $m,k\in\mathbb{N}$ with $m,k>0$ and v is a Mukai vector of type (m,k), then $|v|>0$.

If $v_0=0$, for every pure sheaf E with Mukai vector v and $0\neq F\subseteq E$ with Mukai vector $u=(0,u_1,u_2)$, the *divisor associated with the pair* (E,F) is defined as $D:=u_2v_1-v_2u_1$. The set W_v will then be the set of the nonnumerically trivial divisors associated with all the possible pairs of this type.

A primitive ample divisor H on S will be called a *polarization.*[1] The set W_v is used to define the notion of a v-generic polarization as follows.

Definition 2.1. A polarization H is *v-generic* if $H\cdot D\neq 0$ for every $D\in W_v$.

If $\rho(S)=1$, then the ample generator of the Picard group of S is v-generic for every v.

[1]By a slight abuse of notation, the line bundle $\mathcal{O}_S(H)$ will usually be denoted by H and will still be called a polarization.

If $\rho(S) \geqslant 2$, there can be polarizations which are not v-generic, and to characterize them we introduce v-walls and v-chambers. We let $\mathrm{Amp}(S)$ be the ample cone of S.

DEFINITION 2.2. If $D \in W_v$, the *v-wall associated with D* is

$$D^{\perp} := \{\alpha \in \mathrm{Amp}(S) \,|\, D \cdot \alpha = 0\}\,.$$

The v-wall associated with $D \in W_v$ is a hyperplane in $\mathrm{Amp}(S)$. If $v_0 > 0$, then by [HL97, Theorem 4.C.2], the set of v-walls is locally finite in $\mathrm{Amp}(S)$. If $v_0 = 0$, it is even finite (see [Yos01, Section 1.4]).

DEFINITION 2.3. Suppose $\rho(S) \geqslant 2$. A connected component of $\mathrm{Amp}(S) \setminus \bigcup_{D \in W_v} D^{\perp}$ is called a *v-chamber*.

By definition, a polarization is v-generic if and only if it lies in a v-chamber. Since the family of v-walls is locally finite in $\mathrm{Amp}(S)$, it follows that a v-generic polarization exists for every Mukai vector v verifying the conditions above.

Remark 2.4. If H is v-generic and E is a H-semistable sheaf with Mukai vector v, then any H-destabilizing subsheaf of E has Mukai vector of the form pv for some $p \in \mathbb{Q}$. In particular, if v is primitive and H is v-generic, any H-semistable sheaf with Mukai vector v is H-stable (compare with [KLS06, Section 2.4]).

An important property of generic polarizations is that changing polarization inside a v-chamber does not affect the moduli space. More precisely, we have the following (see [Qin93] or [HL97, Section 4.C]).

PROPOSITION 2.5. *Suppose $\rho(S) \geqslant 2$ and that v is a Mukai vector on S. Let $\mathcal{C}$ be a v-chamber, and suppose $H, H' \in \mathcal{C}$. A sheaf with Mukai vector v is H-(semi)stable if and only if it is H'-(semi)stable. As a consequence, we have natural identifications $M_v(S,H) = M_v(S,H')$ and $M^s_v(S,H) = M^s_v(S,H')$.*

We conclude this section with the behaviour of v-genericity with respect to the tensorization with a line bundle. If v is a Mukai vector on S and $L \in \mathrm{Pic}(S)$, we let $v_L := v \cdot \mathrm{ch}(L)$. If $L = \mathcal{O}_S(D)$ for some divisor D, then we let $v_D := v_L$. Notice that if E is a sheaf such that $v(E) = v$, then $v(E \otimes L) = v_L$.

LEMMA 2.6. *Let v be a Mukai vector and H a polarization on S.*

1. *If $v = (r, \xi, a)$ with $r > 0$ and $L \in \mathrm{Pic}(S)$, we have that H is v-generic if and only if it is v_L-generic.*
2. *If $v = (0, \xi, a)$, where $a \neq 0$ and $d \in \mathbb{Z}$ is such that $d \neq -a/\xi \cdot H$, we have that H is v-generic if and only if it is v_{dH}-generic.*

Proof. If $v = (r, \xi, a)$ with $r > 0$, notice that $v_L = \big(r, \xi + rc_1(L), a + \xi \cdot L + rL^2/2\big)$. Then v and v_L have the same rank, and, as is easily seen, $v^2 = v_L^2$. Hence $|v| = |v_L|$, so $W_v = W_{v_L}$, and we are done.

If $v = (0, \xi, a)$, notice that $v_{dH} = (0, \xi, a + d\xi \cdot H)$. There is a bijection between $W_v \cup \{0\}$ and $W_{v_{dH}} \cup \{0\}$ mapping the divisor associated with a pair (E, F) to the one associated with $(E \otimes \mathcal{O}_S(dH), F \otimes \mathcal{O}_S(dH))$. Indeed, if $D \in W_v$ is associated with (E, F) and $v(F) = (0, \zeta, b)$, we get $D = b\xi - a\zeta$. The divisor associated with $(E \otimes \mathcal{O}_S(dH), F \otimes \mathcal{O}_S(dH))$ is

$$D' = D + d(\xi \cdot H)\zeta - d(\zeta \cdot H)\xi\,,$$

and the bijection maps D to D' (and conversely). Notice that $D \cdot H = D' \cdot H$; hence H is v-generic if and only if it is v_{dH}-generic. □

2.1.2 *General polarizations.* The definition of a v-generic polarization makes perfect sense even for Mukai vectors of type $v = (0, v_1, 0)$, but it is not well adapted to our goals. Indeed, if $v = (0, v_1, 0)$, by defining W_v as before, we see that $D \in W_v$ is of the form $D = bv_1$ for some $b \neq 0$. As v_1 is the first Chern class of an effective divisor, we get $H \cdot D \neq 0$ for all $D \in W_v$, and hence every polarization would be v-generic.

Now, the definition of v-genericity is motivated by the fact that if v is a primitive Mukai vector and H is v-generic, then a H-semistable sheaf with Mukai vector v is H-stable; this holds if $v = (0, v_1, v_2)$, where $v_2 \neq 0$, or if $v = (0, v_1, 0)$ and $\rho(S) = 1$ (as a consequence of Remark 2.4), but it is no longer true if $v = (0, v_1, 0)$ and $\rho(S) \geqslant 2$, as the following example shows (see [Yos01, Lemma 1.2]).

Example 2.7. Let S be a K3 surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot c' \oplus \mathbb{Z} \cdot c''$, where c' and c'' are the classes of two irreducible effective curves C' and C''; for example, S is a generic quartic surface containing a line. We let j' and j'' be the inclusions of C' and C'', respectively, into S. Denote by v the primitive Mukai vector $(0, c' + c'', 0)$, and let M' and M'' be line bundles on C' and C'' with Euler characteristic 0. The sheaves j'_*M' and j''_*M'' are H-stable with respect to any polarization H, and we have $v(j'_*M') = (0, c', 0)$ and $v(j''_*M'') = (0, c'', 0)$. The sheaf $j'_*M' \oplus j''_*M''$ is then H-semistable with Mukai vector $v = (0, c' + c'', 0)$, but it is not H-stable since j'_*M' and j''_*M'' are both H-destabilizing.

The definition of a v-generic polarization we have given in the previous section is then not well adapted to Mukai vectors of the form $(0, v_1, 0)$. Because of this, we introduce a different definition of genericity for polarizations than can be found in [Yos09] (see Definition 1.4 therein for Mukai vectors (v_0, v_1, v_2) with $v_0 > 0$, and Definition 3.1 if $v_0 = 0$) and that is more suitable for Mukai vectors of the form $(0, v_1, 0)$.

Definition 2.8. Let S be a projective K3 surface or an Abelian surface and $v = (v_0, v_1, v_2)$ a Mukai vector on S. A polarization H will be called *general with respect to* v in the following cases:

(1) *Case 1: when* $v_0 > 0$. In this case, H is general with respect to v if for every μ_H-semistable sheaf E such that $v(E) = v$ and every $0 \neq F \subseteq E$, we have that if $\mu_H(E) = \mu_H(F)$, then $c_1(F)/\mathrm{rk}(F) = c_1(E)/\mathrm{rk}(E)$.

(2) *Case 2: when* $v_0 = 0$. In this case, H is general with respect to v if for every H-semistable sheaf E such that $v(E) = v$ and every $0 \neq F \subseteq E$, if $\chi(E)/(c_1(E) \cdot H) = \chi(F)/(c_1(F) \cdot H)$, then $v(F) \in \mathbb{Q}v$.

We first prove that this notion is more general than that of v-genericity if v is not of the form $(0, v_1, 0)$.

Lemma 2.9. *Let S be a projective K3 surface or an Abelian surface and v a Mukai vector on S such that if $\rho(S) \geqslant 2$ and $v = (0, v_1, v_2)$, then $v_2 \neq 0$.*

(1) *If $\rho(S) = 1$, the ample generator of* $\mathrm{Pic}(S)$ *is both v-generic and general with respect to v.*

(2) *If $\rho(S) \geqslant 2$, if a polarization is v-generic, then it is general with respect to v.*

Proof. This is immediate if $\rho(S) = 1$, so we suppose $\rho(S) \geqslant 2$. If $v = (v_0, v_1, v_2)$ and $v_0 > 0$, this is a consequence of [HL97, Theorem 4.C.3]. So we suppose $v_0 = 0$, and hence $v_2 \neq 0$, and consider a v-generic polarization H.

To show that H is general with respect to v, let E be a H-semistable sheaf with Mukai vector v and $0 \neq F \subseteq E$ a subsheaf with Mukai vector $u = (0, u_1, u_2)$. Notice that $\chi(E) = v_2$ and $\chi(F) = u_2$.

If $\chi(E)/(c_1(E) \cdot H) = \chi(F)/(c_1(F) \cdot H)$, then $(u_2 v_1 - v_2 u_1) \cdot H = 0$. Since $u_2 v_1 - v_2 u_1$ is the divisor associated with (E, F), and since H is v-generic, it follows that $u_2 v_1 - v_2 u_1 = 0$, so that

$$v(F) = u = (0, u_1, u_2) = \frac{u_2}{v_2}(0, v_1, v_2) \in \mathbb{Q}v\,,$$

and hence H is general with respect to v. □

If $\rho(S) \geqslant 2$, a polarization H which is general with respect to v is not necessarily v-generic, so it may lie on a v-wall. Anyway, the moduli space $M_v(S, H)$ is equal to a moduli space $M_v(S, H')$, where H' is v-generic. This is the content of the following.

Lemma 2.10. *Let S be a projective K3 surface or an Abelian surface and v a Mukai vector on S. Suppose $\rho(S) \geqslant 2$ and that if $v = (0, v_1, v_2)$, then $v_2 \neq 0$. Suppose moreover that H is a polarization which lies on a v-wall and is general with respect to v, and let $\mathcal{C}$ be a v-chamber such that $H \in \overline{\mathcal{C}}$ (where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in $\mathrm{Amp}(S)$). Then a sheaf E with Mukai vector v is H-(semi)stable if and only if it is H'-(semi)stable for every $H' \in \mathcal{C}$. In particular, we have identifications $M_v(S, H) = M_v(S, H')$ and $M_v^s(S, H) = M_v^s(S, H')$.*

Proof. We present a proof of this for a Mukai vector $v = (v_0, v_1, v_2)$ with $v_0 > 0$, the case $v_0 = 0$ being easier. We will only consider semistable sheaves, the case of stable sheaves being similar.

To do so, suppose that E is H'-semistable but not H-semistable. Then it is μ_H-semistable (since μ-semistability is preserved by passing to limits on a wall), and it has a proper subsheaf F such that $p_H(F) > p_H(E)$; this implies that $\mu_H(E) = \mu_H(F)$, hence $c_1(F)/\mathrm{rk}(F) = c_1(E)/\mathrm{rk}(E)$ (as H is general with respect to v). But since $p_H(F) > p_H(E)$, we get that $p_{H'}(F) > p_{H'}(E)$, which is not possible. As a consequence, if E is H'-semistable, then it is H-semistable.

Conversely, suppose that E is H-semistable but not H'-semistable. Then E has a proper subsheaf F such that $p_{H'}(F) > p_{H'}(E)$. If $\mu_{H'}(F) > \mu_{H'}(E)$, then we must have $\mu_H(F) = \mu_H(E)$ (otherwise, the segment $[H, H']$ would meet a v-wall inside $\mathcal{C}$), so $c_1(F)/\mathrm{rk}(F) = c_1(E)/\mathrm{rk}(E)$. Since $p_H(F) \leqslant p_H(E)$, we get $p_{H'}(F) \leqslant p_{H'}(E)$, which is not possible. If $\mu_{H'}(F) = \mu_{H'}(E)$, by the v-genericity of H', this again implies $c_1(F)/\mathrm{rk}(F) = c_1(E)/\mathrm{rk}(E)$; hence we would get $p_H(F) > p_H(E)$, which is again not possible. As a consequence, if E is H-semistable, then it is H'-semistable. □

As a consequence of Lemmas 2.9 and 2.10, it will be enough to consider v-generic polarizations in order to get results for polarizations which are general with respect to v, at least in the case where $v = (v_0, v_1, v_2)$ with either $v_0 > 0$, or $v_0 = 0$ and $v_2 \neq 0$.

The case $v = (0, v_1, 0)$ was not yet considered, and it is indeed a very special case: while all polarizations are v-generic in this case, it may happen that no general polarizations with respect to v exist at all. An example of this is the following.

Example 2.11. We use the notation of Example 2.7, and we show that if $v = (0, c' + c'', 0)$, then there is no general polarization with respect to v. Indeed, the sheaf $E := j'_* M' \oplus j''_* M''$

is H-semistable for every polarization H, and the Mukai vector of the destabilizing subsheaf $F := j'_*M'$ is $v(F) = (0, c', 0) \notin \mathbb{Q}v$, but

$$\chi(F)/c_1(F) \cdot H = 0 = \chi(E)/c_1(E) \cdot H\,.$$

This shows that H is not general with respect to v.

If $v = (0, v_1, 0)$ and a general polarization H with respect to v_1 exists, the following lemma implies that H is still general with respect to $v_H := (0, v_1, v_1 \cdot H)$ and that $M_v(S, H)$ is isomorphic to $M_{v_H}(S, H)$. By Lemma 2.10, it follows that $M_v(S, H)$ is also isomorphic to $M_{v_H}(S, H')$ for a v_H-generic polarization H'.

Lemma 2.12. *Let S be a projective K3 surface or an Abelian surface and $v = (0, v_1, v_2)$ a Mukai vector on S. A polarization H is general with respect to v if and only if it is general with respect to v_H. Moreover, the tensorization with H induces an isomorphism between $M_v(S, H)$ and $M_{v_H}(S, H)$, and between $M^s_v(S, H)$ and $M^s_{v_H}(S, H)$.*

Proof. The tensorization with a multiple of H preserves the H-(semi)stability; the lemma follows. □

Remark 2.13. As a consequence of Lemmas 2.9, 2.10 and 2.12, if v is a Mukai vector and H is a polarization which is general with respect to v, then $M_v(S, H)$ is either identified with or isomorphic to a moduli space $M_{v'}(S, H')$, where H' is v'-generic. This will allow us to restrict ourselves to v-generic polarizations in the proof of the main results of the present paper, and to prove all the results needed for the proofs of Theorems 1.7 and 1.10 only for v-generic polarizations.

A further reason to consider polarizations which are general with respect to a Mukai vector v instead of v-generic polarizations is the openness, in the algebraic category, of the locus of the base of a deformation where a polarization stays general with respect to a Mukai vector. This is the content of the following proposition.

Proposition 2.14. *Let $f\colon \mathscr{X} \to T$ be a smooth projective family of K3 surfaces or Abelian surfaces over a connected algebraic variety T. Let $\mathscr{L}$, $\mathscr{H}$ be two line bundles on $\mathscr{X}$, and for every $t \in T$, let L_t, H_t be the restrictions of $\mathscr{L}$, $\mathscr{H}$ to the fiber X_t of f over t. Let $v_t := (r, c_1(L_t), a)$ be a Mukai vector on X_t, and suppose that H_t is ample on X_t for every $t \in T$. Then the locus*

$$T_{\mathrm{ng}} := \{t \in T \mid H_t \text{ is not general with respect to } v_t\} \subseteq T$$

is Zariski closed in T.

Proof. Let us first consider the case $r > 0$. As an immediate consequence of the definition, the polarization H_t is not general with respect to v_t if and only if there exists a μ_{H_t}-semistable sheaf E with Mukai vector v_t on X_t admitting a surjective map to a torsion-free sheaf G having the same H_t-slope as E and such that $c_1(G)/\mathrm{rk}(G) \neq c_1(E)/\mathrm{rk}(E)$.

Since the family of μ_{H_t}-semistable sheaves with Mukai vectors v_t on the fibers of f is bounded (see for instance [HL97, Theorem 3.3.7]), there is a variety Y with a morphism $\pi_Y\colon Y \to T$ and a Y-flat sheaf $\mathcal{E}_Y$ on $\mathscr{X}_Y := Y \times_T \mathscr{X}$ such that, for every $t \in T$, every μ_{H_t}-semistable sheaf with Mukai vector v_t appears as the restriction E_y of $\mathcal{E}_Y$ to $\{y\} \times X_t$ for some $y \in \pi_Y^{-1}(t) \subseteq Y$.

By a result of Grothendieck (see [HL97, Lemma 1.7.9]), there exists a finite set $I \subset \mathbb{Q}[x]$ such that $P \in I$ if and only if there are $t \in T$, $y \in \pi_Y^{-1}(t)$ and a torsion-free quotient of E_y whose Hilbert polynomial with respect to H_t is P and whose H_t-slope is $\mu_{H_t}(E_y)$.

For $P \in I$, we let $q_{P,Y}\colon Q_{P,Y} \to Y$ be the relative Quot scheme whose fiber over $y \in Y$ parametrizes the quotients of E_y whose Hilbert polynomial with respect to $H_{\pi_Y(y)}$ is P and let $Q^0_{P,Y} \subset Q_{P,Y}$ be the locus parametrizing quotients $E_y \to G_y$ with $c_1(G_y)/\mathrm{rk}(G_y) \neq c_1(E_y)/\mathrm{rk}(E_y)$. Since L_t is the restriction of the global line bundle $\mathcal{L}$, the locus $Q^0_{P,Y}$ is a union of connected components of $Q_{P,Y}$, and, by construction,

$$T_{\mathrm{ng}} = \bigcup_{P \in I} \pi_Y\big(q_{P,Y}\big(Q^0_{P,Y}\big)\big)\,.$$

As a finite union of images of regular morphisms, T_{ng} is constructible, and we can check its closure in T by the evaluative criterion of properness for the inclusion $T_{\mathrm{ng}} \subset T$.

It then remains to prove that if C is a smooth curve, $C^0 := C \setminus \overline{c}$ is the complement of a point $\overline{c}$ and $\iota\colon C \to T$ is a morphism such that $\iota\big(C^0\big) \subset T_{\mathrm{ng}}$, then we have $\iota(\overline{c}) \in T_{\mathrm{ng}}$ too. Using the universal family of the relative Quot scheme $Q_{P,Y}$ and replacing C^0 with a quasi-finite cover if necessary, we may assume that there exist two C^0-flat families $\mathcal{E}_{C^0}$ and $\mathcal{G}_{C^0}$ of torsion-free sheaves on $\mathscr{X}_{C^0} := C^0 \times_T \mathscr{X}$, together with a surjective morphism $\mathcal{E}_{C^0} \to \mathcal{G}_{C^0}$, such that for every $c \in C^0$, the following two conditions are fulfilled:

(1) The restriction E_c of $\mathcal{E}_{C^0}$ to $X_{\iota(c)}$ has Mukai vector $v_{\iota(c)}$ and is $\mu_{H_{\iota(c)}}$-semistable.

(2) The restriction G_c of $\mathcal{G}_{C^0}$ to $X_{\iota(c)}$ has the same $\mu_{H_{\iota(c)}}$-slope as E_c, but $c_1(G_c)/\mathrm{rk}(G_c) \neq c_1(E_c)/\mathrm{rk}(E_c)$.

As for every coherent sheaf on $\mathscr{X}_{C^0}$, we can extend $\mathcal{E}_{C^0}$ to a coherent sheaf $\mathcal{E}_C$ on $\mathscr{X}_C := C \times_T \mathscr{X}$ and, by the relative version of Langton's theorem [Lan75] (for a proof working in the relative case, see [HL97, Theorem 2.B.1]), we may also assume that its restriction $E_{\overline{c}}$ to $X_{\iota(\overline{c})}$ is $\mu_{H_{\iota(\overline{c})}}$-semistable.

We need to show that there exists a torsion-free quotient $E_{\overline{c}} \to G_{\overline{c}}$ such that $\mu_{H_{\iota(\overline{c})}}(G_{\overline{c}}) = \mu_{H_{\iota(\overline{c})}}(E_{\overline{c}})$ and $c_1(G_{\overline{c}})/\mathrm{rk}(G_{\overline{c}}) \neq c_1(E_{\overline{c}})/\mathrm{rk}(E_{\overline{c}})$.

By flatness, the Hilbert polynomial with respect to $H_{\iota(c)}$ does not depend on c and has to be a fixed element $P \in I$ for every $c \in C^0$. The relative Quot scheme $Q_{P,C} \to C$, whose fiber over $c \in C$ parametrizes quotients of E_c having Hilbert polynomial P, has a connected component $Q^0_{P,C}$ containing the sheaves G_c verifying the above condition (2) and, hence, dominating C; by the projectivity of the relative Quot scheme, the connected component $Q^0_{P,C}$ also surjects onto C.

This implies that there exists a quotient $G_{\overline{c}}$ of $E_{\overline{c}}$ having Hilbert polynomial P and, since $G_{\overline{c}}$ is in the same connected component $Q^0_{P,C}$ of the G_c for $c \neq \overline{c}$, we see that $G_{\overline{c}}$ verifies the above condition (2) as well, so we have $c_1(G_{\overline{c}})/\mathrm{rk}(G_{\overline{c}}) \neq c_1(E_{\overline{c}})/\mathrm{rk}(E_{\overline{c}})$.

If $G_{\overline{c}}$ is torsion-free, we see that $\iota(\overline{c}) \in T_{\mathrm{ng}}$, and we are done. If $G_{\overline{c}}$ is not torsion-free, since $E_{\overline{c}}$ is slope-semistable, by considering its quotient by the torsion subsheaf, we get a torsion-free sheaf with the same rank and first Chern class, and we are done.

We are left with the case $r = 0$. Let $\phi\colon \mathscr{M} \to T$ be the relative moduli space associated with $f\colon \mathscr{X} \to T$, $\mathscr{H}$ and $\mathscr{L}$, and let $\Sigma \subseteq \mathscr{M}$ be the closed subset parametrizing strictly semistable sheaves. Let $\Sigma' \subseteq \Sigma$ be the closed subset parametrizing polystable sheaves of the form $F_1 \oplus \cdots \oplus F_s$ such that $v(F_i) \notin \mathbb{Q}v$ for some $i \leqslant s$. Since ϕ is projective, the image $\phi(\Sigma')$ is closed in T, and, since $T_{\mathrm{ng}} = \phi(\Sigma')$, we are done. $\square$

Proposition 2.14 shows that being general with respect to a Mukai vector is an open property in the Zariski topology; this is remarkable, in particular, in view of the fact that the v-genericity is only open in the analytic topology (see [PR13, Corollary 4.2]).

2.2 Yoshioka's fibration

Here we recall the definition and the main properties of the morphism $a_v\colon M_v(S,H)\to S\times\widehat{S}$ introduced by Yoshioka in [Yos99b] and relate it to another morphism used in [O'G03].

We let S be an Abelian surface, $\widehat{S}$ its dual and $\mathcal{P}$ the Poincaré line bundle on $S\times\widehat{S}$. Fix a Mukai vector v and a polarization H on S which is general with respect to v, and let $M_v(S,H)$ be the moduli space of H-semistable sheaves on S with Mukai vector v.

We recall that for a smooth projective variety X, the Grothendieck group $K(X)$ has the structure of a ring: if F and G are two locally free sheaves, we let $[F]+[G]=[F\oplus G]$ and $[F]\cdot[G]=[F\otimes G]$; if F and G are coherent but not locally free, replace them by a finite locally free resolution of both.

If $f\colon X\to Y$ is a morphism of smooth projective varieties, then we have the pull-back ring morphism $f^*\colon K(Y)\to K(X)$ and the push-forward group morphism $f_!\colon K(X)\to K(Y)$. Moreover, the determinant map $\det\colon K(X)\to\mathrm{Pic}(X)$ is well defined.

2.2.1 *Yoshioka's fibration.* We now define the morphism a_v following Yoshioka (see [Yos99b] and [Yos01]). To do so, fix a coherent sheaf $\mathcal{F}_0$ with Mukai vector v. For every coherent sheaf $\mathcal{F}$ on S with Mukai vector v, we set

$$\delta_v(\mathcal{F}):=\det\left(p_{\widehat{S}!}\left(p_S^*([\mathcal{F}]-[\mathcal{F}_0])\cdot\left([\mathcal{P}]-[\mathcal{O}_{S\times\widehat{S}}]\right)\right)\right)\in\mathrm{Pic}^0\big(\widehat{S}\big)\,,$$

where p_S and $p_{\widehat{S}}$ are the two projections of $S\times\widehat{S}$ onto S and $\widehat{S}$, respectively, and $\mathrm{Pic}^0\big(\widehat{S}\big)$ is the group of topologically trivial line bundles on $\widehat{S}$. Letting

$$F\colon D^b(S)\longrightarrow D^b\big(\widehat{S}\big)\,,\quad F(E^\bullet):=Rp_{\widehat{S}*}\big(p_S^*E^\bullet\otimes\mathcal{P}\big)$$

be the Fourier–Mukai functor with kernel $\mathcal{P}$, we then have

$$\delta_v(\mathcal{F})=\det(F(\mathcal{F}))\otimes\det(F(\mathcal{F}_0))^\vee\in\mathrm{Pic}^0\big(\widehat{S}\big)\,.$$

Notice that we have an isomorphism $\mathrm{Pic}^0\big(\widehat{S}\big)\simeq S$; hence we have a morphism $\delta_v\colon M_v(S,H)\to S$.

We then let

$$a_v\colon M_v(S,H)\longrightarrow S\times\widehat{S}\,,\quad a_v(\mathcal{F}):=\big(\delta_v(\mathcal{F}),\det(\mathcal{F})\otimes\det(\mathcal{F}_0)^\vee\big)\,.$$

Now let $K_v(S,H):=a_v^{-1}(0_S,\mathcal{O}_S)$, where 0_S is the zero of the Abelian group S. If $v^2>0$, the morphism

$$\tau_v\colon K_v(S,H)\times S\times\widehat{S}\longrightarrow M_v(S,H)\,,\quad \tau(\mathcal{E},p,L):=\tau_p^*(\mathcal{E})\otimes L$$

is a finite étale cover (for a proof of this, see [Yos01, Section 4.2]). We will moreover let $K_v^s(S,H):=K_v(S,H)\cap M_v^s(S,H)$.

2.2.2 *O'Grady's fibration.* Another morphism $b_v\colon M_v(S,H)\to S\times\widehat{S}$ was used by O'Grady in [O'G03]. For $\gamma\in CH_0(S)$, we let $\Sigma(\gamma)\in S$ be the sum of the points of the support of a representative of γ, counted with multiplicities (that is, the Albanese image of γ). For a coherent sheaf $\mathcal{F}$ on S, we let $\mathbf{c}_2(\mathcal{F})\in CH_0(S)$ be the second Chern class of $\mathcal{F}$, and we set $\beta(\mathcal{F}):=\Sigma(\mathbf{c}_2(\mathcal{F}))$.

The morphism $b_v\colon M_v(S,H)\to S\times\widehat{S}$ is defined by

$$b_v(\mathcal{F}):=\big(\beta(\mathcal{F}),\det(\mathcal{F})\otimes\det(\mathcal{F}_0)^\vee\big)\,.$$

The relation between a_v and b_v is explained in the following.

LEMMA 2.15. *There is an automorphism $g\colon S \to S$ such that $b_v = (g \times \mathrm{id}_{\widehat{S}}) \circ a_v$.*

Proof. To prove this, we just need to show that for every $\mathcal{F}_1, \mathcal{F}_2 \in M_v(S,H)$, we have $a_v(\mathcal{F}_1) = a_v(\mathcal{F}_2)$ if and only if $b_v(\mathcal{F}_1) = b_v(\mathcal{F}_2)$. Equivalently, we just need to show that for every $\mathcal{F}_1, \mathcal{F}_2 \in M_v(S,H)$ such that $\det(\mathcal{F}_1) \simeq \det(\mathcal{F}_2)$, we have $\delta_v(\mathcal{F}_1) = \delta_v(\mathcal{F}_2)$ if and only if $\beta(\mathcal{F}_1) = \beta(\mathcal{F}_2)$.

First suppose $\det(\mathcal{F}_1) \simeq \det(\mathcal{F}_2)$ and $\delta_v(\mathcal{F}_1) = \delta_v(\mathcal{F}_2)$, and let $\Gamma := [\mathcal{F}_1] - [\mathcal{F}_2] \in K(S)$. As $v(\mathcal{F}_1) = v(\mathcal{F}_2) = v$ and $\det(\mathcal{F}_1) \simeq \det(\mathcal{F}_2)$, the only nontrivial Chern class of Γ (in the Chow ring of S) is $\mathbf{c}_2(\Gamma)$.

Moreover, there is a representative of $\mathbf{c}_2(\Gamma)$ of the form $\overline{\Gamma} := \sum_{i=1}^n p_i - \sum_{i=1}^n q_i$, where $p_1, \dots, p_n$ and $q_1, \dots, q_n$ are points of S. We then notice that $\Gamma \in K(S)$ has the same rank and Chern classes as the class

$$\Gamma' := \left[\oplus_{i=1}^n \mathbb{C}_{p_i} - \oplus_{i=1}^n \mathbb{C}_{q_i}\right] \in K(S)\,.$$

Notice that if we let $\widetilde{F}\colon K(S) \to K\big(\widehat{S}\big)$ be the morphism induced by F on the level of the Grothendieck groups, we have

$$\det\left[\widetilde{F}(\Gamma')\right] = \otimes_{i=1}^n \mathcal{P}_{p_i} \otimes \mathcal{P}_{q_i}^\vee\,.$$

As $\det\big(\widetilde{F}(\Gamma)\big)$ depends only on the rank and the Chern classes of Γ in the Chow ring of $\widehat{S}$, we get

$$\begin{aligned}\det\big(\widetilde{F}([\mathcal{F}_1])\big) \otimes \det\big(\widetilde{F}([\mathcal{F}_2])\big)^\vee &= \det\big(\widetilde{F}(\Gamma)\big) = \det\big(\widetilde{F}(\Gamma')\big)\\ &= \otimes_{i=1}^n \mathcal{P}_{p_i} \otimes \mathcal{P}_{q_i}^\vee = \mathcal{P}_{\sum_{i=1}^n (p_i - q_i)} = \mathcal{P}_{\overline{\Gamma}}\,,\end{aligned}$$

where the equality $\otimes_{i=1}^n \mathcal{P}_{p_i} \otimes \mathcal{P}_{q_i}^\vee = \mathcal{P}_{\sum_{i=1}^n (p_i - q_i)}$ follows from the fact that the map $S \to \widehat{S}$, $p \mapsto \mathcal{P}_p$ is a group isomorphism.

Now, notice that as $\det(\mathcal{F}_1) = \det(\mathcal{F}_2)$, we have that $\delta_v(\mathcal{F}_1) = \delta_v(\mathcal{F}_2)$ if and only if $\det\big(\widetilde{F}([\mathcal{F}_1])\big) = \det\big(\widetilde{F}([\mathcal{F}_2])\big)$. The previous equalities give that this holds if and only if $\mathcal{P}_{\overline{\Gamma}} = \mathcal{O}_S$. But this is equivalent to $\Sigma\big([\overline{\Gamma}]\big) = 0_S$, where $[\overline{\Gamma}]$ is the class of $\overline{\Gamma}$ in $CH_0(S)$.

As this class is $\mathbf{c}_2(\Gamma)$, we finally get that $\delta_v(\mathcal{F}_1) = \delta_v(\mathcal{F}_2)$ if and only if $\Sigma(\mathbf{c}_2(\Gamma)) = 0_S$. This last condition is equivalent to $\Sigma(\mathbf{c}_2(\mathcal{F}_1)) = \Sigma(\mathbf{c}_2(\mathcal{F}_2))$, that is, to $\beta(\mathcal{F}_1) = \beta(\mathcal{F}_2)$, concluding the proof. ☐

As a consequence, we see that b_v is an isotrivial fibration.

2.2.3 *Fibers of the Yoshioka fibration.* The behaviour of moduli spaces of sheaves on an Abelian surface S under changing of polarization in the ample cone of S may now be generalized to the fibers of their Yoshioka (or O'Grady) fibration.

The main result is the following.

LEMMA 2.16. *Suppose that S is an Abelian surface with $\rho(S) \geqslant 2$ and that v is a Mukai vector on S.*

(1) *If $v = (v_0, v_1, v_2)$ is such that $v_0 > 0$, or $v_0 = 0$ and $v_2 \neq 0$, let $\mathcal{C}$ be a v-chamber.*
   (a) *If $H, H' \in \mathcal{C}$, then we have natural identifications $K_v(S,H) = K_v(S,H')$ and $K_v^s(S,H) = K_v^s(S,H')$.*
   (b) *If $H \in \overline{\mathcal{C}}$ is general with respect to v and $H' \in \mathcal{C}$, then we have natural identifications $K_v(S,H) = K_v(S,H')$ and $K_v^s(S,H) = K_v^s(S,H')$.*

(2) *If $v = (0, v_1, 0)$ and H is general with respect to v, then the tensorization with H induces an isomorphism between $K_v(S, H)$ and $K_{v_H}(S, H)$ and an isomorphism between $K^s_v(S, H)$ and $K^s_{v_H}(S, H)$.*

Proof. The first item of the statement is an immediate consequence of Lemma 2.10. For the second, use Lemma 2.12 by noticing that by the very definition of b_v, the tensorization with H maps fibers of b_v to fibers of b_{v_H} and hence, by Lemma 2.15, fibers of a_v to fibers of a_{v_H}. □

2.3 Deformations of a surface with a Mukai vector and a polarization

We introduce the main construction we use in what follows. Let T be a smooth, connected algebraic variety, and use the following notation: if $f: Y \to T$ is a morphism and $\mathscr{L} \in \mathrm{Pic}(Y)$, then for every $t \in T$, we let $Y_t := f^{-1}(t)$ and $\mathscr{L}_t := \mathscr{L}_{|Y_t}$.

Definition 2.17. Let S be a projective K3 or Abelian surface, v a Mukai vector on S and H a polarization on S. Write $v = m(r, \xi, a)$, where $\xi = c_1(L)$. If T is a smooth, connected algebraic variety, a *deformation of* (S, v, H) *along* T is a triple $(\mathscr{X}, \mathscr{L}, \mathscr{H})$, where:

(1) $\mathscr{X}$ is a projective, smooth deformation of S along T; that is, there is a smooth, projective, surjective map $f: \mathscr{X} \to T$ such that $\mathscr{X}_t$ is a projective surface for every $t \in T$, and there is a $0 \in T$ such that $\mathscr{X}_0 \simeq S$;

(2) $\mathscr{L}$ is a line bundle on $\mathscr{X}$ such that $\mathscr{L}_0 \simeq L$;

(3) $\mathscr{H}$ is a line bundle on $\mathscr{X}$ such that $\mathscr{H}_t$ is ample for every $t \in T$ and such that $\mathscr{H}_0 \simeq H$.

For every $t \in T$, we will write $v_t := m(r, c_1(\mathscr{L}_t), a)$.

Remark 2.18. Let S be a projective K3 (respectively, Abelian) surface and $v = (v_0, v_1, v_2)$ a Mukai vector on S. Let T be a smooth, connected variety and $f: \mathscr{X} \to T$ a smooth, projective deformation of S such that $\mathscr{X}_0 \simeq S$ for some $0 \in T$. Suppose that on $\mathscr{X}$, there are two line bundles $\mathscr{H}$ and $\mathscr{L}$, let $H := \mathscr{H}_0$ and $L := \mathscr{L}_0$, and suppose that H is ample and that $c_1(L) = v_1$. Then $(\mathscr{X}, \mathscr{L}, \mathscr{H})$ is a deformation of (S, v, H) along T if and only if $\mathscr{H}_t$ is ample for every $t \in T$. As the set of $t \in T$ such that $\mathscr{H}_t$ is ample is Zariski open in T, by restricting to a nonempty Zariski open subset of T, we may assume that $(\mathscr{X}, \mathscr{L}, \mathscr{H})$ is a deformation of (S, v, H) along T. Moreover, if we assume that H is general with respect to v, thanks to Proposition 2.14, by restricting to a smaller nonempty Zariski open subset of T, we may assume that $\mathscr{H}_t$ is general with respect to v_t for every $t \in T$.

Let S be a projective K3 (respectively, Abelian) surface, v a Mukai vector on S and H a polarization on S that is general with respect to v. If $(\mathscr{X}, \mathscr{L}, \mathscr{H})$ is a deformation of (S, v, H) along a smooth, connected algebraic variety T, we let $\phi: \mathscr{M} \to T$ be the relative moduli space of semistable sheaves and $\phi^s: \mathscr{M}^s \to T$ the relative moduli space of stable sheaves. This means that for every $t \in T$, we have $\mathscr{M}_t = M_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$ and $\mathscr{M}^s_t = M^s_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$.

If S is Abelian, let $\widehat{\mathscr{X}} \to T$ be the dual family, that is, the connected component of the relative Picard variety $g: \mathrm{Pic}_{\mathscr{X}/T} \to T$ containing the section of g corresponding to the family $\mathscr{O}_{\mathscr{X}}$. The dual family is then the smooth projective family whose fiber over $t \in T$ is the dual of $\mathscr{X}_t$. Consider the following condition:

$$\text{The morphism } \phi: \mathscr{M} \to T \text{ has a section, and } \mathscr{X} \to T \text{ is a } T\text{-group scheme}. \qquad (\star)$$

If condition $(\star)$ holds, we have a T-morphism $a_v: \mathscr{M} \to \mathscr{X} \times_T \widehat{\mathscr{X}}$ such that for every $t \in T$, the restriction morphism $a_{v|\mathscr{M}_t}$ is the Yoshioka fibration defined in Section 2.2. If

$$Z := \left\{(0_{\mathscr{X}_t}, \mathcal{O}_{\mathscr{X}_t}) \in \mathscr{X}_t \times \widehat{\mathscr{X}_t} \,|\, t \in T\right\} \subseteq \mathscr{X} \times_T \widehat{\mathscr{X}},$$

we will let $\mathscr{K} := a_v^{-1}(Z)$. Restricting the morphism ϕ to $\mathscr{K}$, we get a morphism $\phi_0\colon \mathscr{K} \to T$, whose fiber over $t \in T$ is $\mathscr{K}_t = K_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$. A similar definition, but using $\mathscr{M}^s$ instead of $\mathscr{M}$, gives the family $\phi_0^s\colon \mathscr{K}^s \to T$.

Remark 2.19. Condition ($\star$) is always verified up to shrinking T and taking a finite étale cover of T thanks to the smoothness of $\mathscr{M}^s$ over T.

The first result we need is that the families $\mathscr{M}$ and $\mathscr{K}$ are T-flat over a Zariski open neighbourhood of any $t \in T$ such that $\mathscr{H}_t$ is general with respect to v_t. This is the content of the following lemma.

LEMMA 2.20. *Let S be a projective K3 (respectively, Abelian) surface, v a Mukai vector on S and H a polarization on S that is general with respect to v. Let T be a smooth, connected algebraic variety and $(\mathscr{X}, \mathscr{L}, \mathscr{H})$ a deformation of (S, v, H) along T, and assume that condition ($\star$) holds if S is Abelian. Suppose that $t \in T$ is such that $\mathscr{H}_t$ is general with respect to v_t.*

(1) *The morphisms $\phi\colon \mathscr{M} \to T$ and $\phi_0\colon \mathscr{K} \to T$ are flat at t.*
(2) *The morphisms $\phi^s\colon \mathscr{M}^s \to T$ and $\phi_0^s\colon \mathscr{K}^s \to T$ are smooth at t.*

Proof. By Remark 2.18, we may suppose that for every $t \in T$, the polarization $\mathscr{H}_t$ is general with respect to v_t.

(1) Notice that $\mathscr{M}$ (respectively, $\mathscr{K}$) is connected (since T and the fibers are connected). Moreover, by [KLS06, Theorem 4.4] (respectively, by [PR14, Remark A.1] for $\mathscr{K}$), and using Lemmas 2.10 and 2.12 (respectively, Lemma 2.16), we see that the fibers of ϕ (respectively, ϕ_0) are reduced, irreducible and equidimensional. Now, by [GD66, Théorème 14.4.4], it follows that ϕ (respectively, ϕ_0) is universally open. Using [GD66, Corollaire 15.2.3], we get that ϕ (respectively, ϕ_0) is flat.

(2) This follows from point (1) since ϕ^s and ϕ_0^s have smooth fibers. □

Let S be a projective K3 (respectively, Abelian) surface, v a Mukai vector on S and H a polarization on S that is general with respect to v. By choosing a nontrivial deformation of (S, v, H) along a smooth, connected variety T, we get a flat, projective deformation $\phi\colon \mathscr{M} \to T$ of M_v and a smooth quasi-projective deformation $\phi^s\colon \mathscr{M}^s \to T$ of M_v^s. Moreover, if S is Abelian, we get a flat, projective deformation $\phi_0\colon \mathscr{K} \to T$ of K_v and a smooth quasi-projective deformation $\phi_0^s\colon \mathscr{K}^s \to T$ of K_v^s. We now prove that this deformation is locally trivial.

LEMMA 2.21. *Let S be a projective K3 (respectively, Abelian) surface, v a Mukai vector on S and H a polarization on S that is general with respect to v. Let T be a smooth connected variety and $(\mathscr{X}, \mathscr{L}, \mathscr{H})$ a deformation of (S, v, H) along T, and assume that condition ($\star$) holds if S is Abelian.*

(1) *If $p \in \mathscr{M}$ and $t := \phi(p)$ is such that $\mathscr{H}_t$ is general with respect to v_t, then $(\mathscr{M}, p) \simeq (\mathscr{M}_t, p) \times (T, t)$ as germs of analytic spaces.*
(2) *If $p \in \mathscr{K}$ and $t := \phi_0(p)$ is such that $\mathscr{H}_t$ is general with respect to v_t, then $(\mathscr{K}, p) \simeq (\mathscr{K}_t, p) \times (T, t)$ as germs of analytic spaces.*

Proof. If v is primitive, then ϕ is a smooth, projective morphism, and there is nothing to prove. If $v = 2w$, where w is primitive and $w^2 = 2$, this is [PR13, Proposition 2.16] if T is a curve (the proof given in [PR13] works for polarizations which are general with respect to the Mukai vector). If T is a smooth connected variety of dimension $d \geqslant 2$, this implies that the statement holds along any smooth connected curve through t. By [FK87, Corollary 0.2], the general statement follows.

For the remaining cases, by [KLS06], the moduli spaces $\mathscr{M}_t = M_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$ and $\mathscr{K}_t = K_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$ are symplectic varieties which are locally factorial, and, by [Nam01, Corollary 1], they have terminal singularities. The Main Theorem of [Nam06] tells us that for every $p \in \mathscr{M}_t$ (respectively, $p \in \mathscr{K}_t$) and for every $n \in \mathbb{N}$, the infinitesimal nth order deformation of $\mathscr{M}_t$ (respectively, of $\mathscr{K}_t$) induced by ϕ (respectively, by ϕ_0), which is flat by Lemma 2.20, is locally trivial at p; the statement follows again by [FK87, Corollary 0.2]. □

As a corollary of this, using the Thom first isotopy lemma (see [Dim92, Theorem 3.5]), we have the following.

Lemma 2.22. *Let S be a projective K3 (respectively, Abelian) surface, v a Mukai vector on S and H a polarization on S that is general with respect to v. Let T be a smooth connected algebraic variety, let $(\mathscr{X}, \mathscr{L}, \mathscr{H})$ be a deformation of (S, v, H) along T, and assume that condition $(\star)$ holds if S is Abelian.*

(1) *If $p \in \mathscr{M}$ and $t := \phi(p)$ is such that $\mathscr{H}_t$ is general with respect to v_t, there is an analytic open neighbourhood $U \subseteq T$ of t such that $\phi^{-1}(U)$ is homeomorphic over U to $\mathscr{M}_t \times U$ and $(\phi^s)^{-1}(U)$ is homeomorphic over U to $\mathscr{M}_t^s \times U$.*
(2) *If $p \in \mathscr{K}$ and $t := \phi_0(p)$ is such that $\mathscr{H}_t$ is general with respect to v_t, there is an analytic open neighbourhood $U \subseteq T$ of t such that $\phi_0^{-1}(U)$ is homeomorphic over U to $\mathscr{K}_t \times U$ and $(\phi_0^s)^{-1}(U)$ is homeomorphic over U to $\mathscr{K}_t^s \times U$.*

2.4 Isomorphisms between moduli spaces

We now describe several isomorphisms that will be frequently used in the proof of Theorem 1.7. All of them are induced by Fourier–Mukai transforms, either the tensorization with a line bundle or the one whose kernel is the ideal sheaf of the diagonal (for K3 surfaces) or the Poincaré bundle (for Abelian surfaces).

2.4.1 *Isomorphisms from tensorization with line bundles.* Let S be a projective K3 or Abelian surface, and let $v = m(r, \xi, a)$ be a Mukai vector. Recall that if $L \in \operatorname{Pic}(S)$, we defined $v_L := v \cdot \operatorname{ch}(L)$ and that if D is a divisor on S, we let $v_D := v_{\mathcal{O}_S(D)}$ (see Section 2.1 and Lemma 2.6).

Definition 2.23. Let $v, v' \in \widetilde{H}(S, \mathbb{Z})$ be two Mukai vectors, and let $v = (r, \xi, a)$, $v' = (r', \xi', a')$.

(1) If H is a polarization on S, we say that v and v' are *H-equivalent* if there is an $s \in \mathbb{Z}$ such that $v' = v_{sH}$.
(2) If $r, r' > 0$, we say that v and v' are *equivalent* if there is an $L \in \operatorname{Pic}(S)$ such that $v' = v_L$.

The following is the main result about isomorphisms induced by tensorization with a line bundle,[2] which shows that moduli spaces of sheaves corresponding to equivalent (or H-equivalent)

[2]In Lemma 2.24, by a slight abuse of notation, we let K_v denote not only the fiber of $a_v \colon M_v \to S \times \widehat{S}$ over $(0_S, \mathcal{O}_S)$ but also any other fiber. This is justified since a_v is an isotrivial fibration, so all its fibers are isomorphic.

Mukai vectors are isomorphic (and the isomorphism is induced by tensorization with a suitable line bundle).

Lemma 2.24. *Let S be a projective K3 or Abelian surface, v a Mukai vector and H an ample line bundle on S.*

(1) *For every $d \in \mathbb{Z}$, the morphism*

$$M_v(S,H) \longrightarrow M_{v_{dH}}(S,H)\,, \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_S(dH)$$

is an isomorphism, which induces isomorphisms $M^s_v(S,H) \simeq M^s_{v_{dH}}(S,H)$. If S is an Abelian surface, it also induces isomorphisms $K_v(S,H) \simeq K_{v_{dH}}(S,H)$ and $K^s_v(S,H) \simeq K^s_{v_{dH}}(S,H)$.

(2) *If $v = (v_0, v_1, v_2)$ and $v_0 > 0$, $L \in \operatorname{Pic}(S)$ and H is v-generic, the morphism*

$$M_v(S,H) \longrightarrow M_{v_L}(S,H)\,, \quad \mathcal{F} \longmapsto \mathcal{F} \otimes L$$

is an isomorphism, which induces isomorphisms $M^s_v(S,H) \simeq M^s_{v_L}(S,H)$. If S is an Abelian surface, it also induces isomorphisms $K_v(S,H) \simeq K_{v_L}(S,H)$ and $K^s_v(S,H) \simeq K^s_{v_L}(S,H)$.

Proof. First, notice that $v(\mathcal{F} \otimes L) = v(\mathcal{F}) \cdot \operatorname{ch}(L)$. To prove the first point of the statement, it is enough to remark that a sheaf $\mathcal{F}$ with Mukai vector v is H-(semi)stable if and only if $\mathcal{F} \otimes \mathcal{O}_S(dH)$ is H-(semi)stable.

For the second point, we need to show that if $\mathcal{F}$ is H-(semi)stable, then $\mathcal{F} \otimes L$ is H-(semi)stable. This is proved for stable sheaves by Yoshioka (see [Yos01, Lemma 1.1]), and the proof goes through for semistable sheaves.

If S is Abelian, by the definition of the morphism a_v (see Section 2.2), we have that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are in the same fiber of a_v, then $\mathcal{F}_1 \otimes L$ and $\mathcal{F}_2 \otimes L$ are in the same fiber of a_{v_L}. As a_v and a_{v_L} are both isotrivial fibrations, the isomorphism between M_v and M_{v_L} obtained by tensorization with L induces an isomorphism between K_v and K_{v_L}. ∎

2.4.2 *Isomorphisms from Fourier–Mukai transforms.* We now recall two basic results, originally due to Yoshioka, about isomorphisms between moduli spaces of sheaves over K3 or Abelian surfaces coming from Fourier–Mukai transforms. Yoshioka's theorems are stated in a more general setting; here we present simplified adapted proofs for the convenience of the reader. We will only consider the Fourier–Mukai transform whose kernel is the ideal of the diagonal (for K3 surfaces) or the Poincaré bundle (for Abelian surfaces).

We need the following notation: if S a projective K3, we let $\Delta \subseteq S \times S$ be the diagonal and $\mathcal{I}$ the ideal of Δ. We have an exact sequence of coherent sheaves on $S \times S$:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{S\times S} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0\,. \tag{2.1}$$

We moreover let $\pi_1, \pi_2 \colon S \times S \to S$ be the two projections.

If S is an Abelian surface and $\widehat{S}$ is its dual, we let $\mathcal{P}$ be the Poincaré line bundle on $S \times \widehat{S}$, $\pi_1 \colon S \times \widehat{S} \to S$ and $\pi_2 \colon S \times \widehat{S} \to \widehat{S}$ the two projections and $\iota \colon S \to S$ the involution acting as -1.

We will moreover consider the functors

$$F_{\mathrm{K3}} \colon D^b(S) \longrightarrow D^b(S)\,, \quad F_{\mathrm{K3}}(E^\bullet) := R\pi_{2*}\big(\pi_1^* E^\bullet \otimes^L \mathcal{I}\big)\,,$$
$$\widehat{F}_{\mathrm{K3}} \colon D^b(S) \longrightarrow D^b(S)\,, \quad \widehat{F}_{\mathrm{K3}}(E^\bullet) := R\mathcal{H}\mathrm{om}_{\pi_1}\big(\mathcal{I}, \pi_2^* E^\bullet\big)$$

if S is a K3 surface and

$$F_{\mathrm{Ab}}\colon D^b(S) \longrightarrow D^b\big(\widehat{S}\big)\,, \quad F_{\mathrm{Ab}}(E^\bullet) := R\pi_{2*}\big(\pi_1^* E^\bullet \otimes \mathcal{P}\big)\,,$$
$$\widehat{F}_{\mathrm{Ab}}\colon D^b\big(\widehat{S}\big) \longrightarrow D^b(S)\,, \quad \widehat{F}_{\mathrm{Ab}}(E^\bullet) := \iota^* R\pi_{1*}\big(\pi_2^* E^\bullet \otimes \mathcal{P}\big)$$

if S is an Abelian surface.

By [Bri99], we know that F_{K3} and F_{Ab} are equivalences of triangulated categories. Moreover, the functor $\widehat{F}_{\mathrm{K3}}[2]$ is the right and left adjoint to F_{K3}, so that $F_{\mathrm{K3}} \circ \widehat{F}_{\mathrm{K3}} = [-2]$ (see [Huy06, Proposition 1.26]), and $\widehat{F}_{\mathrm{Ab}}[2]$ is the right and left adjoint to F_{Ab}, so that $F_{\mathrm{Ab}} \circ \widehat{F}_{\mathrm{Ab}} = [-2]$ (see [Muk81, Theorem 2.2]).

We will make use of the following definition due to Mukai.

Definition 2.25. Let S be a projective K3 or Abelian surface, $\mathcal{G}$ a coherent sheaf on S and F a Fourier–Mukai functor on $D^b(S)$. For $i \in \{0,1,2\}$, we say that $\mathcal{G}$ *verifies* WIT(i) *with respect to* F if $F(\mathcal{G}) = F^i(\mathcal{G})[-i]$.

If S is a projective K3 surface and $\mathcal{G}$ is a coherent sheaf on S, then the functor $R\pi_{2*}\big(\pi_1^*\mathcal{G} \otimes^L \cdot\big)$ applied to the exact sequence (2.1) gives the long exact sequence of coherent sheaves on S

$$\begin{aligned} 0 \longrightarrow F^0_{\mathrm{K3}}(\mathcal{G}) \longrightarrow \mathcal{O}_S \otimes H^0(\mathcal{G}) \xrightarrow{\mathrm{ev}} \mathcal{G} \longrightarrow \\ \longrightarrow F^1_{\mathrm{K3}}(\mathcal{G}) \longrightarrow \mathcal{O}_S \otimes H^1(\mathcal{G}) \longrightarrow 0 \longrightarrow \\ \longrightarrow F^2_{\mathrm{K3}}(\mathcal{G}) \longrightarrow \mathcal{O}_S \otimes H^2(\mathcal{G}) \longrightarrow 0\,, \end{aligned} \tag{2.2}$$

and if $\mathcal{G}$ is torsion-free, the functor $R\mathcal{H}\mathrm{om}_{\pi_1}(\cdot, \pi_2^*\mathcal{G})$ applied to the exact sequence (2.1) gives the long exact sequence of coherent sheaves on S

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_S \otimes H^0(\mathcal{G}) \longrightarrow \widehat{F}^0_{\mathrm{K3}}(\mathcal{G}) \longrightarrow \\ \longrightarrow 0 \longrightarrow \mathcal{O}_S \otimes H^1(\mathcal{G}) \longrightarrow \widehat{F}^1_{\mathrm{K3}}(\mathcal{G}) \longrightarrow \\ \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_S \otimes H^2(\mathcal{G}) \longrightarrow \widehat{F}^2_{\mathrm{K3}}(\mathcal{G}) \longrightarrow 0\,. \end{aligned} \tag{2.3}$$

From now on, we will make use of the following notation.

Notation 2.26. If S is a projective K3 surface and $v = (r, \xi, a)$ is a Mukai vector on S, we let $\widetilde{v} := (a, -\xi, r)$.

If S is an Abelian surface and $L \in \mathrm{Pic}(S)$, we let $\widehat{L} := \det(F(L))^{-1} \in \mathrm{Pic}\big(\widehat{S}\big)$; moreover, if $\xi = c_1(L)$, we let $\widehat{\xi} := c_1\big(\widehat{L}\big)$. Finally, if $v = (r, \xi, a)$ is a Mukai vector on the Abelian surface S, we let $\widetilde{v} := \big(a, -\widehat{\xi}, r\big)$ (which belongs to the Mukai lattice of $\widehat{S}$).

Remark 2.27. If H is an ample line bundle on an Abelian surface S, then $\widehat{H}$ is ample (see [Muk81, Proposition 3.11]).

The first result we need is the following (see [Yos02, Theorem 3.18]).

Lemma 2.28. *Let S be a projective K3 (respectively, Abelian) surface, $v = (v_0, v_1, v_2)$ a Mukai vector on S and H a polarization on S, and let $h := c_1(H)$. Let $r, k \in \mathbb{N}^*$ and $\xi \in \mathrm{NS}(S)$ be effective, and suppose furthermore that we are in one of the following situations:*

(1) *we have $\mathrm{NS}(S) = \mathbb{Z}h$, $v^2 = 2k$ and $v_0 = r$; that is, there are $p, a \in \mathbb{Z}$ such that $v = (r, ph, a)$ and $v^2 = 2k$; or*
(2) *we have $v_0 = 0$, $v_1 = \xi$ and $v^2 = 2k$; that is, there is $p \in \mathbb{Z}$ such that $v = (0, \xi, p)$ and $v^2 = 2k$.*

Then there is a $p_0 \in \mathbb{N}$ such that if $p > p_0$, every H-semistable sheaf $\mathcal{E}$ with Mukai vector v on S verifies WIT(0) *with respect to F_{K3} (respectively, F_{Ab}), and $F^0_{\mathrm{K3}}(\mathcal{E})$ (respectively, $F^0_{\mathrm{Ab}}(\mathcal{E})$) is locally free with Mukai vector $\widetilde{v}$.*

Proof. We let

$$V_{r,k} := \left\{ v' = (r, p'h, a') \in \widetilde{H}(S, \mathbb{Z}) \,|\, (v')^2 = 2k,\ 0 \leqslant p' \leqslant r \right\}$$

and

$$V_{0,\xi,k} := \left\{ v' = (0, \xi, p') \,|\, (v')^2 = 2k,\ 0 \leqslant p' \leqslant \xi \cdot H \right\}.$$

Notice that $V_{r,k}$ and $V_{0,\xi,k}$ are finite sets. As the family of semistable sheaves with fixed Mukai vector v is bounded, by Serre's theorem, there is a $T \in \mathbb{N}$ such that for every $s > T$ and for every H-semistable sheaf $\mathcal{E}'$ with Mukai vector in $V_{r,k}$ or $V_{0,\xi,k}$, we have $H^1(\mathcal{E}' \otimes \mathcal{O}_S(sH)) = H^2(\mathcal{E}' \otimes \mathcal{O}_S(sH)) = 0$, and the evaluation morphism

$$H^0(\mathcal{E}' \otimes \mathcal{O}_S(sH)) \otimes \mathcal{O}_S \longrightarrow \mathcal{E}' \otimes \mathcal{O}_S(sH)$$

is surjective.

Under the hypotheses of case (1), there are $s \in \mathbb{N}$ and $v' \in V_{r,k}$ such that $v'_{sH} = v = (r, ph, a)$. Notice that if $p > p_0 := r + rT$, then we have $s > T$. Since tensorization by sH induces an isomorphism from $M_{v'}(S, H)$ to $M_v(S, H)$ by Lemma 2.24, we conclude that for every H-semistable sheaf $\mathcal{E}$ with $v(\mathcal{E}) = v$, there is an $\mathcal{E}' \in M_{v'}(S, H)$ such that $\mathcal{E} \simeq \mathcal{E}' \otimes \mathscr{O}_S(sH)$, and hence $H^1(\mathcal{E}) = H^2(\mathcal{E}) = 0$, and the evaluation morphism

$$H^0(\mathcal{E}) \otimes \mathcal{O}_S \longrightarrow \mathcal{E}$$

is surjective.

Similarly, under the hypotheses of case (2), there are $s \in \mathbb{N}$ and $v' \in V_{0,\xi,k}$ such that $v'_{sH} = v = (0, \xi, p)$, and if $p > p_0 := \xi \cdot H + (\xi \cdot H)T$, by the same argument, the same conclusion holds for every H-semistable sheaf $\mathcal{E}$ with $v(\mathcal{E}) = v$.

Now, if S is K3, as $H^1(\mathcal{E}) = H^2(\mathcal{E}) = 0$ and the evaluation morphism $H^0(\mathcal{E}) \otimes \mathcal{O}_S \to \mathcal{E}$ is surjective, the exact sequence (2.2) for $\mathcal{E}$ implies that $F^1_{\mathrm{K3}}(\mathcal{E}) = F^2_{\mathrm{K3}}(\mathcal{E}) = 0$, so that $F_{\mathrm{K3}}(\mathcal{E}) = F^0_{\mathrm{K3}}(\mathcal{E})$, which proves that $\mathcal{E}$ verifies WIT(0) with respect to F_{K3}.

If S is Abelian, we not only have that $H^1(\mathcal{E}) = H^2(\mathcal{E}) = 0$ and the evaluation morphism $H^0(\mathcal{E}) \otimes \mathcal{O}_S \to \mathcal{E}$ is surjective, but the same holds for $\mathcal{E} \otimes L$ for every $L \in \widehat{S}$ (since $v(\mathcal{E} \otimes L) = v$). By cohomology and base change, it follows that $F^1_{\mathrm{Ab}}(\mathcal{E}) = F^2_{\mathrm{Ab}}(\mathcal{E}) = 0$, so that $F_{\mathrm{Ab}}(\mathcal{E}) = F^0_{\mathrm{Ab}}(\mathcal{E})$, which proves that $\mathcal{E}$ verifies WIT(0) with respect to F_{Ab}.

We are left with showing that $F^0_{\mathrm{K3}}(\mathcal{E})$ (respectively, $F^0_{\mathrm{Ab}}(\mathcal{E})$) is locally free and its Mukai vector is $\widetilde{v}$.

To show this, let us first consider S to be a K3 surface. As $\mathcal{E}$ verifies WIT(0) with respect to F_{K3}, the exact sequence (2.2) applied to $\mathcal{E}$ gives the exact sequence

$$0 \longrightarrow F^0_{\mathrm{K3}}(\mathcal{E}) \longrightarrow H^0(\mathcal{E}) \otimes \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow 0\,. \tag{2.4}$$

We then see that $v\big(F^0_{\mathrm{K3}}(\mathcal{E})\big) = \widetilde{v}$.

As $\mathcal{E}$ is a coherent sheaf of pure dimension 2 or 1 on the smooth surface S, the projective dimension of $\mathcal{E}$ is at most 1. Since the sequence (2.4) is exact and $H^0(\mathcal{E}) \otimes \mathcal{O}_S$ is locally free, this implies that $F^0_{\mathrm{K3}}(\mathcal{E})$ is locally free too.

This completes the proof when S is a K3 surface. The case of Abelian surfaces is easier: the fact that $v\big(F^0_{\mathrm{Ab}}(\mathcal{E})\big) = \widetilde{v}$ is well known. Moreover, as $H^i(\mathcal{E} \otimes L) = 0$ for every $L \in \widehat{S}$ and for $i = 1, 2$, we see that $\mathcal{E}$ is an IT-sheaf of index 0, and hence $F^0_{\mathrm{Ab}}(\mathcal{E})$ is locally free (see for instance [BL21, Lemma 14.2.1]). □

We are now in the position to prove the main results of this section. The first is the following, showing that if S is a K3 (respectively, Abelian) surface with Picard number 1 and v is a Mukai vector on S with positive rank, then F_{K3} (respectively, F_{Ab}) induces an isomorphism between M_v and $M_{\widetilde{v}}$ (see Notation 2.26 for the definition of $\widetilde{v}$).

Proposition 2.29. *Let S be a K3 or Abelian surface such that $\mathrm{NS}(S) = \mathbb{Z}\cdot h$, and let $h = c_1(H)$ for an ample line bundle H. Let $r, k \in \mathbb{N}^*$, and let $v = (v_0, v_1, v_2)$ be a Mukai vector on S such that $v_0 = r$ and $v^2 = 2k$; that is, there are $n, a \in \mathbb{Z}$ such that $v = (r, nh, a)$ and $v^2 = 2k$.*

(1) *If S is K3, there is an $n_0 \in \mathbb{N}$ such that for every $n > n_0$, the functor F_{K3} induces isomorphisms $M_v(S,H) \simeq M_{\widetilde{v}}(S,H)$ and $M^s_v(S,H) \simeq M^s_{\widetilde{v}}(S,H)$.*
(2) *If S is Abelian, there is an $n_0 \in \mathbb{N}$ such that for every $n > n_0$, the functor F_{Ab} induces isomorphisms $M_v(S,H) \simeq M_{\widetilde{v}}\big(\widehat{S},\widehat{H}\big)$, $M^s_v(S,H) \simeq M^s_{\widetilde{v}}\big(\widehat{S},\widehat{H}\big)$, $K_v(S,H) \simeq K_{\widetilde{v}}\big(\widehat{S},\widehat{H}\big)$ and $K^s_v(S,H) \simeq K^s_{\widetilde{v}}\big(\widehat{S},\widehat{H}\big)$.*

Proof. The surfaces involved in the statements and in the proofs of this proposition and of the related lemmas (that is, a K3 surface S or an Abelian surface S and its dual $\widehat{S}$) all have cyclic Néron–Severi group. In particular, if Σ is such a surface, then $\mathrm{NS}(\Sigma) = \mathbb{Z}\cdot\ell$, where $\ell = c_1(L)$ and L is an ample generator.

This allows us to unify and simplify the notation for the Mukai vectors on such a surface Σ: the Mukai vector $v = (r, n\ell, a)$ will always be written under the simpler form (r, n, a).

Similarly, while discussing (semi)stability, we systematically avoid any explicit reference to the (essentially unique) polarization. In particular, if $\mathcal{E}$ is a coherent sheaf on Σ, we will simply write $p(\mathcal{E})$ for its reduced Hilbert polynomial with respect to the primitive ample divisor.

Finally, we will use the notation F for both F_{K3} and F_{Ab}, and $\widehat{F}$ for both $\widehat{F}_{\mathrm{K3}}$ and $\widehat{F}_{\mathrm{Ab}}$. Making use of this simplified notation, we start the proof of the proposition.

We first notice that by Lemma 2.28, there is an $n_0 \in \mathbb{N}$ such that for every $n > n_0$ and for every semistable sheaf $\mathcal{E}$ with Mukai vector $v = (r, n, a)$ on S, we have that $F(\mathcal{E}) = F^0(\mathcal{E})$ is a locally free sheaf with Mukai vector $\widetilde{v}$.

Our aim is to prove that the locally free sheaf $F^0(\mathcal{E})$ is semistable. Once this is done, it will imply that F induces an injective morphism $f_{\mathrm{K3}}\colon M_v(S,H) \to M_{\widetilde{v}}(S,H)$ if S is K3 and $f_{\mathrm{Ab}}\colon M_v(S,H) \to M_{\widetilde{v}}\big(\widehat{S},\widehat{H}\big)$ if S is Abelian. By [KLS06, Theorem 4.4], these moduli spaces are irreducible of the same dimension, so f_{K3} and f_{Ab} are isomorphisms and induce isomorphisms between the smooth loci of the moduli spaces.

Moreover, if S is Abelian and $\mathcal{E}, \mathcal{E}_0 \in M_v(S,H)$, as $\iota^* \circ \widehat{F}\colon D^b\big(\widehat{S}\big) \to D^b(S)$ is the Fourier–Mukai transform with kernel $\mathcal{P}$, by the definition of $a_{\widetilde{v}}$, we have

$$\begin{aligned} a_{\widetilde{v}}(F(\mathcal{E})) &= \big(\det\big(\iota^*\widehat{F}(F(\mathcal{E}))\big) \otimes \det\big(\iota^*\widehat{F}(F(\mathcal{E}_0))\big)^\vee, \det(F(\mathcal{E})) \otimes \det(F(\mathcal{E}_0))^\vee\big) \\ &= \big(\iota^*\big(\det(\mathcal{E}) \otimes \det(\mathcal{E}_0)^\vee\big), \det(F(\mathcal{E})) \otimes \det(F(\mathcal{E}_0))^\vee\big) = \varepsilon(a_v(\mathcal{E}))\,, \end{aligned}$$

where

$$\varepsilon\colon S \times \widehat{S} \longrightarrow \widehat{S} \times S\,, \quad \varepsilon(p,q) := (\widehat{\iota}(q), p)\,,$$

and $\widehat{\iota}\colon \widehat{S} \to \widehat{S}$ is the involution acting as -1.

It follows that $\mathcal{E}_1$ and $\mathcal{E}_2$ lie in the same fiber of a_v if and only if $F(\mathcal{E}_1)$ and $F(\mathcal{E}_2)$ lie in the same fiber of $a_{\widetilde{v}}$. As a_v and $a_{\widetilde{v}}$ are isotrivial fibrations, it follows that the functor F induces an injection $K_v(S,H) \to K_{\widetilde{v}}(\widehat{S},\widehat{H})$. Since by [PR14, Remark A.1], we know that $K_v(S,H)$ and $K_{\widetilde{v}}(\widehat{S},\widehat{H})$ are irreducible and of the same dimension, the previous injection is an isomorphism and induces an isomorphism between the smooth loci.

Hence it only remains to prove that the sheaf $F^0(\mathcal{E})$ is semistable. The proof will be by contradiction, supposing that $F^0(\mathcal{E})$ is not semistable.

This implies that there is a desemistabilizing subsheaf $\mathcal{G}_1 \subseteq F^0(\mathcal{E})$, that is, a coherent subsheaf such that $p(\mathcal{G}_1) > p\big(F^0(\mathcal{E})\big)$. We may and will choose it to be stable with maximal reduced Hilbert polynomial; such a $\mathcal{G}_1$ is the first term of a Jordan–Hölder filtration of the first term of a Harder–Narasimhan filtration of $F^0(\mathcal{E})$.

We will moreover let $\mathcal{G}_2$ be the quotient of $F^0(\mathcal{E})$ by $\mathcal{G}_1$, so that we have an exact sequence

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow F^0(\mathcal{E}) \longrightarrow \mathcal{G}_2 \longrightarrow 0\,. \tag{2.5}$$

Applying the functor $\widehat{F}$ to it, and using the fact that $\widehat{F} \circ F = [-2]$, we get the two exact sequences

$$0 \longrightarrow \widehat{F}^0(\mathcal{G}_2) \longrightarrow \widehat{F}^1(\mathcal{G}_1) \longrightarrow 0\,, \tag{2.6}$$

$$0 \longrightarrow \widehat{F}^1(\mathcal{G}_2) \longrightarrow \widehat{F}^2(\mathcal{G}_1) \longrightarrow \mathcal{E} \longrightarrow \widehat{F}^2(\mathcal{G}_2) \longrightarrow 0\,. \tag{2.7}$$

The sheaf $\mathcal{G} := \widehat{F}^2(\mathcal{G}_1)/\widehat{F}^1(\mathcal{G}_2)$ is then a subsheaf of $\mathcal{E}$. Our aim is to show that $p(\mathcal{G}) > p(\mathcal{E})$; this would contradict the semistability of $\mathcal{E}$, hence completing the argument.

In order to prove that $p(\mathcal{G}) > p(\mathcal{E})$, we start by collecting in the following lemma some properties of the sheaf $\mathcal{G}_1$.

LEMMA 2.30. *The sheaf $\mathcal{G}_1$ is locally free, verifies* WIT(2) *with respect to $\widehat{F}$, and if $v(\mathcal{G}_1) = (a_1, -n_1, r_1)$, then $a_1, r_1, n_1 > 0$, $a_1 < a$, $n_1 < n$ and either $n_1/a_1 < n/a$, or $n_1/a_1 = n/a$ and $r_1/a_1 > r/a$.*

Proof. The proof of this will be divided in six different steps.

Step 1: the sheaf $\mathcal{G}_1$ is locally free. Indeed, otherwise $\mathcal{G}_1^{**}$ would be a locally free subsheaf of the locally free sheaf $F^0(\mathcal{E})$ with $p(\mathcal{G}_1^{**}) > p(\mathcal{G}_1)$, contradicting the maximality of $p(\mathcal{G}_1)$.

Step 2: if S is K3, then $-c_1(\mathcal{G}_1)$ is effective and nonzero; that is, $n_1 > 0$. As S is K3, by the exact sequence (2.2), there exists an injection $\mathcal{G}_1 \subseteq F^0(\mathcal{E}) \subseteq H^0(\mathcal{E}) \otimes \mathcal{O}_S$. As a consequence, letting V the generic quotient of $H^0(\mathcal{E})$ having rank a_1, we also have an injective morphism $\mathcal{G}_1 \subseteq V \otimes \mathcal{O}_S$.

Since $V \otimes \mathcal{O}_S$ is a trivial vector bundle having the same rank as $\mathcal{G}_1$, we deduce that $-c_1(\mathcal{G}_1)$ is effective and is zero only if $\mathcal{G}_1$ is trivial. Finally, $\mathcal{G}_1$ cannot be trivial since it is contained in $F^0(\mathcal{E})$, which is the kernel of the evaluation morphism $H^0(\mathcal{E}) \otimes \mathcal{O}_S \to \mathcal{E}$ and cannot have nonzero global sections.

Step 3: if S is Abelian, then $-c_1(\mathcal{G}_1)$ is effective and nonzero; that is, $n_1 > 0$. Let $Z \subseteq S$ be a reduced 0-dimensional subscheme of degree $d \gg 0$, that we may and will choose so that for every $L \in \mathrm{Pic}^0(S)$, no section of $\mathcal{E} \otimes L$ vanishes along Z. We let $\mathcal{E}_Z$ be the restriction of $\mathcal{E}$ to Z and consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_Z \longrightarrow 0\,.$$

By construction, for every $L \in \mathrm{Pic}^0(S)$, we have $H^0(\mathcal{K} \otimes L) = 0$.

As a consequence, we have $F^0(K) = 0$, so, applying F to the previous exact sequence, we get an inclusion $F^0(\mathcal{E}) \subseteq F^0(\mathcal{E}_Z)$. Now, notice that, as Z is 0-dimensional, $F^0(\mathcal{E}_Z)$ is a direct sum of line bundles of degree 0 on $\widehat{S}$. As $\mathcal{G}_1 \subseteq F^0(\mathcal{E})$, we then get an inclusion of $\mathcal{G}_1$ in a direct sum of line bundles of degree 0 on $\widehat{S}$. As a consequence, the rank a_1 vector bundle $\mathcal{G}_1$ also admits an injective morphism to a direct sum $\oplus_{i=1}^{a_1} L_i$ of a_1 degree 0 line bundles on $\widehat{S}$. This implies that $-c_1(\mathcal{G}_1)$ is represented by an effective divisor and is zero only if $\mathcal{G}_1 \simeq \oplus_{i=1}^{a_1} L_i$.

Finally, the last isomorphism does not hold since it would imply that $F^0(\mathcal{E}) \otimes L_i^*$ admits a nonzero section. On the other hand, by the projection formula,

$$H^0\big(F^0(\mathcal{E}) \otimes L_i^*\big) \simeq H^0\big(p_S^*(\mathcal{E}) \otimes \mathcal{P} \otimes p_{\widehat{S}}^*(L^*)\big) \simeq H^0\big(\mathcal{E} \otimes \iota^* \widehat{F}^0(L_i^*)\big)\,,$$

and the latter is zero since $\widehat{F}^0(L_i^*) = 0$.

Step 4: the sheaf $\mathcal{G}_1$ *verifies* WIT(2) *with respect to* $\widehat{F}$. First notice that, by the definition of $\mathcal{G}_1$, its slope $\mu(\mathcal{G}_1)$ is maximal among the slopes of the subsheaves of $F^0(\mathcal{E})$. This implies that $\mu(\mathcal{G}_1)$ is an upper bound for the slopes of the subsheaves of $F^0(\mathcal{E})$ and $\mathcal{G}_2$. Since, by Steps 2 and 3, $n_1 > 0$, the class $-c_1(\mathcal{G})$ is effective and nonzero, all subsheaves of $F^0(\mathcal{E})$ and of $\mathcal{G}_2$ have strictly negative slope.

In particular, we see that $H^0(\mathcal{G}_i) = 0$ for $i = 1, 2$. This implies that $\widehat{F}^0(\mathcal{G}_i) = 0$ for $i = 1, 2$. If S is K3, this is a consequence of the exact sequence (2.3) applied to $\mathcal{G}_i$; if S is Abelian, the same argument as before shows that $H^0(\mathcal{G}_i \otimes L) = 0$ for every $L \in \widehat{S}$; hence $\widehat{F}^0(\mathcal{G}_i) = 0$ by cohomology and base change.

Now, the exact sequence (2.6) implies $\widehat{F}^1(\mathcal{G}_1) = 0$, and we conclude that $\mathcal{G}_1$ verifies WIT(2) with respect to $\widehat{F}$.

Step 5: we have $r_1 > 0$. As $\mathcal{G}_1$ verifies WIT(2) with respect to $\widehat{F}$, we see that $v(\widehat{F}^2(\mathcal{G}_1)) = (r_1, n_1, a_1)$ (this follows from the exact sequence (2.3) applied to $\mathcal{G}_1$ if S is K3, and it is well known if S is Abelian), hence $r_1 \geqslant 0$. If $r_1 = 0$, the morphism $\widehat{F}^2(\mathcal{G}_1) \to \mathcal{E}$ in the exact sequence (2.7) would be trivial (since $\mathcal{E}$ is torsion-free and $\widehat{F}^2(\mathcal{G}_1)$ is torsion). As $\widehat{F}$ is fully faithful, this would imply that the inclusion morphism $\mathcal{G}_1 \to F^0(\mathcal{E})$ is trivial, leading to a contradiction; it follows that $r_1 > 0$.

Step 6: conclusion of the proof. We notice that a_1 is the rank of $\mathcal{G}_1$, which is a locally free subsheaf of $F^0(\mathcal{E})$, and $F^0(\mathcal{E})$ is a locally free sheaf of rank a. It follows that $0 < a_1 < a$. As

$$v\big(F^0(\mathcal{E})\big) = \widetilde{v} = (a, -n, r)\,, \quad v(\mathcal{G}_1) = (a_1, -n_1, r_1)$$

and as $p(\mathcal{G}_1) > p\big(F^0(\mathcal{E})\big)$, we have either $n_1/a_1 < n/a$, or $n_1/a_1 = n/a$ and $r_1/a_1 > r/a$. Finally, $n_1 < n$ follows from $n_1/a_1 \leqslant n/a$ and $a_1 < a$. □

We will moreover need the following property of $\widehat{F}^1(\mathcal{G}_2)$.

Lemma 2.31. *If* $\widehat{F}^1(\mathcal{G}_2) \neq 0$*, its first Chern class is strictly negative; that is,* $c_1\big(\widehat{F}^1(\mathcal{G}_2)\big) = mh$ *with* $m < 0$.

Proof. If S is K3, the exact sequence (2.3) applied to $\mathcal{G}_2$ shows that $\widehat{F}^1(\mathcal{G}_2)$ is an extension of a subsheaf of $\mathcal{G}_2$ by $\mathcal{O}_S \otimes H^1(\mathcal{G}_2)$, so if $\widehat{F}^1(\mathcal{G}_2) \neq 0$, its first Chern class is strictly negative.

We now suppose that S is Abelian. First of all, recall that $\widehat{F}^0(\mathcal{G}_2) = 0$ and that all the subsheaves of $\mathcal{G}_2$ have strictly negative first Chern class (see Step 4 of the proof of Lemma 2.30).

Since $\widehat{F}^0(\mathcal{G}_2) = 0$, it follows that $F^i\big(\widehat{F}^0(\mathcal{G}_2)\big) = 0$ for all i. The spectral sequence

$$E_2^{p,q} := F^p\big(\widehat{F}^q(\mathcal{G}_2)\big) \longrightarrow E^{p+q} = \big(F \circ \widehat{F}\big)^{p+q}(\mathcal{G}_2) = \mathcal{G}_2[-2]^{p+q}$$

then provides an inclusion $i\colon F^1\big(\widehat{F}^1(\mathcal{G}_2)\big) \to \mathcal{G}_2$ and the equality $F^0\big(\widehat{F}^1(\mathcal{G}_2)\big) = 0$.

In order to show that if $\widehat{F}^1(\mathcal{G}_2) \neq 0$, its first Chern class is strictly negative, we first prove that $\widehat{F}^1(\mathcal{G}_2)$ is torsion-free.

The sheaf $\widehat{F}^1(\mathcal{G}_2)$ cannot contain a 0-dimensional subsheaf T since $F^0(T)$ would be a nonzero semistable sheaf whose associated polystable sheaf is a direct sum of degree 0 line bundles and $F^0(T) \subseteq F^0\big(\widehat{F}^1(\mathcal{G}_2)\big) = 0$.

It follows that if $\widehat{F}^1(\mathcal{G}_2)$ is not torsion-free, there are a pure 1-dimensional sheaf K such that $c_1(K) = ph$ for a $p > 0$, a torsion-free sheaf Q and an exact sequence

$$0 \longrightarrow K \longrightarrow \widehat{F}^1(\mathcal{G}_2) \longrightarrow Q \longrightarrow 0\,. \tag{2.8}$$

The sheaf K verifies WIT(1) with respect to F. Indeed, applying F to the sequence (2.8), we obtain $F^0(K) \subseteq F^0\big(\widehat{F}^1(\mathcal{G}_2)\big) = 0$ and, since K is 1-dimensional, $F^2(K) = 0$. As a consequence, we get $c_1\big(F^1(K)\big) = p\widehat{h}$.

Moreover, since Q is torsion-free, the argument used for $\mathcal{E}$ in Step 3 of Lemma 2.30 shows that $F^0(Q)$ is a subsheaf of a direct sum of line bundles belonging to $\mathrm{Pic}^0\big(\widehat{S}\big)$; hence we have $c_1\big(F^0(Q)\big) \leqslant 0$.

Applying the functor F to the exact sequence (2.8), we then get the exact sequence

$$0 \longrightarrow F^0(Q) \longrightarrow F^1(K) \longrightarrow F^1\big(\widehat{F}^1(\mathcal{G}_2)\big)\,,$$

which yields an inclusion

$$F^1(K)/F^0(Q) \longrightarrow F^1\big(\widehat{F}^1(\mathcal{G}_2)\big) \xrightarrow{\;i\;} \mathcal{G}_2\,.$$

Since $c_1\big(F^0(Q)\big) \leqslant 0$, we get

$$c_1\big(F^1(K)/F^0(Q)\big) \geqslant c_1\big(F^1(K)\big) \geqslant 0\,,$$

where the last inequality comes from the fact that $c_1\big(F^1(K)\big) = p\widehat{h}$. It follows that $\mathcal{G}_2$ has a nonzero subsheaf with nonnegative first Chern class, but this is not possible (see Step 4 of the proof of Lemma 2.30) and implies that $\widehat{F}^1(\mathcal{G}_2)$ is torsion-free.

It remains to prove that the first Chern class of the torsion-free sheaf $\widehat{F}^1(\mathcal{G}_2)$ is strictly negative. We will show it by distinguishing the case where this sheaf is not μ-semistable from the case where it is μ-semistable.

If $\widehat{F}^1(\mathcal{G}_2)$ is not μ-semistable and $c_1\big(\widehat{F}^1(\mathcal{G}_2)\big) \geqslant 0$, there exist a μ-stable sheaf K with $c_1(K) = ph$ for $p > 0$, a torsion-free sheaf Q and an exact sequence as in (2.8). We can copy the argument used above to show that the torsion of $\widehat{F}^1(\mathcal{G}_2)$ cannot be pure of dimension 1; the only difference is that $F^2(K) = 0$ follows from $\mu(K) > 0$ and the μ-stability of K. As above, we deduce that $\mathcal{G}_2$ has a nonzero subsheaf with nonnegative first Chern class. Since this is absurd, if $\widehat{F}^1(\mathcal{G}_2)$ is not μ-semistable, its first Chern class is strictly negative.

Finally, assume that $\widehat{F}^1(\mathcal{G}_2)$ is μ-semistable and $c_1\big(\widehat{F}^1(\mathcal{G}_2)\big) = mh$ for $m \geqslant 0$. By the μ-stability of $\widehat{F}^1(\mathcal{G}_2)$, if $m > 0$, there are no nonzero morphisms from $\widehat{F}^1(\mathcal{G}_2)$ to any $L \in \mathrm{Pic}^0(S)$, and if $m = 0$, the locus of $\mathrm{Pic}^0(S)$ consisting of line bundles admitting nontrivial morphisms from $\widehat{F}^1(\mathcal{G}_2)$ is, at most, finite. This implies that the support of $F^2\big(\widehat{F}^1(\mathcal{G}_2)\big)$ is empty or finite and, since $F^0\big(\widehat{F}^1(\mathcal{G}_2)\big) = 0$, we obtain $c_1\big(F^1\big(\widehat{F}^1(\mathcal{G}_2)\big)\big) = m\widehat{h}$. Again this is absurd since $i\colon F^1\big(\widehat{F}^1(\mathcal{G}_2)\big) \to \mathcal{G}_2$ is an injection and $\mathcal{G}_2$ cannot contain subsheaves with nonnegative first Chern class. Hence, also in the case where $\widehat{F}^1(\mathcal{G}_2)$ is μ-semistable, its first Chern class is strictly negative. $\square$

We are now ready to conclude the proof of Proposition 2.29. We first notice that as $\mathcal{G} = \widehat{F}^2(\mathcal{G}_1)/\widehat{F}^1(\mathcal{G}_2)$, we have

$$c_1(\mathcal{G}) = c_1\big(\widehat{F}^2(\mathcal{G}_1)\big) - c_1\big(\widehat{F}^1(\mathcal{G}_2)\big)\,;$$

hence, by Lemma 2.31, it follows that $p(\mathcal{G}) \geqslant p\big(\widehat{F}^2(\mathcal{G}_1)\big)$.

Let us now prove that $p\big(\widehat{F}^2(\mathcal{G}_1)\big) > p(\mathcal{E})$. It will then follow that $p(\mathcal{G}) > p(\mathcal{E})$, concluding the contradiction argument.

To show that $p\big(\widehat{F}^2(\mathcal{G}_1)\big) > p(\mathcal{E})$, first recall that $v\big(\widehat{F}^2(\mathcal{G}_1)\big) = (r_1, n_1, a_1)$ and $v(\mathcal{E}) = (r, n, a)$. Moreover, setting $l = h^2/2$ (hence also $l = \widehat{h}^2/2$ if S is Abelian), by hypothesis and Lemma 2.30,

(1) we have $r, n, a > 0$ and $ln^2 - ra = v^2/2 = k > 0$;
(2) we have $r_1, n_1, a_1 > 0$;
(3) we have $a_1 < a$, and either $n_1/a_1 < n/a$, or $n_1/a_1 = n/a$ and $r_1/a_1 > r/a$;
(4) we have $ln_1^2 - r_1a_1 = v(\mathcal{G}_1)^2/2 \geqslant -1$ since $\mathcal{G}_1$ is H-stable (if S is Abelian, we even have $ln_1^2 - r_1a_1 \geqslant 0$).

By Lemma 2.32 below, it follows that there is an $n_0 \in \mathbb{N}$ such that for every $n > n_0$, we have $n_1/r_1 \geqslant n/r$, and if $n_1/r_1 = n/r$, then $a_1/r_1 > a/r$. This exactly means that $p\big(\widehat{F}^2(\mathcal{G}_1)\big) > p(\mathcal{E})$, completing the proof. ☐

We now prove the following, which is used to conclude the proof of Lemma 2.31.

Lemma 2.32. *Fix $k, l, r \in \mathbb{N}$ with $k, l, r > 0$, and let $n, a, r_1, a_1, n_1 \in \mathbb{N}$ with $n, a, r_1, a_1, n_1 > 0$ be such that the following conditions are fulfilled:*

(1) $ln^2 - ra = k$;
(2) $ln_1^2 - r_1a_1 \geqslant -1$;
(3) $a_1 < a$;
(4) $n_1/a_1 < n/a$, or $n_1/a_1 = n/a$ and $r_1/a_1 > r/a$.

If $n > 32r^3k$, then either $n_1/r_1 > n/r$, or $n_1/r_1 = n/r$ and $a_1/r_1 > a/r$.

Proof. We let $k_1 := ln_1^2 - r_1a_1$ so that $k_1 \geqslant -1$. As $n_1/a_1 \leqslant n/a$, it follows that $n_1/n \leqslant a_1/a$. Moreover, as $a_1 = (ln_1^2 - k_1)/r_1$ and $a = (ln^2 - k)/r$, we get the inequality

$$\frac{n_1}{n} \leqslant \frac{r}{r_1} \cdot \frac{n_1}{n} \cdot \frac{n_1 - k_1/ln_1}{n - k/ln}\,.$$

This implies that

$$1 \leqslant \frac{r}{r_1} \cdot \frac{n_1 - k_1/ln_1}{n - k/ln}\,. \tag{2.9}$$

We claim that as $n > 32r^3k$, we have $r > r_1$. Indeed, as $n > 32r^3k \geqslant 3k$ and $k_1 \geqslant -1$, we get

$$\frac{n_1 - k_1/ln_1}{n - k/ln} \leqslant \frac{n_1 + 1/ln_1}{n - 1/3l} \leqslant \frac{n_1 + 1/n_1}{n_1 + 2/3}\,, \tag{2.10}$$

where the last inequality follows from the fact that $n > n_1$ (so that $n - n_1 \geqslant 1$).

As $n > n_1$ and $n > 3k$, the first term of the inequality (2.10) is strictly smaller than 1. This can be checked by analyzing the third term if $n_1 \geqslant 2$ and the second term if $n_1 = 1$ (recall that under our assumption, $n \geqslant 3$). In any case, we get

$$\frac{n_1 - k_1/ln_1}{n - k/ln} < 1\,;$$

hence the inequality (2.9) gives $r > r_1$.

We now write the inequality (2.9) in a different form. More precisely, we have

$$1 \leqslant \frac{n_1/r_1}{n/r} \cdot \frac{1-k_1/ln_1^2}{1-k/ln^2} \leqslant \frac{n_1/r_1}{n/r} \cdot \frac{1+1/ln_1^2}{1-k/ln^2}\,,$$

where the last equality follows from $k_1 \geqslant -1$. As $n > 32r^3k$, we see that $1 - k/ln^2 > 0$; hence the previous inequality becomes

$$\frac{n_1/r_1}{n/r} \geqslant \frac{1-k/ln^2}{1+1/ln_1^2} = 1 - \frac{k/ln^2 + 1/ln_1^2}{1+1/ln_1^2}\,. \tag{2.11}$$

We first want to show that $n_1/r_1 \geqslant n/r$. As the first term of the inequality (2.11) is an integral multiple of $1/r_1n$, by the inequality (2.11), it is enough to show that

$$\frac{k/ln^2 + 1/ln_1^2}{1+1/ln_1^2} \leqslant \frac{1}{r_1 n}\,.$$

To do so, first notice that $1 + 1/ln_1^2 > 1$ and that as $1/ln_1^2 \leqslant 1$, we have

$$\frac{1/ln_1^2}{1+1/ln_1^2} \leqslant \frac{1}{2}\,.$$

Moreover, as $n > 32r^3k$, we get $k/ln^2 < 1/4$. We then find that

$$1 - \frac{k/ln^2 + 1/ln_1^2}{1+1/ln_1^2} > \frac{1}{4}\,,$$

so that, by inequality (2.11), we finally get

$$\frac{n_1}{r_1} > \frac{n}{4r}\,.$$

This implies that $n_1 > n/4r$; hence we get

$$\frac{1}{ln_1^2} < \frac{1}{l \cdot n^2/16r^2} < \frac{1}{ln \cdot 32r^3k/16r^2} = \frac{1}{2lrnk} \leqslant \frac{1}{2rn}\,.$$

Now, using again $n > 32r^3k$, we even get that $k/ln^2 < 1/32r^3n$, hence

$$\frac{k/ln^2 + 1/ln_1^2}{1+1/ln_1^2} < \frac{k}{ln^2} + \frac{1}{ln_1^2} < \frac{1}{32r^3n} + \frac{1}{2rn} < \frac{1}{rn} < \frac{1}{r_1 n}\,,$$

where the last inequality comes from $r > r_1$.

We then have $n_1/r_1 \geqslant n/r$ if $n > 32r^3k$. To complete the proof, notice that if $n_1/r_1 = n/r$, then $r_1/r = n_1/n < a_1/a$. But this means that $a_1/r_1 > a/r$, and we are done. □

We conclude this section with a proposition and a corollary, which allows us to pass from a Mukai vector of rank 0 to a Mukai vector of strictly positive rank (see [Yos02, Proposition 3.14] for a proof for stable sheaves).

We will also have to deal with surfaces whose Néron–Severi group has rank bigger than 1 and, in this case, we need an explicit lower bound on the Euler characteristic of the rank 0 Mukai vector which makes possible checking the genericity of the polarizations for both Mukai vectors.

In order to give this bound, we recall that if S is a smooth projective surface, H is an ample divisor on S and $\xi \in \mathrm{NS}(S)$ is the class of an effective curve, the set of numerical equivalence classes of effective curves C on S such that $C \cdot H \leqslant \xi \cdot H$ is finite.

It follows that the set

$$\left\{ \frac{(C^2+2)(\xi\cdot H)}{2C\cdot H} \,\middle|\, C\subseteq S \text{ is an effective curve } C\cdot H \leqslant \xi\cdot H \right\}$$

is bounded; we denote its maximum by $N_{S,H,\xi}$.

Proposition 2.33. *Let S be a projective K3 or Abelian surface, and set $F := F_{\mathrm{K3}}$ if S is a K3 and $F := F_{\mathrm{Ab}}$ if S is Abelian. Let H a primitive ample line bundle on S, and set $h := c_1(H)$. Let $k \in \mathbb{N}^*$, and let $\xi \in \mathrm{NS}(S)$ be the first Chern class of an effective divisor on S such that $\xi^2 = 2k$. Finally, let $v = (0, \xi, a)$ be a Mukai vector on S, and assume that $\mathrm{WIT}(0)$ holds with respect to F for every H-semistable sheaf $\mathcal{E}$ with Mukai vector v and that $F^0(\mathcal{E})$ is locally free (for example, assume $a > p_0$ for p_0 as in Lemma 2.28(2)).*

(1) *If S is K3, $a > N_{S,H,\xi}$ and H is both v-generic and $\widetilde{v}$-generic, the functor F induces isomorphisms $M_v(S,H) \simeq M_{\widetilde{v}}(S,H)$ and $M^s_v(S,H) \simeq M^s_{\widetilde{v}}(S,H)$.*
(2) *If S is Abelian, $a > N_{S,H,\xi}$, H is v-generic and $\widehat{H}$ is $\widetilde{v}$-generic, the functor F induces isomorphisms $M_v(S,H) \simeq M_{\widetilde{v}}(\widehat{S},\widehat{H})$, $M^s_v(S,H) \simeq M^s_{\widetilde{v}}(\widehat{S},\widehat{H})$, $K_v(S,H) \simeq K_{\widetilde{v}}(\widehat{S},\widehat{H})$ and $K^s_v(S,H) \simeq K^s_{\widetilde{v}}(\widehat{S},\widehat{H})$.*

Proof. As in the proof of Proposition 2.29, in order to get the statement, we just need to prove that for $a > N_{S,H,\xi}$, the sheaf $F^0(\mathcal{E})$ is H-semistable in the K3 case and $\widehat{H}$-semistable in the Abelian case.

The proof is by contradiction: we suppose that $F^0(\mathcal{E})$ is not semistable and contradict in several steps the H-semistability of $\mathcal{E}$.

So, suppose that $\mathcal{E}$ is H-semistable with Mukai vector $v = (0, \xi, a)$, where $a > N_{S,H,\xi}$, and suppose that $F^0(\mathcal{E})$ is not H-semistable if S is K3, and that it is not $\widehat{H}$-semistable if S is Abelian.

Then $F^0(\mathcal{E})$ has a desemistabilizing stable locally free subsheaf $\mathcal{G}_1$ with maximal reduced Hilbert polynomial, and we set $\mathcal{G}_2 := F^0(\mathcal{E})/\mathcal{G}_1$. If S is K3, we write $v(\mathcal{G}_1) = (a_1, -\xi_1, r_1)$; if S is Abelian, we write $v(\mathcal{G}_1) = (a_1, -\widehat{\xi}_1, r_1)$.

Step 1: if S is K3, the class ξ_1 is effective and $\xi_1 \neq 0$. This follows by repeating the argument of Step 2 of Lemma 2.30.

Step 2: if S is Abelian, the class $\widehat{\xi}_1$ is effective and $\widehat{\xi}_1 \neq 0$. This follows as in Step 3 of Lemma 2.30.

Step 3: $r_1 \leqslant 0$ for $a > N_{S,H,\xi}$. First suppose that S is a K3 surface. Since $\mathcal{G}_1$ is H-stable, we have $v(\mathcal{G}_1)^2 \geqslant -2$. Hence if $r_1 \geqslant 1$, we get

$$-2 \leqslant v(\mathcal{G}_1)^2 = (\xi_1)^2 - 2a_1r_1 \leqslant (\xi_1)^2 - 2a_1\,.$$

For $d := \xi\cdot H$ and $d_1 = \xi_1 \cdot H$, the inequality $p_H(\mathcal{G}_1) > p_H\big(F^0(\mathcal{E})\big)$ implies $-d_1/a_1 \geqslant -d/a$ and hence $a_1 \geqslant ad_1/d$, so that

$$-2 \leqslant (\xi_1)^2 - 2a\frac{d_1}{d}\,.$$

Moreover, as $a_1 < a$, the inequality $a_1 \geqslant ad_1/d$ implies $d_1 < d$

As ξ_1 is an effective divisor such that $\xi_1 \cdot H = d_1 < d$, we have $N_{S,H,\xi} \geqslant ((\xi_1^2+2)/2d_1)\cdot d$, and since $a > N_{S,H,\xi}$, we obtain

$$-2 \leqslant (\xi_1)^2 - 2a\frac{d_1}{d} < (\xi_1)^2 - 2\frac{\big((\xi_1)^2+2\big)d}{2d_1}\cdot\frac{d_1}{d} = -2\,,$$

getting a contradiction.

The same argument also works if S is Abelian, simply by replacing any occurrence of ξ, ξ_1, H and S by $\widehat{\xi}$, $\widehat{\xi}_1$, $\widehat{H}$ and $\widehat{S}$.

Step 4: the sheaf $\mathcal{G}_1$ verifies WIT(2) *with respect to $\widehat{F}$, and in particular $r_1 = 0$.* The first part follows as in Step 4 of Lemma 2.30. Since $r_1 = \mathrm{rk}\big(\widehat{F}(\mathcal{G}_1)\big)$, it has to be 0 by Step 3.

Step 5: the sheaf $\widehat{F}^2(\mathcal{G}_1)$ is a subsheaf of $\mathcal{E}$. By the maximality of the reduced Hilbert polynomial of $\mathcal{G}_1$, the sheaf $\mathcal{G}_2 := F^0(\mathcal{E})/\mathcal{G}_1$ is torsion-free. By applying the functor $\widehat{F}$ to the exact sequence

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow F^0(\mathcal{E}) \longrightarrow \mathcal{G}_2 \longrightarrow 0\,,$$

we get the exact sequence

$$0 \longrightarrow \widehat{F}^1(\mathcal{G}_2) \longrightarrow \widehat{F}^2(\mathcal{G}_1) \longrightarrow \mathcal{E} \longrightarrow \widehat{F}^2(\mathcal{G}_2) \longrightarrow 0\,, \tag{2.12}$$

so it suffices to show that $\widehat{F}^1(\mathcal{G}_2) = 0$.

First of all, notice that by Step 4, the rank of $\widehat{F}^2(\mathcal{G}_1)$ is $r_1 = 0$. As $\widehat{F}^1(\mathcal{G}_2) \subseteq \widehat{F}^2(\mathcal{G}_1)$, we know that $\widehat{F}^1(\mathcal{G}_2)$ is a torsion sheaf.

If S is a K3 surface, the exact sequence (2.3) applied to $\mathcal{G}_2$ shows that $\widehat{F}^1(\mathcal{G}_2)$ is an extension of a subsheaf of $\mathcal{G}_2$ by a locally free sheaf. As $\mathcal{G}_2$ is torsion-free, it follows that $\widehat{F}^1(\mathcal{G}_2)$ is torsion-free, and since it is also a torsion sheaf, we obtain $\widehat{F}^1(\mathcal{G}_2) = 0$.

If S is Abelian, the maximality of the reduced Hilbert polynomial of $\mathcal{G}_1$ among the Hilbert polynomials of the subsheaves of $F(\mathcal{E})$ implies that the slope of any subsheaf of $\mathcal{G}_2$ cannot be bigger than the slope of $\mathcal{G}_1$, which is strictly negative by Step 2. Hence any subsheaf of $\mathcal{G}_2$ has strictly negative slope and $H^0(\mathcal{G}_2 \otimes L) = 0$ for every $L \in \mathrm{Pic}^0\big(\widehat{S}\big)$; it follows that $\widehat{F}^0(\mathcal{G}_2) = 0$.

As in the proof of Lemma 2.31, the spectral sequence

$$E_2^{p,q} := F^p\big(\widehat{F}^q(\mathcal{G}_2)\big) \longrightarrow E^{p+q} = \big(F \circ \widehat{F}\big)^{p+q}(\mathcal{G}_2) = \mathcal{G}_2[-2]^{p+q}$$

then provides an inclusion $i\colon F^1\big(\widehat{F}^1(\mathcal{G}_2)\big) \to \mathcal{G}_2$ and the equality $F^0\big(\widehat{F}^1(\mathcal{G}_2)\big) = 0$. Since the support of the torsion sheaf $\widehat{F}^1(\mathcal{G}_2)$ has dimension at most 1, we also have $F^2\big(\widehat{F}^1(\mathcal{G}_2)\big) = 0$, and $\widehat{F}^1(\mathcal{G}_2)$ satisfies WIT(1) with respect to F. The equality $F^0\big(\widehat{F}^1(\mathcal{G}_2)\big) = 0$ implies that $\widehat{F}^1(\mathcal{G}_2)$ has no 0-dimensional torsion; it follows that if $\widehat{F}^1(\mathcal{G}_2) \neq 0$, then $c_1\big(\widehat{F}^1(\mathcal{G}_2)\big)$ would be the class of an effective curve, and the same would hold for $c_1\big(F^1\big(\widehat{F}^1(\mathcal{G}_2)\big)\big)$. But this is impossible since $F^1\big(\widehat{F}^1(\mathcal{G}_2)\big) \subseteq \mathcal{G}_2$ and the slope of every subsheaf of $\mathcal{G}_2$ is negative. It follows that $\widehat{F}^1(\mathcal{G}_2)$=0.

Step 6: conclusion of the proof. Suppose that S is K3. Recall that $v(F(\mathcal{E})) = \widetilde{v} = (a, -\xi, 0)$ and, by Step 4, $v(\mathcal{G}_1) = (a_1, -\xi_1, 0)$ (with ξ_1 effective and nonzero by Step 1). Since $\mathcal{G}_1$ is a desemistabilizing subsheaf of $F(\mathcal{E})$, we have $-d_1/a_1 > -d/a$ or equivalently

$$\frac{a_1}{d_1} > \frac{a}{d}\,. \tag{2.13}$$

By Step 4, the sheaf $\mathcal{G}_1$ verifies WIT(2) with respect to $\widehat{F}$; hence $v\big(\widehat{F}^2(\mathcal{G}_1)\big) = (0, \xi_1, a_1)$, and by Step 5, the sheaf $\widehat{F}^2(\mathcal{G}_1)$ is a subsheaf of $\mathcal{E}$. The inequality (2.13) then implies that $\widehat{F}(\mathcal{G}_1)$ is a desemistabilizing subsheaf for $\mathcal{E}$, which is not possible since $\mathcal{E}$ is semistable.

The proof in the Abelian case is similar; simply replace ξ and ξ_1 by $\widehat{\xi}$ and $\widehat{\xi}_1$, respectively, where necessary. $\square$

Remark 2.34. In the important case where $\rho(S) = 1$, Proposition 2.33 simply says that if $v = (0, \xi, a)$ and $a \gg 0$, the functor F induces an isomorphism $M_v(S, H) \simeq M_{\widetilde{v}}(S, H)$ if S is K3 and $K_v(S, H) \simeq K_{\widetilde{v}}\big(\widehat{S}, \widehat{H}\big)$ if S is Abelian. This easier statement allows us to prove Theorem 1.7 with the exception of the case where the rank of v is 0 and $\rho(S) > 1$. To handle this remaining case, we need the following corollary of Proposition 2.33.

Corollary 2.35. *Let S be a projective K3 or Abelian surface, $k \in \mathbb{N}^*$ and $\xi \in \mathrm{NS}(S)$ the first Chern class of an effective divisor on S such that $\xi^2 = 2k$. For a nonzero $a \in \mathbb{Z}$, let $v = (0, \xi, a)$ be a Mukai vector on S, and let H be a v-generic polarization. For $d \in \mathbb{Z}$, set $v_{dH} := v \cdot \mathrm{ch}(\mathcal{O}(dH)) = (0, \xi, a + d\xi \cdot H)$.*

(1) *If S is K3, there exist a $d_0 \in \mathbb{N}$ such that $a + d_0\xi \cdot H > 0$ and a $\widetilde{v_{d_0H}}$-generic polarization H' such that the functor F_{K3} induces isomorphisms $M_{v_{d_0H}}(S, H) \simeq M_{\widetilde{v_{d_0H}}}(S, H')$ and $M^s_{v_{d_0H}}(S, H) \simeq M^s_{\widetilde{v_{d_0H}}}(S, H')$.*
(2) *If S is Abelian, there exist a $d_0 \in \mathbb{N}$ such that $a + d_0\xi \cdot H > 0$ and a $\widetilde{v_{d_0H}}$-generic polarization $\widehat{H}'$ such that the functor F_{Ab} induces isomorphisms $M_{v_{d_0H}}(S, H) \simeq M_{\widetilde{v_{d_0H}}}(\widehat{S}, \widehat{H}')$, $M^s_{v_{d_0H}}(S, H) \simeq M^s_{\widetilde{v_{d_0H}}}(\widehat{S}, \widehat{H}')$, $K_{v_{d_0H}}(S, H) \simeq K_{\widetilde{v_{d_0H}}}(\widehat{S}, \widehat{H}')$ and $K^s_{v_{d_0H}}(S, H) \simeq K^s_{\widetilde{v_{d_0H}}}(\widehat{S}, \widehat{H}')$.*

Proof. We only deal with the case where S is K3, the Abelian case being very similar.

We first claim that if U is a small compact neighbourhood of H in the ample cone of S, then there is an $N_U \in \mathbb{N}$ such that $N_U > N_{S,L,\xi}$ for every $L \in U$. To prove this, let $\overline{\mathrm{NE}(S)}$ be the closure in $\mathrm{NS}(S) \otimes \mathbb{R}$ of the cone of effective curves of S. For every $L \in \mathrm{Amp}(S)$, we define $F_L : \overline{\mathrm{NE}(S)} \to \mathbb{R}$ as $F_L(\alpha) := \alpha \cdot L$ for every $\alpha \in \overline{\mathrm{NE}(S)}$ and let

$$D_L := \left\{ \alpha \in \overline{\mathrm{NE}(S)} \,|\, F_L(\alpha) \leqslant \xi \cdot L \right\}.$$

As a consequence of Kleiman's ampleness criterion (see [KM98, Corollary 1.19(2)]), the locus D_L is compact for every $L \in U$. As U is compact, it follows that $\bigcup_{L \in U} D_L$ is compact as well, so its subset Y of integral nonzero classes is finite.

Fix now a nonzero class $C \in \overline{\mathrm{NE}(S)}$. The function

$$f_C \colon U \longrightarrow \mathbb{R}, \quad f_C(L) := \frac{(C^2 + 2)(\xi \cdot L)}{2C \cdot L}$$

is continuous. As U is compact, the function f_C has a maximum, and since Y is finite, there exists an $N_U > N_{S,L,\xi}$ for every $L \in U$.

Now choose $d_0 \in \mathbb{N}$ such that $a + d_0\xi \cdot H > N_U$ and such that every H-semistable sheaf $\mathcal{E}$ with Mukai vector v_{d_0H} satisfies WIT(0) with respect to F_{K3} and $F_{\mathrm{K3}}(\mathcal{E})$ is locally free (this is possible by Lemma 2.28(2)).

By Lemma 2.6(2), the polarization H is also v_{d_0H}-generic, so it lies in a v_{d_0H}-chamber $\mathcal{C}$; as the $\widetilde{v_{d_0H}}$-walls are locally finite, there is a $\widetilde{v_{d_0H}}$-generic polarization $H' \in U \cap \mathcal{C}$. But then $M_{v_{d_0H}}(S, H) = M_{v_{d_0H}}(S, H')$ (as $H, H' \in \mathcal{C}$), and as $a + d_0\xi \cdot H > N_{S,H',\xi}$ (since $H' \in U$), by Proposition 2.33, the functor F_{K3} induces an isomorphism between $M_{v_{d_0H}}(S, H')$ and $M_{\widetilde{v_{d_0H}}}(S, H')$, concluding the proof. $\square$

2.5 The proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7. The goal is to show that if S is a projective K3 (respectively, Abelian) surface, v is a Mukai vector on S of type (m, k) and H is a polarization on S which is general with respect to v, the locally trivial deformation equivalence class of $M_v(S, H)$ only depends on (m, k). Before giving the proof, we provide several results we will need.

2.5.1 *Changing polarization and first Chern class.* We first show the following lemma, which allows us, if the rank of the Mukai vector is strictly positive, to suppose that the first Chern class of the Mukai vector is a multiple of the polarization. As a consequence, this will allow us to suppose the Néron–Severi group of S to have rank 1.

LEMMA 2.36. *Let $m, k \in \mathbb{N}$ with $m, k > 0$, and let S be a projective K3 or Abelian surface and v a Mukai vector on S of type (m, k) and of the form $v = m(r, \xi, a)$ with $r > 0$. Let $g := \gcd(r, \xi)$, and suppose that H is a v-generic polarization on S. Moreover, suppose $\rho(S) \geqslant 2$, and let $\mathcal{C}$ be the v-chamber such that $H \in \mathcal{C}$. Then there exist a Mukai vector $v' = m(r, \xi', a')$ and a primitive polarization H' in $\mathcal{C}$ such that:*

(1) *v' is equivalent to v;*
(2) *$\xi' = gc_1(H')$;*
(3) *$(H')^2 \gg 0$.*

In particular, $M_v(S,H) \simeq M_{v'}(S,H')$ and $M^s_v(S,H) \simeq M^s_{v'}(S',H')$. If S is Abelian, we have $K_v(S,H) \simeq K_{v'}(S,H')$ and $K^s_v(S,H) \simeq K^s_{v'}(S,H')$.

Proof. This is a generalization of [O'G97, Lemma II.6]. First, notice that as $g := \gcd(r, \xi)$, there are two coprime integers $s, p \in \mathbb{N}$ and a primitive class $\zeta \in \mathrm{NS}(S)$ such that $r = gs$ and $\xi = gp\zeta$.

Replacing H with another polarization inside $\mathcal{C}$ if necessary, we may suppose $\xi \notin \mathbb{R} \cdot c_1(H)$. Moreover, since $\mathcal{C}$ is an open cone in $\mathrm{NS}(S)$, we may also assume that the sublattice $\Lambda \subseteq \mathrm{NS}(S)$ spanned by $c_1(H)$ and ζ is saturated.

Now, let $d \in \mathbb{N}$, and set $v' := v \cdot \mathrm{ch}(\mathcal{O}_S(dH))$. Then v' is equivalent to v, and

$$\begin{aligned} v' &= m\big(r, \xi + rdc_1(H), a + d\xi \cdot H + rd^2H^2/2\big) \\ &= m\big(gs, gp\zeta + gsdc_1(H), a + dgp\zeta \cdot H + gsd^2H^2/2\big)\,. \end{aligned}$$

Now observe that if $d \gg 0$, then $p\zeta + sdc_1(H) \in \mathcal{C}$. If we let H' be an ample divisor such that $c_1(H') = p\zeta + sdc_1(H)$, we have

$$\xi' := gp\zeta + gsdc_1(H) = g(p\zeta + sdc_1(H)) = gc_1(H')\,.$$

As $\gcd(s, p) = 1$, if we choose d such that $\gcd(d, p) = 1$, the class $c_1(H')$ is primitive in Λ, and since Λ is saturated, it is primitive in $\mathrm{NS}(S)$. The isomorphism $M_v(S,H) \simeq M_{v'}(S,H')$ follows from Lemma 2.24(2).

Finally, as $d \gg 0$, we have

$$(H')^2 = p^2\zeta^2 + 2psd\zeta \cdot H + s^2d^2H^2 \gg 0\,.$$

To conclude the proof, we notice that if S is Abelian, then by Lemma 2.24(1), the tensorization with $\mathcal{O}_S(dH)$ induces an isomorphism between the fibers of the corresponding Yoshioka fibrations. □

2.5.2 *Deformation to elliptic surfaces.* Elliptic surfaces having a section and whose Picard number is 2 prove to be particularly useful, as in this case we have a privileged class of polarizations, called v-suitable. Let Y be an elliptic K3 or Abelian surface such that $\mathrm{NS}(Y) = \mathbb{Z}\cdot f \oplus \mathbb{Z}\cdot\sigma$, where f is the class of a fiber and σ is the class of a section. Let v be a Mukai vector on Y of the form $v = (r, \xi, a)$ with $r > 0$, and recall the following definition (see [O'G97]).

DEFINITION 2.37. A polarization H on Y is called *v-suitable* if H is in the unique v-chamber whose closure contains f.

We have an easy numerical criterion to guarantee that a polarization on Y is v-suitable (see [O'G97, Lemma I.0.3] for K3 surfaces, and point (2) of [PR13, Lemma 2.24] for Abelian surfaces).

Lemma 2.38. *Let Y be a projective elliptic K3 or Abelian surface with $\mathrm{NS}(Y) = \mathbb{Z}\cdot\sigma \oplus \mathbb{Z}\cdot f$, where σ is the class of a section and f is the class of a fiber, and let $v = (r, \xi, a)$ be a Mukai vector on Y such that $r > 0$. Let H be a polarization, and suppose $c_1(H) = \sigma + tf$ for some $t \in \mathbb{Z}$.*

(1) *If Y is K3, then H is v-suitable if $t \geqslant |v| + 1$.*
(2) *If Y is Abelian, then H is v-suitable if $t \geqslant |v|$.*

In the next lemma, by deforming a triple (S, v, H) to a triple (Y, v', H'), where Y is an elliptic surface and H' is v'-suitable, we show that the locally trivial deformation class of $M_v(S, H)$ (respectively, K_v) only depends on numerical data associated with v. In the particular and important case where the rank r is strictly positive and prime to the first Chern class of v, it only depends on r and v^2.

Lemma 2.39. *Let $m, k \in \mathbb{N}$ with $m, k > 0$, and for $i = 1, 2$, let S_i be a projective K3 (respectively, Abelian) surface, v_i a Mukai vector on S_i of type (m, k) and H_i a v_i-generic polarization on S_i. Write $v_i = m(r_i, \xi_i, a_i)$ for $i = 1, 2$, and suppose that the following conditions are verified:*

(1) $r_1 = r_2 =: r > 0$;
(2) $\gcd(r, \xi_1) = \gcd(r, \xi_2) =: g$;
(3) $a_1 \equiv a_2 \bmod g$.

Then $M_{v_1}(S_1, H_1)$ and $M_{v_2}(S_2, H_2)$ (respectively, $K_{v_1}(S_1, H_1)$ and $K_{v_2}(S_2, H_2)$) are locally trivially deformation equivalent.

Proof. The argument we present here was first used by O'Grady in [O'G97] and by Yoshioka in [Yos99a] for primitive Mukai vectors, and by the authors in [PR13] in the case $m = 2$ and $k = 1$.

First, we may assume $\rho(S_i) > 1$. Indeed, consider a deformation $(\mathscr{X}_i, \mathscr{L}_i, \mathscr{H}_i)$ of the triple (S_i, v_i, H_i) over a smooth connected curve C_i inducing a nontrivial deformation of S_i. By [Ogui00, Main Theorem], the locus parametrizing points $t \in C_i$ such that $\rho(\mathscr{X}_{i,t}) > 1$ is dense in the classical topology of C_i. Since the locus of points $t \in C_i$ such that $\mathscr{H}_{i,t}$ is not general with respect to $v_{i,t}$ is finite (see Remark 2.18) and since by Lemma 2.21(1) and Lemma 2.10, we may replace a polarization which is general with respect to Mukai vector v with a v-generic polarization, we may suppose $\rho(S_i) > 1$.

Since the class of a_i modulo g does not change when replacing v_i by an equivalent Mukai vector, by Lemma 2.36, we may even suppose $v_i = m(r, gc_1(H_i), a_i)$, where H_i is ample and $H_i^2 = 2d_i$ with $d_i \gg 0$.

Let Y be a K3 or an Abelian surface with an elliptic fibration and such that $\mathrm{NS}(Y) = \mathbb{Z}\cdot\sigma \oplus \mathbb{Z}\cdot f$, where f is the class of a fiber and σ is the class of a section. For $i = 1, 2$, because of the connectedness of the moduli spaces of polarized K3 or Abelian surfaces, there are a smooth, connected curve T_i and a deformation $(\mathscr{X}_i, \mathscr{L}_i, \mathscr{H}_i)$ over T_i of (S_i, v_i, H_i) such that there is a $t_i \in T_i$ with the property $(\mathscr{X}_{i,t_i}, v_{i,t_i}, H_{i,t_i}) = (Y, v_i', H_i')$, where:

(1) $c_1(H_i') = \sigma + p_i f$;
(2) $v_i' = m(r, gc_1(H_i'), a_i)$.

By Lemma 2.21(1), the varieties $M_{v_i}(S_i, H_i)$ and $M_{v'_i}(Y, H'_i)$ are locally trivially deformation equivalent. Let $\xi'_i := c_1(H'_i)$. Notice that $(v'_1)^2 = (v'_2)^2$ and that they have the same rank. Hence $|v'_1| = |v'_2|$, so by Lemma 2.38 a polarization is v'_1-suitable if and only if it is v'_2-suitable.

Notice that $p_i = d - \sigma^2/2$; that is, $p_i = d_i + 1$ if Y is a K3, and $p_i = d_i$ if Y is Abelian. Hence $p_i \geqslant d \gg 0$ in both cases, and by Lemma 2.38, we have that H'_i is v'_i-suitable for $i = 1, 2$. Hence H'_1 and H'_2 are in the same v'_1-chamber $\mathcal{C}$. Using Proposition 2.5, we then change to a common generic polarization $H \in \mathcal{C}$, which is v'_i-generic for $i = 1, 2$.

The statement for $M_{v_1}(S_1, H_1)$ and $M_{v_2}(S_2, H_2)$ follows if we show that the Mukai vectors $v'_1 = m(r, g(\sigma + p_1 f), a_1)$ and $v'_2 = m(r, g(\sigma + p_2 f), a_2)$ are equivalent since, by Lemma 2.24(2) , this implies that $M_{v'_1}(Y, H) \simeq M_{v'_2}(Y, H)$.

As $(v'_1)^2 = (v'_2)^2$, we have

$$g^2\sigma^2 + 2g^2 p_1 - 2ra_1 = g^2\sigma^2 + 2g^2 p_2 - 2ra_2\,,$$

hence

$$2r(a_2 - a_1) = 2g^2(p_2 - p_1)$$

and since g divides $a_2 - a_1$, we see that r divides $g(p_2 - p_1)$. So there exists an $l \in \mathbb{Z}$ such that $g(p_2 - p_1) = rl$. If we let $L \in \mathrm{Pic}(Y)$ be a line bundle whose first Chern class is lf, the last equality implies that v'_2 and $v'_1 \cdot \mathrm{ch}(L)$ have the same component in $\mathrm{NS}(Y)$. Since they also have the same rank and the same square, we get $v'_2 = v'_1 \cdot \mathrm{ch}(L)$, and v'_2 and v'_1 are equivalent.

If we replace Lemma 2.21(1) with Lemma 2.21(2), then as the tensorization with a line bundle preserves the Yoshioka fibration, the same argument shows that if S_1 and S_2 are both Abelian surfaces, then $K_{v_1}(S_1, H_1)$ and $K_{v_2}(S_2, H_2)$ are locally trivially deformation equivalent. $\square$

2.5.3 *A numerical result on equivalent Mukai vectors.* The following numerical lemma, together with Propositions 2.29 and 2.33, will allow us to show that certain moduli spaces parametrizing semistable sheaves with different ranks, on a K3 or Abelian surface whose Néron–Severi group has rank 1, are isomorphic.

LEMMA 2.40. *Let $m, k \in \mathbb{N}$ with $m, k > 0$, and let S be a projective K3 or Abelian surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot h$, where $h = c_1(H)$ and H is an ample line bundle. Let $v = m(r, nh, a)$ be a Mukai vector of type (m, k), where $r > 0$. For every $s \in \mathbb{Z}$, let*

$$v_{sH} := v \cdot \mathrm{ch}(\mathcal{O}_S(sH)) = m(r, n_s h, a_s)\,.$$

(1) *For every $N \in \mathbb{N}$, there is an $s > N$ such that $n_s \gg 0$ and $\gcd(n_s, a_s) = 1$.*
(2) *If $n = 1$ and $a = 0$, then for every $N \in \mathbb{N}$, there is an $s > N$ such that $n_s \gg 0$, $\gcd(n_s, a_s) = 1$ and $a_s \in 2k\mathbb{Z}$.*

Proof. Write $H^2 = 2l$. A direct computation shows that

$$n_s = n + rs = n_{s-1} + r\,, \quad a_s = a + 2lns + rls^2\,. \tag{2.14}$$

Since $2ln^2 - 2ra = 2ln_s^2 - 2ra_s = 2k$, if there exists an s such that n_s and a_s are not coprime, a common prime divisor p has to divide k.

Let p be a prime factor of k dividing both n_{s_0} and a_{s_0} for some $s_0 \in \mathbb{Z}$. Since the cup product with the Chern character of a line bundle gives an isometry of the Mukai lattice, and since the Mukai vector (r, nh, a) is primitive, the Mukai vector $(r, n_{s_0}h, a_{s_0})$ is primitive as well, so p if divides both n_{s_0} and a_{s_0}, then it cannot divide r.

Now, if p also divides n_s and a_s, then p has to divide the difference $n_s - n_{s_0} = r(s - s_0)$; hence p is a factor of $s - s_0$, and we have $s = s_0 + pm$ for $m \in \mathbb{Z}$.

Since the prime divisors of k are finite, by choosing s outside the union of a finite number of arithmetic sequences, we obtain that $\gcd(n_s, a_s) = 1$; moreover, we may choose $s \gg 0$ in order to obtain $n_s \gg 0$.

If we now suppose $n = 1$ and $a = 0$ (so that $l = k$), then equations (2.14) give $n_s = 1 + rs$ and $a_s = 2ls + rls^2 = 2ks + rks^2$.

If p is a prime number dividing both a_s and n_s, then p divides k. Choosing $s = 2ks'$ for some $s' \in \mathbb{Z}$, then any prime dividing k cannot divide $n_s = 1 + 2ks'r$; hence a_s and n_s are coprime. Moreover, since s is even, $2k$ divides a_s, and, finally, $n_s \gg 0$ if s' is chosen big enough. □

Conclusion of the proof of Theorem 1.7. We start with the proof of point (1). The proof for K3 and for Abelian surfaces is formally the same; to make the notation easier, we only discuss the case of K3 surfaces and leave to the reader the obvious modifications for the Abelian case.

For every $\ell \in \mathbb{N}$ with $\ell > 0$, let X_ℓ be a projective K3 with $\mathrm{NS}(X_\ell) = \mathbb{Z} \cdot h_\ell$, where $h_\ell = c_1(H_\ell)$ and H_ℓ is an ample divisor with $H_\ell^2 = 2\ell$. Let $u_\ell := m(0, h_\ell, 0)$, so u_ℓ is a Mukai vector of type (m, ℓ) and H_ℓ is u_ℓ-generic.

We will show that if S is a K3 surface, $v = m(r, \xi, a)$ is a Mukai vector of type (m, k) on S and H is a polarization on S which is general with respect to v, then $M_v(S, H)$ is locally trivially deformation equivalent to $M_{u_k}(X_k, H_k)$.

We first notice that we can assume H to be a v-generic polarization and $a \neq 0$ if $r = 0$. This follows from Lemma 2.10 if $r > 0$ and from Lemmas 2.10 and 2.12 if $r = 0$. Under these assumptions, we may now start the proof, that will be in several steps: in the first, we reduce to $r > 0$; in the second, we reduce to r and ξ relatively prime; in the third, we reduce to $r \in 2k\mathbb{N}$; the fourth step concludes the proof.

Step 1: reduction to $r > 0$. If $v = m(0, \xi, a)$, where ξ is effective and $a \neq 0$, then by Lemma 2.24(1), for $d \in \mathbb{Z}$, the tensorization by $\mathcal{O}(dH)$ induces an isomorphism between $M_v(S, H)$ and $M_{v_{dH}}(S, H)$. By Corollary 2.35, there exist a $d_0 \in \mathbb{N}$ and a $\widetilde{v_{d_0 H}}$-generic polarization H' such that F_{K3} induces an isomorphism between $M_{v_{d_0 H}}(S, H)$ and $M_{\widetilde{v_{d_0 H}}}(S, H')$ and the rank of $\widetilde{v_{d_0 H}}$ is strictly positive.

Step 2: reduction to $r \gg 0$ and prime with ξ. By Step 1, we may suppose $r > 0$. By Lemma 2.36, we may even suppose $v = m(r, \xi, a)$ with $\xi = nc_1(H)$. We set $\ell := H^2/2$ and consider the Mukai vector $v' = m(r, nh_\ell, a)$ on X_ℓ. Notice that v' is of type (m, k) and that H_ℓ is v'-generic.

As the moduli spaces of polarized K3 are connected, the moduli spaces $M_v(S, H)$ and $M_{v'}(X_\ell, H_\ell)$ are locally trivially deformation equivalent by Lemma 2.21. For $s \in \mathbb{Z}$, write $v'_s := v'_{sH_l} = m(r, n_s h_l, a_s)$. By Lemma 2.24(1), we have $M_{v'}(X_\ell, H_\ell) \simeq M_{v'_s}(X_\ell, H_\ell)$ and, by Lemma 2.40(1), there is an $s \in \mathbb{Z}$ such that $n_s \gg 0$ and $\gcd(n_s, a_s) = 1$.

As $n_s \gg 0$, by Proposition 2.29, we get $M_{v'_s}(X_\ell, H_\ell) \simeq M_{\widetilde{v'_s}}(X_\ell, H_\ell)$, and, moreover, since m, k, r are fixed, we may also assume $a_s \gg 0$. Finally, as $\widetilde{v'_s} = m(a_s, n_s h_l, r)$, $\gcd(a_s, n_s) = 1$ and $a_s \gg 0$, we conclude that it is sufficient to prove the theorem for Mukai vectors of the form $m(r, \xi, a)$ with $r \gg 0$ and prime to ξ.

Step 3: reduction to $r \in 2k\mathbb{Z}$ with $r \gg 0$ and prime to ξ. Since for a nontrivial deformation of a K3, the locus of the base where the rank of the Néron–Severi group jumps is dense in the classical topology (see [Ogui00]) and, by Proposition 2.14, the locus where the polarization is not general is closed, deforming S if necessary, by Lemma 2.21, we may assume $\rho(S) \geqslant 2$.

By Step 2 and Lemma 2.36, we can suppose $v = m(r, c_1(H), a)$, where $r \gg 0$. On the surface X_k, we consider the Mukai vector $v'' = m(r, h_k, 0)$, which is of type (m, k); moreover, H_k is v''-generic. By Lemma 2.39, we know that $M_v(S, H)$ and $M_{v''}(X_k, H_k)$ are locally trivially deformation equivalent.

Now let $s \in \mathbb{Z}$ and $v''_s := v''_{sH_k} = m(r, n_s h_k, a_s)$, where $n_s = 1 + rs$ and $a_s = 2ks + rks^2$. By Lemma 2.24(1), we have $M_{v''}(X_k, H_k) \simeq M_{v''_s}(X_k, H_k)$, and by Lemma 2.40(2), we can choose s such that $n_s \gg 0$, $a_s \in 2k\mathbb{Z}$ and $\gcd(n_s, a_s) = 1$.

Moreover, as $n_s \gg 0$, by Proposition 2.29, we have $M_{v''_s}(X_k, H_k) \simeq M_{\widetilde{v''_s}}(X_k, H_k)$. But $\widetilde{v''_s} = m(a_s, n_s h_k, r)$ and $a_s \in 2k\mathbb{Z}$ and, moreover, $a_s > n_s \gg 0$. In conclusion, we just need to prove the theorem for Mukai vectors of the form $m(r, \xi, a)$ such that $r \gg 0$, r is an even multiple of k, and r and ξ are coprime.

Step 4: conclusion. By Step 3, we may suppose $v = m(r, \xi, a)$, with $r = 2kp$ prime to ξ and $p \gg 0$. We show that $M_v(S, H)$ is locally trivially deformation equivalent to $M_{u_k}(X_k, H_k)$.

By Lemma 2.39, we know that $M_v(S, H)$ is locally trivially deformation equivalent to $M_{v'''}(X_k, H_k)$, where $v''' = m(2kp, h_k, 0)$. As $p \gg 0$, by Proposition 2.33, we have $M_{v'''}(X_k, H_k) \simeq M_{\widetilde{v'''}}(X_k, H_k)$, where $\widetilde{v'''} = m(0, h_k, 2kp)$. But now notice that $\widetilde{v'''} \cdot \mathrm{ch}(\mathcal{O}_{X_k}(-pH_k)) = u_k$, so by Lemma 2.24(1), we have $M_{\widetilde{v'''}}(X_k, H_k) \simeq M_{u_k}(X_k, H_k)$, concluding the proof of point (1).

We are now in the position to prove point (2) of the statement. To do so, notice that the equivalence in point (1) in the Abelian case is obtained using deformations of the moduli spaces induced by deformations of the corresponding triple along smooth, connected varieties, and isomorphisms between moduli spaces induced either by tensor products with line bundles or by the Fourier–Mukai transform whose kernel is the Poincaré line bundle. Since the Yoshioka fibration is preserved by these isomorphisms and, by Lemmas 2.20 and 2.21, a deformation of (S, v, H) along a smooth connected variety also induces a locally trivial deformation of the fibers of the Yoshioka fibrations, point (2) of the statement is implied by point (1). $\square$

3. The moduli spaces are irreducible symplectic varieties

This section is devoted to the proof of Theorem 1.10: if $m, k \in \mathbb{N}$ with $m, k > 0$, S is a projective K3 or Abelian surface, v is a Mukai vector on S of type (m, k) and H is a polarization on S that is general with respect to v, then $M_v(S, H)$, if S is a K3, and $K_v(S, H)$, if S is Abelian, are irreducible symplectic varieties.

To do so, we first show in Section 3.1 that if S is a projective K3 surface and H is general with respect to the Mukai vector v, then $M_v(S, H)$ and $M_v^s(S, H)$ are simply connected. Similarly, if S is an Abelian surface, then $K_v(S, H)$ and $K_v^s(S, H)$ are simply connected (with the exception of the case $(m, k) = (2, 1)$, where $K_v(S, H)$ is still simply connected but the fundamental group of $K_v^s(S, H)$ is $\mathbb{Z}/2\mathbb{Z}$).

This will allow us to show that the exterior algebra of reflexive forms on any finite quasi-étale cover of $M_v(S, H)$ (respectively, of $K_v(S, H)$) is generated by the reflexive pull-back of a symplectic form on $M_v(S, H)$ (respectively, on $K_v(S, H)$): this will be done in Section 3.2, by showing that for a particular choice of S, v and H, there is a rational dominant map from a moduli space $M_u(S, H)$ (respectively, $K_u(S, H)$) with primitive Mukai vector to the moduli space $M_v(S, H)$ (respectively, $K_v(S, H)$).

3.1 Simple connectedness

We first show in this section that for H general with respect to v, the moduli spaces $M_v(S,H)$ and $M_v^s(S,H)$ (respectively, $K_v(S,H)$ and $K_v^s(S,H)$) are simply connected. We will divide the proof of this into two main parts: the first one is devoted to the case of K3 surfaces; in the second, we will consider Abelian surfaces. In both cases, the proof has the same structure: we first show the simple connectedness of the moduli space associated with a particular choice of the surface S, of the Mukai vector v and of the polarization, and then use Theorem 1.7 to conclude.

Before doing this, we state and prove the following result about the codimension of the subset of reducible curves in the linear system of a multiple of the polarization on a K3 surface with Picard number 1.

Lemma 3.1. *Let S be a projective K3 surface or an Abelian surface such that* $\mathrm{NS}(S)=\mathbb{Z}\cdot h$, *where $h=c_1(H)$ and H is an ample line bundle such that $H^2=2k$. Let $m\in\mathbb{N}$ with $m>0$, and consider the subset $R\subseteq|mH|$ parametrizing reducible curves. If $(m,k)\neq(2,1)$, then* $\mathrm{codim}_{|mH|}(R)\geqslant 2$.

Proof. If $C\in|mH|$ is a reducible curve, then if S is K3 (respectively, if S is Abelian), there must be $1\leqslant m_1,m_2\leqslant m$ such that $m=m_1+m_2$ and two curves $C_1\in|m_1H|$ and $C_2\in|m_2H|$ (respectively, $C_1\in|m_1H+L|$ and $C_2\in|m_2H-L|$ for some $L\in\widehat{S}$) such that $C=C_1+C_2$.

For every $1\leqslant m_1,m_2\leqslant m$ such that $m_1+m_2=m$, we let

$$P_{m_1,m_2}:=\begin{cases}|m_1H|\times|m_2H|\,, & S \text{ is K3},\\ \displaystyle\prod_{L\in\widehat{S}}|m_1H+L|\times|m_2H-L|\,, & S\text{ is Abelian}.\end{cases}$$

We then get that

$$R=\bigcup_{\substack{1\leqslant m_1,m_2\leqslant m,\\ m_1+m_2=m}}P_{m_1,m_2}\,.$$

Notice that

$$\dim(|pH|)=\begin{cases}1+kp^2\,, & S\text{ is K3},\\ kp^2-1\,, & S\text{ is Abelian},\end{cases}$$

and that if S is Abelian and $L\in\widehat{S}$, then $\dim(|pH|)=\dim(|pH|\pm L)$. It follows that

$$\dim(P_{m_1,m_2})=\begin{cases}2+k\big(m_1^2+m_2^2\big)\,, & S\text{ is K3},\\ k\big(m_1^2+m_2^2\big)\,, & S\text{ is Abelian},\end{cases}$$

and the codimension of P_{m_1,m_2} in $|mH|$ is $2km_1m_2-1$.

Hence, in order for R to have codimension 1 in $|mH|$, there must be $1\leqslant m_1,m_2\leqslant m$ such that $m_1+m_2=m$ and such that $2m_1m_2k-1=1$. Hence $m_1,m_2,k=1$, so that $m=2$ and $k=1$. Thus, if $(m,k)\neq(2,1)$, we get $\mathrm{codim}_{|mH|}(R)\geqslant 2$. □

Under the hypothesis of Lemma 3.1, we see that the only case where R is a divisor in $|mH|$ is when $(m,k)=(2,1)$.

3.1.1 *The case of K3 surfaces.* Let X be a projective K3 surface with $\mathrm{Pic}(X)=\mathbb{Z}\cdot\mathcal{O}_X(H)$, where H is an ample divisor such that $H^2=2k$. We let $h:=c_1(H)$, and we choose $m\in\mathbb{N}$ with $m>0$.

We let V be the open subset of $|mH|$ of smooth curves and U the open subset of $|mH|$ of integral curves. For $u = m(0,h,0)$, we will consider the morphism $p_u\colon M_u(X,H) \to |mH|$ mapping a sheaf to its Fitting subscheme (see [Eis95, Corollary 20.5] and [LeP93]).

Let $\mathcal{J}_V := p_u^{-1}(V)$ and $\mathcal{J}_U := p_u^{-1}(U)$, which are two open subsets of $M_u(X,H)$. Notice that if $C \in V$, then $\mathcal{F} \in p_u^{-1}(C)$ if and only if there is an $L \in \mathrm{Pic}(C)$ of degree m^2k such that $\mathcal{F} = j_*L$, where $j\colon C \to X$ is the inclusion. In particular, we have an isomorphism[3]

$$p_u^{-1}(C) \to \mathrm{Pic}^{m^2k}(C)$$

obtained by mapping $\mathcal{F} = j_*L$ to L.

Moreover, if $C \in U$, then $\mathcal{F} \in p_u^{-1}(C)$ if and only if $\mathcal{F} = j_*L$, where $j\colon C \to X$ is the inclusion, and L is a rank 1 torsion-free sheaf on C of degree m^2k, that is, such that $\chi(L) = 0$. We notice that all these sheaves are H-stable with Mukai vector u; hence we have

$$\mathcal{J}_V \subseteq \mathcal{J}_U \subseteq M_u^s(X,H) \subseteq M_u(X,H)\,.$$

We start by showing the following (the proof is a generalization of the argument proposed in [O'G99, Section 4]).

PROPOSITION 3.2. *The moduli spaces $M_u(X,H)$ and $M_u^s(X,H)$ are simply connected.*

Proof. Notice that u is a Mukai vector of type (m,k) and that H is u-generic. If $m = 1$, then $M_u(X,H) = M_u^s(X,H)$; this is an irreducible symplectic manifold, and we are done. For $(m,k) = (2,1)$, see [O'G99, Section 4].

For $m \geqslant 2$, we have that $M_u(X,H)$ is a normal, irreducible projective variety (by [KLS06, Theorem 4.4]). Since for normal quasi-projective varieties, the inclusion of an open subvariety induces a surjection on the fundamental groups (see [Kol95, Proposition 2.10]), the chain of inclusions

$$\mathcal{J}_V \xhookrightarrow{j} \mathcal{J}_U \hookrightarrow M_u^s(X,H) \hookrightarrow M_u(X,H)$$

of smooth open subvarieties of $M_u(X,H)$ given before induces a chain of surjections

$$\pi_1(\mathcal{J}_V) \xrightarrow{\pi_1(j)} \pi_1(\mathcal{J}_U) \longrightarrow \pi_1(M_u^s(X,H)) \longrightarrow \pi_1(M_u(X,H))\,.$$

We then just need to show that $\pi_1(j)$ is the trivial map.

To show this, notice that the homotopy exact sequence of the fibration $p_{u|\mathcal{J}_V}\colon \mathcal{J}_V \to V$ gives the exact sequence

$$\pi_1\big(p_u^{-1}(C)\big) \longrightarrow \pi_1(\mathcal{J}_V) \longrightarrow \pi_1(V) \longrightarrow \{1\}\,,$$

where $C \in V$. As remarked above, we have $p^{-1}(C) \simeq \mathrm{Pic}^{m^2k}(C)$; hence the exact sequence is

$$\pi_1\big(\mathrm{Pic}^{m^2k}(C)\big) \xrightarrow{j_C} \pi_1(\mathcal{J}_V) \longrightarrow \pi_1(V) \longrightarrow \{1\}\,. \tag{3.1}$$

We start by proving the following.

LEMMA 3.3. *The morphism $\pi_1(j) \circ j_C\colon \pi_1\big(\mathrm{Pic}^{m^2k}(C)\big) \to \pi_1(\mathcal{J}_U)$ is trivial.*

Proof. Let $\ell \subseteq |mH|$ be a generic line, and suppose that it is generated by two smooth curves intersecting transversally. By Lemma 3.1, we can suppose that all the curves in ℓ are reduced and irreducible.

[3] Here and in what follows, if C is a smooth projective curve and $d \in \mathbb{Z}$, we let $\mathrm{Pic}^d(C)$ be the set of line bundles of degree d on C.

If $b\colon \widetilde{X} \to X$ is the blow-up of X along the base locus $\mathrm{Bs}(\ell)$ of ℓ, then $\widetilde{X}$ is the total space of ℓ. This means that for every $s \in \widetilde{X}$, there is a unique curve C_s of ℓ such that $s \in \widetilde{C}_s$, where $\widetilde{C}_s$ is the proper transform of C_s. We have a natural fibration $p_\ell\colon \widetilde{X} \to \ell$ mapping $s \in \widetilde{X}$ to C_s.

We now define an embedding $g\colon \widetilde{X} \to \mathcal{J}_U$ of fibrations over ℓ. First, fix a $p \in \mathrm{Bs}(\ell)$, and let $d := 1 + m^2k$. Let $s \in \widetilde{X}$; then $b(s) \in C_s$. Consider the rank 1 torsion-free sheaf $L_s := \mathcal{I}_{b(s)} \otimes \mathcal{O}_{C_s}(dp)$, whose degree on C_s is m^2k. If $j_s\colon C_s \to X$ is the inclusion, then $j_{s*}(L_s) \in \mathcal{J}_U$, and we let $g(s) := j_* L_s$. The inclusion g then fits in a commutative diagram (where i is the inclusion)

$$\begin{tikzcd} \widetilde{X} \arrow[r, "g"] \arrow[d, "p_\ell"'] & \mathcal{J}_U \arrow[d, "p_u"] \\ \ell \arrow[r, "i"'] & U\,. \end{tikzcd}$$

Notice that if $t \in \ell$ is a generic point and C is the corresponding curve in ℓ, then $p_\ell^{-1}(t) = \widetilde{C}$, the proper transform of C under b, while $p_u^{-1}(t) \simeq \mathrm{Pic}^{m^2k}(C)$. The restriction $g_t\colon \widetilde{C} \to \mathrm{Pic}^{m^2k}(C)$ of g to $p_\ell^{-1}(t)$ can be identified with the Abel–Jacobi map from C to its Jacobian. It then induces a surjective morphism $\pi_1(g_t)\colon \pi_1\big(\widetilde{C}\big) \to \pi_1\big(\mathrm{Pic}^{m^2k}(C)\big)$.

Now, let $C \in \ell$ be a smooth curve. We have a commutative diagram

$$\begin{tikzcd} \widetilde{C} \arrow[r, "g_t"] \arrow[d, "i"'] & \mathrm{Pic}^{m^2k}(C) \arrow[d, "\widetilde{i}"] \\ \widetilde{X} \arrow[r, "g"'] & \mathcal{J}_U\,, \end{tikzcd}$$

inducing a commutative diagram

$$\begin{tikzcd} \pi_1\big(\widetilde{C}\big) \arrow[r, "\pi_1(g_t)"] \arrow[d, "\pi_1(i)"'] & \pi_1\big(\mathrm{Pic}^{m^2k}(C)\big) \arrow[d, "\pi_1(j)\circ j_C"] \\ \pi_1\big(\widetilde{X}\big) \arrow[r, "\pi_1(g)"'] & \pi_1(\mathcal{J}_U)\,. \end{tikzcd}$$

As $\pi_1\big(\widetilde{X}\big) = \{1\}$ and the morphism $\pi_1(g_t)$ is surjective, it follows that $\pi_1(j) \circ j_C$ is trivial, thus concluding the proof. $\square$

An immediate consequence of Lemma 3.3 is that the surjective morphism $\pi_1(j)$ factors through a surjective morphism

$$\overline{\pi_1(j)}\colon \pi_1(\mathcal{J}_V)/\mathrm{im}(j_C) \longrightarrow \pi_1(\mathcal{J}_U)\,.$$

The exact sequence (3.1) gives an isomorphism between $\pi_1(V)$ and $\pi_1(\mathcal{J}_V)/\mathrm{im}(j_C)$; hence we get a surjective map $\iota\colon \pi_1(V) \to \pi_1(\mathcal{J}_U)$, which is then trivial if and only if $\pi_1(j)$ is trivial. We then just need to show that ι is trivial.

To do so, consider the generic line $\ell \subseteq |mH|$ of the proof of Lemma 3.3. All the curves parametrized by ℓ are reduced and irreducible, and we can suppose that ℓ is transversal to $W := U \setminus V$, where $\ell \cap W := \{x_1, \ldots, x_p\}$ is given by smooth points of W.

As ℓ is generic, by Zariski's main theorem (see [Voi03, Theorem 3.22]), the inclusion of $\ell \setminus W$ in V gives a surjection $\pi_1(\ell \setminus W) \to \pi_1(V)$; hence we finally get a surjective morphism

$\iota_\ell \colon \pi_1(\ell \setminus W) \to \pi_1(\mathcal{J}_U)$, and we just need to show that ι_ℓ is trivial. More precisely, if $\gamma_1, \dots, \gamma_p$ are the generators of $\pi_1(\ell \setminus W)$, we need to show that $\iota_\ell(\gamma_i)$ is trivial.

Now, notice that the fibration $p_\ell \colon \widetilde{X} \to \ell$ has a section σ_ℓ (fixing $p \in \mathrm{Bs}(\ell)$, we let $\sigma_\ell(t) := \pi^{-1}(p) \cap p_\ell^{-1}(t)$). Hence every γ_i has a lifting $\widetilde{\gamma}_i$ in $\pi_1(\widetilde{X})$, and, by construction, its image in $\pi_1(\mathcal{J}_U)$ under $\pi_1(g)$ is $\iota_\ell(\gamma_i)$. But as $\pi_1(\widetilde{X}) = \pi_1(X)$ (since $\pi \colon \widetilde{X} \to X$ is a blow-up) and as $\pi_1(X)$ is trivial (since X is K3), it follows that $\iota_\ell(\gamma_i) = 0$. □

The main consequence of Proposition 3.2 is that the moduli spaces of (semi)stable sheaves associated with (m,k)-triples are simply connected.

Theorem 3.4. *Let $m, k \in \mathbb{N}$ with $m, k > 0$, and let S be a projective K3 surface, v a Mukai vector on S of type (m,k) and H a polarization that is general with respect to v. Then $M_v(S,H)$ and $M_v^s(S,H)$ are simply connected.*

Proof. First suppose $m = 1$, that v is a Mukai vector such that if $v = (0, v_1, v_2)$, then $v_2 \neq 0$, and that H is v-generic. In this case, we then have $M_v(S,H) = M_v^s(S,H)$. By [O'G97] and [Yos99a], we know that $M_v(S,H)$ is an irreducible symplectic manifold, and we are done.

If v is any Mukai vector of type $(1,k)$ and H is general with respect to v, the result follows from the case we considered above and by Lemmas 2.10, 2.12 and 2.16.

Now fix $m \geqslant 2$ and $k \geqslant 1$. By Theorem 1.7(1), the moduli spaces arising from K3 surfaces, Mukai vectors of type (m,k) and polarization which are general with respect to them are all deformation equivalent. As this deformation equivalence is obtained using only isomorphisms of moduli spaces (coming from Fourier–Mukai transforms) and deformations of the moduli spaces induced by deformations of triples, by Lemma 2.22(1), these deformation equivalent moduli spaces are also homeomorphic, and the same holds for their stable (that is, smooth) loci.

It is then enough to prove that $M_v(S,H)$ and $M_v^s(S,H)$ are simply connected for one particular choice of S, v and H, where v is of type (m,k) and H is a general with respect to v. Hence the result follows from Proposition 3.2. □

3.1.2 *The case of Abelian surfaces.* Let A be an Abelian surface with $\mathrm{NS}(A) = \mathbb{Z} \cdot h$, where $h = c_1(H)$ and H is an ample divisor such that $H^2 = 2k$. We let $h := c_1(H)$, $m \in \mathbb{N}$ and $u := m(0,h,0)$.

Let Y_{mH} be the Hilbert scheme of curves on A which are deformations of curves in $|mH|$, and let $p_u \colon M_u(A,H) \to Y_{mH}$ be the morphism mapping a sheaf to its Fitting subscheme. We moreover let $p_u^K \colon K_u(A,H) \to |mH|$ be the restriction of p_u to $K_u(A,H)$.

We first prove simple connectedness in the particular case of $K_v(A,H)$ and $K_v^s(A,H)$.

Proposition 3.5. *If $(m,k) \neq (2,1)$, then $K_u(A,H)$ and $K_u^s(A,H)$ are simply connected.*

Proof. Notice that u is a Mukai vector of type (m,k) and that H is u-generic. If $m = 1$, then $K_u(A,H) = K_u^s(A,H)$, and this is a point (if $k = 1$) or an irreducible symplectic manifold (if $k > 1$), and we are done.

For $m \geqslant 2$, we have that $K_u(A,H)$ is a normal, irreducible projective variety (see [PR14, Remark A.1]). As a consequence, we have a surjective map $\pi_1(K_u^s(A,H)) \to \pi_1(K_u(A,H))$ (see [Kol95, Proposition 2.10]), and it will be sufficient to prove that $K_u^s(A,H)$ is simply connected.

To show this, let $p^K_{u|K_u^s(A,H)} \colon K_u^s(A,H) \to |mH|$ be the restriction of p_u^K to $K_u^s(A,H)$. By the theorem in [GM88, Section 1.1, Part II], the fundamental group of a smooth connected variety admitting a dominant mapping to $\mathbb{P}^N$ (for some N) is generated by the fundamental group of the inverse image of a generic line in $\mathbb{P}^N$. As a consequence, if $\ell \subseteq |mH|$ is a generic line and $K^0 := \big(p^K_{u|K^s_u(A,H)}\big)^{-1}(\ell) \subseteq K^s_u(A,H)$, we have a surjective morphism $\pi_1\big(K^0\big) \to \pi_1(K^s_u(A,H))$. It is then enough to show that K^0 is simply connected.

As ℓ is generic in $|mH|$, by Bertini's theorem, we know that K^0 is smooth. Moreover, by Lemma 3.1, all the curves parametrized by ℓ are reduced and irreducible. It then follows that $K^0 = \big(p^K_u\big)^{-1}(\ell)$.

To show that K^0 is simply connected, we show that K^0 is a fiber of an isotrivial fibration and then use the homotopy exact sequence of this fibration to conclude. The domain of this isotrivial fibration will be $M^0 := p_u^{-1}(\ell)$ (which is a subset of $M_u(A,H)$), that will be identified with the relative compactified Jacobian of ℓ. By construction, there is an inclusion $f\colon K^0 \to M^0$ fitting in a commutative diagram

$$\begin{tikzcd} K^0 \arrow[r, "f"] \arrow[d, "p^0_K"'] & M^0 \arrow[d, "p^0"] \\ \ell \arrow[r, "\mathrm{id}_\ell"'] & \ell\,, \end{tikzcd}$$

where p^0_K is the restriction of p^K_u to K^0 and p^0 is the restriction of p_u to M^0.

We now let $\sigma\colon M^0 \to A$ be the restriction to M^0 of the map $\beta\colon M_u(A,H) \to A$ defined in Section 2.2, mapping a sheaf $\mathcal{F}$ to the Albanese image of $\mathbf{c}_2(\mathcal{F})$. As the determinant of $\mathcal{F} \in M^0$ is represented by the Fitting subscheme of $\mathcal{F}$, which is a divisor in $|mH|$, by Lemma 2.15, we have

$$K^0 = M^0 \cap K_u(A,H) = M^0 \cap b_u^{-1}(0_A, \mathcal{O}_A) = \sigma^{-1}(0_A)\,,$$

where $b_u\colon M_u(A,H) \to A \times \widehat{A}$ is the O'Grady fibration of $M_u(A,H)$ defined in Section 2.2.

Next, we claim that $\sigma\colon M^0 \to A$ is an isotrivial fibration. Indeed, if $L \in \mathrm{Pic}^0(A)$ is represented by a divisor D and δ is a 0-cycle of degree 0 on A representing $mH \cdot D$ in the Chow ring of A, then the tensorization with L induces an automorphism of $M_u(A,H)$ mapping K^0 to $\sigma^{-1}(\delta)$. It follows that the connected algebraic group $\mathrm{Pic}^0(A)$ acts transitively on the fibers of the projective morphism σ; this implies that σ is an isotrivial fibration.

Finally, notice that K^0 is connected. Indeed, it is the inverse image, under the dominant map $p^K_u\colon K_u(A,H) \to |mH|$, of a linear space of the projective space $|mH|$. By [FL81, Theorem 1.1], it follows that K^0 is connected.

To resume, we have an isotrivial fibration $\sigma\colon M^0 \to A$, and K^0 is one of the fibers. The homotopy exact sequence associated with this fibration then gives

$$\pi_2(A) \longrightarrow \pi_1\big(K^0\big) \xrightarrow{\pi_1(f)} \pi_1\big(M^0\big) \xrightarrow{\pi_1(\sigma)} \pi_1(A) \longrightarrow \{1\}\,,$$

where the last term comes from the fact that K^0 is connected. As A is an Abelian surface, we have $\pi_2(A) = \{1\}$; hence in order to show that K^0 is simply connected, we just need to prove that the morphism $\pi_1(\sigma)\colon \pi_1\big(M^0\big) \to \pi_1(A)$ is injective.

To do so, suppose that ℓ is generated by two smooth curves intersecting transversally at a finite number of points. Let $\mathrm{Bs}(\ell)$ be the base locus of ℓ and $\pi\colon \widetilde{A} \to A$ the blow-up of A along $\mathrm{Bs}(\ell)$.

The surface $\widetilde{A}$ is the total space of ℓ: for every $a \in \widetilde{A}$, there is a unique curve $C_a \in \ell$ such that $a \in \widetilde{C}_a$, where $\widetilde{C}_a$ is the proper transform of C_a under π. We then have a fibration $p_\ell\colon \widetilde{A} \to \ell$

mapping $a \in \widetilde{A}$ to the point of ℓ corresponding to C_a.

There is a natural morphism $g\colon \widetilde{A} \to M^0$ of fibrations over ℓ obtained as follows: first, choose $p \in \mathrm{Bs}(\ell)$, and let $d := m^2k+1$. For every $a \in \widetilde{A}$, the rank 1 torsion-free sheaf $\mathcal{I}_{g(a)} \otimes \mathcal{O}_{C_a}(dp)$ has degree m^2k. We then let $g(a) := \mathcal{I}_{g(a)} \otimes \mathcal{O}_{C_a}(dp)$, so we have a commutative diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r, "g"] \arrow[d, "p_\ell"'] & M^0 \arrow[d, "p^0"] \\ \ell \arrow[r, "\mathrm{id}_\ell"'] & \ell\,. \end{tikzcd}$$

If $t \in \ell$ is a generic point, the curve C corresponding to t is smooth, $p_\ell^{-1}(t) = \widetilde{C}$, and $(p^0)^{-1}(t) \simeq \mathrm{Pic}^{m^2k}(C)$. Let $p_1, \dots, p_n \in \ell$ be the points corresponding to singular curves. The fundamental group of M^0 is generated by $\pi_1(\mathrm{Pic}^{m^2k}(C))$ and by liftings $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n$ of the generators $\gamma_1, \dots, \gamma_n$ of $\pi_1(\ell \setminus \{p_1, \dots, p_n\})$.

Moreover, the morphism $g_t\colon \widetilde{C} \to \mathrm{Pic}^{m^2k}(C)$ given by the restriction of g to $p_\ell^{-1}(t)$ can be identified with the Abel–Jacobi map from C to its Jacobian. It then induces a surjective map $\pi_1(\widetilde{C}) \to \pi_1(\mathrm{Pic}^{m^2k}(C))$.

As $\widetilde{C} \subseteq \widetilde{A}$, it follows that $\pi_1(M^0)$ is generated by $\pi_1(\widetilde{A})$ and by $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n$. Now, notice that the fibration $p_\ell\colon \widetilde{A} \to \ell$ has a section. Fixing $p \in \mathrm{Bs}(\ell)$, this section is obtained by mapping $t \in \ell$ to the unique intersection point of $\pi^{-1}(p)$ and $p_\ell^{-1}(t)$.

We can then choose the liftings $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n$ to be in the image of $\pi_1(\widetilde{A})$ in $\pi_1(M^0)$; hence g induces a surjection $\pi_1(g)\colon \pi_1(\widetilde{A}) \to \pi_1(M^0)$. As $\sigma \circ g\colon \widetilde{A} \to A$ induces an isomorphism between $\pi_1(\widetilde{A})$ and $\pi_1(A)$, the morphism $\pi_1(\sigma)$ is injective; this concludes the proof. $\square$

Theorem 1.7 allows us to extend Proposition 3.5 to all Abelian surfaces (provided that we have $(m,k) \neq (2,1)$).

Theorem 3.6. *Let $m,k \in \mathbb{N}$ with $m,k > 0$, and let S be an Abelian surface, v a Mukai vector on S of type (m,k) and H a polarization which is general with respect to v.*

1. *If $(m,k) \neq (2,1)$, then $K_v(S,H)$ and $K_v^s(S,H)$ are simply connected.*
2. *If $(m,k) = (2,1)$, then $K_v(S,H)$ is simply connected and $\pi_1(K_v^s(S,H)) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. First suppose $m=1$, that v is a Mukai vector such that if $v = (0, v_1, v_2)$, then $v_2 \neq 0$, and that H is v-generic. Then $K_v(S,H) = K_v^s(S,H)$, and this is a point (if $k=1$) or an irreducible symplectic manifold (if $k \geqslant 2$). The statement is then clear in this case.

If v is any Mukai vector of type $(1,k)$ and H is general with respect to v, the result follows from the case we considered above and by Lemmas 2.10, 2.12 and 2.16.

Now fix $m \geqslant 2$ and $k \geqslant 1$, and suppose $(m,k) \neq (2,1)$. By Theorem 1.7(2) , the fibers of the Yoshioka fibration of a moduli spaces arising from Abelian surfaces, Mukai vectors of type (m,k) and polarizations which are general with respect to them are all deformation equivalent. As this deformation equivalence is obtained using only isomorphisms of moduli spaces (coming from Fourier–Mukai transforms) and deformations of the moduli spaces induced by deformations of triples, by Lemma 2.22(2), the homeomorphism type of $K_{mw}(S,H)$ and $K_{mw}^s(S,H)$ only depends on m and $k = w^2/2$.

It is then enough to show that $K_v(S,H)$ and $K_v^s(S,H)$ are simply connected for a particular choice of S, v of type (m,k) and H which is general with respect to v. The result follows then from Proposition 3.5.

If $(m,k)=(2,1)$, then by Lemmas 2.10, 2.12 and 2.16, we may suppose that v is a Mukai vector such that if $v=(0,v_1,v_2)$, then $v_2\neq 0$, and that H is v-generic.

In this case, we know that $K_v(S,H)$ admits a symplectic resolution $\widetilde{K}_v(S,H)$, which is an irreducible symplectic manifold by [PR13, Theorem 1.6(2)]. As $K_v(S,H)$ has canonical singularities, by [Tak03], we have $\pi_1(K_v(S,H))=\pi_1\big(\widetilde{K}_v(S,H)\big)$; it follows that $K_v(S,H)$ is simply connected.

By [MRS18, Theorem 4.2 and Proposition 5.3], we know that $K_v^s(S,H)$ has an étale cover of degree 2 from an open subset U of an irreducible symplectic manifold Y which is deformation equivalent to a Hilbert scheme of three points on a K3 surface. This open subset U is obtained by removing from Y 256 copies of $\mathbb{P}^3$ and 1 copy of a desingularization of the singular locus of $K_v(S,H)$. It follows that the complement of U has codimension at least 2 in Y, so that $\pi_1(U)=\pi_1(Y)=\{1\}$. It then follows that the fundamental group of $K_v^s(S,H)$ is $\mathbb{Z}/2\mathbb{Z}$. □

Recall that if X is a normal projective variety having at most rational singularities, it is possible to define the Albanese variety $\mathrm{Alb}(X)$ as the Albanese variety of any desingularization $\widetilde{X}$ of X and construct the Albanese morphism $\mathrm{alb}\colon X\to\mathrm{Alb}(X)$ by descending the usual Albanese morphism of $\widetilde{X}$ (see [Rei83, Proposition 2.3] and [Kaw85, Lemma 8.1]).

As a consequence of Theorem 3.6, we show in the next result that the Yoshioka fibration is the Albanese morphism of the moduli space $M_v(S,H)$.

Corollary 3.7. *Let $m,k\in\mathbb{N}$ with $m,k>0$, and let S be an Abelian surface, v a Mukai vector on S of type (m,k) and H a polarization that is general with respect to v. The morphism $a_v\colon M_v(S,H)\to S\times\widehat{S}$ is the Albanese morphism of $M_v(S,H)$.*

Proof. By Lemmas 2.9, 2.10, 2.12 and 2.24, we may assume that v is not of the form $(0,v_1,0)$ and that H is a v-generic polarization. Under this assumption, for $m=1$, the map a_v is the Albanese map by [Yos01, Theorem 0.1(1)]. We then suppose $m\geqslant 2$.

For $(m,k)=(2,1)$, we know by [LS06, Théorème 1.1] that $M_v(S,H)$ admits a symplectic resolution of the singularities $\pi\colon\widetilde{M}_v(S,H)\to M_v(S,H)$, which is obtained by blowing up the singular locus Σ with reduced structure.

Now, for every $(p,L)\in S\times\widehat{S}$, the fiber $K_{p,L}:=a_v^{-1}(p,L)$ is a singular symplectic variety whose singular locus is $\Sigma_{p,L}:=\Sigma\cap K_{p,L}$, and $\widetilde{K}_{p,L}:=\pi^{-1}(K_{p,L})$ is the symplectic resolution of $K_{p,L}$, which is an irreducible symplectic manifold by [PR13, Theorem 1.6(2)]. It follows that

$$a_v\circ\pi\colon\widetilde{M}_v(S,H)\longrightarrow S\times\widehat{S}$$

is the Albanese morphism of $\widetilde{M}$, so that $a_v\colon M_v(S,H)\to S\times\widehat{S}$ is the Albanese morphism of $M_v(S,H)$.

Let us now finally consider the case $(m,k)\neq(2,1)$ and $m\geqslant 2$. Let $\pi\colon\widetilde{M}\to M_v(S,H)$ be a desingularization of $M_v(S,H)$, where $\widetilde{M}$ is a smooth projective variety. The inclusion $j\colon M_v^s(S,H)\to\widetilde{M}$ induces a surjective morphism $\pi_1(j)\colon\pi_1(M_v^s(S,H))\to\pi_1\big(\widetilde{M}\big)$. If we now let $a_v^s\colon M_v^s(S,H)\to S\times\widehat{S}$ be the restriction of a_v to $M_v^s(S,H)$, then a_v^s is an isotrivial fibration whose fibers are all isomorphic to $K_v^s(S,H)$.

As the fiber $K_v^s(S,H)$ is simply connected (since $(m,k)\neq(2,1)$), the isotrivial fibration a_v^s induces an isomorphism $\pi_1(a_v^s)\colon\pi_1(M_v^s(S,H))\to\pi_1\big(S\times\widehat{S}\big)$.

Now, notice that $a_v \circ \pi \circ j = a_v^s$; hence $\pi_1(j)$ is injective, and hence an isomorphism. But this implies that $\pi_1(a_v \circ \pi)$ is an isomorphism, so that $a_v \circ \pi \colon \widetilde{M} \to S \times \widehat{S}$ is the Albanese morphism for $\widetilde{M}$. It then follows that $a_v \colon M_v(S,H) \to S \times \widehat{S}$ is the Albanese morphism of $M_v(S,H)$, concluding the proof. □

3.2 The proof of Theorem 1.10

We are finally in the position to prove Theorem 1.10, that is, that $M_v(S,H)$ and $K_v(S,H)$ are irreducible symplectic varieties. Before doing this, we calculate the dimension of the space of reflexive p-forms for a particular choice of the surface, of the Mukai vector and of the polarization.

LEMMA 3.8. *Let $m, k \in \mathbb{N}$ with $m, k > 0$, and let X be a projective K3 or Abelian surface such that $\mathrm{NS}(X) = \mathbb{Z} \cdot h_k$, where $h_k^2 = 2k$. Let H_k be a polarization on S such that $c_1(H_k) = h_k$, and let $u_k = m(0, h_k, 0)$.*

(1) *If X is K3 and $p \in \mathbb{N}$ is such that $0 \leqslant p \leqslant \dim(M_{u_k}(X, H_k))$, then*

$$h^0\big(M_{u_k}(X,H_k), \Omega^{[p]}_{M_{u_k}(X,H_k)}\big) = \begin{cases} 1\,, & p \text{ is even,} \\ 0\,, & p \text{ is odd.} \end{cases}$$

(2) *If X is Abelian and $p \in \mathbb{N}$ is such that $0 \leqslant p \leqslant \dim(K_{u_k}(X, H_k))$, then*

$$h^0\big(K_{u_k}(X,H_k), \Omega^{[p]}_{K_{u_k}(X,H_k)}\big) = \begin{cases} 1\,, & p \text{ is even,} \\ 0\,, & p \text{ is odd.} \end{cases}$$

Proof. First suppose that X is K3. We let $u := \big(0, mh_k, 1 - m^2k\big)$, which is a primitive Mukai vector on X.

If $C \in |mH_k|$ is an integral curve and $j \colon C \to X$ is the inclusion, then for every $L \in \mathrm{Pic}^1(C)$, the sheaf j_*L is H_k-stable with Mukai vector u. The sheaves of this type form an open subset U of $M_u(X, H_k)$.

Moreover, if $L \in \mathrm{Pic}^1(C)$, then $L^{\otimes m^2k} \in \mathrm{Pic}^{m^2k}(C)$; hence $j_*\big(L^{\otimes m^2k}\big)$ is an H_k-stable sheaf with Mukai vector v. We then have a rational map

$$g \colon M_u(X,H_k) \dashrightarrow M_{u_k}(X,H_k)\,, \quad g(j_*L) := j_*L^{\otimes m^2k}\,.$$

We first show that g is dominant. To do so, consider the two fibrations $p_u \colon M_u(X,H_k) \to |mH_k|$ and $p_{u_k} \colon M_{u_k}(X,H_k) \to |mH_k|$ mapping a sheaf to its Fitting subscheme. If $C \in |mH_k|$ is smooth, we have $p_u^{-1}(C) \simeq \mathrm{Pic}^1(C)$ and $p_{u_k}^{-1}(C) \simeq \mathrm{Pic}^{m^2k}(C)$, hence $p_u^{-1}(C) \simeq p_{u_k}^{-1}(C) \simeq \mathrm{Pic}^0(C)$, and the restriction of g to $p_u^{-1}(C)$ can be identified with the multiplication by m^2k on $\mathrm{Pic}^0(C)$ and is therefore surjective. This shows that if $V \subseteq |mH_k|$ is the open subset of smooth curves, then g maps $p_u^{-1}(V)$ surjectively to $p_{u_k}^{-1}(V)$. As $M_u(X,H_k)$ and $M_{u_k}(X,H_k)$ are two projective varieties which are both irreducible and of the same dimension, it follows that g is dominant.

Since $M_{u_k}(X,H_k)$ has canonical singularities, letting $\widetilde{M}_{u_k}$ be a resolution of the singularities, by [GKKP11, Theorem 1.4], we have the equality $h^0\big(M_{u_k}(X,H_k), \Omega^{[p]}_{M_{u_k}(X,H_k)}\big) = h^0\big(\widetilde{M}_{u_k}, \Omega^p_{\widetilde{M}_{u_k}}\big)$ for every $p \in \mathbb{N}$. As g is dominant, we also have $h^0\big(\widetilde{M}_{u_k}, \Omega^p_{\widetilde{M}_{u_k}}\big) \leqslant h^0\big(M_u, \Omega^p_{M_u(X,H_k)}\big)$ for every $p \in \mathbb{N}$. Since u is primitive, $M_u(X,H_k)$ is an irreducible symplectic manifold, and we conclude that

$$h^0\big(M_{u_k}(X,H_k),\Omega^{[p]}_{M_{u_k}(X,H_k)}\big)\leqslant\begin{cases}1\,, & p \text{ is even},\\ 0\,, & p\text{ is odd}.\end{cases}$$

Since $M_{u_k}(X,H_k)$ is a symplectic variety, $h^0\big(M_{u_k}(X,H_k),\Omega^{[p]}_{M_{u_k}(X,H_k)}\big)\geqslant 1$ if p is even, and we are done.

If S is Abelian, the same proof works, replacing $M_{u_k}(X,H_k)$ by $K_{u_k}(X,H_k)$ and $M_u(X,H_k)$ by $K_u(X,H_k)$. □

Proof of Theorem 1.10. We first consider S to be a K3 surface. If $m=1$, then $M_v(S,H)$ is an irreducible symplectic manifold by [Yos99a, Theorem 0.1] and by Lemmas 2.10, 2.12 and 2.16 if H is not v-generic or $v=(0,v_1,0)$.

If $m\geqslant 2$, then $M_v(S,H)$ is a symplectic variety; let σ be a symplectic form on it. We have to show that if $f\colon Y\to M_v(S,H)$ is a finite quasi-étale morphism, then the exterior algebra of reflexive forms on the normal variety Y is spanned by $f^{[*]}\sigma$.

Then let $f\colon Y\to M_v(S,H)$ be a finite quasi-étale cover; it induces a finite quasi-étale cover of $M^s_v(S,H)$. But a finite quasi-étale morphism of a smooth variety is étale, and $M^s_v(S,H)$ is simply connected by Theorem 3.4; hence f is an isomorphism.

We then just need to show that the exterior algebra of reflexive forms on $M_v(S,H)$ is spanned by σ. This follows if we show that $h^0\big(M_v(S,H),\Omega^{[p]}_{M_v(S,H)}\big)=1$ if p is even and $h^0\big(M_v(S,H),\Omega^{[p]}_{M_v(S,H)}\big)=0$ if p is odd.

For this, let X be a projective K3 surface with $\mathrm{Pic}(X)=\mathbb{Z}\cdot H_k$, where H_k is an ample line bundle with $(H_k)^2=2k$, and let $u_k:=m(0,h_k,0)$, where $h_k=c_1(H_k)$. By Lemma 3.8, we have

$$h^0\big(M_{u_k}(X,H_k),\Omega^{[p]}_{M_{u_k}(X,H_k)}\big)=\begin{cases}1\,, & p \text{ is even},\\ 0\,, & p\text{ is odd}.\end{cases}$$

By Theorem 1.7, the moduli spaces $M_v(S,H)$ and $M_{u_k}(X,H_k)$ are deformation equivalent, and the deformation is locally trivial. Hence they have resolutions $\widetilde{M}_v$ and $\widetilde{M}_{u_k}$ of the singularities which are deformation equivalent as smooth varieties, so their Hodge numbers are equal. By [GKKP11, Theorem 1.4], we then have

$$h^0\big(M_v(S,H),\Omega^{[2]}_{M_v(S,H)}\big)=h^0\big(\widetilde{M}_v,\Omega^2_{\widetilde{M}_v}\big)=h^0\big(\widetilde{M}_{u_k},\Omega^2_{\widetilde{M}_{u_k}}\big)=h^0\big(M_{u_k}(X,H_k),\Omega^{[2]}_{M_{u_k}(X,H_k)}\big)\,,$$

and we are done.

If S is an Abelian surface, the proof is identical if $(m,k)\neq(2,1)$, replacing moduli spaces of sheaves by the corresponding Albanese fibers and using Theorem 3.6(1) instead of Theorem 3.4.

The case $(m,k)=(2,1)$ has to be treated differently. By Theorem 3.6(2), we have $\pi_1(K^s_v(S,H))=\mathbb{Z}/2\mathbb{Z}$; hence $K_v(S,H)$ has a unique (up to isomorphism) nontrivial connected finite quasi-étale cover Y_v. We need to show that the exterior algebras of reflexive forms on $K_v(S,H)$ and Y_v are spanned by the reflexive pull-back of a symplectic form on $K_v(S,H)$. To do so, it will be enough to show that both $K_v(S,H)$ and Y_v are birational to irreducible symplectic manifolds.

For $K_v(S,H)$, by [PR13, Theorem 1.6(2)], we know that $K_v(S,H)$ has a symplectic resolution which is an irreducible symplectic manifold (in the deformation class OG_6). For Y_v, by [MRS18, Proposition 5.3], we know that it is birational to an irreducible symplectic manifold deformation equivalent to $\mathrm{Hilb}^3(\mathrm{K3})$. This concludes the proof. □

Acknowledgements

We thank the anonymous referees for having pointed out some mistakes in the previous version of the paper. This helped us a lot to improve the main results of the paper and the exposition.

References

BGL22 B. Bakker, H. Guenancia and C. Lehn, *Algebraic approximation and the decomposition theorem for Kähler Calabi–Yau varieties*, Invent. Math. **228** (2022), no. 3, 1255–1308; doi:10.1007/s00222-022-01096-y.

BL21 B. Bakker and C. Lehn, *A global Torelli theorem for singular symplectic varieties*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 3, 949–994; doi:10.4171/jems/1026.

Bea83 A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782; doi:10.4310/jdg/1214438181.

Bea00 ———, *Symplectic singularities*, Invent. Math. **139** (2000), no. 3, 541–549; doi:10.1007/s002229900043.

Bri99 T. Bridgeland, *Equivalences of triangulated categories and Fourier–Mukai transforms*, Bull. London Math. Soc. **31** (1999), no. 1, 25–34; doi:10.1112/S0024609398004998.

Dim92 A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, vol. 1992 (Springer-Verlag, New York, 1992); doi:10.1007/978-1-4612-4404-2.

Dru18 S. Druel, *A decomposition theorem for singular spaces with trivial canonical class of dimension at most five*, Invent. Math. **211** (2018), no. 1, 245–296; doi:10.1007/s00222-017-0748-y.

DG18 S. Druel and H. Guenancia, *A decomposition theorem for smoothable varieties with trivial canonical class*, J. Éc. Polytech. Math. **5** (2018), 117–147; doi:10.5802/jep.65.

Eis95 D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Grad. Texts in Math., vol. 150 (Springer-Verlag, New York, 1995); doi:10.1007/978-1-4612-5350-1.

Fle88 H. Flenner, *Extendability of differential forms on nonisolated singularities*, Invent. Math. **94** (1988), no. 2, 317–326; doi:10.1007/BF01394328.

FK87 H. Flenner and S. Kosarew, *On locally trivial deformations*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 4, 627–665; doi:10.2977/prims/1195176251.

FL81 W. Fulton and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Algebraic Geometry (Chicago, Ill., 1980), Lecture Notes in Math., vol. 862 (Springer, Berlin – New York, 1981), 26–92; doi:10.1007/BFb0090889.

GM88 M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergeb. Math. Grenzgeb. (3), vol. 14 (Springer-Verlag, Berlin, 1988); doi:10.1007/978-3-642-71714-7.

GGK19 D. Greb, H. Guenancia and S. Kebekus, *Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups*, Geom. Topol. **23** (2019), no. 4, 2051–2124; doi:10.2140/gt.2019.23.2051.

GKKP11 D. Greb, S. Kebekus, S. J. Kovács and T. Peternell, *Differential forms on log canonical spaces*, Publ. Math. Inst. Hautes Études Sci. **114** (2011), 87–169; doi:10.1007/s10240-011-0036-0.

GKP16 D. Greb, S. Kebekus and T. Peternell, *Singular spaces with trivial canonical class*, Minimal Models and Extremal Rays (Kyoto, 2011), Adv. Stud. Pure Math., vol. 70 (Math. Soc. Japan, Tokyo, 2016), 67–113; doi:10.2969/aspm/07010067.

GD66 A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966), 5–255; doi:10.1007/BF02684343.

HP19 A. Höring and T. Peternell, *Algebraic integrability of foliations with numerically trivial canonical bundle*, Invent. Math. **216** (2019), no. 2, 395–419; doi:10.1007/s00222-018-00853-2.

Huy06 D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Math. Monogr., vol. 2006 (The Clarendon Press, Oxford Univ. Press, Oxford, 2006); doi:10.1093/acprof:oso/9780199296866.001.0001.

HL97 D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects Math., vol. E31 (Friedr. Vieweg & Sohn, Braunschweig, 1997).

KLS06 D. Kaledin, M. Lehn and C. Sorger, *Singular symplectic moduli spaces*, Invent. Math. **164** (2006), no. 3, 591–614; doi:10.1007/s00222-005-0484-6.

KM18 S. Kapfer and G. Menet, *Integral cohomology of the generalized Kummer fourfold*, Algebr. Geom. **5** (2018), no. 5, 523–567; doi:10.14231/ag-2018-014.

Kaw85 Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. reine angew. Math. **363** (1985), 1–46; doi:10.1515/crll.1985.363.1.

Kol95 J. Kollár, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, vol. 7 (Princeton Univ. Press, Princeton, NJ, 1995); doi:10.1515/9781400864195.

KM98 J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., vol. 134 (Cambridge Univ. Press, Cambridge, 1998); doi:10.1017/CBO9780511662560.

Lan75 S. G. Langton, *Valuative criteria for families of vector bundles on algebraic varieties*, Ann. of Math. (2) **101** (1975), 88–110; doi:10.2307/1970987.

LS06 M. Lehn and C. Sorger, *La singularité de O'Grady*, J. Algebraic Geom. **15** (2006), no. 4, 753–770; doi:10.1090/S1056-3911-06-00437-1.

LeP93 J. Le Potier, *Systèmes cohérents et structures de niveau*, Astérisque **214** (1993); http://numdam.org/item/AST_1993__214__1_0/.

Men15 G. Menet, *Beauville–Bogomolov lattice for a singular symplectic variety of dimension* 4, J. Pure Appl. Algebra **219** (2015), no. 5, 1455–1495; doi:10.1016/j.jpaa.2014.06.012.

Men18 ———, *On the integral cohomology of quotients of manifolds by cyclic groups*, J. Math. Pures Appl. (9) **119** (2018), 280–325; doi:10.1016/j.matpur.2017.11.008.

Men22 ———, *Integral cohomology of quotients via toric geometry*, Épijournal Géom. Algébrique **6** (2022), Art. 3; doi:10.46298/epiga.2022.volume6.5762.

MRS18 G. Mongardi, A. Rapagnetta and G. Saccà, *The Hodge diamond of O'Grady's six-dimensional example*, Compos. Math. **154** (2018), no. 5, 984–1013; doi:10.1112/S0010437X1700803X.

Muk81 S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175; doi:10.1017/S002776300001922X.

Muk84 ———, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math. **77** (1984), no. 1, 101–116; doi:10.1007/BF01389137.

Muk87 ———, *On the moduli space of bundles on $K3$ surfaces. I*, Vector Bundles on Algebraic Varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., vol. 11 (Tata Inst. Fund. Res., Bombay, 1987), 341–413.

Nam01 Y. Namikawa, *A note on symplectic singularities*, 2001, arXiv:math.AG/0101028.

Nam06 ———, *On deformations of $\mathbb{Q}$-factorial symplectic varieties*, J. reine angew. Math. **599** (2006), 97–110; doi:10.1515/CRELLE.2006.079.

O'G97 K. G. O'Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface*, J. Algebraic Geom. **6** (1997), no. 4, 599–644.

O'G99 ———, *Desingularized moduli spaces of sheaves on a $K3$*, J. reine angew. Math. **512** (1999), 49–117; doi:10.1515/crll.1999.056.

O'G03 ———, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), no. 3, 435–505; doi:10.1090/S1056-3911-03-00323-0.

Ogui00 K. Oguiso, *Picard numbers in a family of hyperkähler manifolds – a supplement to the article of R. Borcherds, L. Katzarkov, T. Pantev, N. I. Shepherd-Barron*, 2000, arXiv:math.AG/0011258.

Per20 A. Perego, *Examples of irreducible symplectic varieties*, in *Birational Geometry and Moduli Spaces*, Springer INdAM Ser., vol. 39 (Springer, Cham, 2020), 151–172; doi:10.1007/978-3-030-37114-2_9.

PR13 A. Perego and A. Rapagnetta, *Deformation of the O'Grady moduli spaces*, J. reine angew. Math. **678** (2013), 1–34; doi:10.1515/CRELLE.2011.191.

PR14 ———, *Factoriality properties of moduli spaces of sheaves on abelian and K3 surfaces*, Int. Math. Res. Not. IMRN **2014** (2014), no. 3, 643–680; doi:10.1093/imrn/rns233.

PR20 ———, *The second integral cohomology of moduli spaces of sheaves on Abelian and K3 surfaces*, 2020, arXiv:2012.10649.

Qin93 Z. Qin, *Equivalence classes of polarizations and moduli spaces of sheaves*, J. Differential Geom. **37** (1993), no. 2, 397–415; doi:10.4310/jdg/1214453682.

Rei83 M. Reid, *Projective morphisms according to Kawamata*, 1983, Preprint, University of Warwick; available at http://homepages.warwick.ac.uk/~masda/3folds/Ka.pdf.

Tak03 S. Takayama, *Local simple connectedness of resolutions of log-terminal singularities*, Internat. J. Math. **14** (2003), no. 8, 825–836; doi:10.1142/S0129167X0300196X.

Voi03 C. Voisin, *Hodge theory and complex algebraic geometry. II*, Cambridge Stud. Adv. Math., vol. 77 (Cambridge Univ. Press, Cambridge, 2003); doi:10.1017/CBO9780511615177.

Yos99a K. Yoshioka, *Irreducibility of moduli spaces of vector bundles on K3 surfaces*, 1999, arXiv:math.AG/9907001.

Yos99b ———, *Some notes on the moduli of stable sheaves on elliptic surfaces*, Nagoya Math. J. **154** (1999), 73–102; doi:10.1017/S0027763000025319.

Yos01 ———, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321** (2001), no. 4, 817–884; doi:10.1007/s002080100255.

Yos02 ———, *A note on Fourier–Mukai transform*, 2002, arXiv:math.AG/0112267.

Yos03 ———, *Twisted stability and Fourier–Mukai transform. I*, Compos. Math. **138** (2003), no. 3, 261–288; doi:10.1023/A:1027304215606.

Yos09 ———, *Stability and Fourier–Mukai transform. II*, Compos. Math. **145** (2009), no. 1, 112–142; doi:10.1112/S0010437X08003758.

Arvid Perego perego@dima.unige.it
Dipartimento di Matematica dell'Università di Genova, 16146 Genova, Italy

Antonio Rapagnetta rapagnet@apx.mat.uniroma2.it
Dipartimento di Matematica dell'Università di Roma II - Tor Vergata, 00133 Roma, Italy