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Normal Proofs and Tableaux for the Font-Rius Tetravalent Modal Logic

Abstract. Tetravalent modal logic ($\mathcal{TML}$) was introduced by Font and Rius in 2000. It is an expansion of the Belnap-Dunn four-valued logic $\mathcal{FOUR}$, a logical system that is well-known for the many applications found in several fields. Besides, $\mathcal{TML}$ is the logic that preserves degrees of truth with respect to Monteiro's tetravalent modal algebras. Among other things, Font and Rius showed that $\mathcal{TML}$ has a strongly adequate sequent system, but unfortunately this system does not enjoy the cut-elimination property. However, in a previous work we presented a sequent system for $\mathcal{TML}$ with the cut-elimination property. Besides, in this same work, it was also presented a sound and complete natural deduction system for this logic.

In the present article we continue with the study of $\mathcal{TML}$ under a proof-theoretic perspective. In the first place, we show that the natural deduction system that we introduced before admits a normalization theorem. In the second place, taking advantage of the contrapositive implication for the tetravalent modal algebras introduced by A. V. Figallo and P. Landini, we define a decidable tableau system adequate to check validity in the logic $\mathcal{TML}$. Finally, we provide a sound and complete tableau system for $\mathcal{TML}$ in the original language. These two tableau systems constitute new (proof-theoretic) decision procedures for checking validity in the variety of tetravalent modal algebras.

Keywords: tetravalent modal logic; natural deduction; tableaux; normal proofs; paraconsistent logics; paracomplete logics; Belnap-Dunn logic

1. Introduction

The class **TMA** of tetravalent modal algebras was first considered by Antonio Monteiro (1978), and mainly studied by Loureiro, Figallo, Ziliani

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and Landini. From Monteiro's point of view, in the future these algebras would give rise to a four-valued modal logic with significant applications in Computer Science [see 21]. Later on, Font and Rius were indeed interested in the logics arising from the algebraic and lattice-theoretical aspects of these algebras.

Although such applications have not yet been developed, the two logics considered in [21] are modal expansions of Belnap-Dunn's four-valued logic $\mathcal{FOUR}$, a paraconsistent and paracomplete logical system that is well known for the many applications it has found in several fields. In these logics, the four non-classical epistemic values emerge: $\mathbf{1}$ (true and not false), $\mathbf{0}$ (false and not true), $\mathbf{n}$ (neither true nor false) and $\mathbf{b}$ (both true and false). We may think of them as the four possible ways in which an atomic sentence P can belong to the *present state of information*: either it was told that (1) P is true (and it was not told that P is false); (2) P is false (and it was not told that P is true); (3) P is both true and false (perhaps from different sources, or in different instants of time); or (4) nothing was told about the truth value of P. In this interpretation, it makes sense to consider a modal-like unary operator $\Box$ of epistemic character, such that for any sentence P, the sentence $\Box P$ would mean "the available information confirms that P is true".[1] It is clear that in this setting the sentence $\Box P$ can only be true in the case where we have some information saying that P is true and we have no information saying that P is false, while it is simply false in all other cases (i.e., lack of information or at least some information saying that P is false, disregarding whether at the same time some other information says that P is true); that is, on the set $\{\mathbf{0}, \mathbf{n}, \mathbf{b}, \mathbf{1}\}$ of epistemic values this operator must be defined as $\Box\mathbf{1} = \mathbf{1}$ and $\Box\mathbf{n} = \Box\mathbf{b} = \Box\mathbf{0} = \mathbf{0}$. This is exactly the algebra that generates the variety of tetravalent modal algebras (TMAs).

In [21], Font and Rius studied two logics related to TMAs. One of them is obtained by following the usual "preserving truth" scheme,

[1] This is closely related to the so-called *logics of evidence and truth* (**LET**s), introduced by Carnielli and Rodrigues in [13]. That logics are paraconsistent and paracomplete expansions of $\mathcal{FOUR}$ by the addition of a *classicality* operator $\circ$ which locally recovers the classical properties of negation for the given negation $\neg$. Indeed, it can be proven that $\mathcal{TML}$, the logic preserving degrees of truth asociated to TMAs to be studied in the present paper, is a **LET** in which $\circ\varphi =_{def} \Box(\varphi \vee \neg\varphi)$ (meaning that the information conveyed by φ is reliable). Thus, $\Box\varphi$ is equivalent to $\circ\varphi \wedge \varphi$, the claim that "$\varphi$ is reliably true" under the perspective of **LET**s.

taking $\{\mathbf{1}\}$ as designated set, that is, ψ follows from $\psi_1, \ldots, \psi_n$ in this logic when every interpretation that sends all the ψ_i to $\mathbf{1}$ also sends ψ to 1. The other logic, denoted by $\mathcal{TML}$ (the logic we are interested in), is defined by using the *preserving degrees of truth* scheme, that is, ψ follows from $\psi_1, \ldots, \psi_n$ when every interpretation that assigns to ψ a value that is greater or equal than the value it assigns to the conjunction of the ψ_i's. These authors proved that $\mathcal{TML}$ is not algebraizable in the sense of Blok and Pigozzi, but it is *finitely equivalential* and *protoalgebraic*. However, they confirm that its algebraic counterpart is also the class of TMAs: but the connection between the logic and the algebras is not so good as in the first logic. As a compensation, this logic has a better proof-theoretic behavior, since it has a *strongly adequate Gentzen calculus* [see 21, Theorems 3.6 and 3.19].

In [21], it was proved that $\mathcal{TML}$ can be characterized as a matrix logic in terms of two logical matrices, but later, in [21], it was proved that $\mathcal{TML}$ can be determined by a single logical matrix (see Proposition 2.3 below). Besides, taking profit of the contrapositive implication introduced by Figallo and Landini [18], a sound and complete Hilbert-style calculus for this logic was presented. Finally, the paraconsistent character of $\mathcal{TML}$ was also studied in [14] from the point of view of the *Logics of Formal Inconsistency* (**LFI**s), introduced by Carnielli and Marcos in [12] and afterward developed in [11].[2]

In this paper we continue with the study of $\mathcal{TML}$ from a syntactic point of view. For the reader's convenience, in Section 2 we recall the definition of logic $\mathcal{TML}$ as well as its sequent calculus introduced in [21]. In Section 3 we recall the natural deduction system presented in [17] with introduction and elimination rules for every connective, and we show that every deduction $\mathcal{D}$ in this system reduces to a normal form. In Section 4 we present $\mathcal{TML}$ in a different signature containing the contrapositive implication defined in [18]. In Section 5, by adapting the general technique introduced in [7], a decidable tableau system is defined, in the language $\{\neg, \succ\}$, proving the corresponding soundness and completeness theorems. Finally, in Section 6 we provide a new tableau system for $\mathcal{TML}$ in the original language.

[2] Moreover, as mentioned above, $\mathcal{TML}$ is both paraconsistent and paracomplete, being in fact a **LET**.

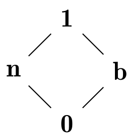
2. The logic $\mathcal{TML}$

Recall that a *De Morgan algebra* is a structure $\langle A, \wedge, \vee, \neg, \mathbf{0}\rangle$ such that $\langle A, \wedge, \vee, \mathbf{0}\rangle$ is a bounded distributive lattice and $\neg$ is a De Morgan negation, i.e., an involution that additionally satisfies De Morgan's laws: for every $a, b \in A$ $\neg\neg a = a$, and $\neg(a \vee b) = \neg a \wedge \neg b$.

A *tetravalent modal algebra* (TMA) is an algebra $\mathbb{A} = \langle A, \wedge, \vee, \neg, \Box, \mathbf{0}\rangle$ of type $(2, 2, 1, 1, 0)$ such that its non-modal reduct $\langle A, \wedge, \vee, \neg, \mathbf{0}\rangle$ is a De Morgan algebra and the unary operation $\Box$ satisfies the following:

$$\Box a \wedge \neg a \approx \mathbf{0},$$
$$\neg\Box a \wedge a \approx \neg a \wedge a.$$

Every TMA $\mathbb{A}$ has a top element $\mathbf{1}$ which is defined as $\neg\mathbf{0}$. These algebras were studied mainly by Loureiro [25, 26], and also by Figallo and Landini [18] and Ziliani [19], at the suggestion of the late Monteiro [see 21]. The class of all tetravalent modal algebras constitute a variety which is denoted by **TMA**. Let $M_4 = \{\mathbf{0}, \mathbf{n}, \mathbf{b}, \mathbf{1}\}$ and consider the lattice given by the following Hasse diagram:



This is a well-known lattice and it is called **L4** [see 1, p. 516]. Then, **TMA** is generated by the above four-element lattice enriched with two unary operators $\neg$ and $\Box$ given by $\neg\mathbf{n} = \mathbf{n}$, $\neg\mathbf{b} = \mathbf{b}$, $\neg\mathbf{0} = \mathbf{1}$ and $\neg\mathbf{1} = \mathbf{0}$ and the unary operator $\Box$ is defined as: $\Box\mathbf{n} = \Box\mathbf{b} = \Box\mathbf{0} = \mathbf{0}$ and $\Box\mathbf{1} = \mathbf{1}$ [see 21]. This tetravalent modal algebra, denoted by $\mathfrak{M}_{4\mathrm{m}}$, has two prime filters, namely, $F_{\mathbf{n}} = \{\mathbf{n}, \mathbf{1}\}$ and $F_{\mathbf{b}} = \{\mathbf{b}, \mathbf{1}\}$. As we said, $\mathfrak{M}_{4\mathrm{m}}$ generates the variety **TMA**, i.e., an equation holds in every TMA iff it holds in $\mathfrak{M}_{4\mathrm{m}}$.

LEMMA 2.1 ([21]). *In every TMA $\mathbb{A}$ and for all $a, b \in A$ the following identities hold*:

(i) $\neg\Box a \vee a \approx \mathbf{1}$,
(ii) $\Box\Box a \approx \Box a$,
(iii) $\Box a \vee \neg a \approx a \vee \neg a$,
(iv) $\Box(a \wedge b) \approx \Box a \wedge \Box b$,
(v) $\Box a \vee \neg\Box a \approx \mathbf{1}$,
(vi) $\Box(a \vee \Box b) \approx \Box a \vee \Box b$,
(vii) $\Box a \wedge \neg\Box a \approx \mathbf{0}$,
(viii) $\Box\neg\Box a \approx \neg\Box a$
(ix) $\Box a \wedge a \approx a$,
(x) $a \wedge \Box\neg a \approx \mathbf{0}$,

(xi) $\Box\mathbf{1} \approx \mathbf{1}$, (xiii) $\Box\mathbf{0} \approx \mathbf{0}$,
(xii) $\Box(\Box a \wedge \Box b) \approx \Box a \wedge \Box b$, (xiv) $\Box(\Box a \vee \Box b) \approx \Box a \vee \Box b$.

Let $\mathscr{L} = \{\vee, \wedge, \neg, \Box\}$ be a propositional language. From now on, we shall denote by $\mathfrak{Fm}$ the absolutely free algebra of type (2,2,1,1,0) generated by some denumerable set of variables, i.e., $\mathfrak{Fm} = \langle Fm, \wedge, \vee, \neg, \Box, \bot\rangle$. As usual, we refer to formulas by lowercase Greek letters α, β, γ, ... and so on; and to finite sets of formulas by uppercase Greek letters Γ, Δ, etc. Then

Definition 2.1 ([21, 14]). The tetravalent modal logic $\mathcal{TML}$ defined over $\mathfrak{Fm}$ is the propositional logic $\langle Fm, \models_{\mathcal{TML}}\rangle$ given as follows: for every finite set $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{\mathcal{TML}} \alpha$ if and only if, for every $\mathbb{A} \in \mathbf{TMA}$ and for every $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathbb{A})$, $\bigwedge\{h(\gamma) : \gamma \in \Gamma\} \leq h(\alpha)$. In particular, $\emptyset \models_{\mathcal{TML}} \alpha$ if and only if $h(\alpha) = \mathbf{1}$ for every $\mathbb{A} \in \mathbf{TMA}$ and for every $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathbb{A})$.

Remark 2.1. Observe that, if $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathbb{A})$ for any $\mathbb{A} \in \mathbf{TMA}$, we have that $h(\bot) = \mathbf{0}$. This follows from the fact that $\bot$ is the 0-ary operation in $\mathfrak{Fm}$, $\mathbf{0}$ is the 0-ary operation in $\mathbb{A}$ and the definition of homomorphism (in the sense of universal algebra).

It is worth mentioning that there are a number of works on modal logics which either share the non-modal fragment with $\mathcal{TML}$ or have non-modal fragments which are characterized by the same four-element matrix. Clearly, these logics have some relation to $\mathcal{TML}$. Some examples of such systems are Priest's KFDE [30], Belnapian modal logics of Odintsov and Wansing [28, 29] and modal bilattice logic [31]. The following result was proved in [14] and will be useful in the sequel.

Lemma 2.2. *Let $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{M}_{4\mathrm{m}})$, $\mathcal{V}' \subseteq Var$ and $h' \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{M}_{4\mathrm{m}})$ such that, for all $p \in \mathcal{V}'$,*

$$h'(p) = \begin{cases} h(p) & \text{if } h(p) \in \{\mathbf{0}, \mathbf{1}\}, \\ \mathbf{n} & \text{if } h(p) = \mathbf{b}, \\ \mathbf{b} & \text{if } h(p) = \mathbf{n} \end{cases}$$

Then for all $\alpha \in \mathfrak{Fm}$ whose variables are in $\mathcal{V}'$,

$$h'(\alpha) = \begin{cases} h(\alpha) & \text{if } h(\alpha) \in \{\mathbf{0}, \mathbf{1}\}, \\ \mathbf{n} & \text{if } h(\alpha) = \mathbf{b}, \\ \mathbf{b} & \text{if } h(\alpha) = \mathbf{n}. \end{cases}$$

Font and Rius proved in [21] that the tetravalent modal logic $\mathcal{TML}$ is a matrix logic. In fact, $\mathcal{TML}$ can be determined by the matrix $\mathcal{M}_{\mathbf{n}} = \langle \mathfrak{M}_{4\mathrm{m}}, \{\mathbf{n}, \mathbf{1}\}\rangle$ and simultaneously, it can be determined by $\mathcal{M}_{\mathbf{b}} = \langle \mathfrak{M}_{4\mathrm{m}}, \{\mathbf{b}, \mathbf{1}\}\rangle$ (both matrices are isomorphic).

Proposition 2.3 ([21]). *$\mathcal{TML}$ is the logic determined by the matrix $\mathcal{M}_{\mathbf{b}} = \langle \mathfrak{M}_{4\mathrm{m}}, \{\mathbf{b}, \mathbf{1}\}\rangle$.*

In order to characterize $\mathcal{TML}$ syntactically, that is, by means of a syntactical deductive system, it was introduced in [21] the Gentzen-style system $\mathfrak{G}$. The sequent calculus $\mathfrak{G}$ is single-conclusion, that is, it deals with sequents of the form $\Delta \Rightarrow \alpha$ such that $\Delta \cup \{\alpha\}$ is a finite subset of Fm. The axioms and rules of $\mathfrak{G}$ are the following:

Axioms

(Structural axiom) $\alpha \Rightarrow \alpha$ (Modal axiom) $\Rightarrow \alpha \vee \neg\Box\alpha$

Structural rules

(Weakening) $\dfrac{\Delta \Rightarrow \alpha}{\Delta, \beta \Rightarrow \alpha}$ (Cut) $\dfrac{\Delta \Rightarrow \alpha \quad \Delta, \alpha \Rightarrow \beta}{\Delta \Rightarrow \beta}$

Logic rules

$(\wedge \Rightarrow)\ \dfrac{\Delta, \alpha, \beta \Rightarrow \gamma}{\Delta, \alpha \wedge \beta \Rightarrow \gamma}$ $(\Rightarrow\wedge)\ \dfrac{\Delta \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Delta \Rightarrow \alpha \wedge \beta}$

$(\vee\Rightarrow)\ \dfrac{\Delta, \alpha \Rightarrow \gamma \quad \Delta, \beta \Rightarrow \gamma}{\Delta, \alpha \vee \beta \Rightarrow \gamma}$

$(\Rightarrow\vee)\ \dfrac{\Delta \Rightarrow \alpha}{\Delta \Rightarrow \alpha \vee \beta} \quad \dfrac{\Delta \Rightarrow \beta}{\Delta \Rightarrow \alpha \vee \beta}$

$(\neg)\ \dfrac{\alpha \Rightarrow \beta}{\neg\beta \Rightarrow \neg\alpha}$ $(\bot)\ \dfrac{\Delta \Rightarrow \bot}{\Delta \Rightarrow \alpha}$

$(\neg\neg\Rightarrow)\ \dfrac{\Delta, \alpha \Rightarrow \beta}{\Delta, \neg\neg\alpha \Rightarrow \beta}$ $(\Rightarrow\neg\neg)\ \dfrac{\Delta \Rightarrow \alpha}{\Delta \Rightarrow \neg\neg\alpha}$

$(\Box\Rightarrow)\ \dfrac{\Delta, \alpha, \neg\alpha \Rightarrow \beta}{\Delta, \alpha, \neg\Box\alpha \Rightarrow \beta}$ $(\Rightarrow\Box)\ \dfrac{\Delta \Rightarrow \alpha \wedge \neg\alpha}{\Delta \Rightarrow \alpha \wedge \neg\Box\alpha}$

The notion of derivation in the sequent calculus $\mathfrak{G}$ is the usual. Besides, for every finite set $\Gamma \cup \{\varphi\} \subseteq Fm$, we write $\Gamma \vdash_{\mathfrak{G}} \varphi$ iff the sequent $\Gamma \Rightarrow \varphi$ has a derivation in $\mathfrak{G}$. We say that the sequent $\Gamma \Rightarrow \varphi$ is provable iff there exists a derivation for it in $\mathfrak{G}$.

In [21], it was proved that $\mathfrak{G}$ is sound and complete with respect to the tetravalent modal logic $\mathcal{TML}$, constituting therefore a proof-theoretic counterpart of it.

Theorem 2.4 (Soundness and Completeness, [21]). *For every finite set* $\Gamma \cup \{\alpha\} \subseteq Fm$,

$$\Gamma \models_{\mathcal{TML}} \alpha \quad \textit{if and only if} \quad \Gamma \vdash_{\mathfrak{G}} \alpha.$$

Moreover,

Proposition 2.5 ([21]). *An arbitrary equation $\psi \approx \varphi$ holds in every TMA iff $\psi \dashv\vdash_{\mathfrak{G}} \varphi$ (that is, $\psi \vdash_{\mathfrak{G}} \varphi$ and $\varphi \vdash_{\mathfrak{G}} \psi$).*

As a consequence of it we have that:

Corollary 2.6 ([21]). (i) *$\psi \approx \mathbf{1}$ holds in every TMA iff $\vdash_{\mathfrak{G}} \psi$.*
(ii) *For all $\psi, \varphi \in Fm$, $\mathfrak{A} \in \mathbf{TMA}$, $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$,*
$\psi \vdash_{\mathfrak{G}} \varphi$ iff $h(\psi) \leq h(\varphi)$.

Corollary 2.6 is a powerful tool for determining whether a given sequent of $\mathfrak{G}$ is provable or not. For instance,

Proposition 2.7 ([17]). *In $\mathfrak{G}$ we have that the sequent $\neg\Box\alpha \Rightarrow \alpha$ is provable iff the sequent $\Rightarrow \alpha$ is provable.*

Recall that a rule of inference is *admissible* in a formal system if the set of theorems of the system is closed under the rule; and a rule is said to be *derivable* in the same formal system if its conclusion can be derived from its premises using the other rules of the system.

A well-known rule for those readers familiar with modal logic is the *Rule of Necessitation*, which states that if φ is a theorem, so is $\Box\varphi$. Formally,

$$(\mathrm{Nec})\ \frac{\Rightarrow \varphi}{\Rightarrow \Box\varphi}$$

Then, we have that:

Lemma 2.8 ([17]). *The Rule of Necessitation is admissible in $\mathfrak{G}$.*

Proposition 2.9 ([17]). *Every proof of $\Rightarrow \Box(\alpha \vee \neg\Box\alpha)$ in $\mathfrak{G}$ uses the cut rule.*

Moreover, we have that, for every $\varphi \in Fm$ such that $\Rightarrow \varphi$ is provable in $\mathfrak{G}$ then $\Rightarrow \Box\varphi$ is provable in $\mathfrak{G}$; and every proof of $\Rightarrow \Box\varphi$ in $\mathfrak{G}$ makes use of the cut rule [see 17]. Consequently,

Theorem 2.10 ([17]). *$\mathfrak{G}$ does not admit cut-elimination.*

In [14], taking advantage of the contrapositive implication $\succ$ introduced by Figallo and Landini in [18], we introduced a sound and complete Hilbert-style calculus for $\mathcal{TML}$. Later, using a general method proposed by Avron, Ben-Naim and Konikowska [3], it was provided an ordinary two-sided multiple-conclusioned sequent calculus, $\mathbf{SC}_{\mathcal{TML}}$, for $\mathcal{TML}$ with the cut-elimination property; and, inspired by the latter, it was presented a *natural deduction* system, sound and complete with respect to $\mathcal{TML}$ [17]. The axioms and rules of $\mathbf{SC}_{\mathcal{TML}}$ are the following:

Axioms

$$\alpha \Rightarrow \alpha$$

Structural rules

$$(\mathrm{w}\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta} \qquad (\Rightarrow\mathrm{w})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$$

$$(\mathrm{cut})\ \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Logic rules

$$(\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \qquad (\vee \Rightarrow)\ \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta}$$

$$(\neg\vee \Rightarrow)\ \frac{\Gamma, \neg\alpha, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta} \qquad (\Rightarrow \neg\vee)\ \frac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)}$$

$$(\wedge \Rightarrow)\ \frac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$(\neg\wedge \Rightarrow)\ \frac{\Gamma, \neg\alpha \Rightarrow \Delta \quad \Gamma, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \wedge \beta) \Rightarrow \Delta} \qquad (\Rightarrow \neg\wedge)\ \frac{\Gamma \Rightarrow \Delta, \neg\alpha, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \wedge \beta)}$$

$$(\neg\neg \Rightarrow)\ \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg\neg\alpha \Rightarrow \Delta} \qquad (\Rightarrow \neg\neg)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\neg\alpha}$$

$$(\Box \Rightarrow)_1\ \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \Box\alpha \Rightarrow \Delta} \qquad (\Box \Rightarrow)_1\ \frac{\Gamma \Rightarrow \Delta, \neg\alpha}{\Gamma, \Box\alpha \Rightarrow \Delta}$$

$$(\Rightarrow \Box)\ \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma, \neg\alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Box\alpha} \qquad (\neg\Box \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma, \neg\alpha \Rightarrow \Delta}{\Gamma, \neg\Box\alpha \Rightarrow \Delta}$$

$$(\Rightarrow \neg\Box)_1\ \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\Box\alpha} \qquad (\Rightarrow \neg\Box)_2\ \frac{\Gamma \Rightarrow \Delta, \neg\alpha}{\Gamma \Rightarrow \Delta, \neg\Box\alpha}$$

3. Normal proofs for $\mathcal{TML}$

From now on, it will be assumed that the reader is acquainted with the basic definitions related to natural deduction systems, such as minor and major premises of a rule, as well as the notion of normal proof. A good reference is the book [33].

In this section, we shall present a natural deduction system for $\mathcal{TML}$. We take our inspiration from the construction made earlier. In particular, it threw some light on how the connective $\Box$ behaves. The proof system $\mathbf{ND}_{\mathcal{TML}}$ will be defined following the notational conventions given in [33].

Definition 3.1. Deductions in $\mathbf{ND}_{\mathcal{TML}}$ are inductively defined as follows:

Basis: The proof tree with a single occurrence of an assumption ϕ with a marker is a deduction with conclusion ϕ from open assumption ϕ.

Inductive step: Let $\mathcal{D}$, $\mathcal{D}_1$,$\mathcal{D}_2$, $\mathcal{D}_3$ be derivations. Then, they can be extended by one of the following rules below. The classes $[\neg\phi]^u$, $[\neg\psi]^v$, $[\phi]^u$, $[\psi]^v$ below contain open assumptions of the deductions of the premises of the final inference, but are closed in the whole deduction.

$$\frac{}{\phi \vee \neg\Box\phi}\,\text{MA}$$

$$\frac{\begin{matrix}\mathcal{D}_1 \\ \phi\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \psi\end{matrix}}{\phi \wedge \psi}\,\wedge\text{I} \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \wedge \psi\end{matrix}}{\phi}\,\wedge\text{E}_1 \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \wedge \psi\end{matrix}}{\psi}\,\wedge\text{E}_2$$

$$\frac{\begin{matrix}\mathcal{D} \\ \neg\phi\end{matrix}}{\neg(\phi \wedge \psi)}\,\neg\wedge\text{I}_1 \qquad \frac{\begin{matrix}\mathcal{D} \\ \neg\psi\end{matrix}}{\neg(\phi \wedge \psi)}\,\neg\wedge\text{I}_2 \qquad \frac{\begin{matrix}\mathcal{D}_1 \\ \neg(\phi \wedge \psi)\end{matrix} \qquad \begin{matrix}[\neg\phi]^u \\ \mathcal{D}_2 \\ \gamma\end{matrix} \qquad \begin{matrix}[\neg\psi]^v \\ \mathcal{D}_3 \\ \gamma\end{matrix}}{\gamma}\,\neg\wedge\text{E},u,v$$

$$\frac{\begin{matrix}\mathcal{D} \\ \phi\end{matrix}}{\phi \vee \psi}\,\vee\text{I}_1 \qquad \frac{\begin{matrix}\mathcal{D} \\ \psi\end{matrix}}{\phi \vee \psi}\,\vee\text{I}_2 \qquad \frac{\begin{matrix}\mathcal{D}_1 \\ \phi \vee \psi\end{matrix} \qquad \begin{matrix}[\phi]^u \\ \mathcal{D}_2 \\ \gamma\end{matrix} \qquad \begin{matrix}[\psi]^v \\ \mathcal{D}_3 \\ \gamma\end{matrix}}{\gamma}\,\vee\text{E},u,v$$

$$\frac{\begin{matrix}\mathcal{D}_1 \\ \neg\phi\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \neg\psi\end{matrix}}{\neg(\phi \vee \psi)}\,\neg\vee\text{I} \qquad \frac{\begin{matrix}\mathcal{D} \\ \neg(\phi \vee \psi)\end{matrix}}{\neg\phi}\,\neg\vee\text{E}_1 \qquad \frac{\begin{matrix}\mathcal{D} \\ \neg(\phi \vee \psi)\end{matrix}}{\neg\psi}\,\neg\vee\text{E}_2$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \phi\end{matrix}}{\neg\neg\phi}\ \neg\neg\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg\neg\phi\end{matrix}}{\phi}\ \neg\neg\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \phi\end{matrix}\qquad \begin{matrix}[\neg\phi]^u\\ \mathcal{D}_2\\ \bot\end{matrix}}{\Box\phi}\ \Box\mathrm{I},u \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}\\ \Box\phi\end{matrix}}{\phi}\ \Box\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \neg\phi\end{matrix}}{\neg\Box\phi}\ \neg\Box\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \neg\Box\phi\end{matrix}\qquad \begin{matrix}\mathcal{D}_2\\ \phi\end{matrix}}{\neg\phi}\ \neg\Box\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \neg\phi\wedge\Box\phi\end{matrix}}{\bot}\ \bot\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{\phi}\ \bot\mathrm{E}$$

Remark 3.1. (i) Actually, in [17], the introduction rule for $\Box$ is

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \psi\vee\phi\end{matrix}\qquad \begin{matrix}[\neg\phi]^u\\ \mathcal{D}_2\\ \psi\end{matrix}}{\psi\vee\Box\phi}\ \Box\mathrm{I}^*,u$$

If we take ψ as $\bot$ in $\Box\mathrm{I}^*$ we get $\Box\mathrm{I}$ as in Definition 3.1.

(ii) The intuition behind rule $\Box$I is the following: "if we have a deduction for ϕ and $\neg\phi$ is not provable, then we have a deduction for $\Box\phi$".

As usual, by application of the rule $\neg\wedge$E ($\vee$E) a new proof-tree is formed from $\mathcal{D}$, $\mathcal{D}_1$, and $\mathcal{D}_2$ by adding at the bottom the conclusion γ while closing the sets $[\neg\phi]^u$ and $[\neg\psi]^u$ of open assumptions marked by u and v, respectively. By application of the rule $\Box$I a new proof-tree is formed from $\mathcal{D}_1$ and $\mathcal{D}_2$ by adding at the bottom the conclusion $\Box\phi$ while closing the set $[\neg\phi]^u$ of open assumptions marked by u. Note that we have introduced the symbol $\bot$: it behaves here as an arbitrary unprovable propositional constant whose negation is provable.

Let $\Gamma\cup\{\alpha\}\subseteq Fm$. We say that the conclusion α is derivable from a set Γ of premises, written $\Gamma\vdash\alpha$, if and only if there is a deduction in $\mathbf{ND}_{\mathcal{TML}}$ of α from Γ. Direct consequences of the I-rules and E-rules for $\wedge$, $\vee$ and $\neg$ we obtain:

LEMMA 3.1. *For all* $\alpha, \beta \in Fm$:

(i) $(\alpha \vee \beta) \wedge \gamma \dashv\vdash (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$,
(ii) $(\alpha \wedge \beta) \vee \gamma \dashv\vdash (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$,
(iii) $\neg(\alpha \vee \beta) \dashv\vdash \neg\alpha \wedge \neg\beta$
(iv) $\neg(\alpha \wedge \beta) \dashv\vdash \neg\alpha \vee \neg\beta$,
(v) $\neg\neg\alpha \dashv\vdash \alpha$.

LEMMA 3.2. *For all* $\alpha, \beta \in Fm$:

(i) $\vdash \Box(\alpha \vee \neg\Box\alpha)$,
(ii) $\neg\Box\alpha \wedge \alpha \dashv\vdash \neg\alpha \wedge \alpha$,
(iii) $\vdash \Box\alpha \vee \neg\Box\alpha$,
(iv) $\Box\alpha \wedge \neg\Box\alpha \dashv\vdash \bot$,
(v) $\Box(\Box\alpha \wedge \Box\beta) \dashv\vdash \Box\alpha \wedge \Box\beta$,
(vi) $\Box(\Box\alpha \vee \Box\beta) \dashv\vdash \Box\alpha \vee \Box\beta$,
(vii) $\Box\alpha \vee \neg\alpha \dashv\vdash \alpha \vee \neg\alpha$,
(viii) $\Box\Box\alpha \dashv\vdash \Box\alpha$,
(ix) $\Box(\alpha \wedge \beta) \dashv\vdash \Box\alpha \wedge \Box\beta$,
(x) $\Box(\alpha \vee \Box\beta) \dashv\vdash \Box\alpha \vee \Box\beta$,
(xi) $\Box\neg\Box\alpha \dashv\vdash \neg\Box\alpha$
(xii) $\alpha \wedge \Box\neg\alpha \dashv\vdash \bot$.

PROOF. We shall only prove (i), (ii), (ix) and (xi).

(i)

$$\dfrac{\dfrac{}{\alpha \vee \neg\Box\alpha}\,\mathrm{MA} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg(\alpha \vee \neg\Box\alpha)^u}{\neg\alpha \wedge \neg\neg\Box\alpha}}{\neg\alpha}\,\wedge\mathrm{E}_1 \qquad \dfrac{\dfrac{\dfrac{\neg(\alpha \vee \neg\Box\alpha)^u}{\neg\alpha \wedge \neg\neg\Box\alpha}}{\neg\neg\Box\alpha}\,\wedge\mathrm{E}_2}{\Box\alpha}\,\neg\neg\mathrm{E}}{\neg\alpha \wedge \Box\alpha}\,\wedge\mathrm{I}}{\bot}\,(\bot)}{\Box(\alpha \vee \neg\Box\alpha)}\,\Box\mathrm{I},u$$

(ii)

$$\dfrac{\dfrac{\dfrac{\neg\alpha \wedge \alpha}{\neg\alpha}\,\wedge\mathrm{E}_1}{\neg\Box\alpha}\,\neg\Box\mathrm{I} \qquad \dfrac{\neg\alpha \wedge \alpha}{\alpha}\,\wedge\mathrm{E}_2}{\neg\Box\alpha \wedge \alpha}\,\wedge\mathrm{I}$$

$$\dfrac{\dfrac{\dfrac{\neg\Box\alpha \wedge \alpha}{\neg\Box\alpha}\,\wedge\mathrm{E}_1 \qquad \dfrac{\neg\Box\alpha \wedge \alpha}{\alpha}\,\wedge\mathrm{E}_2}{\neg\alpha}\,\neg\Box\mathrm{E} \qquad \dfrac{\neg\Box\alpha \wedge \alpha}{\alpha}\,\wedge\mathrm{E}_2}{\neg\alpha \wedge \alpha}\,\wedge\mathrm{I}$$

(ix) See figure 1.

(xi) By $\Box$E we have that $\Box\neg\Box\alpha \vdash \neg\Box\alpha$. For the converse, consider the following deduction

$$\dfrac{\neg\Box\alpha \qquad \dfrac{\dfrac{\neg\neg\Box\alpha^{u}}{\Box\alpha} \qquad \neg\Box\alpha}{\bot}\ \wedge\text{I and item (iv)}}{\Box\neg\Box\alpha}\ \Box\text{I},u$$

$\dashv$

Note that all syntactic proofs displayed in Lemma 3.2 are normal.

Theorem 3.3 (Soundness and Completeness, [17]). *Let $\Gamma, \Delta \subseteq Fm$, Γ finite. The following conditions are equivalent*:

(i) *the sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{SC}_{\mathcal{TML}}$,*
(ii) *there is a deduction of the disjunction of the sentences in Δ from Γ in $\mathbf{ND}_{\mathcal{TML}}$.*

In what follows, the *del-rules* (for "disjunction-elimination-like") of $\mathbf{ND}_{\mathcal{TML}}$ are $\vee$E and $\neg\wedge$E. As usual, a segment (of length n) in a deduction $\mathcal{D}$ of $\mathbf{ND}_{\mathcal{TML}}$ is a sequence $\alpha_1, \ldots, \alpha_n$ of consecutive occurrences of a formula α in D such that for $1 \leqslant n, i < n$,

(a) α_i is a minor premise of a del-rule application in D, with conclusion α_{i+1},
(b) α_n is not a minor premise of a del-rule application, and
(c) α_1 is not the conclusion of a del-rule application.

It is worth observing that, in this paper, the complexity (or degree) of a formula α is defined as follows:

Definition 3.2. Let α be a formula. The complexity (degree) of α, $c(\alpha)$, is the natural number obtained by

(i) if p is a propositional variable then $c(p) = 0$,
(ii) $c(\beta\sharp\gamma) = c(\beta) + c(\gamma) + 1$ for $\sharp \in \{\vee, \wedge\}$,
(iii) $c(\neg\alpha) = c(\alpha) + 1$,
(iv) $c(\Box\alpha) = c(\alpha) + 2$.

A formula occurrence which is neither a minor premise nor the conclusion of an application of a del-rule always belongs to a segment of length 1.

A segment is *maximal*, or a cut (segment) if α_n is the major premise of an E-rule, and either $n > 1$, or $n = 1$ and α_n is the conclusion of

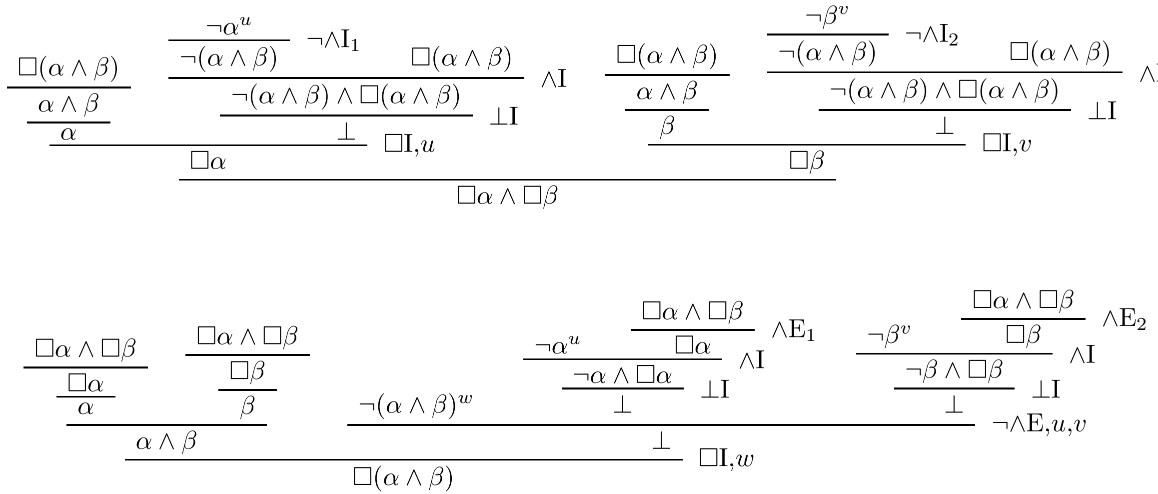


Figure 1. The proof of (ix)

an I-rule. The *cutrank* $cr(\sigma)$ of a maximal segment σ with formula α is the complexity of α. Besides, the cutrank $cr(\mathcal{D})$ of a deduction $\mathcal{D}$ is the maximum of the cutranks of cuts in $\mathcal{D}$. If there is no cut, the cutrank of $\mathcal{D}$ is zero. A *critical cut* of $\mathcal{D}$ is a cut of maximal cutrank among all cuts in $\mathcal{D}$. A deduction without critical cuts is said to be *normal*.

Lemma 3.4. *Let $\mathcal{D}$ be a deduction in $\mathbf{ND}_{\mathcal{TML}}$. Then, $\mathcal{D}$ reduces to a deduction $\mathcal{D}'$ in which the consequence of every application of the $\bot$E rule is a propositional variable p (atomic) or its negation $\neg p$.*

Proof. Consider the following deduction with an application of the $\bot$E

$$\dfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{\alpha}\,\bot\mathrm{E}$$

It is not difficult to check that if α has the shape of $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$, $\neg(\gamma_1 \wedge \gamma_2)$, $\neg(\gamma_1 \vee \gamma_2)$, $\neg\neg\gamma_1$, $\Box\gamma_1$ and $\neg\Box\gamma_1$ we can remove this application of $\bot$E using application(s) of $\bot$E with consequence formula(s) that has complexity strictly less that the complexity of α. Thus, by successively repeating this transformation we can finally obtain a deduction with the required characteristics. ⊣

In what follows, we shall only consider deductions in which the consequence of every application of the $\bot$E rule is a propositional variable p (atomic) or its negation $\neg p$. Observe that, in this kind of deductions, there cannot be occurrences of formulas that are consequences of $\bot$E and the major premise of an E-rule.

We first show how to remove cuts of length 1. Besides, $\wedge$-conversions and $\vee$-conversions are as in the system of natural deduction for intuitionistic (or classical) logic.

$\neg\wedge$-*conversion*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \neg\alpha_i\end{matrix}}{\neg(\alpha_1 \wedge \alpha_2)} \qquad \begin{matrix}[\neg\alpha_1]^u\\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \begin{matrix}[\neg\alpha_2]^v\\ \mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,u,v \quad \text{converts to} \quad \begin{matrix}\mathcal{D}\\ [\neg\alpha_i]\\ \mathcal{D}_i\\ \gamma\end{matrix} \quad \text{for } i = 1, 2.$$

$\neg\vee$-*conversion*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \neg\alpha_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \neg\alpha_2\end{matrix}}{\neg(\alpha_1 \wedge \alpha_2)}}{\neg\alpha_i} \quad \text{converts to} \quad \begin{matrix}\mathcal{D}_i\\ \neg\alpha_i\end{matrix} \quad \text{for } i = 1, 2.$$

¬¬-*conversion*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \alpha\end{matrix}}{\neg\neg\alpha}}{\alpha} \quad\text{converts to}\quad \begin{matrix}\mathcal{D}\\ \alpha\end{matrix}$$

□-*conversion*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \alpha\end{matrix} \qquad \begin{matrix}[\neg\alpha]^u\\ \mathcal{D}_2\\ \bot\end{matrix}}{\Box\alpha}}{\alpha} \quad\text{converts to}\quad \begin{matrix}\mathcal{D}_1\\ \alpha\end{matrix}$$

¬□-*conversion*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \neg\alpha\end{matrix}}{\neg\Box\alpha} \qquad \begin{matrix}\mathcal{D}_2\\ \alpha\end{matrix}}{\neg\alpha} \quad\text{converts to}\quad \begin{matrix}\mathcal{D}_1\\ \neg\alpha\end{matrix}$$

In order to remove cuts of length > 1, we permute E-rules upwards over minor premises of del-rules:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \alpha_1 \vee \alpha_2\end{matrix} \qquad \begin{matrix}[\alpha_1]^u\\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \begin{matrix}[\alpha_2]^v\\ \mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,\vee\text{E} \qquad \mathcal{D}'}{\gamma'}\,\text{E-rule}$$

converts to

$$\dfrac{\begin{matrix}\mathcal{D}\\ \alpha_1 \vee \alpha_2\end{matrix} \qquad \dfrac{\begin{matrix}[\alpha_1]^u\\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \mathcal{D}'}{\gamma'}\,\text{E-rule} \qquad \dfrac{\begin{matrix}[\alpha_2]^v\\ \mathcal{D}_2\\ \gamma\end{matrix} \qquad \mathcal{D}'}{\gamma'}}{\gamma'}\ ,$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \neg(\alpha_1 \wedge \alpha_2)\end{matrix} \qquad \begin{matrix}[\neg\alpha_1]^u\\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \begin{matrix}[\neg\alpha_2]^v\\ \mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,\vee\text{E} \qquad \mathcal{D}'}{\gamma'}\,\text{E-rule}$$

converts to

$$\dfrac{\begin{matrix}\mathcal{D}\\ \neg(\alpha_1 \vee \alpha_2)\end{matrix} \qquad \dfrac{\begin{matrix}[\neg\alpha_1]^u\\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \mathcal{D}'}{\gamma'}\,\text{E-rule} \qquad \dfrac{\begin{matrix}[\neg\alpha_2]^v\\ \mathcal{D}_2\\ \gamma\end{matrix} \qquad \mathcal{D}'}{\gamma'}}{\gamma'}\ ,$$

Applications of $\vee$E ($\neg\wedge$E) with major premise $\alpha_1 \vee \alpha_2$ ($\neg(\alpha_1 \wedge \alpha_2)$), where at least one of $[\alpha_1]$, $[\alpha_2]$ ($[\neg\alpha_1]$, $[\neg\alpha_2]$) is empty in the deduction of the first or second minor premise, are redundant and can be removed easily. We are now ready to prove the main result of this section.

Theorem 3.5 (Normalization). *Every deduction $\mathcal{D}$ in $\mathbf{ND}_{\mathcal{TML}}$ reduces to a normal deduction.*

Proof. We assume that, in every application of an E-rules, the major premise is always to the left of the minor premise(s), if there are any minor premises. We use a main induction on the cutrank n of $\mathcal{D}$, with a subinduction on m, the sum of lengths of all critical cuts (= cut segments) in $\mathcal{D}$. By a suitable choice of the critical cut to which we apply a conversion we can achieve the result that either n decreases (and we can appeal to the main induction hypothesis), or that n remains constant but m decreases (and we can appeal to the subinduction hypothesis). Let σ be a top critical cut in $\mathcal{D}$ if no critical cut occurs in a branch of $\mathcal{D}$ above σ. Now apply a conversion to the rightmost top critical cut of $\mathcal{D}$; then the resulting $\mathcal{D}'$ has a lower cutrank (if the segment treated has length 1, and is the only maximal segment in D), or has the same cutrank, but a lower value for m. ⊣

4. The contrapositive implication in $\mathcal{TML}$

The original language of the logic of TMAs — in particular, the language of logic $\mathcal{TML}$ — does not have an implication operator as a primitive connective. It is a natural question to ask how to define a binary operator in TMAs, in terms of the others, with the behavior of some kind of implication. Such operators are useful in order to characterize the lattice of congruences of a given class of algebras.

Some proposal for an implication operator in TMAs have appeared in the literature. For instance, Loureiro proposed in [25] the following implication for TMAs:

$$x \to y = \neg\Box x \vee y.$$

In turn, by considering the operator

$$x \mapsto y = (x \to y) \wedge (\neg\Box\neg y \vee \neg x),$$

Figallo and Landini introduced in [18] an interesting implication operator for TMAs defined as follows:

$$x \succ y = (x \mapsto y) \wedge ((\neg x \vee y) \to (\Box \neg x \vee y)).$$

This operator was called in [34] *contrapositive implication* for TMAs.

It is easy to see that the contrapositive implication can be written in a simpler form:

$$x \succ y = (x \to y) \wedge (\neg y \to \neg x) \wedge ((\neg x \vee y) \to (\Box \neg x \vee y)).$$

The main feature of contrapositive implication is that it internalizes the consequence relation (whenever just one premise is considered), as we shall see in Theorem 4.5. Another important aspect of contrapositive implication is that all the operations of the TMAs can be defined in terms of $\succ$ and $\mathbf{0}$. In fact:

Proposition 4.1 ([18]). *In every TMA it holds*:

(i) $\mathbf{1} \approx (\mathbf{0} \succ \mathbf{0})$,
(ii) $\neg x \approx (x \succ \mathbf{0})$,
(iii) $x \vee y \approx (x \succ y) \succ y$,
(iv) $x \wedge y \approx \neg(\neg x \vee \neg y)$,
(v) $\Box x \approx \neg(x \succ \neg x)$.

Therefore, $\succ$ and $\mathbf{0}$ are enough to generate all the operations of a given TMA.

Additionally, from Proposition 4.1 an axiomatization for TMAs was given in [18] in terms of $\succ$ and $\mathbf{0}$ as follows.

Proposition 4.2 ([18]). *In every TMA it can be defined a binary operation $\succ$ and an element $\mathbf{0}$ such that the following holds:*

(C1) $(\mathbf{1} \succ x) \approx x$,
(C2) $(x \succ \mathbf{1}) \approx \mathbf{1}$,
(C3) $(x \succ y) \succ y \approx (y \succ x) \succ x$,
(C4) $x \succ (y \succ z) \approx \mathbf{1}$ *implies* $y \succ (x \succ z) \approx \mathbf{1}$,
(C5) $((x \succ (x \succ y)) \succ x) \succ x \approx \mathbf{1}$,
(C6) $(\mathbf{0} \succ x) \approx \mathbf{1}$,
(C7) $(x \succ \mathbf{0}) \approx \neg x$,
(C8) $((x \wedge y) \succ z) \succ ((x \succ z) \vee (y \succ z)) = \mathbf{1}$.

Conversely, if an algebra with a binary operation $\succ$ and an element 0 *satisfies* (C1)–(C6) *and* (C8), *where* $\mathbf{1} := \mathbf{0} \succ \mathbf{0}$, $x \vee y := (x \succ y) \succ y$

and $x \wedge y := \neg(\neg x \vee \neg y)$ *such that* $\neg x := (x \succ \mathbf{0})$*, then the resulting structure is a TMA, where* $\Box x := \neg(x \succ \neg x)$*.*

Definition 4.1. A *contrapositive tetravalent modal algebra* is an algebra $\langle A, \succ, 0\rangle$ of type $(2, 0)$ that satisfies items (C1)–(C6) and (C8) of Proposition 4.2 (with the abbreviations defined therein). We shall denote the class of these algebras by $\mathbf{TMA}^{\mathrm{c}}$.

Observe that the classes **TMA** and $\mathbf{TMA}^{\mathrm{c}}$ are termwise equivalent. The main differences reside in the underlying language defining both classes and the fact that the characterization of the latter does not allow to see that in fact it is a variety. As it was showed in [14], the contrapositive implication operator $\succ$ is very useful when describing a Hilbert-style system for $\mathcal{TML}$.

It is worth noting that in $\mathfrak{M}_{4\mathrm{m}}$, the canonical TMA, the contrapositive implication operator $\succ$ has the following truth-table:

$\succ$	**0**	**n**	**b**	**1**
0	**1**	**1**	**1**	**1**
n	**n**	**1**	**b**	**1**
b	**b**	**n**	**1**	**1**
1	**0**	**n**	**b**	**1**

Remark 4.1. Clearly, the logic of the contrapositive tetravalent modal algebras $\models_{\mathcal{TMA}^{\mathrm{c}}}$ can be defined by analogy with Definition 2.1.

The new connective $\succ$ has some nice properties, displayed below. As usual, we define the dual connective $\Diamond$ as an abbreviation of $\neg\Box\neg$. By the simple inspection of the truth-tables we obtain:

Proposition 4.3. *For all* $\alpha, \beta \in Fm$*, the following holds in* $\mathcal{TML}$*:*

(i) $\models_{\mathcal{TML}} \bot \succ \alpha$,
(ii) $\models_{\mathcal{TML}} \alpha \succ \top$,
(iii) $\models_{\mathcal{TML}} \alpha \succ (\beta \succ \alpha)$,
(iv) $\models_{\mathcal{TML}} (\alpha \vee \beta) \succ (\beta \vee \alpha)$,
(v) $\models_{\mathcal{TML}} \neg\neg\alpha \succ \alpha$,
(vi) $\models_{\mathcal{TML}} \alpha \succ \neg\neg\alpha$,
(vii) $\models_{\mathcal{TML}} \Box\alpha \succ \Box\Box\alpha$,
(viii) $\models_{\mathcal{TML}} \Box\alpha \succ \alpha$,
(ix) $\models_{\mathcal{TML}} \alpha \succ \Box\Diamond\alpha$,
(x) $\models_{\mathcal{TML}} \Box\alpha \succ \Diamond\alpha$,
(xi) $\models_{\mathcal{TML}} \Box(\alpha \succ \beta) \succ (\Box\alpha \succ \Box\beta)$,
(xii) $\models_{\mathcal{TML}} (\Diamond\alpha \wedge \Diamond\beta) \succ (\Diamond(\alpha \wedge \Diamond\beta) \vee \Diamond(\alpha \wedge \beta) \vee \Diamond(\beta \wedge \Diamond\alpha))$.

Remark 4.2. Theorem (xi) is the **(K)** axiom [see 6]. Theorems (vii), (viii), (ix), (x) and (xii) correspond to the axioms (**4**), (**T**), (**B**), (**D**) and

(**.3**), respectively [see 6]. Therefore $\mathcal{TML}$ satisfies all the modal axioms of the classical modal logic **S5**. Nevertheless, we cannot affirm that $\mathcal{TML}$ is a normal modal logic since the implication operator $\succ$ does not satisfy some properties of classical implication [see 6].

There exist interesting similarities between Łukasiewicz's $Ł_3$ (seen as a modal logic) and $\mathcal{TML}$. In both logics, $\Box\alpha$ and $\Diamond\alpha$ are defined by the same formulas, namely $\neg(\alpha \succ \neg\alpha)$ and $\neg\alpha \succ \alpha$, respectively (in the case of $Ł_3$, $\neg$ and $\succ$ denote the respective negation and implication operators). Moreover, both implications ($Ł_3$'s implication and the contrapositive implication) do not satisfy the contraction law: $\alpha \succ (\alpha \succ \beta)$ is not equivalent to $(\alpha \succ \beta)$. From this follows that both logics satisfy the following modal principle: $\alpha \succ (\alpha \succ \Box\alpha)$, which is not valid in the classical modal logic **S5**.

Let $\Box^0\alpha =_{def} \alpha$ and $\Box^{n+1}\alpha =_{def} \Box^n\Box\alpha$ for any $n \in \mathbb{N}$. $\Diamond^n\alpha$ is defined analogously. Then, from Proposition 4.3 we obtain:

Proposition 4.4. *$\mathcal{TML}$ satisfies the following well-known instance of the Lemmon-Scott schemes* [cf. 24] *for any $n, l, k, m \in \mathbb{N}$,*

$$\models_{\mathcal{TML}} \Diamond^k\Box^l\alpha \succ \Box^m\Diamond^n\alpha$$

but $\not\models_{\mathcal{TML}} \Box\Diamond\alpha \succ \Diamond\Box\alpha$.

Finally, in $\mathcal{TML}$ we have a weak version of the *Deduction Metatheorem* with respect to the contrapositive implication.

Theorem 4.5 ([18]). *Let $\alpha, \beta \in Fm$. The following conditions are equivalent*:

(i) $\alpha \models_{\mathcal{TML}} \beta$,
(ii) $\models_{\mathcal{TML}} \alpha \succ \beta$.

This last result shows that contrapositive implication $\succ$ internalizes the consequence relation of $\mathcal{TML}$ whenever just one premise is considered. In algebraic terms, $\succ$ internalizes the partial order $\leq$ of TMAs.

It is worth noting that it is not possible to improve Theorem 4.5 in the following sense:

Proposition 4.6. *In $\mathcal{TML}$ both directions of the deduction metatheorem, with respect to $\succ$, fail if more than one premise are allowed. Specifically*:

(i) $\alpha, \beta \models_{\mathcal{TML}} \gamma$ *does not imply that* $\alpha \models_{\mathcal{TML}} \beta \succ \gamma$,
(ii) $\alpha \models_{\mathcal{TML}} \beta \succ \gamma$ *does not imply that* $\alpha, \beta \models_{\mathcal{TML}} \gamma$.

Proof. (i) Observe that $\mathbf{n} \wedge \mathbf{b} \leq \mathbf{0}$, but $\mathbf{n} \not\leq \mathbf{b} \succ \mathbf{0} = \mathbf{b}$. In order to find an example of this, consider $\alpha = (\bullet p \wedge \bullet q \wedge \bullet(p \succ q) \wedge p)$, $\beta = q$ and $\gamma = \bot$, where p, q are two different propositional variables, and $\bullet\delta =_{def} \Diamond(\delta \wedge \neg\delta)$ is the *inconsistency* operator. Then $h(\alpha \wedge \beta) = \mathbf{0} = h(\gamma)$ for every $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{M}_{4\mathrm{m}})$. That is, $\alpha, \beta \models_{\mathcal{TML}} \gamma$. Now, let h such that $h(p) = \mathbf{n}$ and $h(q) = \mathbf{b}$. Then $h(\alpha) = \mathbf{n}$ and $h(\beta) = \mathbf{b}$ and so $h(\alpha) \not\leq h(\beta \succ \gamma) = \mathbf{b} \succ \mathbf{0} = \mathbf{b}$. Therefore, $\alpha \not\models_{\mathcal{TML}} \beta \succ \gamma$.

(ii) Note that $\mathbf{n} \leq \mathbf{n} \succ \mathbf{0} = \mathbf{n}$, but $\mathbf{n} \wedge \mathbf{n} = \mathbf{n} \not\leq \mathbf{0}$. For instance, consider $\alpha = p$, $\beta = \neg p$ and $\gamma = \bot$, where p is a propositional variable. Then $\alpha \models_{\mathcal{TML}} \beta \succ \gamma$, since $\alpha \models_{\mathcal{TML}} \neg\neg\alpha$. Consider $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{M}_{4\mathrm{m}})$ such that $h(p) = \mathbf{n}$. Then $h(\alpha \wedge \beta) = \mathbf{n} \not\leq \mathbf{0} = h(\gamma)$ and so $\alpha, \beta \not\models_{\mathcal{TML}} \gamma$. ⊣

In particular, the contrapositive implication does not satisfy local *modus ponens* (in the sense of [10]).

The importance of the contrapositive implication was confirmed in [14], where a Hilbert-style axiomatization for $\mathcal{TML}$ was given in terms of $\succ$ and $\bot$.

5. A tableau system for $\mathcal{TML}$ in the language $\{\neg, \succ\}$

In this section, by adapting the general techniques introduced in [7], we define a decidable tableau system adequate to check validity in the logic $\mathcal{TML}$. This constitutes a new (proof-theoretic) decision procedure for checking validity in the variety of tetravalent modal algebras, besides the four-valued truth-tables of $\mathcal{TML}$ and the one available in terms of the cut-free sequent system introduced in [17].

The procedure for finding a set of tableau-rules for $\mathcal{TML}$ is based on the general method presented in [7] for obtaining bivalued semantics and tableau rules for a wide class of finite matrix logics [see 8 for further development of this technique]. The given matrix logic must satisfy just one condition: to be expressive enough to "separate" among the different truth-values of the same kind, namely distinguished and non-distinguished.

Since we are here interested in just one example, the logic $\mathcal{TML}$, we will simplify the procedure for obtaining the tableau rules without entering in the (rather involved) details of the general construction presented in [7, 8]. Moreover, for the sake of the reader, we will present original proofs of soundness and completeness of the generated tableau system, generalizing the classical ones of [32], and so this section will be self-contained.

For simplicity, and making use of contrapositive implication, we will use the language $\mathfrak{Fm}'' = \langle Fm'', \succ, \neg\rangle$. The use of $\neg$ instead of $\bot$ as primitive will be convenient in order to simplify the presentation of the rules, besides the fact that the negation $\neg$ will play a fundamental role in the sequel. It is worth noting that the "expressive power" of $\mathfrak{Fm}''$ is the same as that of $\mathfrak{Fm}'$, since $\bot$ can be defined in the former as $\neg(\alpha \succ \alpha)$ for any α. Additionally, the tableau rules will be extended to the usual language $\mathfrak{Fm}$ of tetravalent modal algebras.

In the following subsections, we will assume that the reader is acquainted with the definition of tableaux, as well as with the related notions such as closure rules, closed and open branches etc. The reader unfamiliar with such concepts can consult [32].

5.1. Separating the truth-values of $\mathcal{TML}$

From now, $\mathcal{TML}$ will be seen as the matrix logic $\mathcal{M}_N = \langle \mathfrak{M}_{4\mathrm{m}}, \{\mathbf{b}, 1\}\rangle$. Consider the function $f\colon M_4 \to \{\mathrm{T}, \mathrm{F}\}$ given by $f(\mathbf{1}) = f(\mathbf{b}) = \mathrm{T}$ and $f(\mathbf{0}) = f(\mathbf{n}) = \mathrm{F}$. This function splits the truth-values into two classes: the distinguished and the non-distinguished ones.

Consider now the formula $\neg p$ in Fm''. This formula (seen as an operator over M_4) "separates" the truth-values of $\mathcal{TML}$ as follows: for $x \in M_4$,

$$\begin{aligned} x = \mathbf{1} &\text{ iff } f(x) = \mathrm{T} \text{ and } f(\neg x) = \mathrm{F};\\ x = \mathbf{b} &\text{ iff } f(x) = \mathrm{T} \text{ and } f(\neg x) = \mathrm{T};\\ x = \mathbf{n} &\text{ iff } f(x) = \mathrm{F} \text{ and } f(\neg x) = \mathrm{F};\\ x = \mathbf{0} &\text{ iff } f(x) = \mathrm{F} \text{ and } f(\neg x) = \mathrm{T}.\end{aligned}$$

From this it follows:

$$(\dagger)\begin{cases} x \in \{\mathbf{1}, \mathbf{b}\} & \text{iff } f(x) = \mathrm{T};\\ x \in \{\mathbf{1}, \mathbf{n}\} & \text{iff } f(\neg x) = \mathrm{F};\\ x \in \{\mathbf{0}, \mathbf{b}\} & \text{iff } f(\neg x) = \mathrm{T};\\ x \in \{\mathbf{0}, \mathbf{n}\} & \text{iff } f(x) = \mathrm{F}.\end{cases}$$

5.2. Describing the truth-table of $\succ$ in terms of T/F

By inspection of the truth-table of $\succ$ and by using (†) it follows, for all $x, y \in M_4$:

$$f(x \succ y) = \mathrm{T} \ \text{ iff } \begin{cases} & f(y) = \mathrm{T} & \text{OR} \\ f(\neg x) = \mathrm{T}, & f(y) = \mathrm{F},\, f(\neg y) = \mathrm{T} & \text{OR} \\ f(x) = \mathrm{F} & f(y) = \mathrm{F},\, f(\neg y) = \mathrm{F} & \end{cases}$$

$$f(x \succ y) = \mathrm{F} \ \text{ iff } \begin{cases} f(x) = \mathrm{T}, & f(y) = \mathrm{F},\, f(\neg y) = \mathrm{F}, & \text{OR} \\ f(\neg x) = \mathrm{F}, & f(y) = \mathrm{F},\, f(\neg y) = \mathrm{T}. & \end{cases}$$

$$f(\neg(x \succ y)) = \mathrm{T} \ \text{iff } \begin{cases} f(x) = \mathrm{T}, & f(y) = \mathrm{F},\, f(\neg y) = \mathrm{T} & \text{OR} \\ f(\neg x) = \mathrm{F}, & f(y) = \mathrm{T},\, f(\neg y) = \mathrm{T}. & \end{cases}$$

$$f(\neg(x \succ y)) = \mathrm{F} \ \text{iff } \begin{cases} & f(\neg y) = \mathrm{F} & \text{OR} \\ f(\neg x) = \mathrm{T}, & f(y) = \mathrm{T},\, f(\neg y) = \mathrm{T} & \text{OR} \\ f(x) = \mathrm{F}, & f(y) = \mathrm{F},\, f(\neg y) = \mathrm{T} & \end{cases}$$

5.3. Obtaining the tableau rules for $\mathcal{TML}$

By substituting in the expressions above the truth-values x, y by formulas α, β of Fm'', and by substituting equations "$f(x) = \mathrm{T}$" and "$f(x) = \mathrm{F}$" by signed formulas $T(\alpha)$ and $F(\alpha)$, respectively, we obtain automatically the following tableau rules for $\mathcal{TML}$:

$$\frac{T(\alpha \succ \beta)}{T(\beta) \,|\, T(\neg\alpha), F(\beta), T(\neg\beta) \,|\, F(\alpha), F(\beta), F(\neg\beta)}$$

$$\frac{F(\alpha \succ \beta)}{T(\alpha), F(\beta), F(\neg\beta) \ | \ F(\neg\alpha), F(\beta), T(\neg\beta)}$$

$$\frac{T(\neg(\alpha \succ \beta))}{T(\alpha), F(\beta), T(\neg\beta) \ | \ F(\neg\alpha), T(\beta), T(\neg\beta)}$$

$$\frac{F(\neg(\alpha \succ \beta))}{F(\neg\beta) \,|\, T(\neg\alpha), T(\beta), T(\neg\beta) \,|\, F(\alpha), F(\beta), T(\neg\beta)}$$

$$\frac{T(\neg\neg\alpha)}{T(\alpha)} \qquad \frac{F(\neg\neg\alpha)}{F(\alpha)}$$

Closure rule:

$$\frac{T(\alpha), F(\alpha)}{\star}$$

Let $\mathbb{T}$ be the tableau system defined by the rules above. Given a signed formula η, a completed tableau starting from η is called *a tableau for* η. We say that a formula α in Fm'' is *provable in* $\mathbb{T}$, written as $\vdash_{\mathbb{T}} \alpha$, if there exists a closed tableau for the signed formula $F(\alpha)$.

Given a tableau system, it is in general convenient to define derived rules in order to get shorter proofs. So straightforward, by using the rules of $\mathbb{T}$. we state a fundamental set of derived rules:

PROPOSITION 5.1. *The following rules can be derived in* $\mathbb{T}$:

$$\frac{T(\alpha \succ \beta), T(\neg(\alpha \succ \beta))}{F(\neg\alpha), T(\beta), T(\neg\beta) \mid T(\alpha), T(\neg\alpha), F(\beta), T(\neg\beta)}$$

$$\frac{F(\alpha \succ \beta), F(\neg(\alpha \succ \beta))}{T(\alpha), F(\beta), F(\neg\beta) \mid F(\alpha), F(\neg\alpha), F(\beta), T(\neg\beta)}$$

$$\frac{F(\alpha \succ \beta), T(\neg(\alpha \succ \beta))}{T(\alpha), F(\neg\alpha), F(\beta), T(\neg\beta)}$$

$$\frac{T(\alpha \succ \beta), F(\neg(\alpha \succ \beta))}{F(\alpha), T(\neg\alpha) \mid F(\alpha), F(\neg\alpha), F(\beta), F(\neg\beta) \mid T(\alpha), T(\neg\alpha), T(\beta), T(\neg\beta) \mid T(\beta), F(\neg\beta)}$$

5.4. Soundness and completeness of $\mathbb{T}$

Now we will prove the adequacy of $\mathbb{T}$, that is, that $\vdash_{\mathbb{T}} \alpha$ if and only if $\models_{\mathcal{TML}} \alpha$, for every formula α. We begin by introducing some definitions and previous results.

Given a formula $\alpha \in Fm''$, the *degree* of α, denoted by $d(\alpha)$, is a natural number defined as follows: $d(p) = 1$ (for $p \in Var$); $d(\alpha \succ \beta) = d(\alpha) + d(\beta) + 1$; $d(\neg\alpha) = d(\alpha) + 1$.

It is clear that the degree of the formulas occurring in the conclusion of a rule of $\mathbb{T}$ is strictly less than the degree of the premise of the rule. From this, it is easy to prove the following useful result:

PROPOSITION 5.2. *Given a signed formula* η, *it is always possible to build a (open or closed) completed tableau in* $\mathbb{T}$ *for* η.

Given a homomorphism $h\colon \mathfrak{Fm}'' \to \mathfrak{M}_{4\mathrm{m}}$ and a signed formula η, we say that h *satisfies* η if

- $\eta = T(\alpha)$ and $h(\alpha) \in \{\mathbf{1}, \mathbf{b}\}$;
- $\eta = F(\alpha)$ and $h(\alpha) \in \{\mathbf{0}, \mathbf{n}\}$;

It follows that h satisfies $T(\neg\alpha)$ iff $h(\alpha) \in \{\mathbf{0}, \mathbf{b}\}$, and h satisfies $F(\neg\alpha)$ iff $h(\alpha) \in \{\mathbf{1}, \mathbf{n}\}$.

Let Υ be a set of signed formulas. Then h *satisfies* Υ if h satisfies η for every $\eta \in \Upsilon$. By a straightforward proof by cases we obtain:

LEMMA 5.3. *Let $h : \mathfrak{Fm}'' \to \mathfrak{M}_{4\mathrm{m}}$ be a homomorphism, and let*

$$\frac{\eta}{\Upsilon_1 \mid \ldots \mid \Upsilon_n}$$

be a rule of $\mathbb{T}$. If h satisfies η then h satisfies Υ_i for some $1 \leqslant i \leqslant n$.

PROPOSITION 5.4. *If $\models_{\mathcal{TML}} \alpha$ then every completed tableau for $F(\alpha)$ is open.*

PROOF. Assume that $\models_{\mathcal{TML}} \alpha$ and suppose that there exists a completed closed tableau $\mathcal{T}$ for $F(\alpha)$. Since $\not\models_{\mathcal{TML}} \alpha$, there is a homomorphism h such that $h(\alpha) \in \{\mathbf{0}, \mathbf{n}\}$. Thus, h satisfies $F(\alpha)$. By the previous lemma, h must satisfy the set of signed formulas occurring in some branch θ of $\mathcal{T}$. Since $\mathcal{T}$ is closed, the branch θ is closed, that is, the closure rule was used in θ. But it is an easy task to verify that no homomorphism can satisfy simultaneously both premises of the closure rule. This leads to a contradiction, and then every completed tableau for $F(\alpha)$ must be open. ⊣

COROLLARY 5.5 (Soundness of $\mathbb{T}$). *If $\vdash_{\mathbb{T}} \alpha$ then $\models_{\mathcal{TML}} \alpha$.*

In order to prove completeness, we need to state the following result.

PROPOSITION 5.6. *Let θ be an open branch of a completed open tableau $\mathcal{T}$, and let Υ be the set of signed formulas occurring in θ. Let h be a homomorphism such that, for every $\alpha \in Var$:*

$$(\ddagger)\begin{cases} h(\alpha) \in \{\mathbf{1}, \mathbf{b}\} & \text{if } T(\alpha) \in \Upsilon; \\ h(\alpha) \in \{\mathbf{1}, \mathbf{n}\} & \text{if } F(\neg\alpha) \in \Upsilon; \\ h(\alpha) \in \{\mathbf{0}, \mathbf{n}\} & \text{if } F(\alpha) \in \Upsilon; \\ h(\alpha) \in \{\mathbf{0}, \mathbf{b}\} & \text{if } T(\neg\alpha) \in \Upsilon. \end{cases}$$

In any other case $h(\alpha)$ is arbitrary, for $\alpha \in Var$. Then ($\ddagger$) holds for any complex formula α.

PROOF. By induction on the degree of α.

(i) $\alpha \in Var$. The result is clearly true.

(ii) $\alpha = \neg\beta$. If $T(\alpha) \in \Upsilon$ then $T(\neg\beta) \in \Upsilon$ and so $h(\beta) \in \{\mathbf{0}, \mathbf{b}\}$, by the induction hypothesis. Thus, $h(\alpha) \in \{\mathbf{1}, \mathbf{b}\}$. If $T(\neg\alpha) \in \Upsilon$ then $T(\neg\neg\beta) \in \Upsilon$ and so $T(\beta) \in \Upsilon$, since $\mathcal{T}$ is completed. Thus $h(\beta) \in \{\mathbf{1}, \mathbf{b}\}$, by induction hypothesis and then $h(\alpha) \in \{\mathbf{0}, \mathbf{b}\}$. The other cases are proved analogously.

(iii) $\alpha = \beta \succ \gamma$.

(iii.1) $T(\alpha) \in \Upsilon$. Since $\mathcal{T}$ is completed then the rule for $T(\beta \succ \gamma)$ was eventually used, splitting into three branches. One of them is a sub-branch of θ, thus one of the following cases hold:

(iii.1.1) $T(\gamma) \in \Upsilon$. Then $h(\gamma) \in \{\mathbf{1}, \mathbf{b}\}$, by the induction hypothesis. Then, $h(\alpha) \in \{\mathbf{1}, \mathbf{b}\}$, by the definition of $\succ$.

(iii.1.2) $T(\neg\beta), F(\gamma), T(\neg\gamma) \in \Upsilon$. Then $h(\beta) \in \{\mathbf{0}, \mathbf{b}\}$ and $h(\gamma) = \mathbf{0}$, by the induction hypothesis, and so $h(\alpha) \in \{\mathbf{1}, \mathbf{b}\}$.

(iii.1.3) $F(\beta), F(\gamma), F(\neg\gamma) \in \Upsilon$. Then $h(\beta) \in \{\mathbf{0}, \mathbf{b}\}$, by the induction hypothesis, and so $h(\alpha) = \mathbf{1} \in \{\mathbf{1}, \mathbf{b}\}$.

The proof of the remaining sub cases for (iii) (namely: $F(\alpha) \in \Upsilon$, $T(\neg\alpha) \in \Upsilon$ and $F(\neg\alpha) \in \Upsilon$) are analogous. ⊣

PROPOSITION 5.7. *Assume that there is a completed open tableau for $F(\alpha)$. Then $\not\models_{\mathcal{TML}} \alpha$.*

PROOF. Consider, by hypothesis, an open branch θ of a completed open tableau $\mathcal{T}$ for $F(\alpha)$, and let Υ be the set of signed formulas occurring in θ. Let h be a homomorphism defined as in Proposition 5.6. Then $h(\alpha) \in \{\mathbf{0}, \mathbf{n}\}$, since $F(\alpha) \in \Upsilon$, and so $\not\models_{\mathcal{TML}} \alpha$. ⊣

THEOREM 5.8 (Completeness of $\mathbb{T}$). *If $\models_{\mathcal{TML}} \alpha$ then $\vdash_{\mathbb{T}} \alpha$.*

PROOF. If $\models_{\mathcal{TML}} \alpha$ then, by Proposition 5.7, every completed tableau for $F(\alpha)$ is closed, and so there exists (by Proposition 5.2) a closed tableau for $F(\alpha)$. That is, $\vdash_{\mathbb{T}} \alpha$. ⊣

COROLLARY 5.9. *Let α be a formula. Then every completed tableau for $F(\alpha)$ is open, or every completed tableau for $F(\alpha)$ is closed.*

PROOF. Suppose that there exists a completed open tableau $\mathcal{T}$ for $F(\alpha)$, as well as a completed closed tableau $\mathcal{T}'$ for $F(\alpha)$. By Proposition 5.7, $\not\models_{\mathcal{TML}} \alpha$. On the other hand, by Corollary 5.5, it follows that $\models_{\mathcal{TML}} \alpha$, a contradiction. ⊣

PROPOSITION 5.10. *Suppose that* $\vdash_{\mathbb{T}} \alpha$. *Then every completed tableau for* $T(\neg\alpha)$ *is closed.*

PROOF. If $\vdash_{\mathbb{T}} \alpha$ then $\models_{\mathcal{TML}} \alpha$, by Corollary 5.5. Thus, $h(\alpha) \in \{\mathbf{1}, \mathbf{b}\}$ for every homomorphism h. Suppose that there exists a completed open tableau $\mathcal{T}$ for $T(\neg\alpha)$, and let Υ be the set of signed formulas obtained from an open branch θ of $\mathcal{T}$. Define a homomorphism h as in Proposition 5.6. Then $h(\alpha) \in \{\mathbf{0}, \mathbf{b}\}$ since $T(\neg\alpha) \in \Upsilon$. But $h(\alpha) \in \{\mathbf{1}, \mathbf{b}\}$ and so $h(\alpha) = \mathbf{b}$. Using Lemma 2.2 there exists a homomorphism h' such that $h'(\alpha) = \mathbf{n}$, a contradiction. Therefore every completed tableau for $T(\neg\alpha)$ is closed. ⊣

The last result shows from the tableau perspective the fact that, in $\mathcal{TML}$, the tautologies just get the truth-value $\mathbf{1}$ by means of homomorphisms.

It is worth noting that the tableau system $\mathbb{T}$ allows us to decide whether a given formula is valid or not in $\mathcal{TML}$, and so it decides the validity in the variety **TMA** of equations of the form $\alpha \approx \mathbf{1}$. With respect to inferences of the form $\alpha \vdash_{\mathcal{TML}} \beta$, they can be recovered in $\mathbb{T}$ by means of tableaux for $F(\alpha \succ \beta)$. Thus, $\mathbb{T}$ decides the validity in the variety **TMA** of equations $\alpha \approx \beta$. Finally, as it happens with the classical case [cf. 32], the set of signed formulas of an open branch of a completed open tableau allows us to find a model for that set of formulas: in particular, it finds a counter-model for a non-valid formula.

6. A tableau system for $\mathcal{TML}$ in the original language

In this short section, we use the results exposed in the previous section to present a tableau system for $\mathcal{TML}$ in the original language.

DEFINITION 6.1. Let $\mathbb{T}'$ be the tableau system defined by the following set of rules:

$$\frac{T(\alpha \vee \beta)}{T(\alpha) \mid T(\beta)} \qquad \frac{T(\neg(\alpha \vee \beta))}{T(\neg\alpha), T(\neg\beta)}$$

$$\frac{F(\alpha \vee \beta)}{F(\alpha), F(\beta)} \qquad \frac{F(\neg(\alpha \vee \beta))}{F(\neg\alpha) \mid F(\neg\beta)}$$

$$\frac{T(\alpha \wedge \beta)}{T(\alpha), T(\beta)} \qquad \frac{T(\neg(\alpha \wedge \beta))}{T(\neg\alpha) \mid T(\neg\beta)}$$

$$\frac{F(\alpha \wedge \beta)}{F(\alpha) \mid F(\beta)} \qquad \frac{F(\neg(\alpha \wedge \beta))}{F(\neg\alpha), F(\neg\beta)}$$

$$\frac{T(\neg\neg\alpha)}{T(\alpha)} \qquad \frac{F(\neg\neg\alpha)}{F(\alpha)}$$

$$\frac{T(\Box\alpha)}{T(\alpha), F(\neg\alpha)} \qquad \frac{F(\Box\alpha)}{F(\alpha) \mid T(\neg\alpha)} \qquad \frac{T(\neg\Box\alpha)}{F(\Box\alpha)} \qquad \frac{F(\neg\Box\alpha)}{T(\Box\alpha)}$$

Closure rules:

$$\frac{T(\alpha), F(\alpha)}{\star} \qquad \frac{T(\bot)}{\star} \qquad \frac{F(\neg\bot)}{\star}$$

The satisfaction of a signed formula η by a homomorphism $h\colon \mathfrak{Fm} \to \mathfrak{M}_{4\mathrm{m}}$ is defined as in Subsection 5.4. Then, by a straightforward proof by cases, we obtain:

Lemma 6.1. *Let $h : \mathfrak{Fm} \to \mathfrak{M}_{4\mathrm{m}}$ be a homomorphism, and let*

$$\frac{\eta}{\Upsilon_1 \mid \ldots \mid \Upsilon_n}$$

be a rule of $\mathbb{T}'$ for $n = 0, 1, 2$. If h satisfies η then h satisfies Υ_i for some $1 \leqslant i \leqslant n$.

By a similar argument to the one given for $\mathbb{T}$, the following result is clearly valid:

Proposition 6.2. *Given a signed formula η, it is always possible to build a (open or closed) completed tableau in $\mathbb{T}'$ for η.*

By analogous proof to the proof of Proposition 5.4 we have:

Proposition 6.3. *If $\not\models_{\mathcal{TML}} \alpha$ then every completed tableau for $F(\alpha)$ is open.*

Corollary 6.4 (Soundness of $\mathbb{T}'$). *If $\vdash_{\mathbb{T}'} \alpha$ then $\models_{\mathcal{TML}} \alpha$.*

Next, we prove a version of Proposition 5.6 for this setting.

Proposition 6.5. *Let θ be an open branch of a completed open tableau $\mathcal{T}$ in $\mathbb{T}'$, and let Υ be the set of signed formulas occurring in θ. Let h be a homomorphism such that, for every $\alpha \in Var$:*

$$(\ddagger)\begin{cases} h(\alpha) \in \{\mathbf{1}, \mathbf{b}\} & \text{if } T(\alpha) \in \Upsilon; \\ h(\alpha) \in \{\mathbf{1}, \mathbf{n}\} & \text{if } F(\neg\alpha) \in \Upsilon; \\ h(\alpha) \in \{\mathbf{0}, \mathbf{n}\} & \text{if } F(\alpha) \in \Upsilon; \\ h(\alpha) \in \{\mathbf{0}, \mathbf{b}\} & \text{if } T(\neg\alpha) \in \Upsilon. \end{cases}$$

In any other case $h(\alpha)$ is arbitrary, for $\alpha \in Var$. Then (‡) holds for any complex formula α.

Proof. By induction on the complexity of α (see Definition 3.2). We analyze just some cases, the others are analogous.

(i) $\alpha \in Var$ or $\alpha = \bot$. The result is clearly true.

(ii) $\alpha = \neg\beta$. If $T(\alpha) \in \Upsilon$ then $T(\neg\beta) \in \Upsilon$ and so $h(\beta) \in \{\mathbf{0}, \mathbf{b}\}$, by the induction hypothesis. Thus, $h(\alpha) \in \{\mathbf{1}, \mathbf{b}\}$. If $T(\neg\alpha) \in \Upsilon$ then $T(\neg\neg\beta) \in \Upsilon$ and so $T(\beta) \in \Upsilon$, since $\mathcal{T}$ is completed. Thus $h(\beta) \in \{\mathbf{1}, \mathbf{b}\}$, by the induction hypothesis and then $h(\alpha) \in \{\mathbf{0}, \mathbf{b}\}$. The other cases are proved analogously.

(iii) $\alpha = \beta \wedge \gamma$.

Suppose that $T(\alpha) \in \Upsilon$. Since $\mathcal{T}$ is completed then the rule for $T(\beta\wedge\gamma)$ was eventually used. Then $T(\beta), T(\gamma) \in \Upsilon$, and by the induction hypothesis, $h(\beta), h(\gamma) \in \{\mathbf{1}, \mathbf{b}\}$, hence $h(\beta \wedge \gamma) \in \{\mathbf{1}, \mathbf{b}\}$.

If $T(\neg\alpha) \in \Upsilon$, then the rule for $T(\neg(\beta \wedge \gamma))$ was eventually used splitting into two branches. One of them is a sub-branch of θ, thus one of the following conditions hold:

(iii.1) $T(\neg\beta) \in \Upsilon$. Then $h(\neg\beta) \in \{\mathbf{1}, \mathbf{n}\}$, by the induction hypothesis. Then, $h(\neg\alpha) \in \{\mathbf{1}, \mathbf{n}\}$, by the definition of of the operations in $\mathcal{TML}$ and by the fact that $h(\neg\alpha) = h(\neg\beta \vee \neg\gamma)$.

(iii.2) $T(\neg\gamma) \in \Upsilon$. Analogous to (iii.1).

The proof of the remaining sub cases, namely $F(\alpha) \in \Upsilon$ and $F(\neg\alpha) \in \Upsilon$), are analogous.

(iv) $\alpha = \Box\beta$.

We just analyze the case where $F(\Box\beta) \in \Upsilon$. Since $\mathcal{T}$ is completed, the rule $F(\Box\beta)$ was used splitting into two branches. Then, one of the following conditions hold.

(iv.1) $F(\beta) \in \Upsilon$. By the induction hypothesis, $h(\beta) \in \{\mathbf{0}, \mathbf{n}\}$ and then $h(\Box\beta) = \mathbf{0} \in \{\mathbf{0}, \mathbf{n}\}$.

(iv.2) $T(\neg\beta) \in \Upsilon$. By the induction hypothesis, $h(\beta) \in \{\mathbf{0}, \mathbf{b}\}$ and therefore $h(\Box\beta) = \mathbf{0} \in \{\mathbf{0}, \mathbf{n}\}$. ⊣

Finally,

Theorem 6.6 (Completeness of $\mathbb{T}'$). *If $\models_{\mathcal{TML}} \alpha$ then $\vdash_{\mathbb{T}'} \alpha$.*

Proof. Let α be a formula. Note that (as in Proposition 5.7), if there is a complete open tableau in $\mathbb{T}'$ for $F(\alpha)$, then $\not\models_{\mathcal{TML}} \alpha$. So, if $\models_{\mathcal{TML}} \alpha$ then every tableau in $\mathbb{T}'$ for $F(\alpha)$ is closed and therefore, by Proposition 6.2, there exists a closed tableau for $F(\alpha)$. That is, $\vdash_{\mathbb{T}'} \alpha$. ⊣

COROLLARY 6.7. *Let α be a formula. Then every completed tableau in $\mathbb{T}'$ for $F(\alpha)$ is open, or every completed tableau in $\mathbb{T}'$ for $F(\alpha)$ is closed.*

PROOF. Analogous to the proof of Corollary 5.9, but now by using Proposition 6.5. ⊣

PROPOSITION 6.8. *Suppose that $\vdash_{\mathbb{T}'} \alpha$. Then every completed tableau for $T(\neg\alpha)$ is closed.*

PROOF. It is similar to the proof of Proposition 5.10, but now by using Corollary 6.4 and Proposition 6.5. ⊣

Now it can be proved, only by tableau tools, the admissibility of the Rule of Necessitation (Nec) (recall Lemma 2.8). In order to see this, suppose that $\models_{\mathcal{TML}} \alpha$, and start a tableau in $\mathbb{T}'$ for $F(\Box\alpha)$. By Definition 6.1, two branches are created: one with $F(\alpha)$ and the other with $T(\neg\alpha)$. By Proposition 6.2, both tableaux will eventually terminate. Using Theorem 6.6, Corollary 6.7 and Proposition 6.8, both tableaux are closed. From this, the original tableau in $\mathbb{T}'$ for $F(\Box\alpha)$ is closed. This shows that $\models_{\mathcal{TML}} \Box\alpha$, by Corollary 6.4.

7. Concluding remarks

In this paper, we have continued the study of $\mathcal{TML}$ under a proof-theoretic perspective. First, we showed that the natural deduction system $\mathbf{ND}_{\mathcal{TML}}$ introduced in [17] admits a normalization theorem. Later, taking advantage of the contrapositive implication for the tetravalent modal algebras introduced in [18], we defined a decidable tableau system for $\mathcal{TML}$. The original language of the logic of TMAs — in particular, the language of logic $\mathcal{TML}$ — does not have an implication connective as a primitive connective. However, using the contrapositive implication for $\mathcal{TML}$ as a primitive connective and following a general techniques introduced in [7], we defined a sound an complete tableau system for this logic. Finally, inspired by this last system, we provided a sound and complete tableau system for $\mathcal{TML}$ in the original language. These two tableau systems constitute new (proof-theoretic) decision procedures for checking validity in the variety of tetravalent modal algebras, besides the four-valued truth-tables of $\mathcal{TML}$ and the one available in terms of the cut-free sequent system introduced in [17].

An interesting task (for future work) would be to provide a natural deduction system for $\mathcal{TML}$ in terms of negation and implication $(\neg, \succ)$. This would involve to find proper introduction/elimination rules for $\succ$ which does not seem an easy job in the light of the unusual properties of this implication.

Also, we propose to extend $\mathcal{TML}$ to first-order languages. This would provide a suitable context for studying and developing its potential applications in computer science as envisaged by Antonio Monteiro fifty years ago.

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References

[1] Anderson, A. R., and N. D. Belnap (with contributions by thirteen others), *Entailment: the Logic of Relevance and Necessity*, Vol. II, Princeton University Press, 1992.

[2] Arieli, O., and A. Avron, "The value of the four values", *Artificial Intelligence* 102(1), 1998: 97–141.

[3] Avron, A., J. Ben-Naim, and B. Konikowska, "Cut-free ordinary sequent calculi for logics having generalized finite-valued semantics", *Logica Universalis* 1(1), 2006: 41–69.

[4] Belnap, N., "How computers should think", pages 30–56 in G. Ryle (ed.), *Contemporary Aspects of Philosophy*, Oriol Press, 1976.

[5] Béziau, J.-Y., "A new four-valued approach to modal logic", *Logique et Analyse* 54(213), 2011: 109–121.

[6] Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic*, Cambridge University Press, 2001.

[7] Caleiro, C., W. A. Carnielli, M. Coniglio, and J. Marcos, "Two's company: 'The humbug of many logical values' ", pages 169–189 in J.-Y. Béziau (ed.),*Logica Universalis*, Basel: Birkhäuser, 2005.

[8] Caleiro, C., and J. Marcos, " Classic-like analytic tableaux for finite-valued logics", pages 268–280 in H. Ono, M. Kanazawa and R. de Queiroz (eds.), *Logic, Language, Information And Computation*, Lecture Notes in Computer Science, vol. 5514, Springer, 2009.

[9] Cantú, L. M., and M. Figallo, "Cut-free sequent-style systems for a logic associated to involutive Stone algebras", *Journal of Logic and Computation* 33(7), 2023: 1684–1710. DOI: 10.1093/logcom/exac061

[10] Carnielli, W. A., M. E. Coniglio, D. Gabbay, P. Gouveia and C. Sernadas, *Analysis and Synthesis of Logics*, vol. 35 in the Applied Logic Series, Springer, 2008.

[11] Carnielli, W. A., M. E. Coniglio and J. Marcos, J., "Logics of formal inconsistency", pages 1–93 in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, vol. 14, Springer, 2007.

[12] Carnielli, W. A., and J. Marcos, "A taxonomy of **C**-systems", pages 1–94 in W. A. Carnielli, M. E. Coniglio and I. M. L. D'Ottaviano (eds.), *Paraconsistency: The Logical Way to the Inconsistent*, volume 228 of "Lecture Notes in Pure and Applied Mathematics", Marcel Dekker, New York, 2002.

[13] Carnielli,W. A., and A. Rodrigues, "An epistemic approach to paraconsistency: A logic of evidence and truth", *Synthese* 196(9), 2017: 3789–3813.

[14] Coniglio, M., and M. Figallo, "Hilbert-style presentations of two logics associated to tetravalent modal algebras", *Studia Logica* 102(3), 2014: 525–539.

[15] da Costa, N. C. A., *Inconsistent Formal Systems* (in Portuguese), Habilitation Thesis, 1963. Republished by Editora UFPR, Curitiba, 1993.

[16] da Costa, N. C. A., "Calculs propositionnel pour les systèmes formels inconsistants", *Comptes Rendus de l'Académie de Sciences de Paris*, série A, vol. 257, 1963: 3790–3792.

[17] Figallo, M., "Cut-free sequent calculus and natural deduction for the tetravalent modal logic", *Studia Logica* 109(6), 2021: 1347–1373. DOI: 10.1007/s11225-021-09944-3

[18] Figallo, A. V., and P. Landini, "On generalized I-algebras and 4-valued modal algebras", *Reports on Mathematical Logic* 29, 1995: 3–18.

[19] Figallo, A. V., and A. Ziliani, "Symmetric tetra-valued modal algebras", *Notas Soc. Mat. Chile* 10(1), 1991: 133–141.

[20] Font, J. M., and M. Rius, "A four-valued modal logic arising from Monteiro's last algebras", pages 85–92 in *Proc. 20th Int. Symp. Multiple-Valued Logic* (Charlotte, 1990), The IEEE Computer Society Press, 1991.

[21] Font, J. M., and M. Rius, "An abstract algebraic logic approach to tetravalent modal logics", *The Journal of Symbolic Logic* 65(2), 2000: 481–518.

[22] Gastaminza, M. L., and S. Gastaminza, "Characterization of a De Morgan lattice in terms of implication and negation", *Proc. Japan Acad.* 44(7), 1968: 659–662.

[23] Jansana, R., "Propositional consequence relations and algebraic logic, in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy* (Spring 2011 Edition). http://plato.stanford.edu/archives/spr2011/entries/consequence-algebraic/

[24] Lemmon, E. J., and D. Scott, *An Introduction to Modal Logic. The Lemmon Notes*, K. Segerberg (ed.), vol. 11 of American Philosophical Quarterly Monograph series, Basil Blackwell, Oxford, 1977.

[25] Loureiro, I., "Álgebras modais tetravalentes", PhD thesis, Faculdade de Ciências de Lisboa, 1983.

[26] Loureiro, I., "Homomorphism kernels of a tetravalent modal algebra" *Portugaliae Mathematica* 39, 1980: 371–377.

[27] Montgomery, H., and R. Routley, "Contingency and non-contingency bases for normal modal logics", *Logique et Analyse* 9(35/36), 1966: 318–328.

[28] Odintsov, S. P., and H. Wansing, "Modal logics with Belnapian truth values", *Journal of Applied Non-Classical Logics* 20(3), 2010: 279–301.

[29] Odintsov, S. P., and H. Wansing, 'Disentangling FDE-based paraconsistent modal logics, *Studia Logica* 105(6), 2017: 1221-–1254.

[30] Priest, G.. "Many-valued modal logics: A simple approach", *The Review of Symbolic Logic* 1(2), 2008: 190–203.

[31] Rivieccio, U., A. Jung A and R. Jansana, "Four-valued modal logic: Kripke semantics and duality", *Journal of Logic and Computation*, 27(1), 2015: 155–199.

[32] Smullyan, R. M., *First-Order Logic*, Berlin: Springer-Verlag, 1968. Corrected republication by Dover Publications, New York, 1995.

[33] Troelstra, A. S., and H. Schwichtenberg, *Basic Proof System*, Cambridge, UK: Cambridge University Press, 1996.

[34] Ziliani, A., "Algebras de De Morgan modales 4-valuadas monádicas", PhD thesis, Universidad Nacional del Sur, Bahía Blanca, Argentina, 2001.

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