PROSPECTIVE TEACHER'S SPECIALIZED CONTENT KNOWLEDGE ON DERIVATIVE

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Mathematics teacher knowledge has been widely studied, and recently a remarkable advancement has been reach with the proposal of the "Mathematical Knowledge for Teaching" (MKT) model for describing the complex of knowledge that a teacher should have to teach a specific mathematics topic. Nonetheless there are still questions to be addressed, such as, how or under what criteria can the MKT be assessed? How the teacher educators can help the prospective teacher to develop the different components of the MKT? How are related the different components of MKT? In this report, we have tackled, though partially, such questions, by advancing specific criteria to explore the prospective teachers' knowledge about the notion of derivative: common, specialized and extended knowledge. In this report we inform specifically on the specialized content knowledge.

BACKGROUND

The mathematical and didactical education of prospective teachers is a very pressing issue for teachers' educators. One of the problems that have raised a great deal of interest is to identify the didactic-mathematical knowledge required for the prospective teachers to teach mathematics. On this regard, a great deal of research has been conducted to identify the components of the web of knowledge that a teacher should know in order to develop his/her practice efficiently and to facilitate the students' learning.

Some researchers have proposed a number of alternatives where some features that make up the teachers' knowledge can be identified. The works of Shulman (1986), Fennema & Franke (1992) and Ball (2000), show a multifaceted vision on the construction of the knowledge required to teach. More recent researches such as those of Ball, Lubienski & Mewborn (2001), Llinares & Krainer (2006), Ponte & Chapman (2006), Philipp (2007), Sowder (2007), Ball, Thames & Phelps (2008), Hill, Ball & Schilling (2008) and Sullivan & Wood (2008), show the nonexistence of an universal agreement on a theoretical frame to describe the mathematics' teacher knowledge (Rowland & Ruthven, 2011). This fact is a cause of concern not only for the prospective teachers' education and for the professional development of inservice teachers, but also for the researchers community, because it is important to establish a general understanding on what meanings entail the content knowledge and how it affects the practice of teaching. It is difficult to have a coherent approach for a program of teacher education if the role of the teacher knowledge, the features implied and how they interact in the mathematics teaching process (Petrou & Goulding, 2011) are not well understood. The question is: how to determine such didactic-mathematical knowledge based on models that include categories too "wide"? As Godino (2009) points out, the various models on the mathematical knowledge for teaching, informed by the researches in mathematics education, include categories too "wide" and disjoint, that calls for models that allow conducting a more precise analysis of each knowledge component that are put into effect in an effective teaching of mathematics. Besides, the latter will allow orienting the design of formative actions and the elaboration of tools to assess the mathematics teachers' knowledge.

In this report we offer a partial answer to questions such as: How or under which criteria can the MKT be assessed? How teachers' educators can help the prospective teachers to develop the different MKT components? How the different MKT components are related among them? To propose an answer to these questions, we use the theoretic tools provided by the Onto-Semiotic Approach (OSA) (Godino, Batanero & Font, 2007) to knowledge and instruction. We have designed and applied a questionnaire to explore some relevant features of the epistemic facet of the didactic-mathematical knowledge of prospective teachers, on the derivative, which includes, according to the Ball and colleagues' model, the common content knowledge, specialized content knowledge and extended content knowledge. Specifically, we focus on the specialized content knowledge, for which we propose two levels of analysis and different categories of analysis.

THE DIDACTIC-MATHEMATICAL MODEL

In this research we use the Didactic-Mathematical Knowledge Model (DMK) proposed by Godino (2009) within the Ontosemiotic approach to cognition and instruction (Godino, Batanero & Font, 2007). This model for the DMK includes six facets or dimensions for the didactic-mathematical knowledge, which are involved in the teaching and learning of mathematics specific topics: 1) *Epistemic*: components of the institutional implemented meaning (problems, languages, procedures, definitions, properties, justifications); 2) *Cognitive:* development of the personal meanings (learning); 3) *Affective*: the emotional states (attitudes, emotions, motivations) of each student regarding not only the mathematics objects but also the planed study process, and its distribution over time; 4) *Interactional*: sequence of interactions between the teachers and students, oriented at the fixation and negotiation of meanings; 5) *Mediational*: distribution over time of the technological resources used and distribution of time for the actions and processes involved; and 6) *Ecological*: system of relations with the social, political, economic context that underlies and affects the study process.

For each of the above facets, four levels of analysis are considered. These levels allow the analysis of the teacher's DMK according to the type of information required to take instructional decisions. The aforementioned levels are: 1) *Mathematical and didactical practices*; description of the actions performed to solve the mathematics tasks proposed to contextualize the content and to promote learning. The general lines of action of the teacher and students are also described; 2) *Configuration of objects and processes* (mathematical and didactical), description of mathematics objects and processes that intervene during the mathematic practices, as well as those which emerge out of them. The purpose of this level is to describe the complexity of objects and meanings that intervene in the mathematics and didactics practices. Such complexity is an explanatory factor not only for the meaning conflicts but also for the learning progression; 3) *Norms and meta-norms*, identification of the web of rules, habits, norms that regulate and facilitate the study process, and that affect each facet and its interactions; and 4) *Suitability*, identification of possible improvements of the study process, that increment the didactic suitability.

Due to the fact that our research was carried out with prospective teachers we focus the analysis on the epistemic facet and some aspects of the cognitive facet, based on the students' answers to a task. Investigating the levels three and four of the analysis described above is beyond the scope of this report.

Subjects and Context

The questionnaire was administered to a sample of 53 students enrolled in the final modules (sixth and eighth semester) of the degree in mathematics teaching offered by the Universidad Autónoma de Yucatán (UADY) in Mexico. This is a four-year degree (8 semesters). The School of Mathematics of the UADY is responsible for training teachers to work at higher secondary or university level in the state of Yucatan (Mexico). The 53 students who responded to the questionnaire had studied differential calculus in the first semester of their degree course, and they had subsequently completed other modules related to mathematical analysis (integral calculus, vector calculus, differential equations, etc.). They had also studied subjects related to the teaching of mathematics. These 53 students constituted the entire population of students with these characteristics in the University of Yucatán.

The EF-DMK-Derivative Questionnaire

The questionnaire, which we have call *EF-DMK-Derivative* (Pino-Fan, Godino, Font and Castro, 2012, pp. 298-299), is made up of seven tasks, and has been designed to assess certain relevant features of the epistemic facet of prospective secondary teachers' didactic-mathematical knowledge (DMK) on the derivative. According to Ball and colleagues model (Ball, Lubienski & Mewborn, 2001; Hill, Ball & Schilling, 2008) this epistemic facet comprises three types of knowledge: common content knowledge, specialized content knowledge and extended content knowledge.

Three criteria were considered for the selection of the seven tasks that make up the questionnaire. The first states that the tasks must include a wide range of meanings related to the derivative; the second states that, for the resolutions, some representation means should be use, and the third states that the type of knowledge must include: common content knowledge, specialized knowledge and extended knowledge. The description of the features and content assessed in every task included in the questionnaire, can be seen in Pino-Fan, et al., (2012, pp.299-301). The methodological choice to build and implement a written questionnaire has the advantage of being applied to a relatively large sample, compared with study based in interviews to a few students, although it is not possible to deepen in the evaluation of the others facets and nuances of the didactic-mathematical knowledge.

The Specialized Content Knowledge

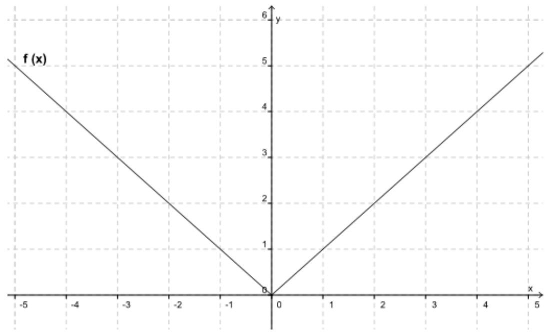
Ball, Thames and Phelps (2008) propose that the specialized content knowledge is the "mathematical knowledge and skill unique to teaching" (p. 400). This knowledge includes "how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill, Ball y Schilling, 2008, p. 377-378). We agree with this approach to the specialized content knowledge; nonetheless the question that arises is: what specific criteria allow us to analyze and to improve such knowledge required by the prospective teachers? One of the fundamental features of the specialized content knowledge tasks, included in the questionnaire, is the reflection carried out by the prospective teachers, on mathematical objects, its meanings and the complex relations among them. These web of complex relations are put into effect while teaching and learning mathematics. The relationship between objects and meanings are fixed and operationalized by means of the notion *configuration of objects and meanings* (Godino, et al., 2007). Such notion favors not only the systematic identification of a number of procedures to solve the mathematic tasks (including the identification of representations, concepts and properties), but also the identification of both, procedures justifications and properties used in solving them. Additionally, the aforementioned analysis, not only of the tasks but also of the didactic variables that intervene and orient the reflection, on both the possible generalizations, or particularizations, and the connections to other mathematical contents (Godino, 2009).

Thus, in our model for the didactic-mathematical knowledge, two levels are proposed for the specialized content knowledge. In the first level, the prospective teachers *should use not only, different* representations, concepts, propositions, procedures, and justifications, but the range of mathematical object' meanings of the mathematic concept under study – the notion of derivative –. The second level refers to the *teachers' competency to identify knowledge* (language elements, concepts/ definitions, properties/propositions, procedures and justifications) put into effect during the resolution of tasks on the derivative. It is clear that the specialized content knowledge implies the common content knowledge and some features of the extended knowledge.

The item a) of Task 2 (Fig. 1) assesses the common content knowledge, and the item b) assesses the extended content knowledge. The common content knowledge is used when the prospective teacher answers this item without providing any justifications nor using any representation. The extended content knowledge is assessed when the prospective teacher has to generalize the initial task about the derivability of the absolute value function at $x=0$, on the basis of valid justifications for the proposition "the graph of a derivable function cannot have peaks" by defining the derivative as the limit of the increment quotient. On one side, items b) and c) refer to the first level of the specialized content knowledge, because prospective teachers may solve them making use of both different representations (graphic, symbolic and verbal), providing valid justifications for their procedures. On the other side, item e) explores the second level of the specialized knowledge, because the prospective teacher must both, solve the aforementioned tasks and identify the web of knowledge that are put into effect in its resolution.

Task 2

Consider the function $f(x)=|x|$ and its graph.



a) For what values of x is $f(x)$ derivable?
b) If it is possible, calculate $f'(2)$ and draw a graph of your solution? If it is not possible, explain why.
c) If it is possible, calculate $f'(0)$ and draw a graph of your solution? If it is not possible, explain why.
d) Based on the definition of the derivative, justify why the graph of a derivable function cannot have 'peaks' (corners, angles).
e) What knowledge is put in to play when solving the above items of this task?

Figure 1: Task 2 from the *EF-DMK-Derivative* Questionnaire

Figure 2 shows Task 5, that assess the first level of the specialized knowledge, for prospective teachers must use different derivative meanings in its resolution: slope of a tangent line and, instant rate of change. At first glance it seems to be one of the "drill exercises" that are commonly found in the high school calculus text books, where it suffices to apply some theorems and propositions on the derivative to solve it. Due to the latter, both item a) and b), individually, assess features of the prospective teachers' common content knowledge. Nonetheless, the main task objectives are double: first, to explore, globally, the mathematical activity carried out by the prospective teachers, and, second, to test the connections or associations among the different derivative meanings established by them.

Task 5

Given the function $y=x^3-\dfrac{x^2}{2}-2x+3$

a) Find the points on the graph of the function for which the tangent is horizontal.
b) At what points is the instantaneous rate of change of y with respect to x equal to zero?

Figure 2: Task 5 from the *EF-DMK-Derivative* Questionnaire

RESULTS: ANALYSIS AND DISCUSSION

We will present the analysis of an answer provided by a prospective teacher (A), which exemplifies one of the solution typologies identified on the prospective teachers set of solutions to Task 2. On such analysis the primary objects (language, concepts, properties, procedures and justifications), and processes than intervene on both, the statement and on the tasks solutions, are identified. We base our analysis on the levels 1 and 2 of teacher's knowledge described above, which refer, both to *the didactical and mathematical practices*, and to the configuration of objects, respectively. In this section we do not present the "global" results of the entire questionnaire, not even the complete results for the two tasks that we presented. It is because we do pretend to show the usefulness of both the criteria and methodology of analysis proposed; on the other hand, the space constraints make it impossible to present the results in its entire extension.

Analysis to Cognitive Configurations Subjacent of the Task 2

In regard to cognitive configuration used by the prospective teachers to provide a solution to Task 2, three types of resolutions were identified. Each type of configuration is associated to a specific configuration of objects and processes. We have named these three types of cognitive configuration as: *graphic-verbal, technic and formal.* A high percentage of prospective teachers, 88,6% and 54,7%, respectively, provided a configuration *graphic-verbal* to items a) and c) (e.g., "…it is not derivable at *x=0* because an infinite number of tangent lines can be traced, to the function on that point"). For section b), the majority of prospective teachers, 62, 3%, provided a *technic* configuration (using both the derivative rules and the definition of the absolute value function). One preservice teacher (1,9%) provided a *formal* solution, using the derivative meaning as an instant rate of change (limit of the increment quotient), for the first four items in the task.

Figure 3 shows the solution provided by the prospective teacher that has a *graphic-verbal configuration* associated. In regard to the mathematic practice performed by the prospective teacher (A), it can be observed in Fig.3 that he begins his solution process with a visual justification of the property "the derivability of an absolute value function". Generally speaking, his solution is made of verbal descriptions based on the graph of the function $f(x)=|x|$.

Cognitive Configuration:

In the solution provided by the prospective teacher (A), it is possible to identify the use of a great deal of linguistic features such as: the use of natural language (verbal descriptions), some symbolic entries such as "$\Re-\{0\}$" or the rule of derivation by parts $f'(x)=\begin{cases} x & si\, x>0 \\ -x & si\, x<0 \\ 0 & si\, x=0 \end{cases}$ of an absolute value function [item c)]. In the same venue, student (A) uses graphic elements [items b) and c)] to "explain" his analysis. These linguistic elements refer to a group of concepts/definitions, propositions/properties that are illustrated in what follows.

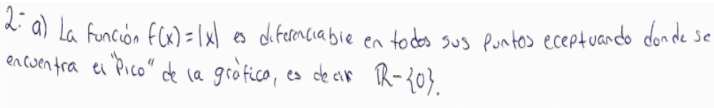
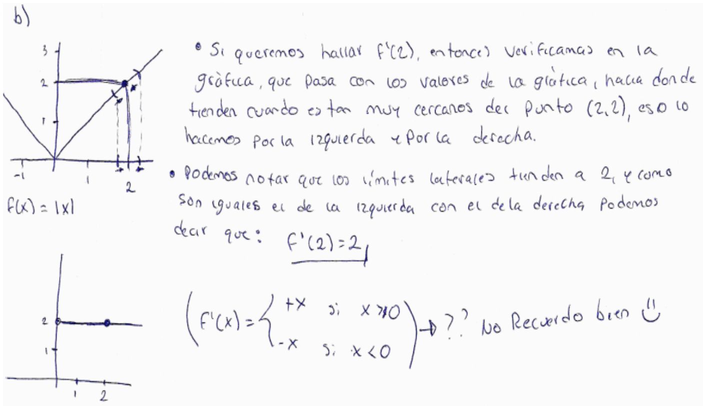
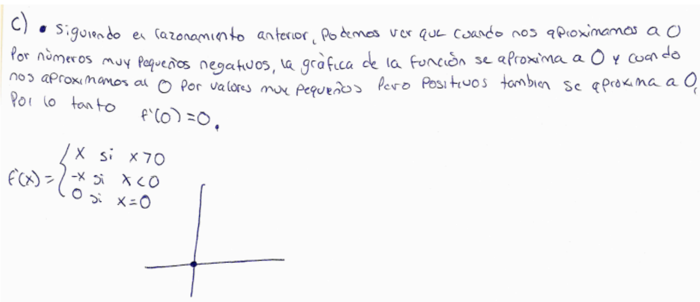
	a) The function $f(x)=\|x\|$ is differentiable at all points except where finds the "peak" of the graph, that is $\Re-\{0\}$.
	b) If we want to find $f'(2)$ then we verify in the graph, what happens to the values of the graph, towards where it tends [the values] when they are very close of the point (2,2), we do it that for the left and for the right. We can note [in the graph] that the lateral limits tend to 2, and as they are equal, both to the left and the right, we can say that: $f'(2)=2$ $\left(f'(x)=\begin{cases}+x & si\, x\geq 0\\ -x & si\, x<0\end{cases}\right)\rightarrow$?? I do not remember.
	c) Following the above reasoning, we can see that when we approach to 0 by very smal negative numbers, the graphic of the function approaches to 0 and when we approach to 0 by very small positive numbers it also approaches to 0. Therefore $f'(0)=0$.

Figure 3: Graphic-verbal resolution of task 2. Student A

Among the concepts and definitions used by the prospective teacher, we can underline those of function (absolute value), domain (of the derivative function and represented by $\Re-\{0\}$), approximation (to a specific point of the absolute value function, in this case to *x = 2* and *x = 0,* taking values close to those points), and the derivative of the absolute value function (wrongly taken as $f'(x)=\begin{cases} x & si\, x>0\\ -x & si\, x<0\\ 0 & si\, x=0\end{cases}$). The one side limits, and the bilateral limits, to the points *x = 2* and *x = 0,* though calculated correctly (if the question were to calculate the limits to the absolute function on such points), based on the graph, were incorrectly used by the prospective teacher (A), when calculating the derivative of the function in the points *x = 2* and *x = 0.* This misuse of one side limits, to calculate the derivative, seems to be based on the misinterpretation or misunderstanding of a proposition that refers to the relationship between continuity and derivability: a derivable function is always a continuous function, but a continuous function could be not derivable. This is put in evidence with the procedure and the ensuing justification to calculate *f' (2)*: "If we want to find *f'(2)*=2, then we verify in the graph what happens to the graph values, where do they tend when they are very close to (2, 2); we do it for both sides: left and right. We can see that the lateral limits tend to 2, and due they are the same, both on the left and on the right, we can say that f '(2) = 2".

The procedure and justification provided by the prospective teacher to calculate *f'(2)*=2 can also be seen in the graphic representation given to item b) (Fig.3). The misunderstanding is made more evident when the prospective teacher points out both the procedure and the justification to solve item c), which ask for a verification to be carried out on the derivability of the absolute value function at *x*=0. The student says: "Following the preceding reasoning, we can see that when we get close to 0, using very small negative numbers, the graph of the function gets close to 0; and when we get close to 0, using very small but positive numbers, the graph of the function also gets close to 0, thus $f'(0)=0$". Among other properties used in the student's solution we can highlight the derivability at zero of the absolute value function, which is visually justified, as follows, "The function $f(x)=|x|$ is derivable in every point except in those points where a "corner" is found on the graphic …".

The answers that we have included in this type of cognitive configuration, focus on procedures and justifications based on the visual analysis of the graphic features of the function, as in the example provided (Task 2). Another answer type, quite common, were those where the no derivability of the absolute value function, at *x*= 0, is justified by tracing an "infinity" number of tangents to the function at that point. Regarding item e), that assesses level two of the specialized content knowledge, it is clear that, as no previous instruction has been offered, to the prospective teachers, on the identification of previous and emerging knowledge involved in the task, only a limited number of concepts, such as function, absolute value and derivative at a point, were provided by the prospective teachers.

FINAL REFLECTIONS

The results obtained through the implementation of the questionnaire *EF-DMK-Derivative* show that the prospective teachers manifest difficulties to solve tasks related, not only to the specialized and extended content knowledge but also, with the common content knowledge. It is clear that the prospective teachers have a better performance when solving tasks that entail the use of the derivative as the slope of a tangent line. This was confirmed when the prospective teachers solve tasks such as the fifth (Fig.2) where their answers show a disconnection among the different derivative meanings. The manifested inadequacies of knowledge, justify the pertinence of designing specific formative actions in order to develop the epistemic facet of the didactic-mathematical knowledge on the derivative. The development could be accomplished first, by designing a teaching process for the derivative, which stresses the derivative global meaning (Pino-Fan, Godino y Font, 2011). Secondly, the two levels of the specialized content knowledge should be considered, both in its *application level* (use of linguistic elements, concepts, properties, procedures and justifications, as well as the use of different derivative partial meanings to solve the tasks) and in its *identification level*. The latter refers to the competency to identify mathematics objects, their meanings and the relation among them. This prospective teacher's competence would allow a suitable learning management of their future students.

These two levels for the specialized content knowledge are closely related, within the model DMK, to other facets for the teacher's knowledge. The level one, related to the application, is connected to the interactional and mediational facets (knowledge of content and teaching), because the mastery of this level of specialized content knowledge about specific topics, such as the derivative, gives the teacher the resources to perform efficiently his future professional tasks. The level two, identification, is related to the cognitive and affective facets (knowledge of content and students), because it entitles the teacher to detect (previously, during and after the teaching activity) features such as: the mathematical knowledge involved, mathematical objects and meanings, conflicts and mistakes that can arise to his/her future students. This identification competence may lead the prospective teacher to manage the students' learning in a more effective way. Finally, the Didactic-Mathematical Knowledge Model (DMK), offers the tool "configuration of primary mathematics objects" that allows analysing and categorizing some features of the epistemic facet of didactic-mathematical knowledge manifested by prospective teachers. In this report we have focused on the specialized content knowledge, how to analyse it and we have offered suggestions to improve it. The analysis presented herein, as a way of example, is what prospective teachers are expected to demonstrate at the (beginning of, during, and end of) second stage, and its development should begin since the early stages of teacher's professional training.

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