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Type A fusion rules

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Abstract. In this paper we will define fusion algebras and give the general construction to obtain them from affine lie algebras. We also give several known methods to compute the structure constants for fusion algebras of type A.

1. Preliminaries

It is well known that the classical finite-dimensional Lie algebras are classified into four infinite families known as types A ($sl_n(\mathbb{C})$), B ($so_{2n+1}(\mathbb{C})$), C ($s_{2n}(\mathbb{C})$) y D ($so_{2n}(\mathbb{C})$) and some finite families known as types $E_6 - E_8$, F_4 y G_2, also called sporadic.

Since all classical Lie algebras are subalgebras of gl_n, it is possible to define a bilinear form $(\cdot, \cdot)$ given by

$$(A, B) = \mathrm{tr}(ad_A ad_B),$$

where $ad_A : L \to L$ is the function defined by $ad_A(B) = [A, B]$.

It is also known that each of these Lie algebras contains an abelian subalgebra, known as the Cartan subalgebra, which is usually denoted by $\mathfrak{h}$. This subalgebra plays an important role in the theory of classical Lie algebras, since it allows us to decompose the Lie algebra as a direct sum of eigen-subspaces, i.e., in the form

$$\mathfrak{g} = \oplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha, \tag{1}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}$. If $\mathfrak{g}_\alpha \neq 0$, we say that α is a root. The decomposition (1) is known as the root space decomposition. It can be shown that

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there are special roots $\alpha_1, \ldots, \alpha_{n-1}$, known as simple roots, so that any other root α is an integral linear combination of them, i.e., if α is a root then

$$\alpha = \sum_{i=1}^{n-1} c_i \alpha_i,$$

with $c_i \in \mathbb{Z}$. Moreover, the c_i's are all non-negative or all non-positive.

The bilinear form $(\cdot, \cdot)$ can be extended to $\mathfrak{g}^*$ and its restriction to $\mathfrak{h}^*$ is non-degenerate and positive definite; this introduce a geometry in the vector space generated by the simple roots $\alpha_1, \ldots, \alpha_{n-1}$.

Now, let's take linear functional $\lambda_1, \ldots, \lambda_{n-1} \in \mathfrak{h}^*$ so that

$$(\lambda_i, \alpha_j) = \delta_{ij};$$

this functionals are called the fundamental weights of the algebra $\mathfrak{g}$.

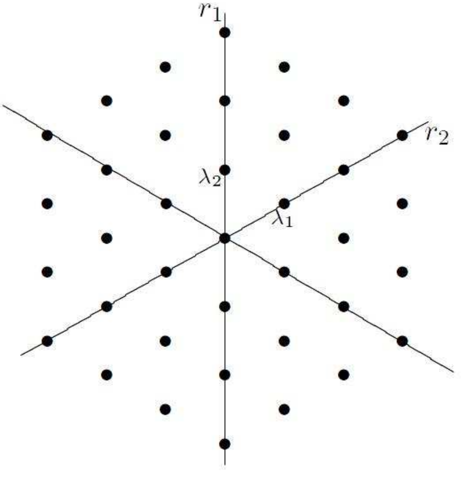
We define the weight lattice as the set

$$P = \left\{ \lambda \;\middle|\; \lambda = \sum_{i=1}^{n-1} c_i \lambda_i, \text{ with } c_1, \ldots, c_{n-1} \in \mathbb{Z} \right\}.$$

The set of dominant weights, denoted by P^+, is defined by

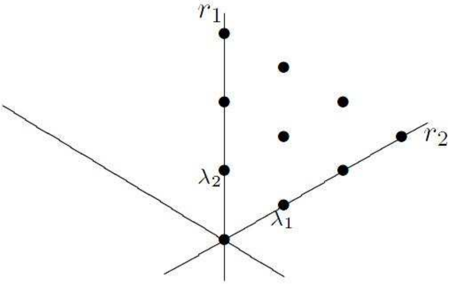
$$P^+ = \left\{ \sum_{i=1}^{N-1} a_i \lambda_i \;\middle|\; 0 \leq a_i \in \mathbb{Z} \right\}.$$

The diagram below shows the type A weight lattice



where λ_1 and λ_2 are the fundamental weights.

The diagram below shows the set of dominant weights, P^+, of type A.



The weights are very important in the representation theory of the Lie algebra, since every irreducible module is uniquely determined, up to isomorphism, by an integral dominant weight, and therefore there is a one-to-one correspondence between dominant weights and irreducible finite-dimensional modules. If $\lambda \in P^+$, we will denote by V^λ the unique irreducible module determined by λ. Furthermore, every irreducible module V^λ decomposes as a direct sum of the form

$$V^\lambda = \oplus_{\beta \in P} V_\beta,$$

where $V_\beta = \{v \in V^\lambda \mid h.v = \beta(h)v, \text{ for all } h \in \mathfrak{h}\}$.

Example 1.1. *Let* $\mathfrak{g} = sl_3(\mathbb{C}) = \mathrm{Span}\{e_{12}, e_{13}, e_{23}, h_1 = e_{11} - e_{22}, h_2 = e_{22} - e_{33}, e_{21}, e_{31}, e_{32}\}$ *where* e_{ij} *is the* 3×3 *matrix whose entries are all zero but the* ij*-entry which is a one. The maximal abeliam subalgebra is* $\mathfrak{h} = \langle h_1, h_2 \rangle$*, the simple roots are linear functionals* α_1 *and* $\alpha_2 \in \mathfrak{h}^*$ *so that the decomposition into root subspaces of* $\mathfrak{g}$ *is given by*

$$\mathfrak{g} = \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}.$$

It can be seen that this subspaces are all 1*-dimensional except* $\mathfrak{g}_0$*, which is* 2*-dimensional; more specifically, we have:*

$$\mathfrak{g}_{\alpha_1+\alpha_2} = \langle e_{13} \rangle, \quad \mathfrak{g}_{\alpha_1} = \langle e_{12} \rangle, \quad \mathfrak{g}_{\alpha_2} = \langle e_{23} \rangle, \quad \mathfrak{g}_{-\alpha_1} = \langle e_{21} \rangle,$$

$$\mathfrak{g}_{-\alpha_2} = \langle e_{32} \rangle, \quad \mathfrak{g}_{-\alpha_1-\alpha_2} = \langle e_{13} \rangle, \quad y \quad \mathfrak{g}_0 = \langle h_1, h_2 \rangle.$$

It can be shown that in this case the fundamental weights λ_1 *and* λ_2 *form the dual base for* $\mathfrak{g}^*$ *with respect to the base* $\{h_1, h_2\}$*. The weight diagram for this module is displayed in Example* 2.1 *below.*

1.1. Affine Lie algebras

Affine Lie algebras are classified into two groups: twisted and untwisted. Here we will only present the construction of the untwisted.

The affine untwisted Lie algebras can be obtained from finite dimensional Lie algebras as follows. If $\mathfrak{g}$ is a finite-dimensional Lie algebra, we define

$$\hat{\mathfrak{g}} \cong \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d, \tag{2}$$

and define the bracket product $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$ by

$$[x(m), y(n)] = [x, y](m+n) + m\delta_{m+n,0}(x, y)c,$$

$$[d, x(m)] = mx(m), \quad \text{and} \quad [c, x(m)] = [c, d] = 0,$$

where $x(m) = t^m \otimes x$. We say that $\hat{\mathfrak{g}}$ is of type $X_n^{(1)}$ if $\mathfrak{g}$ is of type X_n, where $X = A, B, C, D, E, F$ or G.

The symmetric bilinear form $(\cdot, \cdot)$ can be extended from $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ by defining

$$(x(m), y(n)) = \delta_{m+n,0}(x, y), \quad (x(m), c) = (y(n), d) = (c, c) = (d, d) = 0,$$

and

$$(c, d) = 1.$$

The affine Weyl group $\widehat{W}$ of $\hat{\mathfrak{g}}$ is the subgroup of $GL(\mathfrak{H}^*)$ generated by the simple reflections

$$r_i(\Lambda) = \Lambda - \Lambda(\check{\alpha}_i)\alpha_i, \quad \text{for} \quad i = 0, \ldots, N-1,$$

and $W = \text{Span}\{r_i \mid 1 \leq i \leq N-1\}$ is a subgroup of $\widehat{W}$. For each $0 \leq k \in \mathbb{Z}$ the affine Weyl group $\widehat{W}$ acts on the weight lattice P of $\mathfrak{g}$ under the usual action of simple reflections of W and with

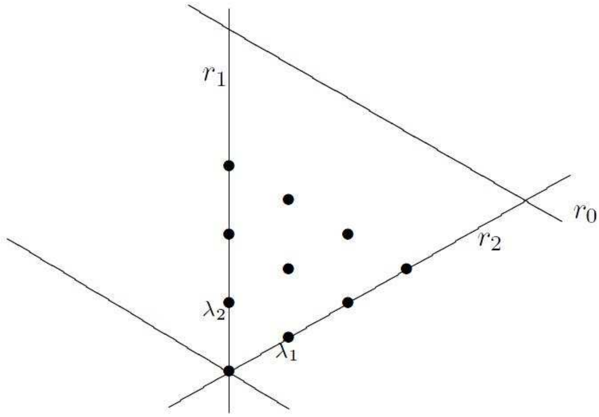
$$r_0(\beta) = r_\theta(\beta) + (k + \check{h})\theta, \tag{3}$$

where $r_\theta(\lambda) = \lambda - \dfrac{2(\lambda, \theta)}{(\theta, \theta)}\theta$ is the reflection with respect to the maximal root θ and $\check{h}$ is the dual Coxeter number. The set of weights of level k of $\mathfrak{g}$ is the set

$$P_k^+ = \left\{ \sum_{j=1}^{N-1} a_j \lambda_j \in P^+ \;\middle|\; \sum_{j=1}^{N-1} a_j \leq k \right\},$$

and the fundamental region under the action of $\widehat{W}$ on P is $P_{k+\check{h}}^+$.

The diagram below shows the set of weights of level 3, P_3^+, for $\widehat{sl_3}$. The fundamental region under the action of $\widehat{W}$ is the region enclosed by the reflection lines r_1, r_2 and r_0.



The representation theory of $\hat{\mathfrak{g}}$ is very similar to the one for $\mathfrak{g}$ if we restrict to category $\mathcal{O}$, which satisfy certain special properties, in particular the property of having a maximal weight $\Lambda \in \hat{P}$ and the fact that irreducible modules are uniquely determined by this maximal weight. The canonical central element c acts on irreducible modules in this category as a scalar k and we have that

$$k = \Lambda(c) = \sum_{i=0}^{N-1} \Lambda_i(c) = \sum_{i=0}^{N-1} u_i \frac{(\alpha_i, \alpha_i)}{2}.$$

So, for a fixed k there exists a finite number of $\Lambda \in \hat{P}^+$ with $\Lambda(c) = k$. Moreover, it can be shown that if $\Lambda \in \hat{P}$ then $\Lambda = \lambda + kc$, where $\lambda \in P_k^+$ and c is the canonical central element. Therefore, irreducible modules of level k for $\mathfrak{g}$ will be denoted by V^λ, where $\lambda \in P_k^+$.

Example 1.2. *The affine type A Lie algebra $\widehat{sl}_N$ obtained by the previous construction with $\mathfrak{g} = sl_N$, has simple roots*

$$\alpha_0, \dots, \alpha_{N-1}.$$

The central canonical element c and the null root d^* are given by

$$c = \sum_{i=0}^{N-1} \check{\alpha}_i \quad \textit{and} \quad d^* = \sum_{i=0}^{N-1} \alpha_i.$$

The fundamental weights are given by the equations

$$\Lambda_0 = c^*, \quad \Lambda_i = c^* + \lambda_i, \quad \textit{for} \quad 1 \le i \le N-1, \tag{4}$$

where $\lambda_1, \dots, \lambda_{N-1}$ are the fundamental weights for $\mathfrak{g} = sl_N$.

Every highest weight irreducible module $\hat{V}$ for $\widehat{sl}_N$ of level k is uniquely determined, up to isomorphism, by a dominant weight $\Lambda = \sum_{i=0}^{N-1} n_i\Lambda_i$ that satisfies the condition $k = \sum_{i=0}^{N-1} n_i$. From the equation (4) it follows that

$$\Lambda = n_0 c^* + \sum_{i=1}^{N-1}(n_i c^* + n_i\lambda_i) = kc^* + \lambda,$$

where $\lambda = \sum_{i=1}^{N-1} n_i \leq k \in P_k^+$.

1.2. Fusion algebras

The original definition of fusion algebras is due to J. Fuchs [9], (see also [5] or [16]) and we give it below:

Definition 1.3. A fusion algebra is a finite dimensional associative and commutative algebra over $\mathbb{Q}$, that satisfies the following statements:

1. There exists a distinguished basis

   $$B = \{x_a \mid a \in A\}, \quad \text{for some indexing set } A,$$

   so that the product of the elements of the basis are given by

   $$x_a \cdot x_b = \sum_{c\in A} N_{a,b}^c x_c,$$

   where $0 \leq N_{a,b}^c \in \mathbb{Z}$.

2. There exists an element $\omega \in A$ so that the function

   $$\mathcal{C}(x_a) = \sum_{b\in A} N_{a,b}^{\omega} x_b \tag{5}$$

   is an involution, i.e., the matrix $C = (C_{ab})_{a,b\in A}$ for $\mathcal{C}$, where $C_{ab} = N_{a,b}^{\omega}$, satisfies the equation $C^2 = I$ (the identity matrix). The function $\mathcal{C}$ is called the **conjugate function**.

1.3. Fusion algebras from affine Lie algebras

There exists a way of constructing fusion algebras from affine Lie algebras, which can be obtained by defining a truncated tensor product, $\otimes_k$, on the level k irreducible modules.

This truncated tensor product endows the algebra generated by irreducible modules of $\hat{\mathfrak{g}}$ of level k with the structure of a fusion algebra. For affine Lie algebras obtained from classical simple finite dimensional Lie algebras, the truncated tensor product is defined as follows:

Given two irreducible modules of level V^λ y V^μ of level k for $\hat{\mathfrak{g}}$, we define

$$V^\lambda \otimes_k V^\mu = \sum_{\nu \in P_k^+} N_{\mu,\lambda}^{(k)\nu} V^n u, \tag{6}$$

where

$$N_{\mu,\lambda}^{(k)\nu} = \sum_{w \in \widehat{W}} (-1)^{l(w)} Mult_\lambda(w(\nu+\rho) - \mu - \rho). \tag{7}$$

This truncated tensor product is known as the level k fusion product.

1.4. The connection

It is known that algebra generated by the level k modules for $\hat{\mathfrak{g}}$ under the fusion product (6) is a fusion algebra, and the coefficients (7) are known as the fusion coefficients. Hence, for each Lie algebra type A-G, there is a two parameter family of fusion algebras $\mathcal{F}(X_N, k)$, where $X = A, \ldots, G$ is the type of the algebra, N is the rank and k is the level.

For type A Lie algebras there are several known algorithms to compute the fusion products; among them we have the Kac-Walton algorithm (which actually works for all types of Lie algebras), and two other methods which only work for type A one using S_k-orbits of $\mathbb{Z}_N^k$, and other using Young diagrams. The main goal of this work is to present these three methods which will be developed in the following sections as follows: Kac-Walton algorithm, Section 2; the method S_k-orbits of $\mathbb{Z}_N^k$, Section 3; and the algorithm using Young diagrams, Section 4.

2. Kac-Walton algorithm

The Kac-Walton algorithm is suggested by Formula (7). This algorithm computes the full fusion product of two level k modules for $\hat{\mathfrak{g}}$, where $\mathfrak{g}$ is a finite dimensional Lie algebra of any type. We describe this algorithm next.

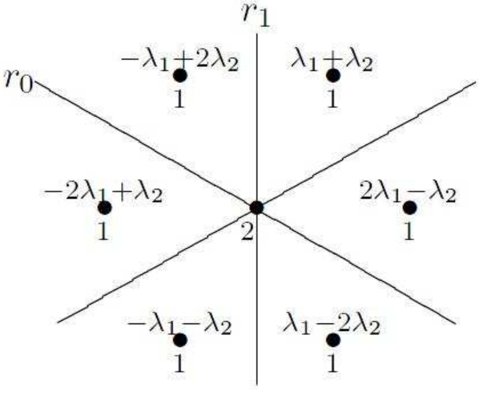
Fix a level k and let V^λ and V^μ two highest weight irreducible modules for $\hat{\mathfrak{g}}$ of level k.

Step 1 Shift the weight diagram of V^λ by adding $\mu + \rho$ to each weight.

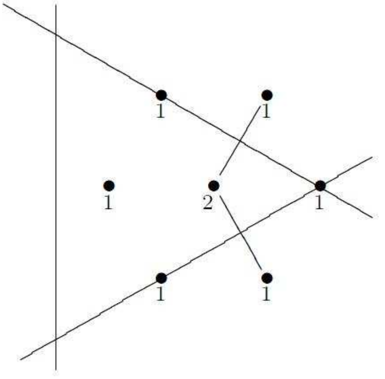
Step 2 Use the level k action of the affine Weyl group $\widehat{W}$ on P to move all shifted weights into the fundamental domain, P_k^+, bounded by the reflection walls of all simple reflections r_i, for $1 \leq i \leq N-1$, and of the affine reflection r_0. That affine reflection is a hyperplane perpendicular to θ going through the point $(k+\check{h})\theta$. The reflected weights counted with inner multiplicities accumulate as an alternating sum of inner multiplicities of V^λ, adding if the required w is even, subtracting if it is odd.

Step 3 The resulting pattern of numbers will be non-negative integers, zero if on a reflection wall, and after shifting the pattern back by subtracting ρ, we get the level k fusion product multiplicities.

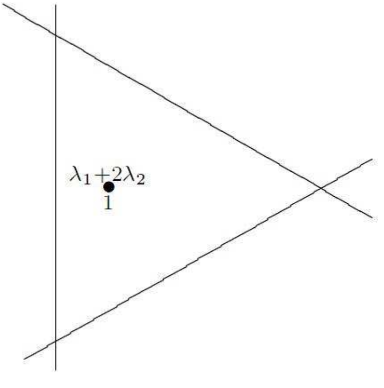
Example 2.1. *Consider the modules V^λ and V^μ of sl_3, where $\lambda = \lambda_1 + \lambda_2$ and $\mu = 2\lambda_1$, and let $k = 2$. Below we can see the weight diagram of V^λ:*



Adding $\mu + \rho = 2\lambda_1 + \lambda_1 + \lambda_2 = 3\lambda_1 + \lambda_2$ to each weight on this diagram, we get:



where the lines connecting weights show where the weights outside the fundamental region reflect inside it. In this case, two weights outside the fundamental region reflect onto the weight $3\lambda_1 + \lambda_2$, reducing its multiplicity by 2. Since weights over the reflection walls do not contribute to the fusion product, we obtain the following pattern of weights:



Subtracting $\rho = \lambda_1 + \lambda_2$ to this weight we get the fusion product

$$V^{\lambda_1+\lambda_2} \otimes_2 V^{2\lambda_1} = V^{\lambda_2}. \tag{8}$$

3. S_k-orbits of $\mathbb{Z}_N^k$

This method of computing fusion coefficients was first posed by Feingonld and Weiner, (see [6]), who managed to prove that it worked for ranks 2 and 3, and completed by Saldarriaga [16], who proved that it works for all ranks. We describe this method below.

Let $\hat{\mathfrak{g}}$ be the affine algebra of type $A_{N-1}^{(1)}$ built from $\mathfrak{g} = sl_N$.

Let G be the group $\mathbb{Z}_N^k$ and let S_k act on it by permuting the k-tuples, so that every orbit of $\mathbb{Z}_N^k$ under this action has a unique standard representative in the form

$$((N-1)^{a_{N-1}},\dots,1^{a_1},0^{a_0}),$$

where the exponent indicates the number of repetitions of the base.

We get a one-to-one correspondence between S_k-orbits of $\mathbb{Z}_N^k$ and N-tuples whose sum is k, by:

$$((N-1)^{a_{N-1}},\dots,1^{a_1},0^{a_0}) \longleftrightarrow (a_0,a_1,\dots,a_{N-1}). \tag{9}$$

Therefore, we get a correspondence between weights $\lambda \in P_k^+$ and orbits of $\mathbb{Z}_N^k$ under the described action of S_k. The orbit corresponding to λ will be denoted by $[\lambda]$, and the correspondence is given by

$$\lambda = \sum_{j=1}^{N-1} a_j\lambda_j \in P_k^+ \longmapsto [\lambda] = [((N-1)^{a_{N-1}},\dots,1^{a_1},0^{a_0})], \tag{10}$$

where $a_0 = k - \sum_{j=1}^{N-1} a_j$ and $[((N-1)^{a_{N-1}},\dots,1^{a_1},0^{a_0})]$ denotes the orbit of $\mathbb{Z}_N^k$ whose standard representative is $((N-1)^{a_{N-1}},\dots,1^{a_1},0^{a_0})$.

Now, for each N-tuple of nonnegative integers

$$(a_0,a_1,\dots,a_{N-1}), \qquad \text{such that} \qquad a_0+a_1+\cdots+a_{N-1}=k,$$

we define the subset of G

$$[(a_0,a_1,\dots,a_{N-1})] = \left\{x \in \mathbb{Z}_N^k \mid j \text{ occurs } a_j \text{ times in } x,\ 0 \le j \le N-1\right\}. \tag{11}$$

Then G is a disjoint union of these subsets.

Note that the symmetric group S_k acts on G by permuting k-tuples. The **set of orbits under this action**, $\mathcal{O} = \mathcal{O}(N,k)$, consists of the subsets (11) defined above, and each orbit contains a unique representative in **standard form**

$$((N-1)^{a_{N-1}},\dots,1^{a_1},0^{a_0}), \tag{12}$$

where the exponent indicates the number of repetitions of the base.

Notation. Given $x \in \mathbb{Z}_N^k$, we will denote the **orbit of x** by $[x]$, and the **representative in standard form of this orbit** will be denoted by $\hat{x}$.

For orbits $[a]$,$[b]$ and $[c]$, we define the set

$$T([a],[b],[c]) = \{(x,y,z) \in [a] \times [b] \times [c] \mid x+y=z\}.$$

Note that $\sigma \in S_k$ acts on $(x,y,z) \in T([a],[b],[c])$ by $\sigma(x,y,z) = (\sigma x, \sigma y, \sigma z)$.

Definition 3.1. Denote by $\mathbf{M}^{(\mathbf{k})[\mathbf{c}]}_{[\mathbf{a}],[\mathbf{b}]}$ **the number of** $\mathbf{S_k}$**-orbits of** $\mathbf{T}([\mathbf{a}],[\mathbf{b}],[\mathbf{c}])$.

We have the following theorem due to Saldarriaga [16]

Theorem 3.2. *For $m \le k$ let $[m\lambda_1]$ and $[k\lambda_1]$ be the orbits in $\mathbb{Z}_N^k$ corresponding to the weights $m\lambda_1$ and $k\lambda_1$, respectively. For $0 \le t \le N-1$ let λ be the weight whose corresponding orbit in $\mathbb{Z}_N^k$ is given by $[\lambda] = [m\lambda_1] \times [k\lambda_1]^t$, and let μ be any other weight on level k. Then for every weight ν on level k we have*

$$N^{(k)\nu}_{\lambda,\mu} = M^{(k)[\nu]}_{[\lambda],[\mu]}.$$

This theorem states that the type A fusion coefficients can be computed via S_k-orbits of $\mathbb{Z}_N^k$. But before we give examples, we next describe how to compute the product of two S_k-orbits of $\mathbb{Z}_N^k$.

Definition 3.3. Let $[a]$, $[b] \in \mathcal{O}$, and assume that $[b] = \{y_1, \ldots, y_t\}$. For $1 \le i \le t$ set

$$z_i = \hat{a} + y_i. \tag{13}$$

We say that the equation $z_j = \hat{a} + y_j$ in the list (13) is **redundant**, if for some $i < j$ and $\sigma \in S_k$ we have

$$\sigma\hat{a} = \hat{a},\ \sigma y_j = y_i \text{ and } \sigma z_j = z_i,$$

that is, if the triples $(\hat{a}, y_i, z_i)$ and $(\hat{a}, y_j, z_j)$ are in the same S_k-orbit of $T([a],[b],[z_i])$.

Removing all redundant equations from (13), and reordering them as $z_i = \hat{a} + y_i$ for $1 \le i \le s$, so that no two triples $(\hat{a}, y_i, z_i)$, $1 \le i \le s$, are in the same S_k-orbit, and noticing that several z_i's could be in the same orbit, and for every $[c] \in \mathcal{O}$, we have

$$M^{(k)[c]}_{[a],[b]} = \mathrm{Card}\{1 \le i \le s \mid z_i \in [c]\}. \tag{14}$$

We can also observe the following.

Remark 3.4. From Equation (14), we also get that $M^{(k)[c]}_{[a],[b]}$ can be computed by removing all redundancies from the list of equations

$$z = \hat{a} + y,$$

where $y \in [b]$ and $z \in [c]$.

Next we give some examples to illustrate this remark.

Example 3.5. *Let $a = (2,1)$ and $b = (1,1)$ be elements in $\mathbb{Z}_3^2$. Then we have that $\hat{a} = (2,1)$ and $[b] = \{(1,1)\}$. Now to compute $[a] \times [b]$, we remove all redundancies from the list*

$$(2,1) + (1,1) = (0,2),$$

from which we get that

$$[(2,1)] \times [(1,1)] = [(2,0)]. \tag{15}$$

From this equation and relation (10), we get the fusion product

$$V^{\lambda_1+\lambda_2} \otimes_2 V^{2\lambda_1} = V^{\lambda_2}.$$

Notice that this result matches the result obtained in (8) computed by using the Kac-Walton algorithm.

The next example shows that the fusion product depends on the level, as we will compute the same fusion product of the previous example, but in level three we will see how the result is different.

Example 3.6. *Let $a = (2,1,0)$ and $b = (1,1,0)$ be elements in $\mathbb{Z}_3^3$. Then we have that $\hat{a} = (2,1,0)$ and $[b] = \{(1,1,0),(1,0,1),(0,1,1)\}$. Now, to compute $[a] \times [b]$, we remove all redundancies from the list:*

$$(2,1,0) + (1,1,0) = (0,2,0),$$
$$(2,1,0) + (1,0,1) = (0,1,1),$$
$$(2,1,0) + (0,1,1) = (2,2,1).$$

Since there are no redundant equations in that list, we get that

$$[(2,1,0)] \times [(1,1,0)] = [(2,2,1)] + [(1,1,0)] + [(2,0,0)]. \tag{16}$$

From this equation and Relation (10), we get the fusion product

$$V^{\lambda_1+\lambda_2} \otimes_3 V^{2\lambda_1} = V^{\lambda_1+2\lambda_2} \oplus V^{2\lambda_1} \oplus V^{\lambda_2}.$$

Another important feature of this method, is that allows us to compute type A fusion products for any rank in a piece of paper. If we were to compute a rank 3 fusion product using Kac-Walton, we would have to draw a three dimensional weight diagram. Here we do not need any graph at all, as the next two examples illustrates.

Example 3.7. *Now let* $a = (3, 2, 1)$ *and* $b = (1, 1, 0)$ *be elements in* $\mathbb{Z}_4^3$*. Then we have that* $\hat{a} = (3, 2, 1)$ *and* $[b] = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$*. To compute* $[a] \times [b]$*, we remove all redundancies from the list:*

$$(3, 2, 1) + (1, 1, 0) = (0, 3, 1),$$
$$(3, 2, 1) + (1, 0, 1) = (0, 2, 2),$$
$$(3, 2, 1) + (0, 1, 1) = (3, 3, 2).$$

There are no redundant equations, so we get

$$[(2, 2, 1)] \times [(1, 1, 0)] = [(3, 1, 0)] + [(2, 2, 0)] + [(3, 3, 2)]. \tag{17}$$

Hence, following Relation (10) *we get the fusion product*

$$V^{\lambda_1+\lambda_2+\lambda_3} \otimes_3 V^{2\lambda_1} = V^{\lambda_1+\lambda_3} \oplus V^{2\lambda_2} \oplus V^{\lambda_2+2\lambda_3}.$$

Example 3.8. *Now let* $a = (3, 2, 1)$ *and* $b = (2, 2, 1)$ *be elements in* $\mathbb{Z}_4^3$*. Then we have that* $\hat{a} = (3, 2, 1)$ *and* $[b] = \{(2, 2, 1), (2, 1, 2), (1, 2, 2)\}$*. To compute* $[a] \times [b]$*, we remove all redundancies from the list:*

$$(3, 2, 1) + (2, 2, 1) = (1, 0, 2),$$
$$(3, 2, 1) + (2, 1, 2) = (1, 3, 3),$$
$$(3, 2, 1) + (1, 2, 2) = (0, 0, 3).$$

We can see that the there are no redundancies, so we get

$$[(3, 2, 1)] \times [(2, 2, 1)] = [(2, 1, 0)] + [(3, 3, 1)] + [(3, 0, 0)].$$

Hence, we get the fusion product in A_3 *level 3*

$$V^{\lambda_1+\lambda_2+\lambda_3} \otimes_3 V^{\lambda_1+2\lambda_2} = V^{\lambda_1+\lambda_2} \oplus V^{\lambda_1+2\lambda_3} \oplus V^{\lambda_3}.$$

4. Young diagrams

A partition is a finite sequence of non-negative integers $(\mu_1, \dots, \mu_n, \dots)$ so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots$. The length of the partition μ, $l(\mu)$, is the number of non-zero μ_i's.

There is also a map between partitions of length at most N and dominant integral weights of A_{N-1} given by

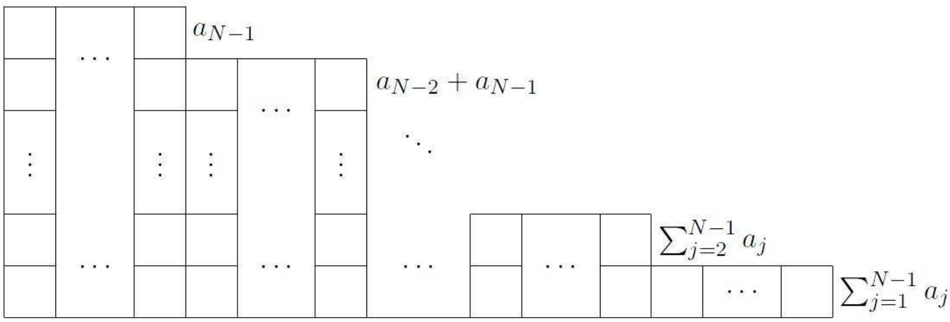
$$(\mu_1, \dots, \mu_N) \longmapsto \lambda = \sum_{j=1}^{N-1} (\mu_j - \mu_{j+1})\lambda_j \in P^+, \tag{18}$$

and if $\mu_1 - \mu_N \leq k$, then $\lambda \in P_k^+$.

Also, given a dominant integral weight of $\mathfrak{g} = sl_N$, $\mu = \sum_{j=1}^{N-1} a_j \lambda_j$, we have associated to it a partition, denoted by (μ), by:

$$(\mu) = \left(\sum_{j=1}^{N-1} a_j, \sum_{j=2}^{N-1} a_j, \dots, a_{N-1} \right) = (\mu_1, \mu_2, \dots, \mu_{N-1}), \tag{19}$$

where μ_t, $1 \leq t \leq N-1$, is the t^{th} part of the partition (μ). To such partition we can associate a Young diagram which is defined as the set of unit squares centered at the points $(s,t) \in \mathbb{Z}^2$ for $1 \leq s \leq \mu_t$ and $1 \leq t \leq l(\mu)$, where $l(\mu)$ denotes the length of (μ), the largest value of t such that $\mu_t \neq 0$. The Young diagram associated to the partition (19) is given below:

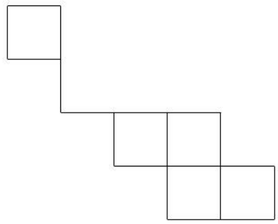


Here the label at the end of every row means the length of the row. The **conjugate** of the partition (μ) is the partition whose Young diagram is obtained by interchanging the rows and the columns of the Young diagram of (μ).

The following definitions are important to describe the product of symmetric polynomials.

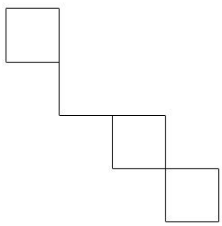
Definition 4.1. If (ν) and (μ) are Young diagrams so that $\mu_i \leq \nu_i$ for all i, the set difference between (ν) and (μ) is called a **skew diagram**, and we denoted it by $(\nu)/(\mu)$.

Example 4.2. *Let $(\mu) = (3,2,1,0)$ and $(\nu) = (5,4,1,1)$; the Young diagram below corresponds to $(\nu)/(\mu)$.*



Definition 4.3. A skew diagram is called an **m-column strip** if it has m boxes with at most one box in each row, and it is called an **m-row strip** if it has m boxes with at most one box in each column.

Example 4.4. *Let* $(\mu) = (3,2,1,0)$ *and* $(\nu) = (4,3,1,1)$



The skew diagram $(\nu)/(\mu)$ *in this example is both a 3-row strip and a 3-column strip.*

The algebra of symmetric polynomials in N variables is the algebra of polynomials $f \in \mathbb{Q}[x_1, x_2, \ldots, x_N]$ invariant under the action of the symmetric group S_N that permutes the variables. This algebra is denoted by $\mathbf{\Lambda_N} = \mathbb{Q}[\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_N}]^{\mathbf{S_N}}$.

For $m > 0$ we define the homogeneous symmetric polynomial

$$h_m = \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq N} x_{i_1} \cdots x_{i_m} ,$$

and we define $e_m = 0 = h_m$ for $m < 0$. We also define $e_0 = 1 = h_0$.

A basis for Λ_N is given by the Schur polynomials $S_{(\mu)}$ indexed by partitions $(\mu) = (\mu_1, \mu_2, \ldots, \mu_N)$ with $l(\mu) \leq N$, defined by

$$S_{(\mu)} = \det(h_{\mu_i - i + j}), \tag{20}$$

if $1 \leq l(\mu) \leq N$. Note that $S_{(\mu)} = h_m$ if $(\mu) = (m)$ consists of a single part.

Equation (20) is known in the literature as the Jacobi-Trudi determinant.

It was proved by Goodman and Wenzl [13] that the fusion algebra $\mathcal{F}(A_{N-1},k)$ associated to $\widehat{sl}_N$ on level k is isomorphic to the quotient algebra of symmetric polynomials $\Lambda_N/I^{(N,k)}$, where

$$I^{(N,k)} = \langle S_{(1^N)} - 1, S_{(\mu)} \mid \mu_1 - \mu_N = k+1 \rangle.$$

The isomorphism $\Lambda_N/I^{(N,k)} \cong \mathcal{F}(A_{N-1},k)$ gives the following dictionary:

$$\begin{gathered}\Lambda_N^{S_N}/I^{(N,k)} \longleftrightarrow \mathcal{F}(A_{N-1},k), \\ S_{(\mu)} \longleftrightarrow V^{\mu}, \\ S_{(\mu)}S_{(\lambda)} = \sum_{(\nu)} N^{(k)(\nu)}_{(\mu),(\lambda)} S_{(\nu)} \longleftrightarrow V^{\mu} \otimes_k V^{\lambda} = \bigoplus_{\nu} N^{(k)\nu}_{\mu,\lambda} V^{\nu},\end{gathered} \tag{21}$$

where the left hand side is indexed by partitions $(\mu) = (\mu_1,\ldots,\mu_N) \subset N \times k$, and the right hand side is indexed by weights $\mu = \sum_{j=1}^{N-1}(\mu_j - \mu_{j+1})\lambda_j \in P_k^+$. The correspondence (18) allows us to move from left to right in the dictionary, and the correspondence (19) allows us to move in the opposite direction in the dictionary. We also get an equality of structure constants in both rings, i.e., $N^{(k)(\nu)}_{(\mu),(\lambda)} = N^{(k)\nu}_{\mu,\lambda}$.

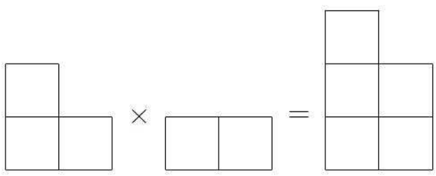
We now present a modification of an important result of Goodman and Wenzl, (see [13], [16], or [17]) which provide another way of computing the multiplication in the fusion algebra $\mathcal{F}(A_{N-1},k)$.

Theorem 4.5 (Fusion Pieri rule for multiplication by h_m). *Let $(\mu) \subseteq (N-1) \times k$, and let $m \leq k$. Then, in $\Lambda_N/I^{(N,k)}$ we have*

$$S_{(\mu)}h_m = \sum_{\substack{(\nu)\subseteq N\times k, \\ (\nu)/(\mu) \text{ is an } m\text{-row strip}}} S_{(\nu)}. \tag{22}$$

Next, we give examples using this result to compute type A fusion products.

Example 4.6. *Let $(\mu) = (2,1)$, $N = 3$ and $k = 2$. Using Theorem 4.5 to compute $S_{(\mu)}h_2$ we get*



,

that is, in the algebra $\Lambda_3/I^{(3,3)}$*, we have that*

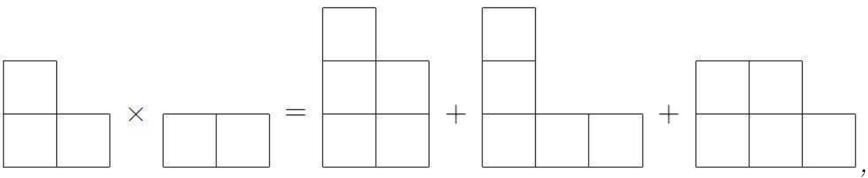
$$S_{(2,1)}h_2 = S_{(2,2,1)}.$$

Now, by the correspondence (18)*, the weight associated to the partition* $(\mu) = (2,1)$ *is* $\mu = \lambda_1 + \lambda_2$*, the weight associated to* $h_2 = S_{(2,0,0)}$ *is* $2\lambda_1$*, and the weight associated to* $(2,2,1)$ *is* λ_2*. Then, the above equation translates into the fusion product in* $\mathcal{F}(A_2, 3)$

$$V^{\lambda_1+\lambda_2} \otimes_3 V^{2\lambda_1} = V^{\lambda_2}.$$

Notice that this calculation matches the one we got in Example 3.5, which was done via S_3*-orbits of* $\mathbb{Z}_3^2$*.*

Example 4.7. *Let* $(\mu) = (2,1)$*,* $N = 3$ *and* $k = 3$*. Using Theorem 4.5 to compute* $S_{(\mu)}h_2$ *we get*



that is, in the algebra $\Lambda_3/I^{(3,3)}$*, we have that:*

$$S_{(2,1)}h_2 = S_{(2,2,1)} + S_{(3,1,1)} + S_{(3,2)}.$$

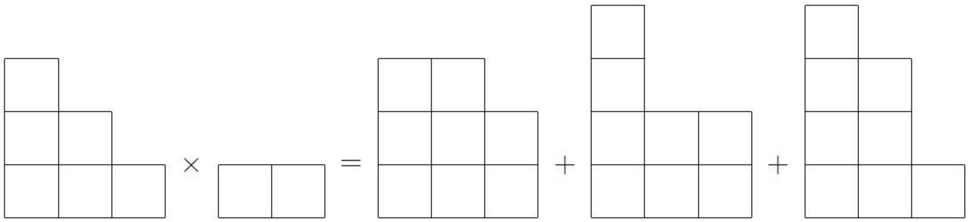
Now, using the dictionary (21)*, the above equation translates into the fusion product in* $\mathcal{F}(A_2, 3)$

$$V^{\lambda_1+\lambda_2} \otimes_3 V^{2\lambda_1} = V^{\lambda_2} \oplus V^{2\lambda_1} \oplus V^{\lambda_1+2\lambda_2}.$$

Notice that this calculation matches the one we got in Example 3.6, which was done via S_3*-orbits of* $\mathbb{Z}_3^3$*.*

In the next example we see another fusion product done via Pieri rules, whose outcome agrees with an orbit calculation for $N = 4$.

Example 4.8. *Let* $(\mu) = (3,2,1)$*,* $N = 4$ *and* $k = 3$*. Using the Theorem* 4.5 *we get*



This shows that $S_{(3,2,1)}h_2 = S_{(3,3,2)} + S_{(3,3,1,1)} + S_{(3,2,2,1)}$, *where the computation is done in the algebra* $\Lambda_4/I^{(4,3)}$. *By taking away columns of length 4 we get*

$$S_{(3,2,1)}h_2 = S_{(3,3,2)} + S_{(2,2)} + S_{(2,1,1)}.$$

By the dictionary (21) *we get the fusion product*

$$V^{\lambda_1+\lambda_2+\lambda_3} \otimes_3 V^{2\lambda_1} = V^{\lambda_2+2\lambda_3} \oplus V^{2\lambda_2} \oplus V^{\lambda_1+\lambda_3}.$$

Notice that this calculation matches the one we got in Example 3.7, *which was done via* S_3*-orbits of* $\mathbb{Z}_4^3$.

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