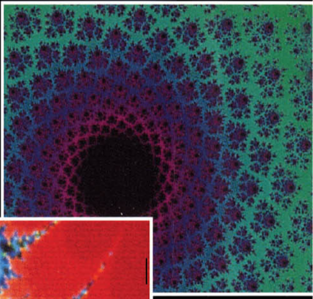
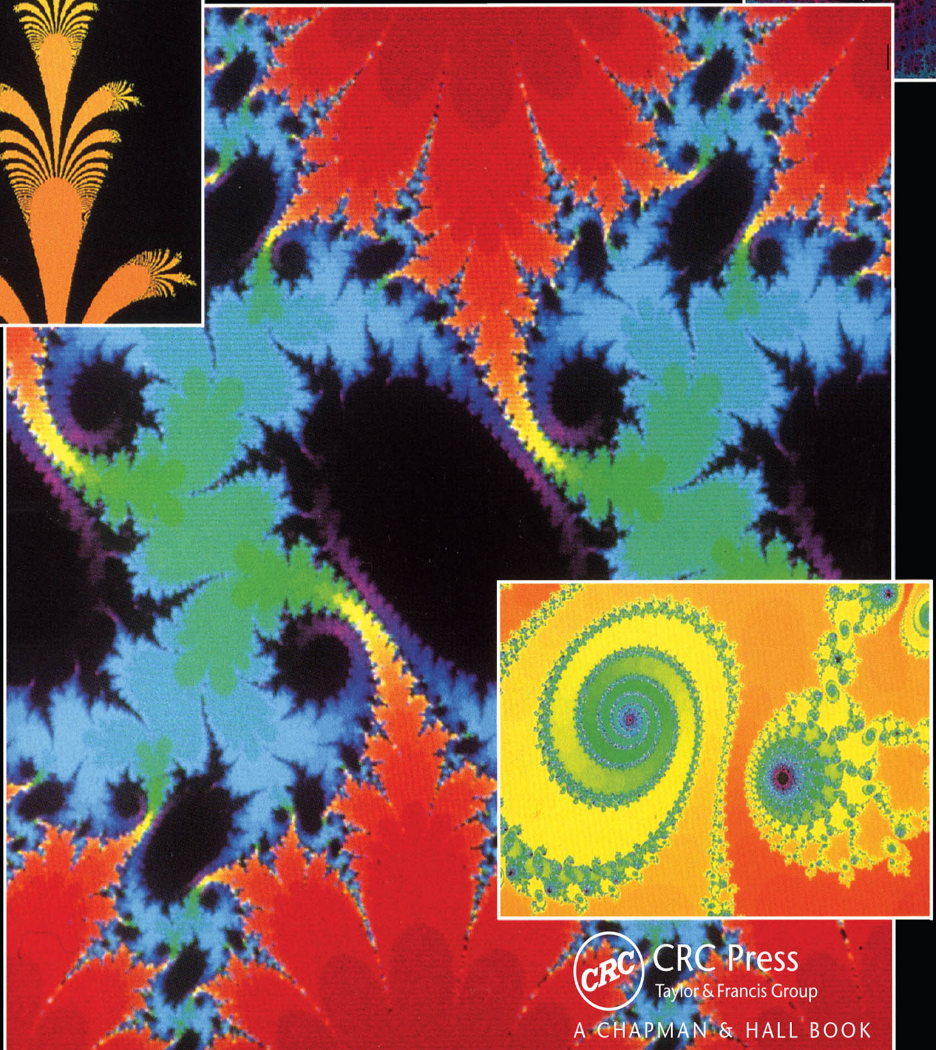
STUDIES IN NONLINEARITY

A FIRST COURSE IN CHAOTIC DYNAMICAL SYSTEMS

Theory and EXPERIMENT





CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

ROBERT L. DEVANEY

A FIRST COURSE IN
CHAOTIC DYNAMICAL SYSTEMS
THEORY AND EXPERIMENT

Studies in Nonlinearity

Series Editor: Robert L. Devaney

Ralph H. Abraham and Christopher D. Shaw, *Dynamics: The Geometry of Behavior* (1992)

Robert L. Devaney, James F. Georges, Delbert L. Johnson, *Chaotic Dynamical Systems Software* (1992)

Nicholas B. Tufillaro, Tyler Abbott, Jeremiah Reilly, *An Experimental Approach to Nonlinear Dynamics and Chaos* (1992)

A FIRST COURSE IN CHAOTIC DYNAMICAL SYSTEMS

THEORY AND EXPERIMENT

Robert L. Devaney
Boston University

THE ADVANCED BOOK PROGRAM

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

First published 1992 by Westview Press

Published 2018 by CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

CRC Press is an imprint of the Taylor & Francis Group, an informa business

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Cover Designer: *Nancy Brescia*

Library of Congress Cataloging-in-Publication Data

Devaney, Robert L., 1948–
A first course in chaotic dynamical systems: theory and experiment / Robert L. Devaney.
p. cm. – (Studies in nonlinearity)
Includes bibliographical references and index.
1. Differentiable dynamical systems. 2. Chaotic behavior in systems. I. Title. II. Series: Addison-Wesley studies in nonlinearity.
QA614.8.D49 1992 515'.352–dc20 91-38310

ISBN 13: 978-0-201-55406-9 (pbk)

The Advanced Book Program

Preface

This book is an undergraduate text in dynamical systems. It is aimed at students who have taken at least one year of calculus, but not necessarily any higher level mathematics courses. The book is an outgrowth of a one-semester course taught by the author at Boston University for the past several years. Students in the course ranged from beginning mathematics majors to senior level science and engineering students, from English majors who take mathematics courses "because they are fun" to prospective secondary school mathematics teachers. With the possibility of including computer experimentation, laboratory reports, group projects, together with some wonderfully accessible mathematics, a course in dynamical systems can easily be tailored to such a diverse audience.

I feel that it is desirable to introduce students to this field at an early stage in their mathematical careers. There are a number of reasons for this. First, the study of dynamics offers mathematicians an opportunity to expose students to contemporary ideas in mathematical research. Many of the ideas and theorems in this book were first discovered within the students' lifetimes; many of the pictures included herein were first viewed within the past decade. To emphasize the contemporary nature of the field, I have included snapshots and brief biographies of a number of individuals who have made recent contributions to the ideas in this book.

Much of the current interest in dynamics centers around the chaotic behavior that occurs when a simple function is iterated. In this book, the emphasis is on the simplest possible setting in which this occurs, namely iteration of real or complex quadratic polynomials. By dealing mainly with this special case, the material becomes accessible to students who do not have a background in topology or analysis. For example, with only the knowledge of how to multiply complex numbers, students can comprehend the basic mathematical ideas behind such topics as the Julia set or the Mandelbrot set.

A second pedagogical reason to introduce dynamics early in the curriculum is that the course may serve as a bridge between the low–level, often nonrigorous calculus courses and the much more demanding real analysis courses. All too often, students see no connection between the calculus courses that occupy their early years as an undergraduate and the more advanced analysis courses they take later. Dynamics offers students an opportunity to use and build upon their knowledge of calculus and, at the same time, to see in a very concrete setting many of the important topics from basic analysis. I have found that students begin to appreciate the need for abstract metric spaces when they first encounter an object like the space of sequences in symbolic dynamics. They realize the importance of an ϵ-δ definition of continuity when they try to analyze the shift map. Cantor sets become natural objects to study when students see how often they arise in dynamics. Indeed, students who have studied analysis prior to dynamics often remark that they now know why all that abstraction is important!

To accommodate beginning students, this book is structured so that students are gradually introduced to more and more sophisticated ideas from analysis as the chapters unfold. It starts with only a few elementary notions that can be explained using graphical methods or differential calculus. Proofs are introduced slowly at first, and plenty of routine exercises are included. Later come concepts such as dense sets and metric spaces. These concepts arise naturally in the setting of simple dynamical systems, so they can be introduced in a manner that is both concrete and accessible. I feel that this approach is beneficial to those students who do not contemplate future graduate study in mathematics—they see some of the principal ideas of analysis but not in the setting of an intensive course designed for prospective PhDs.

One of the unique aspects of a dynamics course is the possibility of including an experimental component. My students make weekly trips to the computer lab to perform numerical experiments related to the topics covered in class. These experiments range from observations of the rate of convergence to attracting vs. neutral fixed points to a reenactment of Feigenbaum's celebrated discovery of the universality of the period–doubling route to chaos. Students are asked to perform a detailed analysis of the placement of the windows in the orbit diagram as well as an assessment of the meaning of the decorations on the Mandelbrot set. They go to the lab to gather data; they formulate hypotheses and conjectures; they write up their findings in a lengthy lab report. Given the incredible beauty of many of the images the students investigate as well as the open-ended nature of many of the investigations, this portion of the course is always great fun! Moreover, the possibility of combining rigorous mathematics with experimental ideas is a unique opportunity.

Included in the book are a number of sections marked "Experiment." These are the laboratory assignments that my students complete outside of regular class hours. Many of these labs require two weeks to complete, so there are many more experiments included in the text than are possible to complete in a one-semester course. In addition, many of the later labs (especially in the chapter on the Mandelbrot set) demand access to sophisticated computers (certainly including a numeric co-processor).

For this reason, it may be beneficial for instructors to perform some of the labs as classroom demonstrations, asking students to react verbally to what they observe and to discuss what unfolds on the screen. I always use a computer in the classroom to motivate dynamical ideas and to illustrate in "dynamic" fashion what the theorems in the course mean. Of course, the computer does not always give the correct answer. To make students cognizant of this fact and to make sure that they remain suspicious during the course, the first experiments the students perform are entitled "the computer may lie." These experiments should be the first performed, or at least witnessed, by the students.

Software and Solutions Manual

As part of my course, I make extensive use of software for Macintosh computers developed specifically for the course with the assistance of James Georges and Del Johnson. This software, called *A First Course in Chaotic Dynamical Systems Software*, is available from Addison-Wesley and parallels both the experiments and the problems in this book. The software runs on any Macintosh computer with 2 Mb RAM, running System 6.0.5 or higher, and including Color QuickDraw. Site licenses are also available (call Addison-Wesley at 800-447-2226).

The software has been an invaluable aid as both a laboratory and demonstration tool. However, several caveats are in order. The software is not designed as a research tool. Rather, its capabilities are limited by the scope of the experiments and projects in the text. Second, many of the experiments demand a significant amount of computational power or elaborate graphics such as those found on the Macintosh II series of computers. Run times on computers without mathematics coprocessors may be unreasonably long.

On the other hand, the software has been designed so that no prior computer experience on the part of the user is necessary. Indeed, we have segregated various labs into separate programs that progress in order of difficulty of use. Users are not confronted by a vast array of menu items or options that do everything but wash the dirty dishes in the sink. Rather, each lab has a specific purpose, and the ease of use allows students to concentrate on the mathematics during each lab, rather than the mechanics of making the software work. This means that the students have a significant mathematical experience in the lab rather than a frustrating bout with over-powerful software.

A privately published solutions manual compiled by Thomas R. Scavo is also available for $12.50 ($15.00 outside US and Canada) by writing to the author, Robert L. Devaney, at the Mathematics Department, Boston University, 111 Cummington Street, Boston, MA 02215. The manual contains detailed solutions to approximately 75% of the exercises in the text (not including experiments).

Acknowledgments

It is a pleasure to acknowledge the invaluable assistance of Ed Packel, Bruce Peckham, Mark Snavely, Michèle Taylor, and Benjamin Wells, all of whom read and made many fine comments about the manuscript. I am particularly indebted to Tom Scavo for many excellent suggestions concerning both the manuscript and the software. Many of the color plates in this book were produced at Boston University using a program written by Paul Blanchard, Scott Sutherland, and Gert Vegter. Scott Sutherland also assisted with many of the other figures in the book. Thanks are also due Stefen Fangmeier, Chris Mayberry, Chris Small, Sherry Smith, and Craig Upson for help with the computer graphics images. Jim Georges and Del Johnson worked many long hours putting the software for the course into a very usable and enjoyable format; their excellent design for the user interface has made laboratory assignments quite enjoyable and beneficial for my students. Eℓwood Devaney verified the mathematical accuracy of the entire text; all errors that remain are due to him. Finally, I must also thank my friends Vincenzo, Gaetano, Wolfgang, Giacomo, Gioacchino, Richard, Giuseppe, and Richard for providing me with many hours of enjoyment while this book was taking shape.

Robert L. Devaney
Boston, Massachusetts
January 1993

Contents

A FIRST COURSE IN CHAOTIC DYNAMICAL SYSTEMS

THEORY AND EXPERIMENT

CHAPTER 1

A Mathematical and Historical Tour

Rather than jump immediately into the mathematics of dynamical systems, we will begin with a brief tour of some of the amazing computer graphics images that arise in this field. One of our goals in this book is to explain what these images mean, how they are generated on the computer, and why they are important in mathematics. We will do none of the mathematics here. For now, you should simply enjoy the images. We hope to convince you, in the succeeding chapters, that the mathematics behind these images is even prettier than the pictures. In the second part of the chapter, we will present a brief history of some of the developments in dynamical systems over the past century. You will see that many of the ideas in dynamics arose fairly recently. Indeed, none of the computer graphics images from the tour had been seen before 1980!

1.1 Images from Dynamical Systems

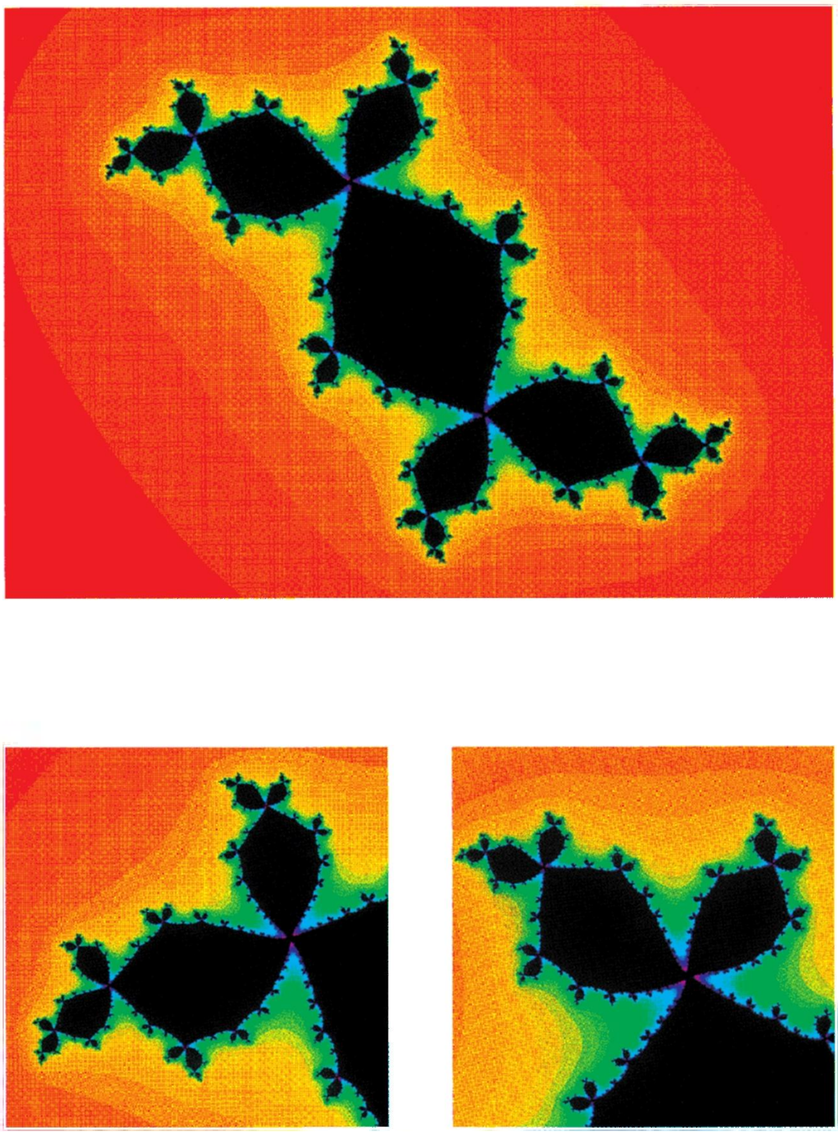
This book deals with some very interesting, exciting, and beautiful topics in mathematics—topics which, in many cases, have been discovered only in the last decade. The main subject of the book is *dynamical systems,* the branch of mathematics that attempts to understand processes in motion. Such processes occur in all branches of science. For example, the motion of the stars and the galaxies in the heavens is a dynamical system, one that has been studied for centuries by thousands of mathematicians and scientists. The stock market is another system that changes in time, as is the world's weather. The changes chemicals undergo, the rise and fall of populations, and the motion of a simple pendulum are classical examples of dynamical systems in chemistry, biology, and physics. Clearly, dynamical systems abound.

What does a scientist wish to do with a dynamical system? Well, since the system is moving or changing in time, the scientist would like to predict where the system is heading, where it will ultimately go. Will the stock market go up or down? Will it be rainy or sunny tomorrow? Will these two chemicals explode if they are mixed in a test tube?

Clearly, some dynamical systems are predictable, whereas others are not. You know that the sun will rise tomorrow and that, when you add cream to a cup of coffee, the resulting "chemical" reaction will not be an explosion. On the other hand, predicting the weather a month from now or the Dow Jones average a week from now seems impossible. You might argue that the reason for this unpredictability is that there are simply too many variables present in meteorological or economic systems. That is indeed true in these cases, but this is by no means the complete answer. One of the remarkable discoveries of twentieth-century mathematics is that very simple systems, even systems depending on only one variable, may behave just as unpredictably as the stock market, just as wildly as a turbulent waterfall, and just as violently as a hurricane. The culprit, the reason for this unpredictable behavior, has been called "chaos" by mathematicians.

Because chaos has been found to occur in the simplest of systems, scientists may now begin to study unpredictability in its most basic form. It is to be hoped that the study of these simpler systems will eventually allow scientists to find the key to understanding the turbulent behavior of systems involving many variables such as weather or economic systems.

In this book we discuss chaos in these simple settings. We will see that chaos occurs in elementary mathematical objects—objects as familiar as quadratic functions—when they are regarded as dynamical systems. You may feel at this point that you know all there is to know about quadratic functions—after all, they are easy to evaluate and to graph. You can differentiate and integrate them. But the key words here are "dynamical systems." We will treat simple mathematical operations like taking the square root, squaring, or cubing as dynamical systems by repeating the procedure over and over, using the output of the previous operation as the input for the next. This process is called *iteration*. This procedure generates a list of real or complex numbers that are changing as we proceed—this is our dynamical system. Sometimes we will find that, when we input certain numbers into the process, the resulting behavior is completely predictable, while other numbers yield results that are often bizarre and totally unpredictable.



Plates 1, 2, and 3. Douady's Rabbit and several magnifications

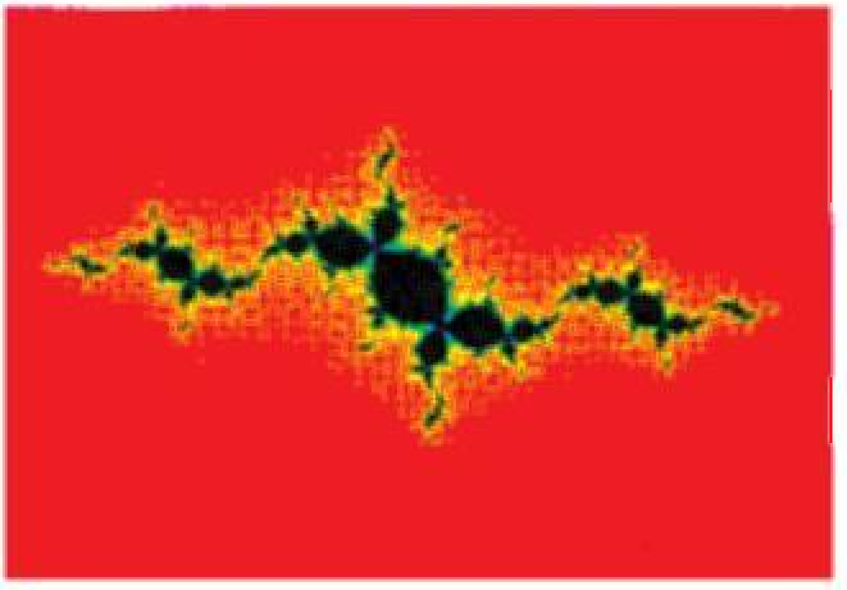


Plate 4. Dancing rabbits

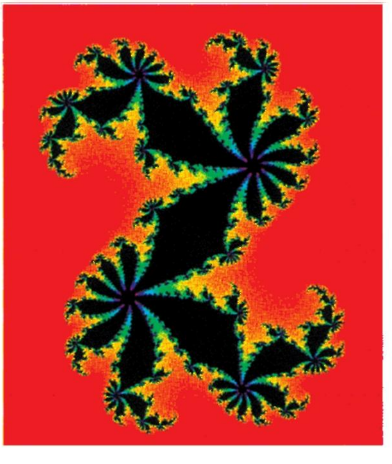


Plate 5. A dragon

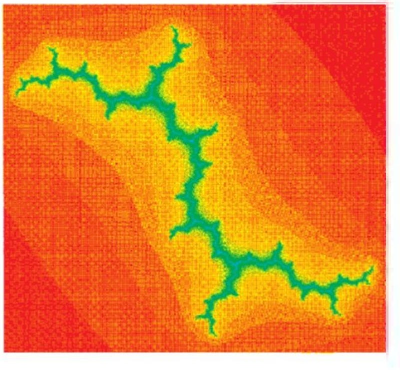
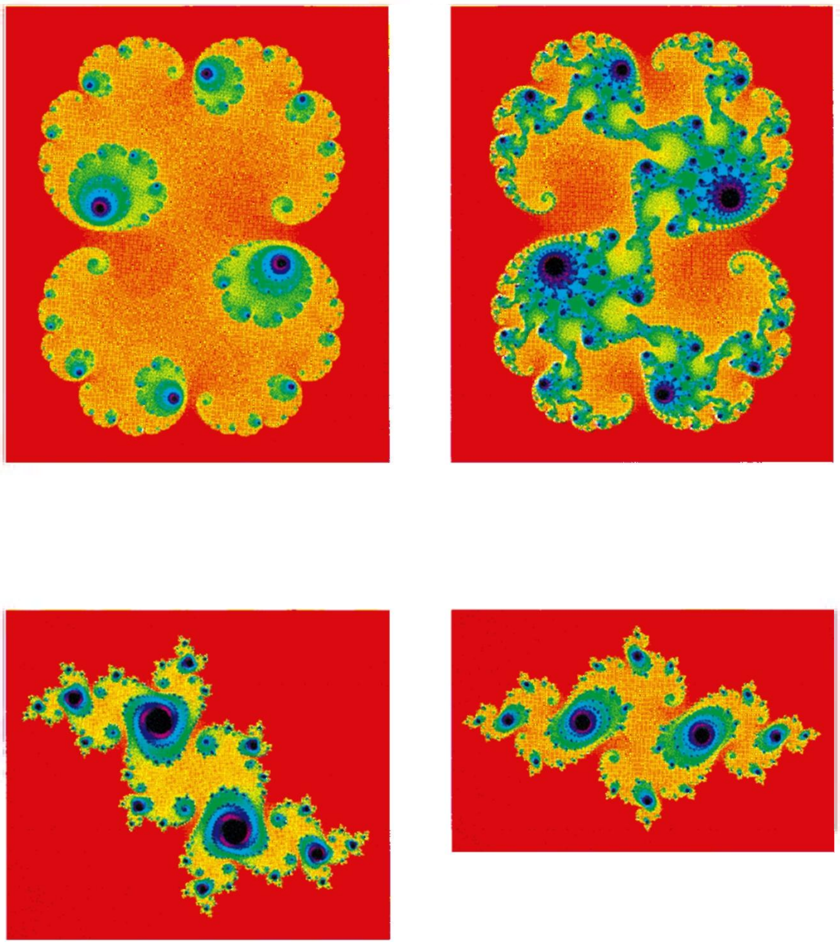


Plate 6. A dendrite



Plates 7–10.

Filled Julia sets for quadratic functions may be Cantor sets.

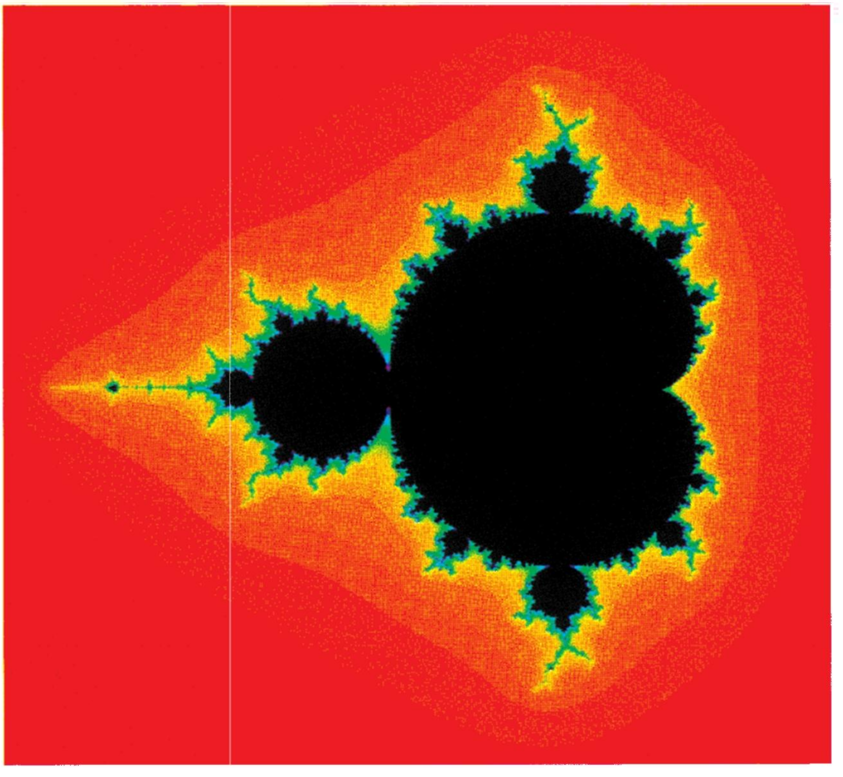


Plate 11. The Mandelbrot set

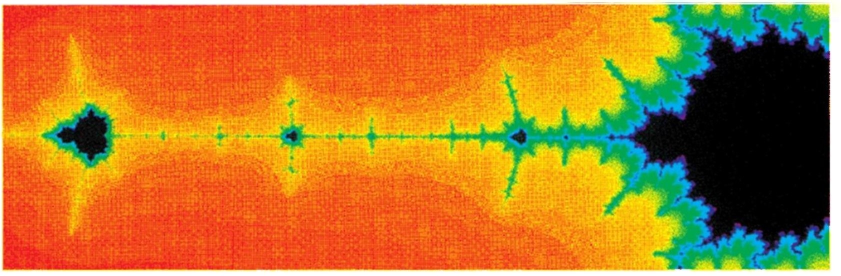
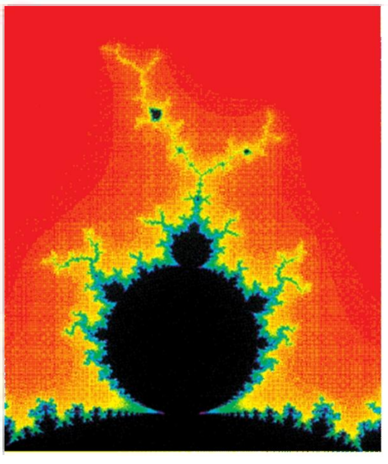
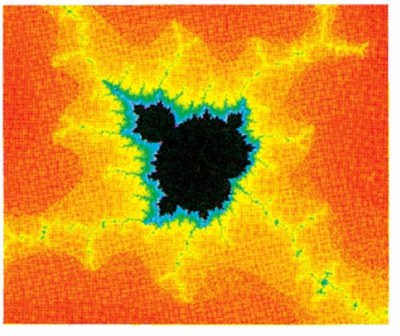
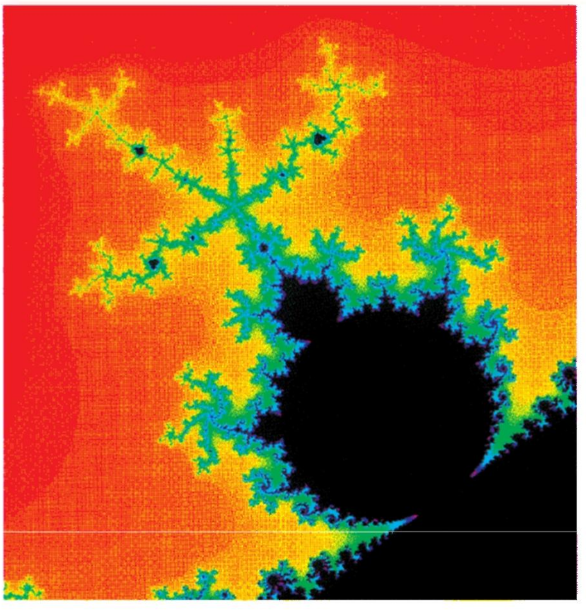


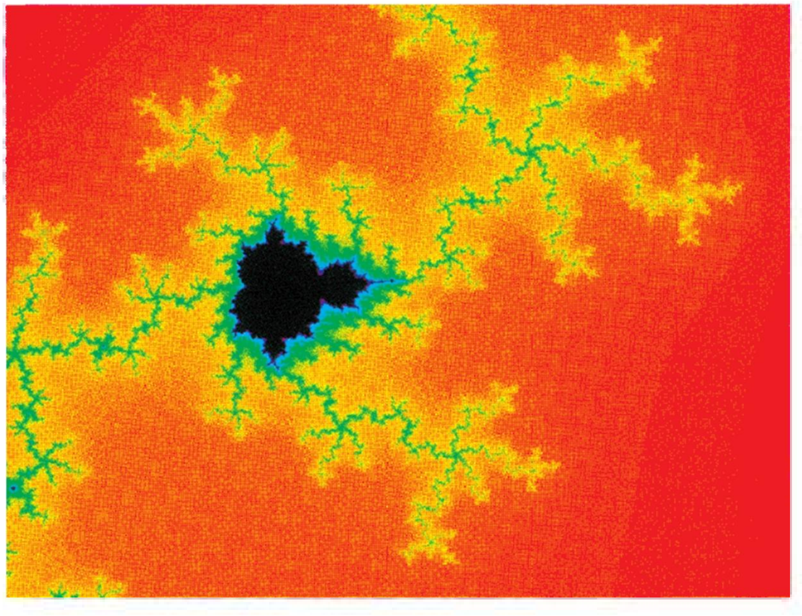
Plate 12. Tail of the Mandelbrot set



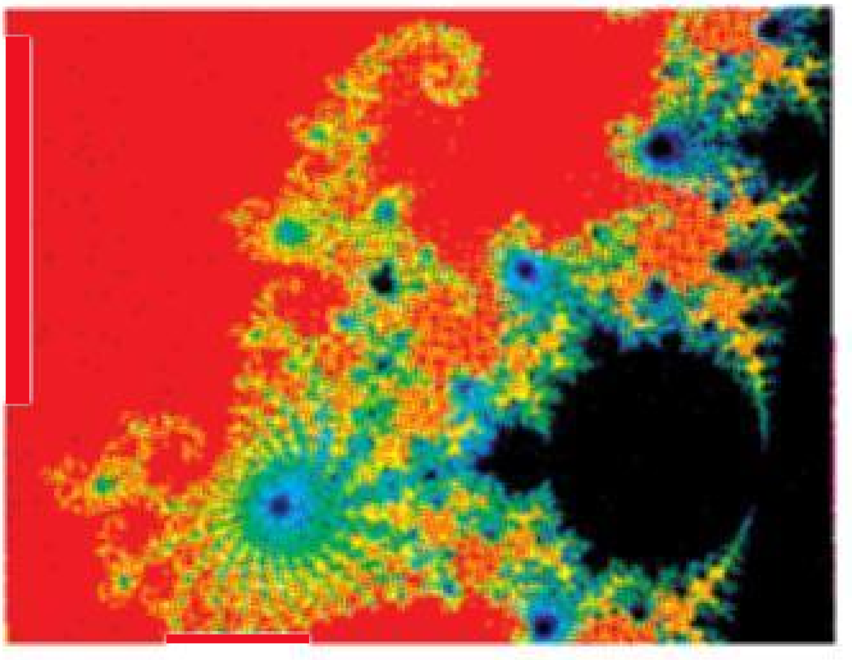


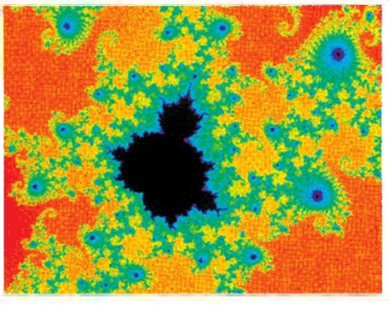
Plates 13, 14. The period 3 bulb and a magnification

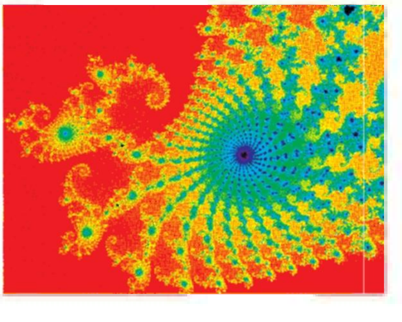




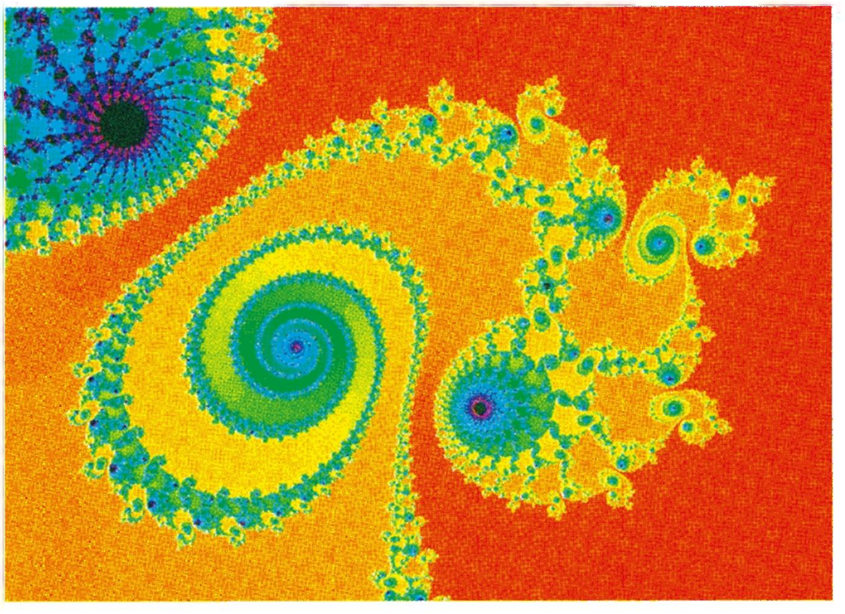
Plates 15, 16. The period 5 bulb and a magnification

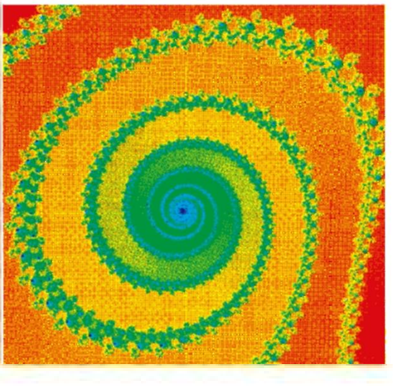


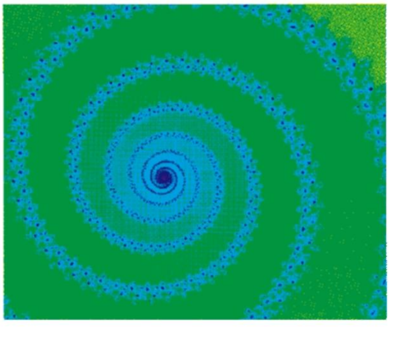




Plates 17–19. The period 25 bulb and several magnifications







Plates 20–22. Fine detail in the Mandelbrot set

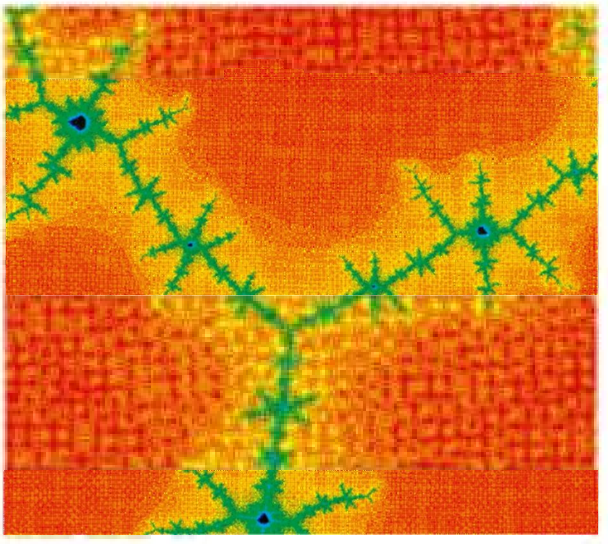


Plate 23.
Antenna attached to the period 3 bulb in the Mandelbrot set

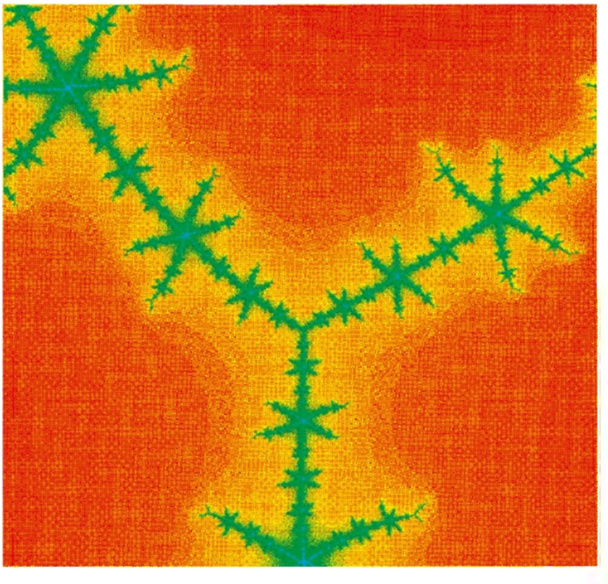
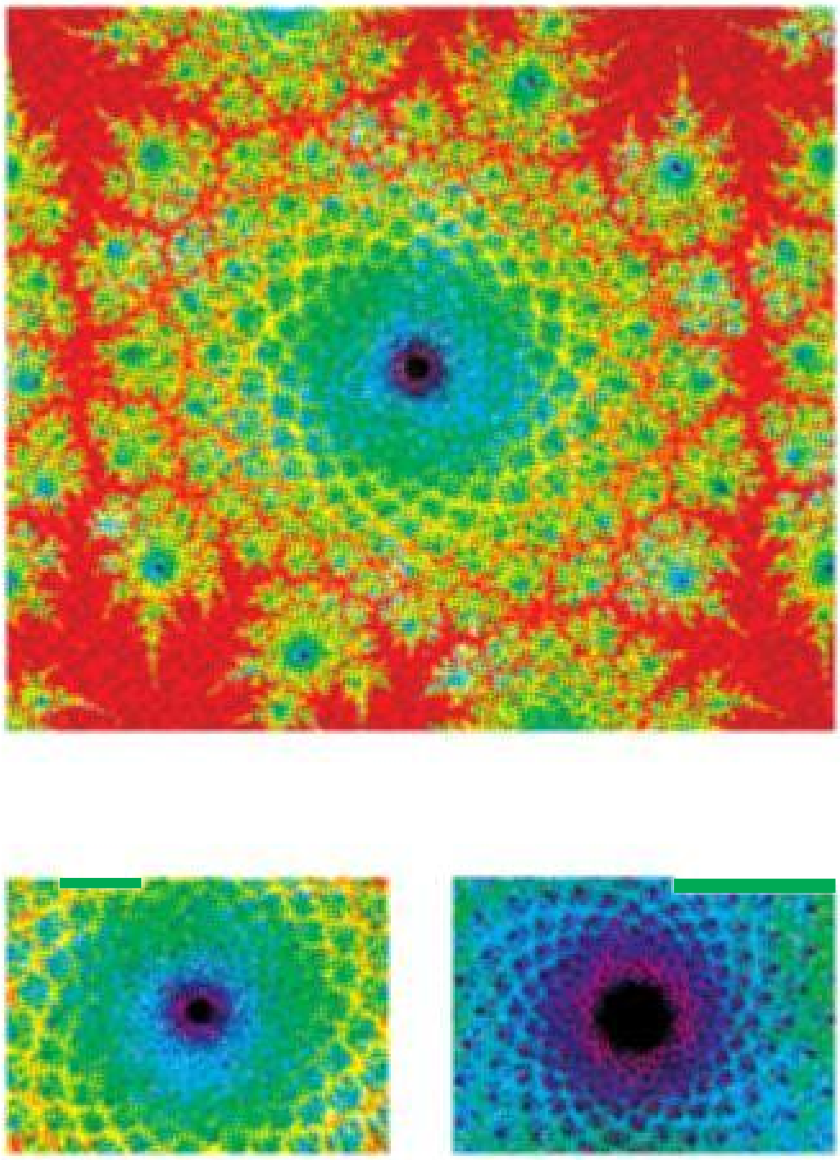


Plate 24.
Julia set corresponding to the junction point in Plate 23



Plates 25–27. Julia set of $(.61 + .81i)\sin(z)$ and several magnifications

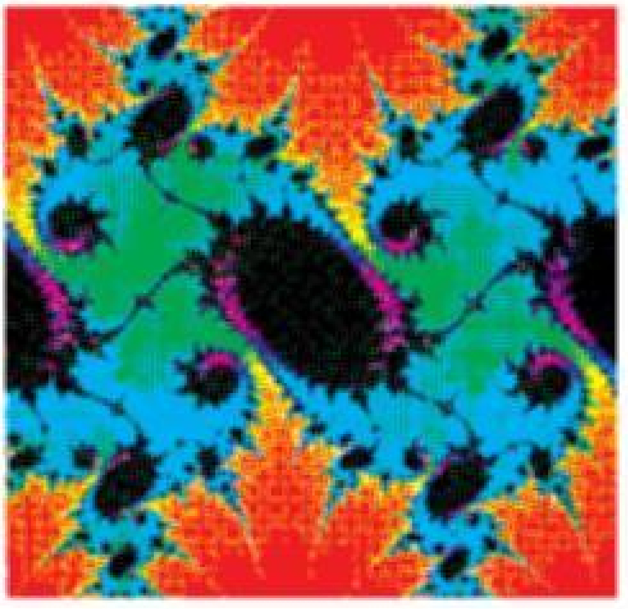


Plate 28. Julia set of $(1 + 0.2\imath)\sin(z)$

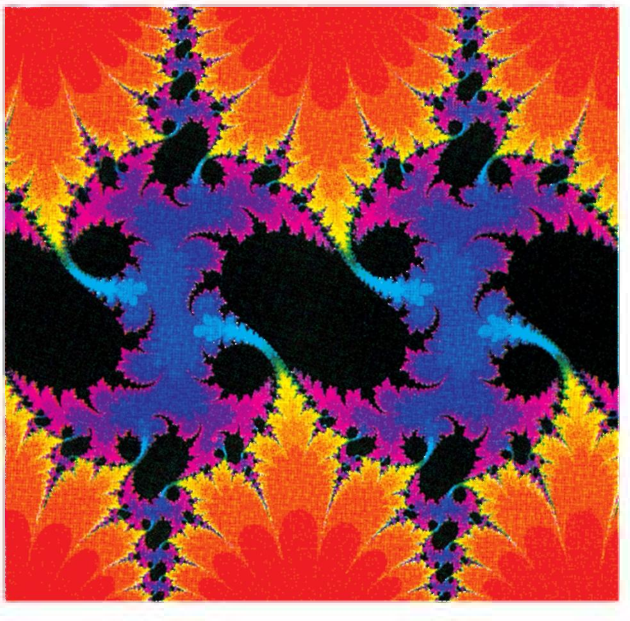
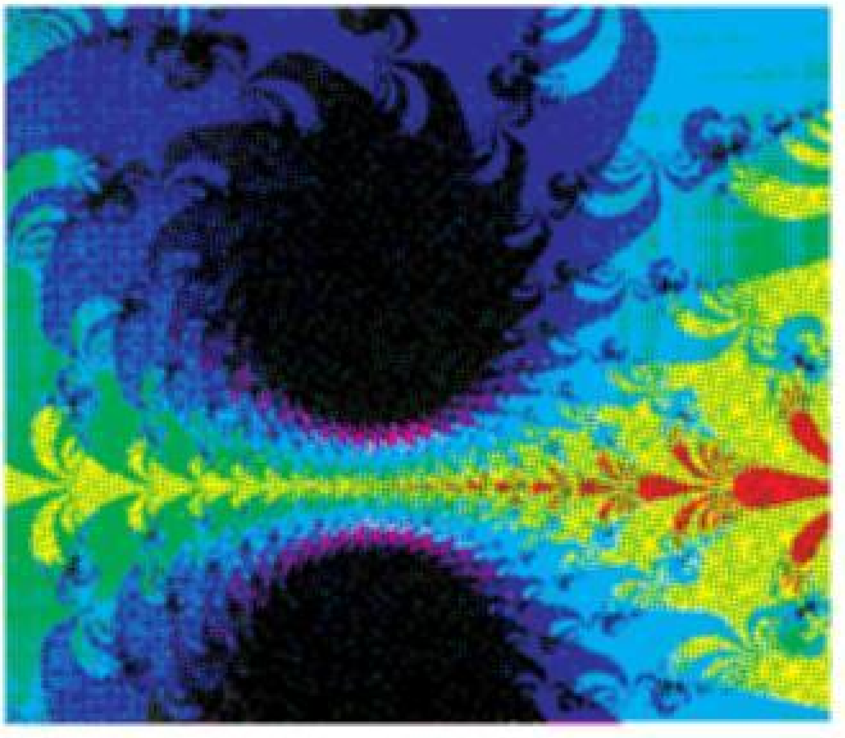
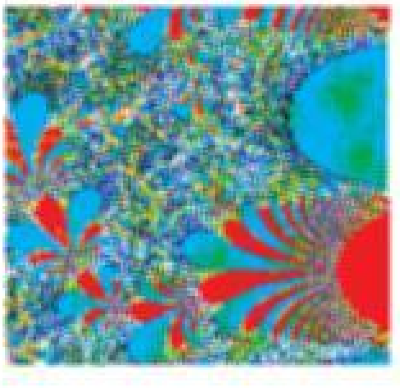


Plate 29. Julia set of $(1 + 0.1\imath)\sin(z)$

Plate 30. Julia set of $2\pi i \exp(z)$





Plates 31–33. Julia sets of complex exponentials

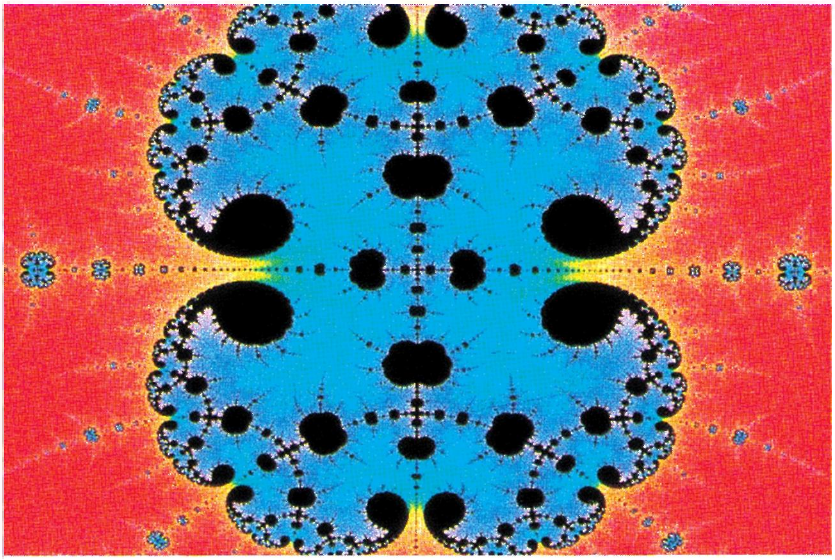


Plate 34. Julia set of $2.95\cos(z)$

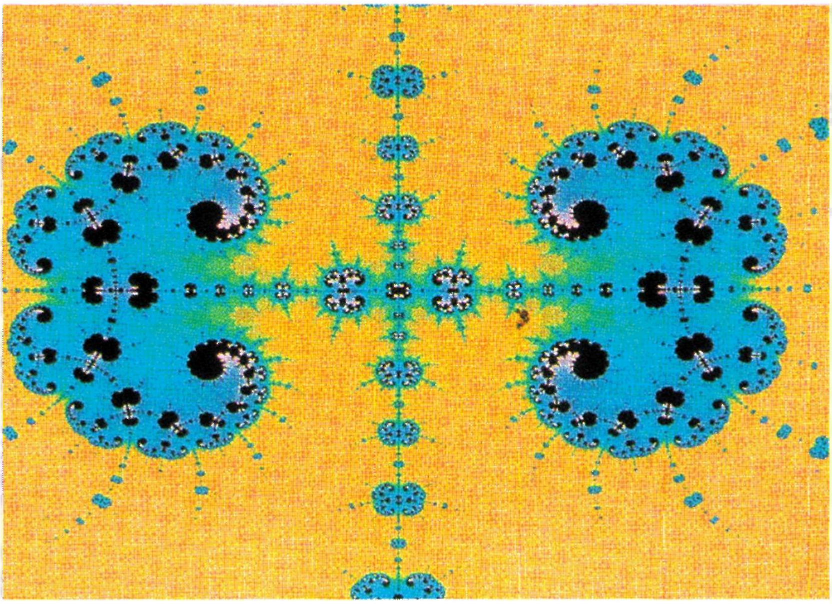
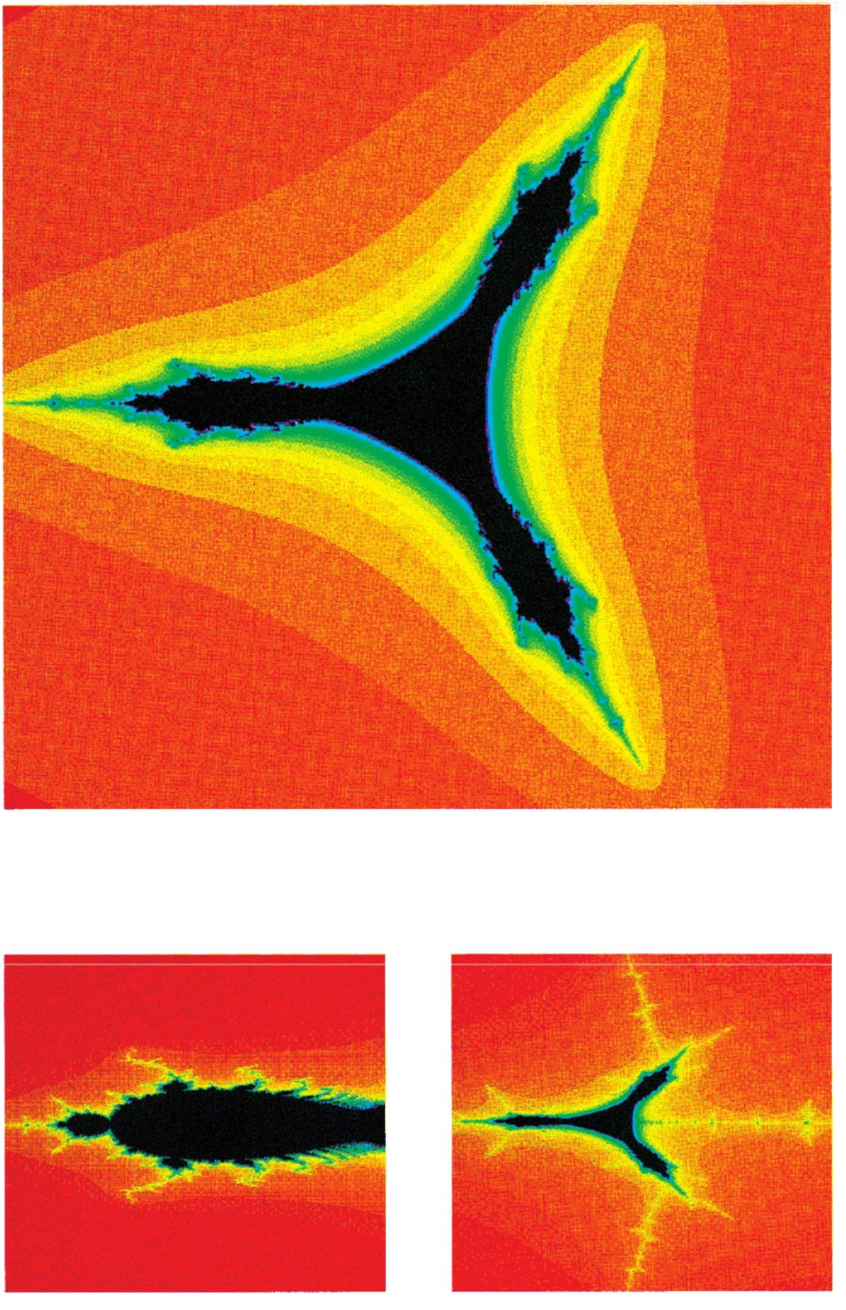


Plate 35. Julia set of $2.96\cos(z)$



Plates 36–38. The tricorn and several magnifications

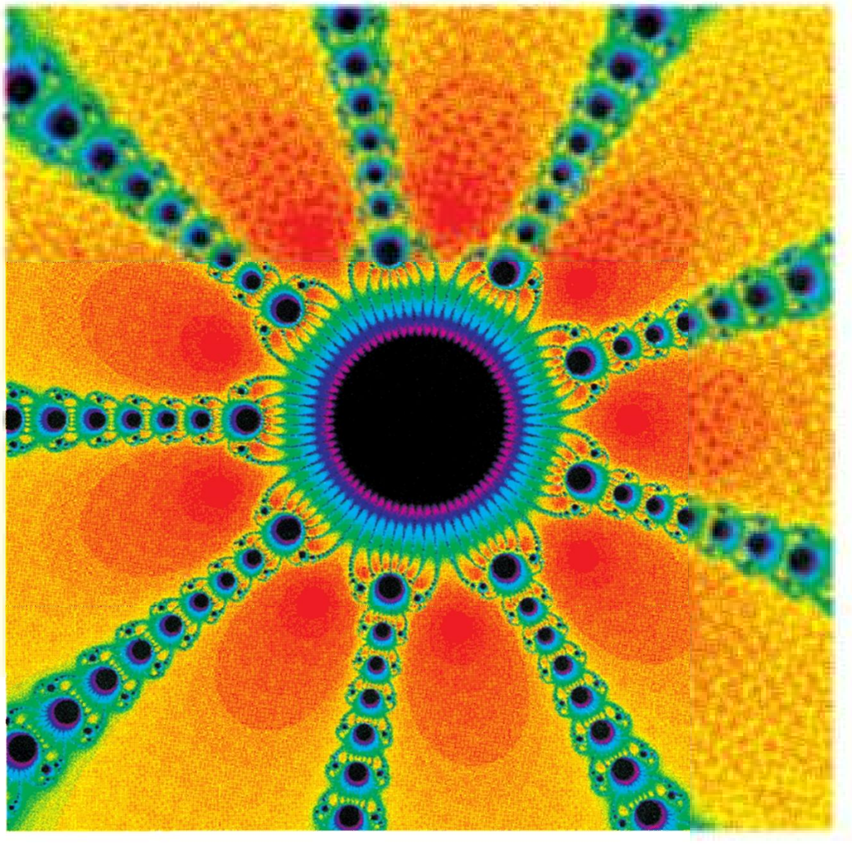


Plate 39. Julia set for Newton's method

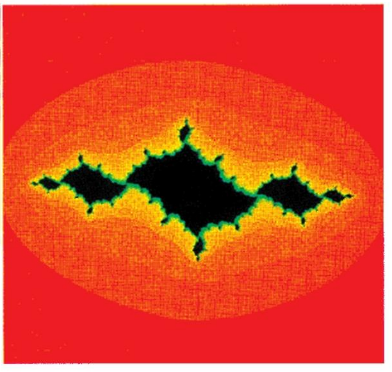


Fig. 17.8a

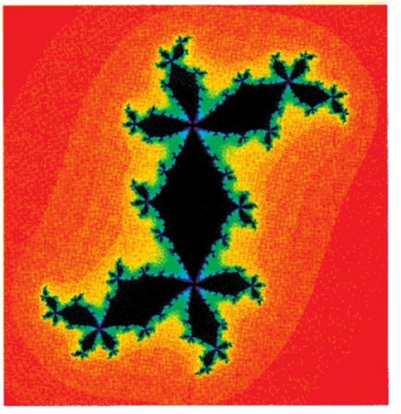


Fig. 17.8b

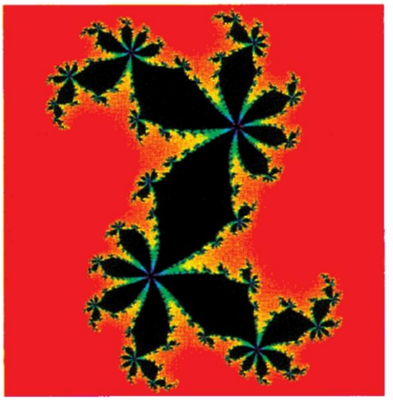


Fig. 17.8c

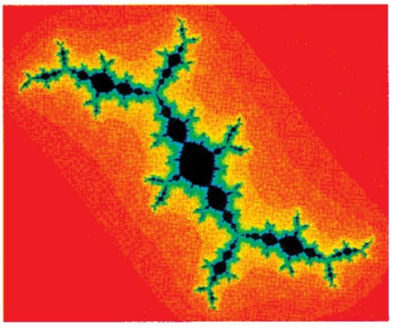


Fig. 17.8d

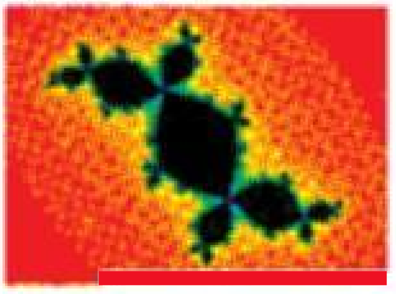


Fig. 17.8e


Fig. 17.8f

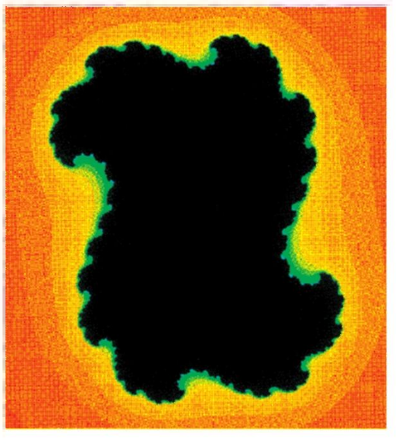


Fig. 17.8g

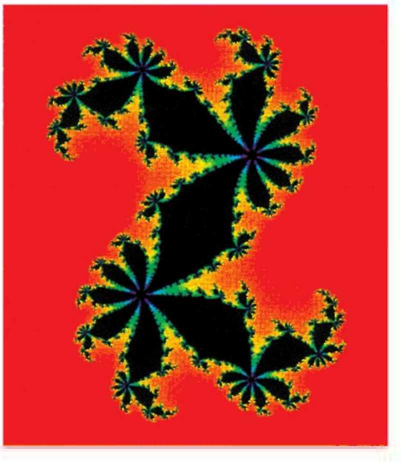


Fig. 17.8h

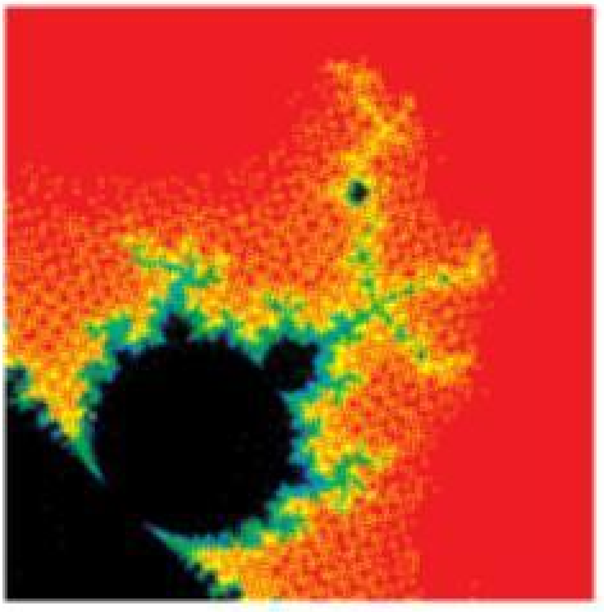


Fig. 17.9a Period 4 bulb

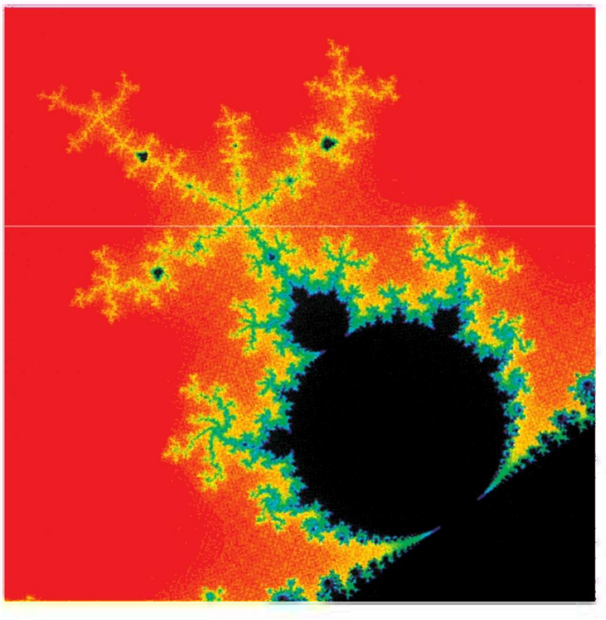


Fig. 17.9b Period 5 bulb

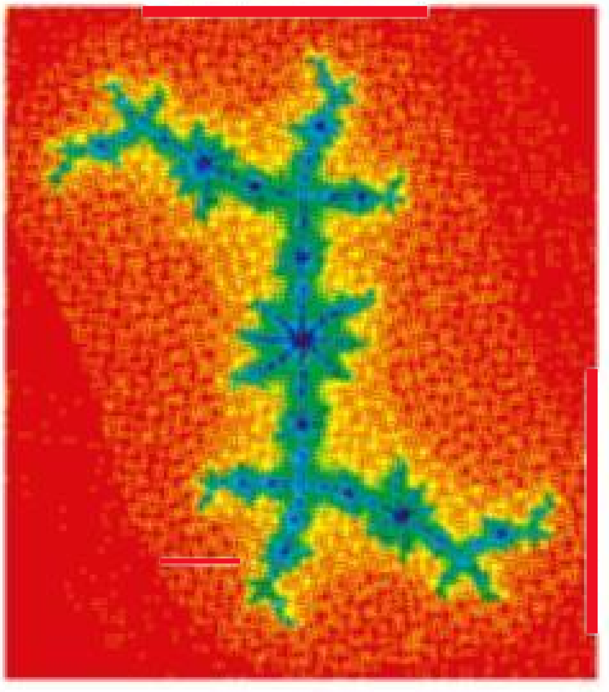


Fig. 17.10a.
Julia set corresponding to junction point in Fig. 17.9a

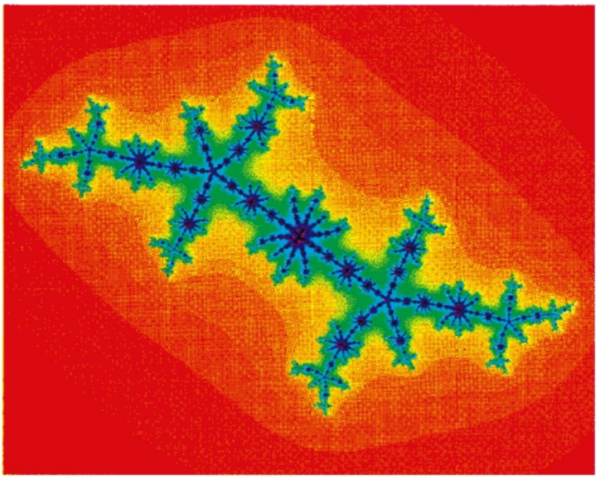


Fig. 17.10b.
Julia set corresponding to junction point in Fig. 17.9b

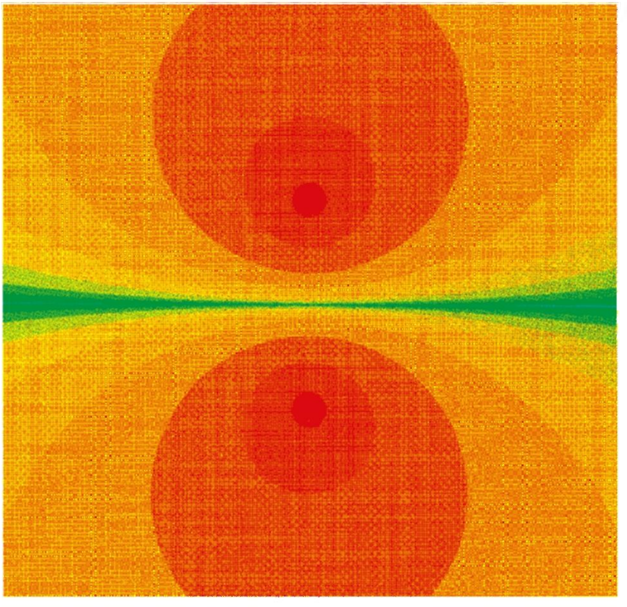


Fig. 18.4a. Newton's method for $f(z) = z^2 + 1$

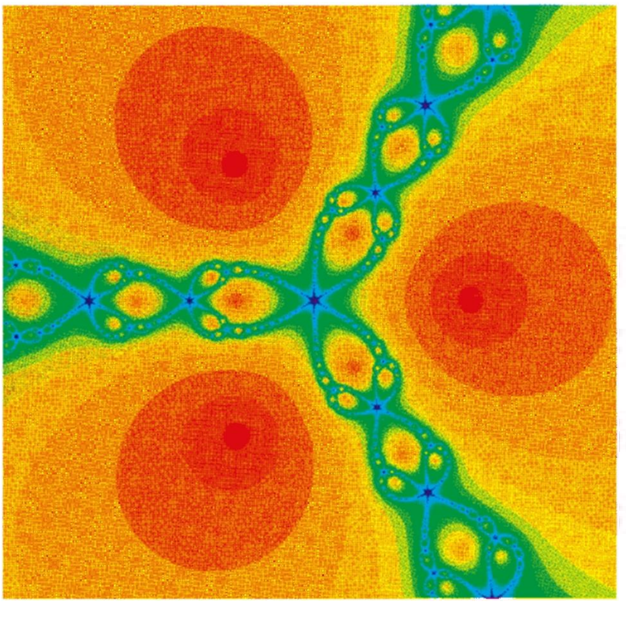


Fig. 18.4b. Newton's method for $f(z) = z^3 - 1$

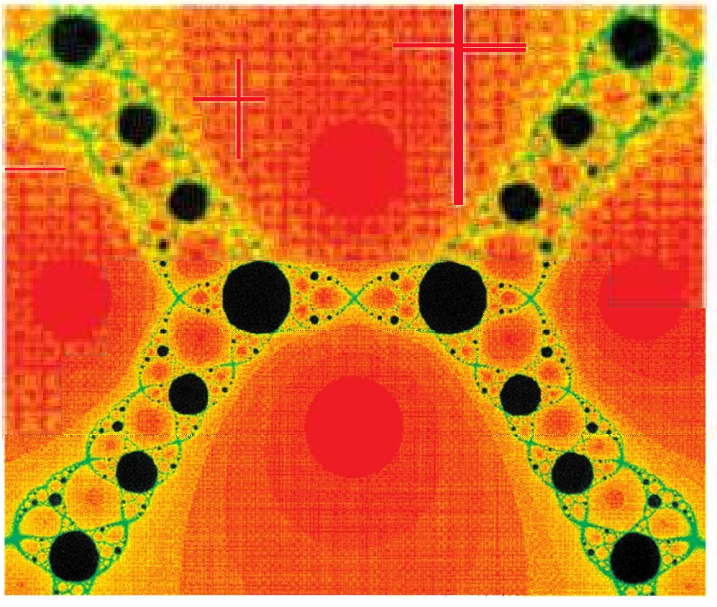


Fig. 18.5a

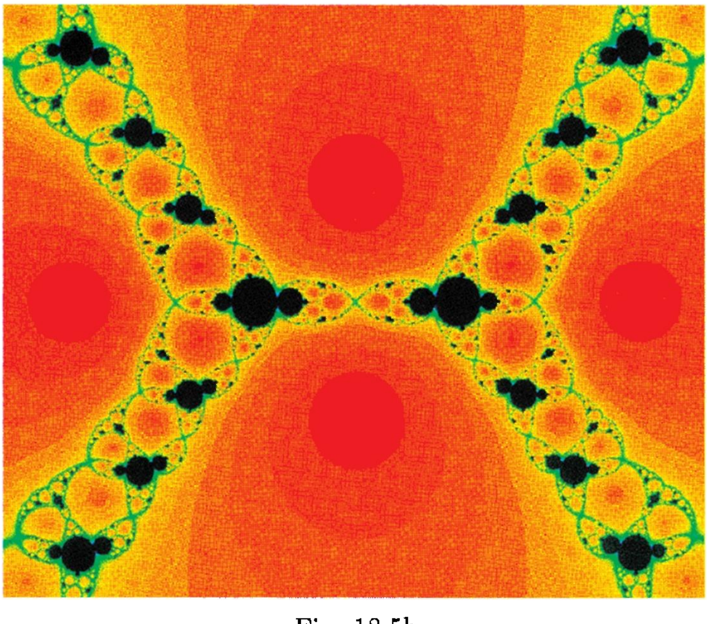


Fig. 18.5b

For the types of functions we will consider, the set of numbers that yield chaotic or unpredictable behavior in the plane is called the *Julia set* after the French mathematician Gaston Julia, who first formulated many of the properties of these sets in the 1920's. These Julia sets are spectacularly complicated, even for quadratic functions. They are examples of *fractals*. These are sets which, when magnified over and over again, always resemble the original image. The closer you look at a fractal, the more you see exactly the same object. Moreover, fractals naturally have a dimension that is not an integer, not 1, not 2, but often somewhere in between, such as dimension 1.4176, whatever that means! We will discuss these concepts in more detail in Chapter 14.

Here are some examples of the types of images that we will study. In Plate 1 we show the Julia set of the simple mathematical expression $z^2 + c$, where both z and c are complex numbers. In this particular case, $c = -.122 + .745i$. This image is called *Douady's rabbit*, after the French mathematician Adrien Douady whose work we will discuss in Chapter 17. The black region in this image resembles a "fractal rabbit." Everywhere you look, you see a pair of ears. In Plates 2 and 3 we have magnified portions of the rabbit, revealing more and more pairs of ears.

As we will describe later, the black points you see in these pictures are the non-chaotic points. They are points representing values of z that, under iteration of this quadratic function, eventually tend to cycle between three different points in the plane. As a consequence, the dynamical behavior is quite predictable. All of this is by no means apparent right now, but by the time you have read Chapter 16, you will consider this example a good friend. Points that are colored in this picture also behave predictably: They are points that "escape," that tend to infinity under iteration. The colors here simply tell us how quickly a point escapes. The boundary between these two types of behavior—the interface between the escaping and the cycling points—is the Julia set. This is where we will encounter all of the chaotic behavior for this dynamical system.

In Plates 4-10 we have displayed Julia sets for other quadratic functions of the form $z^2 + c$. Each picture corresponds to a different value of c. For example, Plate 6 is a picture of the Julia set for $z^2 + i$. As we see, these Julia sets may assume a remarkable variety of shapes. Sometimes the images contain large black regions as in the case of Douady's rabbit. Other times the Julia set looks like an isolated scatter of points, as in Plates 7–10. Many of these Julia sets are *Cantor sets*. These are very complicated sets that arise often in the study of dynamics. We will begin our study of Cantor sets in Chapter 7 when we introduce the most basic fractal of all, the *Cantor middle-thirds set*.

All these Julia sets correspond to mathematical expressions that are of the form z^2+c. As we see, when c varies, these Julia sets change considerably in shape. How do we understand the totality of all of these shapes, the collection of all possible Julia sets for quadratic functions? The answer is called the ***Mandelbrot set***. The Mandelbrot set, as we will see in Chapter 17, is a dictionary, or picture book, of all possible quadratic Julia sets. It is a picture in the c-plane that provides us with a road map of the quadratic Julia Sets. This image, first viewed in 1980 by Benoit Mandelbrot and others, is quite important in dynamics. It completely characterizes the Julia sets of quadratic functions. It has been called one of the most intricate and beautiful objects in mathematics.

Plate 11 shows the full Mandelbrot set. Note that it consists of a basic central cardioid shape, with smaller balls or decorations attached. Plates 13, 15, and 17 are magnifications of some of these decorations. Note how each decoration differs from the others. Buried deep in various regions of the Mandelbrot set, we also find what appear to be small copies of the entire set, as shown in Plates 12, 14, 16, and 18. The set possesses an amazing amount of complexity, as illustrated by Plates 19–22. Nonetheless, each of these small regions has a distinct dynamical meaning, as we will discuss in Chapter 17.

In Plate 23 we have magnified a small area near the "antenna" in the portion of the Mandelbrot set shown in Plate 14. Note that there is a junction point where this antenna seems to branch. In Plate 24 we have displayed a portion of the Julia set for the c-value corresponding to this junction point. Note the remarkable similarity of these two images. This is by no means an accident. As we will see in some of the experiments in Chapter 17, there is an amazing resemblance between certain areas of the Mandelbrot set and the corresponding Julia sets.

In this book we will also investigate the chaotic behavior of many other functions. For example, in Plates 25–27 we have displayed the Julia set of $(0.61 + 0.81i)\sin z$ and several magnifications. Plates 28 and 29 give other examples of Julia sets for functions of the form $c\sin z$. If we investigate exponential functions, we find Julia sets that look quite different as, for example, those in Plates 30-33. Plates 34-35 depict the Julia sets for several cosine functions, while the images in Plates 36-38 are called the *tricorn*, a very different geometric object arising in the study of quadratics. Finally, in Plate 39, we have included a Julia set for Newton's method. This iterative process, familiar from elementary calculus, surprisingly leads to considerable chaotic behavior, as we will see in Chapter 13.

The images in this mathematical tour show quite clearly the great beauty of mathematical dynamical systems theory. But what do these pictures mean and why are they important? These are questions that we will answer in the remainder of this book.

1.2 A Brief History of Dynamics

Dynamical systems has a long and distinguished history as a branch of mathematics. Beginning with the fundamental work of Isaac Newton, differential equations became the principal mathematical technique for describing processes that evolve continuously in time. In the eighteenth and nineteenth centuries, mathematicians devised numerous techniques for solving differential equations explicitly. These methods included Laplace transforms, power series solutions, variation of parameters, linear algebraic methods, and many other techniques familiar from the basic undergraduate course in ordinary differential equations.

There was one major flaw in this development. Virtually all of the analytic techniques for solving differential equations worked mainly for linear differential equations. Nonlinear differential equations proved much more difficult to solve. Unfortunately, many of the most important processes in nature are inherently nonlinear.

An example of this is provided by Newton's original motivation for developing calculus and differential equations. Newton's laws enable us to write down the equations that describe the motion of the planets in the solar system, among many other important physical phenomena. Known as the n-body problem, these laws give us a differential equation whose solution describes the motion of n "point masses" moving in space subject only to their own mutual gravitational attraction. If we know the initial positions and velocities of these masses, then all we have to do is solve Newton's differential equation to be able to predict where and how these masses will move in the future.

This turns out to be a formidable task. If there are only one or two planets, then these equations may be solved explicitly, as is often done in a freshman or sophomore calculus or physics class. For three or more masses, the problem today remains completely unsolved, despite the efforts of countless mathematicians during the past three centuries. It is true that numerical solutions of differential equations by computers have allowed us to approximate the behavior of the actual solutions in many cases, but there are still regimes in the n-body problem where the solutions are so complicated or chaotic that they defy even numerical computation.

Although the explicit solution of nonlinear ordinary differential equations has proved elusive, there have been three landmark events over the past century that have revolutionized the way we study dynamical systems. Perhaps the most important event occurred in 1890. King Oscar II of Sweden announced a prize for the first mathematician who could solve the n-body problem and thereby prove the stability of the solar system. Needless to say, nobody solved the original problem, but the great French mathematician Henri Poincaré came closest. In a beautiful and far-reaching paper, Poincaré totally revamped the way we tackle nonlinear ordinary differential equations. Instead of searching for explicit solutions of these equations, Poincaré advocated working qualitatively, using topological and geometric techniques, to uncover the global structure of all solutions. To him, a knowledge of all possible behaviors of the system under investigation was much more important than the rather specialized study of individual solutions.

Poincaré's prize winning paper contained a major new insight into the behavior of solutions of differential equations. In describing these solutions, mathematicians had previously made the tacit assumption that what we now know as stable and unstable manifolds always match up. Poincaré questioned this assumption. He worked long and hard to show that this was always the case, but he could not produce a proof. He eventually concluded that the stable and unstable manifolds might not match up and could actually cross at an angle. When he finally admitted this possibility, Poincaré saw that this would cause solutions to behave in a much more complicated fashion than anyone had previously imagined. Poincaré had discovered what we now call ***chaos***. Years later, after many attempts to understand the chaotic behavior of differential equations, he threw up his hands in defeat and wondered if anyone would ever understand the complexity he was finding. Thus, "chaos theory," as it is now called, really dates back over 100 years to the work of Henri Poincaré.

Poincaré's achievements in mathematics went well beyond the field of dynamical systems. His advocacy of topological and geometric techniques opened up whole new subjects in mathematics. In fact, building on his ideas, mathematicians turned their attention away from dynamical systems and toward these related fields in the ensuing decades. Areas of mathematics such as algebraic and differential topology were born and flourished in the twentieth century. But nobody could handle the chaotic behavior that Poincaré had observed, so the study of dynamics languished.

There were two notable exceptions to this. One was the work of the French mathematicians Pierre Fatou and Gaston Julia in the 1920's on the dynamics of complex analytic maps. They too saw chaotic behavior, this time on what we now call the *Julia set*. Indeed, they realized how tremendously intricate these Julia sets could be, but they had no computer graphics available to see these sets, and as a consequence, this work also stopped in the 1930's.

At the same time, the American mathematician G. D. Birkhoff adopted Poincaré's qualitative point of view on dynamics. He advocated the study of iterative processes as a simpler way of understanding the dynamical behavior of differential equations, a viewpoint that we will adopt in this book.

The second major development in dynamical systems occurred in the 1960's. The American mathematician Stephen Smale reconsidered Poincaré's crossing stable and unstable manifolds from the point of view of iteration and showed by an example that the chaotic behavior that baffled his predecessors could indeed be understood and analyzed completely. The technique he used to analyze this is called *symbolic dynamics* and will be a major tool for us in this book. At the same time, the American meteorologist E. N. Lorenz, using a very crude computer, discovered that very simple differential equations could exhibit the type of chaos that Poincaré observed. Lorenz, who actually had been a Ph.D. student of Birkhoff, went on to observe that his simple meteorological models exhibited what is now called *sensitive dependence on initial conditions*. For him, this meant that long-range weather forecasting was all but impossible and showed that the mathematical topic of chaos was important in other areas of science.

This led to a tremendous flurry of activity in nonlinear dynamics in the 1970's. The ecologist Robert May found that very simple iterative processes that arise in mathematical biology could produce incredibly complex and chaotic behavior. The physicist Mitchell Feigenbaum, building on Smale's earlier work, noticed that, despite the complexity of chaotic behavior, there was some semblance of order in the way systems became chaotic. Physicists Harry Swinney and Jerry Gollub showed that these mathematical developments could actually be observed in physical applications, notably in turbulent fluid flow. More recently, other systems, such as the motion of the planet Pluto or the beat of the human heart, have been shown to exhibit similar chaotic patterns. In mathematics, meanwhile, new techniques were developed to help understand chaos. John Guckenheimer and Robert

F. Williams employed the theory of strange attractors to explain the phenomenon that Lorenz had observed a decade earlier. And tools such as the Schwarzian derivative, symbolic dynamics, and bifurcation theory—all topics we will discuss in this book—were shown to play an important role in understanding the behavior of dynamical systems.

The third and most recent major development in dynamical systems was the availability of high-speed computing and, particularly, computer graphics. Foremost among the computer-generated results was Mandelbrot's discovery in 1980 of what is now called the ***Mandelbrot set***. This beautiful image immediately reawakened interest in the old work of Julia and Fatou. Using the computer images as a guide, mathematicians such as Adrien Douady, John Hubbard, and Dennis Sullivan greatly advanced the classical theory. Other computer graphics images such as the orbit diagram and the Lorenz attractor generated considerable interest among mathematicians and led to further advances.

One of the most interesting side effects of the availability of high speed computing and computer graphics has been the development of an experimental component in the study of dynamical systems. Whereas the old masters had to rely solely on their imagination and their intellect, now mathematicians have an invaluable additional resource to investigate dynamics: the computer. This tool has opened up whole new vistas for dynamicists, some of which we will sample in this book. In a series of sections called "Experiments," you will have a chance to rediscover some of these wonderful facts yourself.

References

Arnol'd, V. Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1978.
Arnol'd, V. Geometrical Methods in the Theory of Ordinary Differential Equations, Springer-Verlag, New York, 1983.
Baker, G. L. and Gollub, J. P. Chaotic Dynamics: An Introduction, Cambridge University Press, Cambridge, 1990.
Beardon, A. Iteration of Rational Functions, Springer-Verlag, New York, 1991.
Collet, P. and Eckmann, J.-P. Iterated Maps on the Interval as Dynamical Systems, Birkhäuser, Boston, 1980.
Devaney, R. L. An Introduction to Chaotic Dynamical Systems, Second Edition. Addison-Wesley, Redwood City, Calif., 1989.
Devaney, R. L. and Keen, L. , eds. Chaos and Fractals: The Mathematics Behind the Computer Graphics, American Mathematical Society, Providence, 1989.
Guckenheimer, J. and Holmes, P. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.
Hirsch, M. W. and Smale, S. Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, New York, 1974.
Milnor, J. Dynamics in One Complex Variable: Introductory Lectures, SUNY Stony Brook, Institute for Mathematical Sciences, Preprint #1990/5, 1990.
Palis, J. and deMelo, W. Geometric Theory of Dynamical Systems, Springer-Verlag, New York, 1982.
Ruelle, D. Elements of Differentiate Dynamics and Bifurcation Theory, Academic Press, San Diego, 1989.
Shub, M. Global Stability of Dynamical Systems, Springer-Verlag, New York, 1986.
Tufillaro, N. , Abbott, T. , and Reilly, J. An Experimental Approach to Nonlinear Dynamical Systems, Addison-Wesley, Redwood City, CA, 1992.
Wiggins, S. Global Bifurcations and Chaos, Springer-Verlag, New York, 1988.
Georges, J. , Johnson, D. and Devaney, R. L. A First Course in Chaotic Dynamical Systems: Laboratory, Addison-Wesley Co., Reading, MA, 1992.
Parker, T. and Chua, L. Practical Numerical Algorithms for Chaotic Systems, Springer-Verlag, New York, 1989.
Peitgen, H.-O. and Richter, P. The Beauty of Fractals, Springer-Verlag, Heidelberg, 1986.
Peitgen, H.-O. and Sajupe, D. The Science of Fractal Images, New York, Springer-Verlag, 1989.
Fatou, P. "Sur les Équationes Fonctionelles." Bull Soc. Math. France 48 (1920), 33–94, 208–314.
Feigenbaum, M. J. "Quantitative Universality for a Class of Nonlinear Transformations." J. Stat. Phys. 19 (1978), 25–52.
Julia, G. "Mémoire sur l'Itération des Fonctions Rationelles." J. Math. Pures Appl. 4 (1918), 47–245.
Li, T.-y. , and Yorke, J. , "Period Three Implies Chaos." American Mathematical Monthly 82 (1975), 985–992.
Lorenz, E. N. "Deterministic Nonperiodic Flows." J. Atmospheric Sci. 20 (1963), 130–141.
Mandelbrot, B. "Fractal Aspects of the Iteration of $z \rightarrow \lambda z(1-z)$." Nonlinear Dynamics, Annals of the New York Academy of Science 357 (1980), 249–259.
Sarkovskii, A. N. "Coexistence of Cycles of a Continuous Map of a Line into Itself." Ukrain. Mat. Z. 16 (1964), 61–71 (In Russian).
Smale, S. "Diffeomorphisms with Many Periodic Points." Differential and Combinatorial Topology, Princeton Univ. Press, (1964), 63–80.
Gleick, J. Chaos: Making a New Science, Viking, New York, 1987.
McGuire, M. An Eye for Fractals, Addison-Wesley, Redwood City, CA, 1991.
Peterson, I. The Mathematical Tourist, W. H. Freeman, New York, 1988.
Schroeder, M. Fractals, Chaos, Power Laws, W. H. Freeman, New York, 1991.
Barnsley, M. Fractals Everywhere, Boston, Academic Press, 1988.
Edgar, G. A. Measure, Topology, and Fractal Geometry, Springer-Verlag, New York, 1990.
Falconer, K. Fractal Geometry, Wiley, Chichester, 1990.
Falconer, K. The Geometry of Fractal Sets, Cambridge University Press, Cambridge, 1985.
Kaye, B. A Random Walk Through Fractal Dimensions, VCH Publishers, New York, 1989.
Mandelbrot, B. The Fractal Geometry of Nature, W. H. Freeman, New York, 1983.
Prusinkiewicz, P. and Lindenmayer, A. The Algorithmic Beauty of Plants, Springer-Verlag, New York, 1990.
Abraham, R. and Shaw, C. Dynamics: The Geometry of Behavior, Addison-Wesley, Redwood City, CA, 1992.
Devaney, R. L. Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics, Addison-Wesley, Menlo Park, 1990.
Tuck, E. O. and de Mestre, N. J. Computer Ecology and Chaos, Longman Cheshire, Melbourne, 1991.
Sandefur, J. Discrete Dynamical Systems: Theory and Applications, Clarendon Press, Oxford, 1990.