Iterated forcing with finitely additive measures: applications of probability to forcing theory

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Forcing iterado con medidas finitamente aditivas: aplicaciones de la probabilidad en la teoría de forcing

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There is a concept which corrupts and upsets all others. I refer not to evil, whose limited empire is that of ethics; I refer to the infinite.

Jorge Luis Borges, [Bor84]

Contents

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Abstract

The method of finitely additive measures along finite support iterations was introduced by *Saharon Shelah* in 2000 (see [She00]) to show that, consistently, $\mathrm{cov}(\mathcal{N})$ may have countable cofinality. In 2019, *Jakob Kellner*, *Saharon Shelah* and *Anda Tănasie* (see [KST19]) improved the method: they achieved some new generalizations and applications, such as separating the left side of Cichoń's diagram with $\mathfrak{b} < \mathrm{cov}(\mathcal{N})$. In this thesis, based on probability theory tools and the articles cited above, we develop a general theory of iterated forcing using finitely additive measures. For this purpose, we introduce two new notions: on the one hand, we define a new linkedness property, which we call "μ-FAM-*linked*" and, on the other hand, we generalize the notion of *intersection number* to forcing notions, which justifies the limit steps of our iteration theory. Finally, we apply our theory to prove in detail the consistency of $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0$, and some separations of Cichoń's diagram where $\mathrm{cov}(\mathcal{N})$ is singular. In particular, we obtain a new constellation of Cichón's diagram separating the left side with $\mathrm{cov}(\mathcal{N})$ singular.

Keywords: iterated forcing, probability, finitely additive measure, consistency results, null set, intersection number, cardinal invariant, singular cardinal, Cichoń's diagram.

Resumen

El método que utiliza medidas finitamente aditivas a lo largo de iteraciones de soporte finito fue introducido por *Saharon Shelah* en el año 2000 (véase [She00]) para demostrar que, consistentemente, $\mathrm{cov}(\mathcal{N})$ puede tener cofinalidad contable. En el año 2019, *Jakob Kellner*, *Saharon Shelah* y *Anda Tănasie* (véase [KST19]) mejoraron dicho método: lograron algunas generalizaciones y aplicaciones nuevas, como separar el lado izquierdo del diagrama de Cichoń con $\mathfrak{b} < \mathrm{cov}(\mathcal{N})$. En esta tesis, basados en las herramientas de la teoría de la probabilidad y los artículos citados anteriormente, desarrollamos una teoría general de forcing iterado utilizando medidas finitamente aditivas. Para ello, introducimos dos nociones nuevas: por un lado, definimos una nueva propiedad de ligadura, a la que llamamos "μ-FAM-*linked*" y, por otro lado, generalizamos la noción de *número de intersección* a nociones de forcing, lo que justifica el paso límite de nuestra teoría de iteraciones. Finalmente, aplicamos dicha teoría para probar en detalle la consistencia de $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0$, y algunas separaciones del diagrama de Cichoń donde $\mathrm{cov}(\mathcal{N})$ es singular. En particular, obtenemos una nueva constelación del diagrama de Cichón separando el lado izquierdo con $\mathrm{cov}(\mathcal{N})$ singular.

Palabras clave: forcing iterado, probabilidad, medida finitamente aditiva, resultados de consistencia, conjunto nulo, número de intersección, cardinal invariante, cardinal singular, diagrama de Cichoń.

Introduction

Background and previous work

The *forcing method* is a technique that was created by *Paul Cohen* in 1963 to show that the *continuum hypothesis* is independent of ZFC, the standard axiomatic system where the current mathematics is formalized (see [Coh66]). About a decade later, *Robert. M. Solovay* and *Stanley Tennenbaum* (see [SS71]) constructed a technique in which Cohen's method is iterated along a transfinite sequence, which was called *iterated forcing method* and which was used to solve problems of infinite combinatorics, such as the consistency of *Martin's axiom* and problems in general topology, such as the consistency of *Suslin's hypothesis*. From there, the study of forcing iterations became a broad branch of set theory research, since creating iterated forcing methods allows tackling very complex problems in various areas of mathematics, particularly in the combinatorics of real numbers.

In the combinatorics of real numbers, there is a sub-branch devoted to the study of the so-called *cardinal invariants*. *Cardinal invariants* are cardinal numbers that capture combinatorial properties. For example, $\mathrm{cov}(\mathcal{N})$ is defined as the minimum cardinal of a family of Lebesgue-null sets, whose union is the set of real numbers. There are other cardinals $\mathrm{add}(\mathcal{N})$, $\mathrm{add}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$, $\mathrm{cof}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$, that result from abstracting properties of the measure and category of the real numbers (see Section 1.3). The order relations between these cardinals are given according to a diagram known as *Cichoń's diagram* (see Figure 1.3), which has been an essential object of study by set theorists in recent years, as it has encouraged the creation of increasingly sophisticated iterated forcing methods to prove the consistency of different *separations* of the diagram, that is, of divisions where the cardinals take different values. For instance, perhaps the most remarkable result in this respect (see [GKS19]) is the so-called *Cichoń's maximum*, proved by *Martin Goldstern*, *Jakob Kellner* and *Saharon Shelah*, which achieves a complete separation of Cichoń's diagram, that is, a separation where ten cardinals take different values. However, this result used large cardinals. Later, the same authors and *Diego Mejía* succeeded in obtaining the same result, but without the need to resort to large cardinals (see [GKMS22]).

By the late 1980s, it was known that all the cardinals in Cichoń's diagram have uncountable cofinality, except one: $\mathrm{cov}(\mathcal{N})$. In the Seventies, *David Fremlin* (see Historical Remark 5.1.8) conjectured that $\mathrm{cov}(\mathcal{N})$ has uncountable cofinality, which, due to the behavior of the other cardinals, seemed reasonable. However, this was an open problem until the year 2000, when *Saharon Shelah* in [She00] constructed a finite-support iteration to prove that, consistently, Fremlin's conjecture was false, that is, it is consistent with ZFC that $\mathrm{cov}(\mathcal{N})$ have countable cofinality.

The iteration was not a usual iteration, although it used partial random forcing, that is, random forcing restricted to certain suitable small models, the iteration at each step was accompanied by a sequence of finitely additive measures satisfying certain properties. One of the key points in the work of *Saharon Shelah* was to study extensions of finitely additive measures in order to establish the necessary extension theorems in order to construct the iteration, where by *extension theorems* we mean the following. Suppose that $\mathcal{K}$ is the collection of all finite support iterations of partial random forcing and sufficiently small posets, using finitely additive measures, defined by *Saharon Shelah* in [She00]. Then:

- **Extension theorem at successor steps**: If $\mathbb{K} \in \mathcal{K}$ has length π, then there exists some $\mathbb{K}^\bullet \in \mathcal{K}$ of length $\pi + 1$, such that $\mathbb{K}^\bullet$ extends $\mathbb{K}$.
- **Extension theorem at limit steps**: If γ is a limit ordinal, and $\langle \mathbb{K}_\alpha \colon \alpha < \gamma \rangle$ is a sequence of iterations in $\mathcal{K}$ such that, for any $\alpha < \beta < \gamma$, $\mathbb{K}_\beta$ extends $\mathbb{K}_\alpha$, then there exists some $\mathbb{K} \in \mathcal{K}$ such that, for any $\alpha < \gamma$, $\mathbb{K}$ extends to $\mathbb{K}_\alpha$.

In general, being able to establish these theorems boils down to properly extend the finitely additive measures sequences. Particularly, in the iteration constructed by *Saharon Shelah*, the structure of random forcing is fundamental to prove the extension theorems, where many tools of probability theory were also used.

Subsequently, until 2019, there were at least two papers related to iterated forcing using finitely additive measures method: on the one hand, *Tomek Bartoszyński* in [BJ10] deals with a reformulation of the method[1] and, on the other hand, in [KS19], *Ashutosh Kumar* and *Saharon Shelah* presented some new interesting applications. In both papers only restricted random forcing is used, that is, no new forcing notions are presented with which the iteration can be built.

The most important subsequent work for the development of this thesis is [KST19]: in 2019, *Jakob Kellner*, *Saharon Shelah* and *Anda Tănasie*, resumed the study of the method using finitely additive measures and succeeded in making new significant contributions, not only in the applications but also in the development of the method. For example, they introduced the notion of *strong-*FAM*-limit for intervals*, which made it possible to establish conditions for generalizing the extension theorem at successor steps. Moreover, they succeeded in proving that the forcing $\tilde{\mathbb{E}}$ that *H. Horowitz* and *Saharon Shelah* introduced in [HS16], also has a suitable structure to be able to construct the iteration defined in [She00], replacing random forcing by $\tilde{\mathbb{E}}$. This allowed them to obtain new applications of the method. For example, they succeeded in forcing a constellation of Cichoń's diagram, where the entire left-hand side of the diagram is separated and $\mathfrak{b} < \mathrm{cov}(\mathcal{N})$. So, in [KST19] they not only succeed in generalizing the extension theorems in successor steps, but they also found one more example of a forcing notion other than random forcing, that allows constructing such iterations, which was very valuable for the development of this work.

However, the problem of finding conditions for generalizing the extension theorem at limit steps was still open. If such conditions were found, a general theory of iterated forcing with finitely additive measures could be formalized, and this is precisely what this work deals with. So this thesis extends and improves the work developed in [She00] and [KST19].

[1] However, this was not one of our references, since, in the opinion of professor *Diego Mejía*, there are mistakes in Bartoszyński's approach.

Our work, new ideas and new results

Initially, when we set out the objectives of this thesis, the idea was simply to study the method of iterations with finitely additive measures defined by *Saharon Shelah* in [She00] and, supported by the relatively recent work of *Jakob Kellner*, *Saharon Shelah* and *Anda Tănasie*, to obtain a simplified version of [She00]. We knew it was going to be a hard and difficult work, as [She00] is a very difficult article to read, in fact, in a personal communication, *Saharon Shelah* told *Diego Mejia* that [She00] was one of his, literally thousands of articles, that had taken the most work. However, as the meetings went on, we noticed, with the help of [KST19], that the iteration could be generalized. As we said in the previous section, all that remained was to find conditions to generalize the extension theorem at the limit step. Thanks to the work of *John L. Kelley* (see [Kel59]), professor *Diego Mejia* had the brilliant idea of applying the intersection number to forcing notions, which finally allowed us to find conditions for generalizing the extension theorem at limit steps, formalize the method of iterated forcing using finitely additive measures, and to define a general theory. It was thus that the idea of the intersection number for forcing notions led us to the notion of μ-FAM-*linked* (see Definition 4.2.8), which is the key to the formalization of the method.

Once we noticed that the generalization of the method was possible, our goal changed: it was no longer only to study [She00] or [KST19], but we proposed to formalize a general theory of iterated forcing with finitely additive measures in a complete and detailed way, although this implied hard work, much more than we initially imagined.

To develop the iteration in detail (independent of whether we generalize Shelah's method), it would be necessary for us to formalize also some concepts of probability theory that lack good bibliography, and to study finitely additive measures on Boolean algebras. This implied that the length of the thesis would be much longer than a usual master's thesis in mathematics, but it was a price we were willing to pay.

Several new notions, ideas, and results appear in this thesis: we present a definition of the *intersection number for forcing notions*, and results in Section 4.1 are results that we note and prove, except for Theorem 4.1.7 and Theorem 4.1.9, which came from the original work for Boolean algebras of *John L. Kelley*. In particular, Crucial Lemma 4.1.10 is a new and very important result for us since it is the one that allowed us to prove Main Lemma 4.3.17 and to be able to define conditions for generalizing the extension theorems in the limit step. The linkedness properties $(\Xi, \bar{I}, \varepsilon)$-linked and μ-FAM-linked introduced in Chapter 4 are new, although the first one is based on ideas from [KST19], specifically, we managed to capture and generalize the idea of *strong fam limit for intervals* from [KST19], as a new linkedness property. We should also mention that, although the notion of *probability tree* is widely known, the definition we present in Definition 2.3.1 is our definition, which is not intended to be a standard definition since we could not find any bibliography for it. Moreover, all definitions and results that appear in the Section 2.3 are ours and resulted from the need to be able to apply probability theory in the context of forcing.

Finally, it is important to mention that in this thesis two problems that remained open are solved. On the one hand, after the work of *Jakob Kellner*, *Saharon Shelah* and *Anda Tănasie* (see [KST19]), the problem of finding conditions to establish an extension theorem at limit steps remained open. In this thesis, these conditions are found and it is possible to prove an extension theorem at limit steps (see Theorem 4.3.18), **which allowed defining a general theory of iterated forcing with finitely additive measures**. On the other hand, in terms of the applications, in this thesis, a new constellation of the Cichoń's diagram is obtained, where the left side is separated with $\mathrm{cov}(\mathcal{N})$ singular (see Subsection 5.7.1), which opens up a whole spectrum of questions about the possibility of forcing singular cardinals in the Cichoń's diagram.

The structure of this thesis

We should mention that, if the reader is interested only in the formalization of the theory of iterated forcing using finitely additive measures and has the necessary background in forcing and combinatorics of the real numbers, it is possible to start from Chapter 4 directly, without the need to read the previous chapters. We have taken care to carefully reference all the results of Chapter 1, Chapter 2, and Chapter 3 that are used in Chapter 4 and Chapter 5. So, if the reader decides to read Chapter 4 directly, he/she should return to the previous chapters only when deemed necessary.

To achieve the objectives of this work, we structured the thesis as follows:

In Chapter 1 we present, without going into details, the preliminaries that we consider most important for the development of this work. In particular, we define Polish spaces and some relevant examples such as ${}^{\omega}2$, which will be the space we will use the most throughout the thesis. Afterward, we define cardinal invariants and study some of their properties through *Tukey connections*. After that, we establish the basic notation of Boolean algebras and elementary concepts. We develop some theory on atoms of Boolean algebras because we need some results about that. Finally, we introduce the theory of forcing, from the definition of *forcing notion* to the definition of *finite support iteration*, passing through the generic extension, nice names, and *linkedness properties*, where the notion of μ-Fr-*linked* from [BCM21] will be particularly interesting for us since it is related to our new notion of μ-FAM-*linked*.

In Chapter 2 we define and study probability trees. Initially, this chapter was going to be part of the preliminaries, however, throughout the development of the thesis, we realized that it would be necessary to formalize some concepts of probability theory that lack a good bibliography, such as a probability tree. For this, in the first two sections, we introduce the elementary notions of probability theory such as probability space, conditional probability, events, and random variables. In the third section, where all the results and definitions presented are our own, we define probability trees and prove some results relevant to us. Finally, we introduce a new notion: that of *relative expected value* in probability trees, which will facilitate some calculations in Chapter 2.

In Chapter 3 we take care of a detailed study of finitely additive measures on Boolean algebras since they are so important in our work. In the first section, we introduce the basic definitions and study some fundamental properties. In the second section, we show a connection that exists between finitely additive $\{0, 1\}$-valued measures and ultrafilters, which will allow us, in particular, to construct an example of a finitely additive measure that is not a measure. In Section 3.3 and Section 3.4, we study some criteria for extending finitely additive measures that we will use in Chapter 4. Finally, since finitely additive measures have been extensively studied in the context of real analysis, but not so much in the Boolean algebras context, in Section 3.5 we develop a detailed integration theory for finitely additive measures on Boolean algebras. We conclude by showing that the integral we will define is a generalization of the *Riemann integral* and that both notions are absolute for transitive models of ZFC.

In Chapter 4 we define a general theory of iterated forcing with finitely additive measures. For this, we begin by generalizing the definition of *intersection number*, which was originally defined for Boolean algebras, to forcing notions and study some of its basic properties like its preservation under complete embeddings, and a crucial result that will allow us to generalize the iterations using finitely additive measures. In the second section, we introduce the notion of μ-FAM-linked and study some of its fundamental properties. We then prove, in great detail, that the random forcing notion is σ-FAM-linked, which is too long a proof, since the original proof in [She00] was quite difficult, lacked detail, and had important details to be clarified. In the third section, we develop the iteration theory with finitely additive measures for restricted μ-FAM-linked forcing notions and prove the extension theorems. We conclude by introducing uniform Δ-systems, which will be fundamental for applications of the theory.

In Chapter 5 we study some applications of the theory defined in Chapter 4 to combinatorial problems of real numbers. In particular, we prove in great detail the result of *Saharon Shelah* that establishes the consistency of $\mathrm{cov}(\mathcal{N})$ with countable cofinality. In addition, we show some effects of iterating with finitely additive measures in Cichoń's diagram. In particular, we show the consistency of some separations of Cichoń's diagram where $\mathrm{cov}(\mathcal{N})$ is singular.

Finally, in Chapter 6 we propose some open problems that resulted throughout the development of the thesis, and that we consider relevant for future applications of the iterations method with finitely additive measures.

Papers in progress

This thesis will result in three papers that are under development:

1. A paper about probability trees, based in Chapter 2, where we will also present a generalization of the definition of probability tree, which due to time constraints we do not present in this thesis, and some applications.

2. A paper about finitely additive measures on Boolean algebras, based in Chapter 3, where we will also present a recent proof of the *compatibility theorem of finitely additive measures* (see Theorem 3.4.2), and we develop the idea of $\mathscr{B}$-measurability (see Definition 3.5.9) for a Boolean algebra $\mathscr{B}$.

3. A paper where we are going to present the theory of finitely additive measures developed in Chapter 4, where we are also going to present some applications that for time constraints we could not include in Chapter 5.

List of figures

CHAPTER 1

Preliminaries

During the past decade, many new axioms of set theory have appeared. The principal object of these axioms is to settle important problems which cannot be settled without new axioms. Thus Gödel and Cohen have shown that the Continuum Hypothesis cannot be settled on the basis of the presently accepted axioms; so it is natural to look for reasonable new axioms which will settle it. A second purpose of new axioms is more mundane: to assist in the proof of independence results.

Joseph R. Shoenfield[1]

In this chapter, we will introduce the necessary concepts and results to be able to develop the subsequent chapters. In general, we omit proofs and details, except for some results that we consider to be of special relevance or that do not have clear references.

It is important to mention that we consider two types of preliminaries: *essential preliminaries* and *specific preliminaries*. On the one hand, the *essential preliminaries* are those that are part of a first undergraduate course in mathematics, such as real analysis, general topology, set theory, measure theory, probability theory, mathematical logic, and functional analysis. On the other hand, the *specific preliminaries* are related to topics that are covered, either in a second undergraduate course or in master's courses, such as those related to forcing theory and its iterations, descriptive set theory, combinatorics of real numbers, etc.

Regarding the *essential preliminaries*, we are only going to define the necessary notation. In contrast, with respect to the *specific preliminaries*, we are going to develop everything that we are going to use throughout the thesis, although it is important to mention that, in terms of the preliminaries, this is not intended to be a self-contained work: we assume the reader's experience with forcing

[1] See [Sho75].

iterations and in combinatorics of real numbers, as our discussions in this chapter are concise and without detail.

We begin by defining the notation related to the essential preliminaries.

1.1 Preliminary notation

We denote by $\mathbb{Z}$, Q and $\mathrm{I\!R}$ the sets of integers, rational and real numbers respectively. Notice the difference between the symbols "Q" and "$\mathbb{Q}$", and "$\mathrm{I\!R}$" and "$\mathbb{R}$", since $\mathbb{Q}$ and $\mathbb{R}$ will be used to denote forcing notions. If $I \subseteq \mathrm{I\!R}$ is an interval, we define $I_{\mathbb{Q}} := I \cap \mathbb{Q}$. We usually use lowercase Latin letters x, y, z, etc. to denote real numbers.

We use the symbol ω to denote the set of natural numbers, Ord denotes the proper class of ordinal numbers, and Card denotes the proper class of the cardinal numbers. We usually use Greek letters: α, β, γ, ξ, ζ, etc. to refer to ordinals. Similarly, we use Greek letters: $\kappa, \lambda, \mu, \chi, \theta$, etc. to refer to cardinals. The cardinal of the real numbers $2^{\aleph_0}$ is denoted by $\mathfrak{c}$. To refer to natural numbers we use j, k, ℓ, m, n, k, ℓ, etc. and very frequently n^*.

Let A, B be sets, $\alpha \in \mathrm{Ord}$ and $\kappa, \lambda \in \mathrm{Card}$. Define

$$[A]^{\kappa} := \{X \subseteq A \colon |X| = \kappa\} \text{ and } [A]^{<\kappa} := \{X \subseteq A \colon |X| < \kappa\}.$$

We denote by ${}^{A}B$ the set of functions f from A into B and ${}^{<\alpha}A := \bigcup\{{}^{\xi}A \colon \xi < \alpha\}$. Similarly, ${}^{\leq\alpha}A := {}^{<\alpha+1}A$. For any $t \in {}^{<\alpha}A$ we define its *length* by $\lg(t) := \mathrm{dom}(t)$. We use the symbols "$\langle$" and "$\rangle$" to denote sequences and "$\langle\,\rangle$" to denote the empty sequence. With respect to cardinality, we define $\kappa^{<\lambda} := |{}^{<\lambda}\kappa|$. It is known (see [Kun11, Lem. I.13.17]) that, if λ is infinite and $\kappa \geq 2$, then $\kappa^{<\lambda} := \sup\{\kappa^{\theta} \colon \theta \in \mathrm{Card} \wedge \theta < \lambda\}$. If A, B are non-empty, then $\mathrm{Fn}(A, B)$ is the set of *finite partial functions* from A into B, that is, the functions $f \colon X \to B$, such that $X \in [A]^{<\aleph_0}$. The functions $\pi_1 \colon A \times B \to A$ such that $\pi_1(a, b) := a$, and $\pi_2 \colon A \times B \to B$ such that $\pi_2(a, b) := b$, are called *projections* on the first and second component, respectively.

Let X be a set and R a binary relation on X. We say that (X, R) is a *preoder* if R is reflexive and transitive. Also, (X, R) is a *partial order* or simply *poset*, if R is reflexive, transitive and anti-symmetric. When R is symmetric, reflexive and transitive, we say that it is an *equivalence relation on* X. In this case, we define, for any $x \in X$, $[x]_R := \{y \in X \colon xRy\}$ and call it the R*-equivalence class of* x. If every non-empty subset of X has a minimum element with respect to R, we say that (X, R) is well-ordered, in which case, there exists a unique ordinal $\mathrm{otp}((X, R))$ isomorphic to (X, R). If (X, R) is a preorder and $Y \subseteq X$, define $Y^{\uparrow} := \{x \in X \colon \exists y \in Y(yRx)\}$.

For any set X, we denote by $\mathrm{trcl(X)}$ to its *transitive closure*. Furthermore, for any cardinal κ, $\mathcal{H}(\kappa) := \{X \colon |\mathrm{trcl(X)}| < \kappa\}$. It is known that, if κ is uncountable regular, then $\mathcal{H}(\kappa)$ is a model for all the ZFC axioms except possibly for the *power set axiom* (see [Kun12, Sec. I.14]).

If X is a topological space and $A \subseteq X$, then $\mathrm{cl_X(A)}$ denotes the closure of A in X.

Finally, if $(X, \mathcal{A}, \mathbf{m})$ is a measure space, $A, B \in \mathcal{A}$ and $\mathbf{m}(A) \neq 0$, then we define *the* $\mathbf{m}$*-measure of* B *relative to* A by:

$$\mathbf{m}_A(B) := \frac{\mathbf{m}(A \cap B)}{\mathbf{m}(A)}.$$

1.2 Basic descriptive set theory

In this section, we present the necessary concepts of descriptive set theory. In particular, we are going to define trees, Polish spaces, and Borel sets. Our main references on this topic are [UADP20], [Kec95], [Jec03], and [Mos80].

1.2.1 Trees

Let Z be a non-empty set. A *tree on* ${}^{<\omega}Z$ is a set $\mathcal{T} \subseteq {}^{<\omega}Z$ such that, for any $\rho, \eta \in {}^{<\omega}Z$, if $\rho \subseteq \eta$ and $\eta \in \mathcal{T}$, then $\rho \in \mathcal{T}$. For instance, for any $n^* < \omega$, ${}^{n^* \geq}2$ is a tree on ${}^{<\omega}2$ called the *complete binary tree* of height $n^* + 1$.

Now, fix a tree $\mathcal{T}$ and $\rho, \eta \in \mathcal{T}$. Elements in $\mathcal{T}$ are called *nodes*, and if $\rho \subseteq \eta$, we say that η is an *extension of* ρ. We say that ρ and η are *compatible* if they are compatible as functions. The *height of* ρ *in* $\mathcal{T}$ is $\mathrm{ht}_{\mathcal{T}}(\rho) := \mathrm{dom}(\rho)$ and the *height of* $\mathcal{T}$ is

$$\mathrm{ht}(\mathcal{T}) := \sup\{\mathrm{ht}_{\mathcal{T}}(\rho) + 1 \colon \rho \in \mathcal{T}\}.$$

When the context is clear, we simply denote $\mathrm{ht}_{\mathcal{T}}(\rho)$ as $\mathrm{ht}(\rho)$. An *infinite branch* of $\mathcal{T}$ is an element of $z \in {}^{\omega}Z$ such that, for any $n < \omega$, $z \restriction n \in \mathcal{T}$. The set of branches of $\mathcal{T}$ is denoted by $[\mathcal{T}]$. Also, for any $\rho \in {}^{<\omega}Z$, we define $[\rho] := \{x \in {}^{\omega}Z \colon \rho \subseteq x\}$.

Below we define another notions related to tress:

Definition 1.2.1. Let $\mathcal{T}$ be a tree on ${}^{<\omega}Z$ and $\rho \in \mathcal{T}$. We define:

1. For any $h < \omega$, the *h-th level of* $\mathcal{T}$ is $\mathcal{L}_h(\mathcal{T}) := \mathcal{T} \cap {}^{h}Z$,
2. $\mathcal{T}_{\geq \rho} := \{\eta \in \mathcal{T} \colon \rho \subseteq \eta\}$ is the set of *successors of* ρ in $\mathcal{T}$.
3. $\mathrm{succ}_\rho(\mathcal{T}) := \mathcal{T}_{\geq \rho} \cap \mathcal{L}_{\mathrm{ht}_{\mathcal{T}}(\rho)+1}(\mathcal{T})$, that is, $\mathrm{succ}_\rho(\mathcal{T})$ is the set of *immediate successors of* ρ in $\mathcal{T}$.
4. A *splitting node of* $\mathcal{T}$ is a node $\rho \in \mathcal{T}$ such that, $|\mathrm{succ}_\rho(\mathcal{T})| > 1$.
5. If $\mathcal{T}$ has some splitting node, we define $\mathrm{trunk}(T)$ as the splitting node of shortest length.
6. $\max(\mathcal{T}) := \{\rho \in \mathcal{T} \colon \mathrm{succ}_\rho(\mathcal{T}) = \emptyset\}$, that is, it is the set of *maximal nodes* of $\mathcal{T}$.

When the context is clear we simply write "succ_ρ" instead of "$\mathrm{succ}_\rho(\mathcal{T})$".

Notice that, for any $\rho \in \mathcal{T}$, $\mathrm{ht}(\rho) < \omega$ and, if $[\mathcal{T}] \neq \emptyset$, then $\mathrm{ht}(\mathcal{T}) = \omega$. Also, $\rho \in \mathcal{L}_j(\mathcal{T})$ if, and only if, $\mathrm{ht}(\rho) = j$.

Also, we need the notions of *well pruned* and *perfect* trees:

Definition 1.2.2. Let $\mathcal{T}$ a tree on ${}^{<\omega}Z$. Then

1. We say that $\mathcal{T}$ is *well-pruned tree* of height $n^* < \omega$ if $\mathcal{T} \neq \emptyset$ and, for any $\rho \in \mathcal{T} \setminus \max(\mathcal{T})$, $\mathrm{succ}_\rho(\mathcal{T}) \neq \emptyset$ and $\max(\mathcal{T}) = \mathcal{L}_{n^*}(\mathcal{T})$. Similarly, $\mathcal{T}$ is a *well-pruned tree* of height ω if $\mathcal{T} \neq \emptyset$ and, for any $\rho \in \mathcal{T}$, $\mathrm{succ}_\rho(\mathcal{T}) \neq \emptyset$.

2. We say that $\mathcal{T}$ is *perfect* if $\mathcal{T} \neq \emptyset$ set and, for any $\rho \in \mathcal{T}$, there exists some $\eta \in \mathcal{T}$, such that $\rho \subseteq \eta$ and η is a splitting node in $\mathcal{T}$.

Notice that every perfect tree is a well-pruned tree of height ω. For example, the *binary complete tree* ${}^{<\omega}2$ is perfect.

Lemma 1.2.3. *If $\mathcal{T}_0, \dots, \mathcal{T}_{n-1}$ are trees on ${}^{<\omega}Z$, then:*

1. $\bigcup_{i<n}[\mathcal{T}_i] = \left[\bigcup_{i<n} \mathcal{T}_i\right]$.
2. $\bigcap_{i<n}[\mathcal{T}_i] = \left[\bigcap_{i<n} \mathcal{T}_i\right]$.

From the definition is clear that any non-empty perfect tree is infinite, moreover:

Lemma 1.2.4. *If $\mathcal{T}$ is a perfect tree on ${}^{<\omega}Z$, then* $\sup_{k<\omega} |\mathcal{L}_k(\mathcal{T})| \geq \omega$.

In Figure 1.1 we present an example of a well-pruned tree of length 3. There, we have that $\mathcal{L}_3(\mathcal{T}) = \mathrm{succ}_{\mathrm{trunk}(\mathcal{T})}(\mathcal{T}) = \max(\mathcal{T})$.

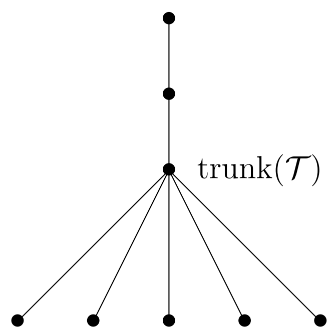

Figure 1.1: A graphic example of a tree $\mathcal{T}$ and its trunk.

1.2.2 Polish spaces and Borel sets

A topological space $\mathcal{X}$ is a *Polish space* if it is separable and there exists some metric d on $\mathcal{X}$ compatible with the topology of $\mathcal{X}$, such that $(\mathcal{X}, d)$ is a complete space. This notion generalizes the combinatorial structure of real numbers. If $\mathcal{X}$ and $\mathcal{Y}$ are Polish spaces, then $\mathcal{X} \times \mathcal{Y}$ is a Polish space with the product topology. In fact, this is true for countable products of Polish spaces.

For any Polish space $\mathcal{X}$, we define $\mathcal{B}(\mathcal{X})$ as the $\subseteq$-minimal σ-algebra on X containing the open sets of $\mathcal{X}$. Elements in $\mathcal{B}(\mathcal{X})$ are called *Borel sets* of $\mathcal{X}$. Also, we say that $A \subseteq \mathcal{X}$ is an *analytic set on* $\mathcal{X}$ if there are another Polish space $\mathcal{Y}$ and $B \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$, such that $A = \pi_1(B)$.

Now, assume that $(\mathcal{X}, \mathcal{A}, \mathbf{m})$ is a measure space, where $\mathcal{X}$ is a Polish space and $\mathbf{m}$ is a complete measure. We define the set of $\mathbf{m}$*-null sets of* $\mathcal{X}$ by $\mathcal{N}(\mathcal{X}) \coloneqq \{A \in \mathcal{A} \colon \mathbf{m}(A) = 0\}$.

In an analogous way, we can define a similar notion with respect to the topology: we say that a set $F \subseteq \mathcal{X}$ is *nowhere dense* if, for every non-empty open set $B \subseteq \mathcal{X}$, there exists a non-empty open set $A \subseteq B$, such that $A \cap F = \emptyset$. Now, a subset of $\mathcal{X}$ is *meager* or of *first category*, if it is a countable union of nowhere sets. Finally, we denote the collection of all meager sets of $\mathcal{X}$ by $\mathcal{M}(\mathcal{X})$.

Recall that $P \subseteq \mathcal{X}$ is a *perfect set* if it does not contain isolated points. *Cantor-Bendixson theorem* allows us to decompose closed sets in terms of a perfect set and a countable set:

Theorem 1.2.5. *Let $\mathcal{X}$ be a metric space with a countable basis. For any closed $C \subseteq \mathcal{X}$, there are a perfect set P and a countable set C, such that $A = P \cup C$.*

Below, we present two familiar examples of Polish spaces:

Example 1.2.6.

1. The canonical example of Polish space is the set of reals numbers endowed with the usual topology. In this case, the metric $d(x, y) := |x - y|$ is a complete metric compatible with the topology of $\mathbb{R}$. We consider $\mathrm{Leb}_{\mathbb{R}} \colon \mathcal{L}(\mathbb{R}) \to [0, \infty]$ as the usual *Lebesgue measure* on $\mathbb{R}$. Recall that elements in $\mathcal{L}(\mathbb{R})$ are called *Lebesgue measurable sets* of $\mathbb{R}$ and $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$.

2. The *Baire space* is the space of functions ${}^{\omega}\omega$, endowed with the product topology, where ω has the discrete topology. If we define, for any $x, y \in {}^{\omega}\omega$,

$$d_{\omega}(x, y) := 2^{-\min\{n<\omega \colon x(n) \neq y(n)\}},$$

then d_{ω} is a complete metric compatible with the topology of ${}^{\omega}\omega$. Also,

$$\mathbb{Q}_{\omega} := \{x \in {}^{\omega}\omega \colon \exists N < \omega \, \forall n > N(x(n) = 0)\}$$

is a countable dense subset of ${}^{\omega}\omega$. Thus, ${}^{\omega}\omega$ is a Polish space.

Now we present with more detail an example of a Polish space that we will use throughout this work: the *Cantor space*.

The Cantor space

The *Cantor space* is the space of functions ${}^{\omega}2$, endowed with the product topology, where ω has the discrete topology. It is clear that $\{[s] \colon s \in {}^{<\omega}2\}$ is a basis for the topology of ${}^{\omega}2$. If we define for $x, y \in {}^{\omega}2$,

$$d_2(x, y) := 2^{-\min\{n<\omega \colon x(n) \neq y(n)\}},$$

then d_2 is a complete metric on ${}^{\omega}2$ compatible with its topology. Also, the subset of functions eventually equals zero is a countable dense subset of ${}^{\omega}2$. Thus, it is a Polish space.

We can also consider ${}^{\omega}2$ as a probability space by endowing it with the Lebesgue measure as follows:

For any $A \subseteq {}^{\omega}2$, we define

$$\mathcal{F}_A := \left\{ \langle s_n \colon n < \omega \rangle \colon A \subseteq \bigcup_{n<\omega} [s_n] \wedge \forall n < \omega \, (s_n \in {}^{<\omega}2) \right\}.$$

Also, we set

$$\mathrm{Leb}^*(A) \coloneqq \inf\left\{\sum_{n<n^*} 2^{-\operatorname{lg}(s_n)} \colon \langle s_n \colon n < n^* \rangle \in \mathcal{F}_A\right\}.$$

So Leb^* is an outer measure on $\mathcal{P}({}^\omega 2)$ and therefore, there are a set $\mathcal{L}({}^\omega 2) \subseteq \mathcal{P}({}^\omega 2)$ and some complete measure $\mathrm{Leb}\colon \mathcal{L}({}^\omega 2) \to [0,\infty]$, such that $\mathcal{B}({}^\omega 2) \subsetneq \mathcal{L}({}^\omega 2)$, where $\mathcal{L}({}^\omega 2)$ is the collection of *Lebesgue measurable* sets of ${}^\omega 2$. Also, for any $s \in {}^{<\omega}2$, $\mathrm{Leb}([s]) = 2^{-\operatorname{lg}(s_n)}$.

Remark 1.2.7. We simply denote $\mathcal{N}({}^\omega 2)$ by $\mathcal{N}$ and $\mathcal{M}({}^\omega 2)$ by $\mathcal{M}$.

The following is the so-called *Lebesgue density theorem* (see [Oxt13, Thm. 3.20]) in the context of ${}^\omega 2$

Theorem 1.2.8. *Let $A \in \mathcal{B}({}^\omega 2)$. If $\mathrm{Leb}(A) > 0$ then*

$$\mathrm{Leb}\left(\{x \in A \colon \lim_{n\to\infty} \mathrm{Leb}_{[x\restriction n]}(A) = 1\}\right) = \mathrm{Leb}(A).$$

Finally, we present the *Mostowski's absoluteness theorem* for ${}^\omega 2$ (see [Jec03, Thm. 25.4] and [MRM19] for a more general context). It is known that we can study the absoluteness of formulas according to their complexity. For example, every Δ_0 formula, that is, every formula with bounded quantifiers is absolute for transitive models of ZFC ([Kun12, Sec. II.17]). However, we can define much more complex formulas for Polish spaces, for example:

Definition 1.2.9. Let $\mathcal{X}$ be a Polish space and φ a formula. We say that φ is a Σ^1_1*-property on* $\mathcal{X}$ when $\{x \in \mathcal{X} \colon \varphi(x)\} \in \Sigma^1_1(\mathcal{X})$.

Mostowski's absoluteness theorem guarantees that every Σ^1_1 property is absolute for transitive models of ZFC:

Theorem 1.2.10. *If φ is a Σ^1_1 property on ${}^\omega 2$, then φ is absolute for every transitive model of* ZFC.

Likewise, Σ^1_1 properties on ${}^\omega\omega$ are absolute for transitive models of ZFC.

1.3 Combinatorics of real numbers

In this section, we are going to study the definitions and necessary results related to combinatorics of the real numbers. In particular, we define the so-called *cardinal invariants associated with an ideal* and study some of their relations using *Tukey relations*. Our main references in this section are [CM22], [BJ95], and [Bla10].

1.3.1 Cardinal invariants associated with an ideal and Cichoń's diagram

Cardinal Invariants or *cardinal characteristics* are cardinals intended to “capture” *combinatorial properties*. For example, consider the following natural question: *how many null sets are necessary to cover the real line?* This combinatorial question can be translated into the language of cardinals:

let κ be the minimum cardinal of a family of null sets whose union is equal to $\mathbb{R}$. So some combinatorial facts are translated into properties of κ. For instance, that the union of countable many null sets is a null set again implies that $\kappa > \aleph_0$, and that $\bigcup_{x\in\mathbb{R}}\{x\} = \mathbb{R}$, implies that $\kappa \leq \mathfrak{c}$. So, in general, cardinal invariants allow us to capture combinatorial properties through relations between cardinals.

There are many cardinal characteristics (see, for example, [Hal19, Ch. 9]), however, for the development of this work it will only be necessary to consider four cardinals with respect to two ideals, and the so-called the *bounding number* ($\mathfrak{b}$), the *dominating number* ($\mathfrak{d}$) (see Definition 1.3.5) and $\mathfrak{c}$.

Definition 1.3.1. Let X be a non-empty set. An *ideal on X* is a set $\mathcal{I} \subseteq \mathcal{P}(X)$ such that:

1. $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$
2. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,
3. If $A, B \in \mathcal{I}$, then $A \cup B \in I$.

Now we define the so-called *characteristic cardinals associated with an ideal*:

Definition 1.3.2. Consider an ideal $\mathcal{I}$ on a set X containing all singletons from X. We define the following cardinals:

1. $\mathrm{cov}(\mathcal{I}) \coloneqq \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} = X\}$, called the *covering of $\mathcal{I}$*.
2. $\mathrm{add}(\mathcal{I}) \coloneqq \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} \notin \mathcal{I}\}$, called the *additivity of $\mathcal{I}$*.
3. $\mathrm{non}(\mathcal{I}) \coloneqq \min\{|Y| \colon Y \subseteq X \wedge Y \notin \mathcal{I}\}$, called the *uniformity of $\mathcal{I}$*.
4. $\mathrm{cof}(\mathcal{I}) \coloneqq \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{I} \wedge \forall A \in \mathcal{I}\, \exists B \in \mathcal{F}\, (A \subseteq B)\}$, called the *cofinality of $\mathcal{I}$*.

We can establish some relations between these cardinals. For example, all of them are infinite. In particular, we show this for $\mathrm{add}(\mathcal{I})$: if $\mathrm{add}(\mathcal{I}) < \aleph_0$, then there exists some $\mathcal{F} \subseteq \mathcal{I}$ such that $|\mathcal{F}| < \aleph_0$, and $\bigcup \mathcal{F} \notin \mathcal{I}$, that is, the union of finite many elements of $\mathcal{I}$ are not in $\mathcal{I}$, which contradicts that $\mathcal{I}$ is an ideal.

Theorem 1.3.3. *If $\mathcal{I}$ is an ideal on X containing all its singletons, then $\mathrm{add}(\mathcal{I})$ is regular, and the order relations are given by Figure 1.2.*

$$\begin{tikzcd}
 & & \mathrm{cov}(\mathcal{I}) \arrow[r] \arrow[ddr] & \mathrm{cof}(\mathcal{I}) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \mathrm{add}(\mathcal{I}) \arrow[ur] \arrow[dr] & & & {|\mathcal{I}|} \arrow[r] & 2^{|X|} \\
 & & \mathrm{non}(\mathcal{I}) \arrow[r] \arrow[uur] & {|X|} \arrow[ur] & &
\end{tikzcd}$$

Figure 1.2: Order relationships between the cardinal invariants associated with an ideal. Each arrow indicates the corresponding inequality.

According to his definition, we can characterize some properties of cardinal invariants in simple ways, for example:

Lemma 1.3.4. *Let λ an infinite cardinal. Then $\operatorname{cov}(\mathcal{N}) \geq \lambda$ if, and only if, no family $\mathcal{F} \subseteq \mathcal{N}$ of size $< \lambda$ covers ${}^{\omega}2$.*

To define the well-known *dominating* and *bounding* numbers, we first define the preorder $\leq^*$ on ${}^{\omega}\omega$ as follows:

$$x \leq^* y :\Leftrightarrow \exists N < \omega\, \forall n > N\, (x(n) \leq y(n)).$$

When $x \leq^* y$ we say that y *dominates* x. Notice that if $x, y \in {}^{\omega}\omega$, then we can find $z \in {}^{\omega}\omega$ such that $x \leq^* z$ and $y \leq^* z$.

Definition 1.3.5. Let $f, g \in {}^{\omega}\omega$. We define the following cardinals:

1. $\mathfrak{b} \coloneqq \min\{|B| \colon F \subseteq {}^{\omega}\omega \wedge \neg[\exists y \in {}^{\omega}\omega \forall x \in B(x \leq^* y)]\}$, called the *bounding number*.
2. $\mathfrak{d} \coloneqq \min\{|D| \colon D \subseteq {}^{\omega}\omega \wedge \forall x \in {}^{\omega}\omega \exists y \in D(x \leq^* y)\}$, called the *dominating number*.

That is, $\mathfrak{b}$ is the smallest cardinal of a unbounded family in ${}^{\omega}\omega$ and $\mathfrak{d}$ is is the smallest cardinal of a dominating family of ${}^{\omega}\omega$. It is not difficult to show that:

Theorem 1.3.6. $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^{\aleph_0}$.

Cichoń's diagram

The cardinals in Figure 1.2 have been extensively studied for the ideals $\mathcal{M}$ and $\mathcal{N}$ (see Section 1.2.2). There is a diagram that is responsible for representing a complete comparison between these cardinals, which is known as *Cichoń's diagram*, presented in Figure 1.3, where each arrow in the diagram represents a provable inequality in ZFC and the dotted lines mean that the equalities

$$\operatorname{add}(\mathcal{M}) = \min\{\mathfrak{b}, \operatorname{cov}(\mathcal{M})\} \text{ and } \operatorname{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \operatorname{non}(\mathcal{M})\},$$

holds. Also, the diagram is *complete* in the sense that no more arrows can be added.
The proof of these inequalities and the completeness appear in [BJ95].

1.3.2 Relational systems and Tukey connections

The core idea of the proofs of the inequalities in Cichoń's diagram is the notion of *Tukey connection*, which we define in this section. But first, we must to define *relational systems*:

Definition 1.3.7. $\mathcal{R} = (X, Y, \lhd)$ is a *relational system* if X, Y are non-empty sets and $\lhd$ is a relation.

Definition 1.3.8. Let $\mathcal{R} = (X, Y, \lhd)$ a relational system. We say that:

1. $B \subseteq X$ is $\mathcal{R}$*-bounded* if, and only if, there is some $y \in B$ such that, for any $x \in X$, $x \lhd y$.

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r, dashed] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u, dashed] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u, dashed] \arrow[r, dashed] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 1.3: Cichoń's diagram.

2. $D \subseteq Y$ is $\mathcal{R}$-*dominating* if, and only if, for any $x \in X$, there is some $d \in D$ such that $x \lhd y$.

These notions of bounding and dominating allow us to define an analog of $\mathfrak{b}$ and $\mathfrak{d}$ for relational systems:

Definition 1.3.9. Let $\mathcal{R}$ a relational system. We define the following cardinals:

1. $\mathfrak{b}(\mathcal{R}) \coloneqq \min\{|B| \colon B \subseteq X$ and B is $\mathcal{R}$-unbounded$\}$, called the *bounding number of* $\mathcal{R}$
2. $\mathfrak{d}(\mathcal{R}) \coloneqq \min\{|D| \colon D \subseteq X$ and D is $\mathcal{R}$-dominating$\}$, called the *dominating number of* $\mathcal{R}$

Below we present some examples of relational systems. But first, we need the notion of *directed pre-order*:

Definition 1.3.10. We say that $(S, \leq_S)$ is a *directed preorder* if it is a pre-order and, for all $s, r \in S$, there exists some $t \in S$ such that $r \leq_S t$ and $s \leq_S t$.

For example, it is clear that $(\omega, \leq^*)$ is a direct preorder. Also, notice that every linear order is a directed pre-order, so in particular, if α is an ordinal, then $(\alpha, \in)$ is a directed preorder.

Definition 1.3.11. If $(S, \leq_S)$ is a directed preoder and $S \neq \emptyset$, then $S \coloneqq (S,\ S,\ \leq_S)$ is a relational system. In this case, we define $\mathrm{cf}(S) \coloneqq \mathfrak{d}(S)$.

So $\mathfrak{b} = \mathfrak{b}(({}^{\omega}\omega, {}^{\omega}\omega,\ \leq^*)$ and $\mathfrak{d} = \mathfrak{d}(({}^{\omega}\omega,\ {}^{\omega}\omega,\ \leq^*)$.

Theorem 1.3.12. *If $(S, \leq_S)$ is a directed preorder without maximum, then*

$$\mathrm{cf}(\mathfrak{b}(S)) = \mathfrak{b}(S) \leq \mathrm{cf}(\mathfrak{d}(S)) \leq \mathfrak{d}(S) \leq |S|.$$

Moreover, if S is a linear order, then $\mathfrak{b}(S) = \mathfrak{d}(S) = \mathrm{cof}(S)$.

The main point in this proof is the following lemma:

Lemma 1.3.13. *Let κ be a cardinal. If $B, D \subseteq S$ are witness of $\mathfrak{b}(S)$ and $\mathfrak{d}(S)$ respectively, then:*

1. *If $B = \bigcup_{\alpha<\kappa} B_\alpha$ and $\forall \alpha < \kappa\, (|B_\alpha| < \mathfrak{b}(S))$, then $\kappa \geq \mathfrak{b}(S)$.*
2. *If $D = \bigcup_{\alpha<\kappa} D_\alpha$ and $\forall \alpha < \kappa\, (|D_\alpha| < \mathfrak{d}(S))$, then $\kappa \geq \mathfrak{b}(S)$.*

Now we present another some examples of relational systems:

Example 1.3.14. Let $\mathcal{I}$ be an ideal on a set X containing all its singletons.

1. We can consider $\mathcal{I} \coloneqq (\mathcal{I}, \subseteq)$ as a directed preorder and therefore $\mathcal{I} \coloneqq (\mathcal{I}, \mathcal{I}, \subseteq)$ is a relational system. In this case, we have that:
   (a) $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$.
   (b) $\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$.
2. $\mathcal{C}_\mathcal{I} \coloneqq (X, \mathcal{I}, \in)$ is a relational system. In this case we have that:
   (a) $\mathfrak{d}(\mathcal{C}_\mathcal{I}) = \mathrm{cov}(\mathcal{I})$,
   (b) $\mathfrak{b}(\mathcal{C}_\mathcal{I}) = \mathrm{non}(\mathcal{I})$.

Similar to Lemma 1.3.4, we have that:

Lemma 1.3.15. *Let λ be an infinite cardinal. Then $\mathfrak{b} \geq \lambda$ if, and only if, any $F \subseteq \omega^\omega$ of size $< \lambda$ is bounded in ω^ω.*

When θ is an infinite cardinal and X is a set with $|X| \geq \theta$, we have that $[X]^{<\theta}$ is an ideal very useful in the applications because, to compute cardinal invariants, one usually compares relational systems with such ideals. In particular, we need the following result:

Lemma 1.3.16. *Let θ be an infinite cardinal and X a set such that $|X| \geq \theta$. Then, $\mathrm{non}([X]^{<\theta}) = \theta$.*

In a natural way, we can define the *dual* of a relational system:

Definition 1.3.17. We define the *dual* of a relational system $\mathcal{R} = (X, Y, \lhd)$ by $\mathcal{R}^\perp \coloneqq (Y, X, \lhd^\perp)$, where the relation is given by: for any $y \in Y$ and $x \in X$, $y \lhd^\perp x :\Leftrightarrow \neg(x \lhd y)$.

For example, $\mathcal{C}_\mathcal{I}^\perp = (\mathcal{I}, X, \not\ni)$ and $\mathcal{I}^\perp = (\mathcal{I}, \mathcal{I}, \not\supseteq)$.
The following are some basic dual properties:

Lemma 1.3.18. *Let $\mathcal{R} = (X, Y, \lhd)$ a relational system. Then*

1. *$(\mathcal{R}^\perp)^\perp = \mathcal{R}$.*
2. *Being $\mathcal{R}^\perp$-unbounded and being $\mathcal{R}$-dominating are equivalent.*
3. *Being $\mathcal{R}^\perp$-dominating and $\mathcal{R}$-unbounded are equivalent.*

4. $\mathfrak{d}(\mathcal{R}^\perp) = \mathfrak{b}(\mathcal{R})$ *and* $\mathfrak{b}(\mathcal{R}^\perp) = \mathfrak{d}(\mathcal{R})$.

Now we introduce the notion of *Tukey connections*, which can be thought of as homomorphisms between relational systems:

Definition 1.3.19. Let $\mathcal{R} = (X, Y, \lhd)$, $\mathcal{R}' = (X', Y', \lhd')$ be relational systems. We say that $(\psi_-, \psi_+)\colon \mathcal{R} \to \mathcal{R}'$ is a *Tukey connection* from $\mathcal{R}$ to $\mathcal{R}'$ if $\psi_-\colon X \to X'$ and $\psi_+\colon Y' \to Y$ are functions such that

$$\forall x \in X\, \forall y' \in Y'\, (\psi_-(x) \lhd' y' \Rightarrow x \lhd \psi_+(y')).$$

In this case, we write $\mathcal{R} \preceq_{\mathrm{T}} \mathcal{R}'$ and we say that $\mathcal{R}$ *is Tukey-below* $\mathcal{R}'$. When $\mathcal{R} \preceq_{\mathrm{T}} \mathcal{R}'$ and $\mathcal{R}' \preceq_{\mathrm{T}} \mathcal{R}$ we say that $\mathcal{R}$ and $\mathcal{R}'$ are *Tukey equivalent* and we denote it by $\mathcal{R} \equiv_{\mathrm{T}} \mathcal{R}'$.

As expected, Tukey connections preserve the basic properties of relational systems:

Theorem 1.3.20. *Let $\mathcal{R} = (X, Y, \lhd)$, $\mathcal{R}' = (X', Y', \lhd')$ be relational systems and a Tukey connection $(\psi_-, \psi_+)\colon \mathcal{R} \to \mathcal{R}'$. Then:*

1. *If $D' \subseteq Y'$ is $\mathcal{R}'$-dominating, then $\psi_+[D'] \subseteq Y$ is $\mathcal{R}$-dominating.*
2. *$(\psi_+, \psi_-)\colon (\mathcal{R}')^\perp \to \mathcal{R}^\perp$ is a Tukey connection.*
3. *If $B \subseteq X$ is $\mathcal{R}$-unbounded, then $\psi_-[B]$ is $\mathcal{R}'$-unbounded.*

As mentioned before, the inequalities in Cichoń's diagram were proved using Tukey connections. Figure 1.4 illustrates these connections.

$$\begin{tikzcd}
 & \mathbf{C}_{\mathcal{N}} \arrow[r] & \mathbf{C}_{\mathcal{M}}^\perp \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{N} \arrow[r] & \mathbf{C}_{[\mathbb{R}]^{<\aleph_1}} \\
 & & ({}^\omega\omega)^\perp \arrow[u] \arrow[r] & {}^\omega\omega \arrow[u] & & \\
\mathbf{C}_{[\mathbb{R}]^{<\aleph_1}}^\perp \arrow[r] & \mathcal{N}^\perp \arrow[uu] \arrow[r] & \mathcal{M}^\perp \arrow[u] \arrow[r] & \mathbf{C}_{\mathcal{M}} \arrow[u] \arrow[r] & \mathbf{C}_{\mathcal{N}}^\perp \arrow[uu] &
\end{tikzcd}$$

Figure 1.4: Cichoń's diagram via Tukey connections. Any arrow represents a Tukey connection in the given direction.

Strongly unbounded families

One notion that will be really useful in Chapter 5 is that of *strongly unbounded family*:

Definition 1.3.21. Let $\mathcal{R} = \langle X, Y, \lhd \rangle$ be a relational system, I an index set and θ a cardinal such that $\aleph_0 \leq \theta \leq |I|$. We say that a family $\{x_i \colon i \in I\} \subseteq X$ is *strongly θ-$\mathcal{R}$-unbounded*, when for any $y \in Y$, $|\{i \in I \colon x_i \, R \, y\}| < \theta$.

If $\mathcal{R}$ is a relational system, we can characterize "$\mathcal{C}_{[I]^{<\theta}} \preceq_{\mathrm{T}} \mathcal{R}$" in terms of existence of a strongly θ-$\mathcal{R}$-unbounded family $\{x_i \colon i \in I\} \subseteq X$:

Theorem 1.3.22. *Let θ be an infinite cardinal, I a set of size $\geq \theta$ and let $\mathcal{R} = (X, Y \lhd)$ be a relational system. Then, $\mathcal{C}_{[I]^{<\theta}} \preceq_{\mathrm{T}} \mathcal{R}$ if, and only if, there exists some strongly θ-$\mathcal{R}$-unbounded family $\{x_i \colon i \in I\}$. In this case, $\mathfrak{b}(\mathcal{R}) \leq \theta$.*

1.4 Boolean algebras

Boolean algebras will be important for us fundamentally for two reasons: on the one hand, the domain of finitely additive measures (see Definition 3.1.1), will be Boolean algebras. On the other hand, although there are different alternatives, we are going to define the completion of a forcing notion as a complete Boolean algebra (Subsection 1.5.4). For more on Boolean algebras, [BM77] and [GP09] are recommended.

Definition 1.4.1. $\mathscr{B} \coloneqq \langle \mathscr{B}, \wedge, \vee, \sim, 0_{\mathscr{B}}, 1_{\mathscr{B}} \rangle$ is a *Boolean algebra* if $\mathscr{B}$ is a non-empty set, $\wedge$, $\vee$ are binary operations on $\mathscr{B}$, $\sim$ is a unary operation on $\mathscr{B}$, $1_{\mathscr{B}}, 0_{\mathscr{B}} \in \mathscr{B}$ and it satisfies the following properties for all $a, b, c \in \mathscr{B}$:

1. *Commutativity*:
   (a) $a \vee b = b \vee a$,
   (b) $a \wedge b = b \wedge a$.
2. *Associativity*:
   (a) $a \vee (b \vee c) = (a \vee b) \vee c$,
   (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
3. *Absorption*:
   (a) $(a \vee b) \wedge b = b$
   (b) $(a \wedge b) \vee b = b$.
4. *Distributivity*:
   (a) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
   (b) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$.
5. *Identity*:
   (a) $a \wedge 1_{\mathscr{B}} = a$,
   (b) $a \vee 0_{\mathscr{B}} = a$.
6. *Complements*:
   (a) $a \vee \sim a = 1_{\mathscr{B}}$,
   (b) $a \wedge \sim a = 0_{\mathscr{B}}$.

The operations $\wedge$ and $\vee$ are known as *meet* and *join* respectively. Also $\wedge, \vee$ and $\sim$ are known as the *Boolean operations of $\mathscr{B}$*.

Notation 1.4.2. We will use calligraphic letters $\mathscr{B}, \mathscr{C}, \mathscr{D}$, etc. to denote Boolean algebras.

Example 1.4.3. The canonical example of a Boolean algebra is the power set: consider a non-empty set X. Then $\langle \mathcal{P}(X), \cap, \cup, {}^{\mathrm{c}}, \emptyset, X \rangle$ is a Boolean algebra.

Let $\mathscr{B}$ be a Boolean algebra. A *Boolean subalgebra* of $\mathscr{B}$ is a non-empty subset $\mathscr{C} \subseteq \mathscr{B}$ which is closed under the Boolean operations of $\mathscr{B}$. So, it is clear that $\mathscr{C}$, is itself, a Boolean algebra and $0_{\mathscr{B}}, 1_{\mathscr{B}} \in \mathscr{C}$.

Let $B \subseteq \mathscr{B}$. It is easy to prove that the intersection of all Boolean subalgebras containing B is a Boolean subalgebra of $\mathscr{B}$. Indeed, it is the $\subseteq$-minimum Boolean subalgebra of $\mathscr{B}$ containing B, which is called the *Boolean subalgebra generated by* B and denoted by $\langle B\rangle_{\mathscr{B}}$ or, simply $\langle B\rangle$, when the context is clear. In this case, B is called the *generator set* of $\langle B\rangle_{\mathscr{B}}$ If B is a finite set, we say that $\langle B\rangle_{\mathscr{B}}$ is a *finitely generated Boolean algebra*.

We can characterize $\langle B\rangle_{\mathscr{B}}$ in terms of finite Boolean combinations in B:

Lemma 1.4.4. *Let $\mathscr{B}$ a Boolean algebra and $B \subseteq \mathscr{B}$. Then $\langle B\rangle$ consists of all elements of the form*

$$\bigvee_{j=1}^{n} \bigwedge_{k=1}^{m_j} b_{jk},$$

where $n < \omega$, for each $j \leq n$, $m_j < \omega$, and for all $j, k < \omega$, either $b_{jk} \in B$ or $\sim b_{jk} \in B$.

Every Boolean algebra induces a partial order on itself:

Lemma 1.4.5. *Every Boolean algebra $\mathscr{B}$ can be endowed with a partial order structure as follows: for any $a, b \in \mathscr{B}$,*

$$a \leq_{\mathscr{B}} b :\Leftrightarrow a \wedge b = a.$$

This will allow us to consider Boolean algebras as forcing notions (see Section 1.5), however, note that Definition 1.4.1(5)(b) implies that $0_{\mathscr{B}}$ is the least element of $(\mathscr{B}, \leq_{\mathscr{B}})$, which would trivialize the forcing properties of $\mathscr{B}$. So, we define:

Definition 1.4.6. When $\mathscr{B}$ is a Boolean algebra, we define $\mathscr{B}^+ := \mathscr{B} \setminus \{0_{\mathscr{B}}\}$.

Notice that, if $\mathscr{B}$ is a Boolean algebra and $a, b \in \mathscr{B}$, then $a \wedge b$ and $a \vee b$ are the *infimum* and the *supreme* of $\{a, b\}$ respectively, in the sense that they are the minimum upper bound and the maximum lower bound, respectively, with respect to the order $\leq_{\mathscr{B}}$. Furthermore, if I is a finite set and $B := \{b_i \colon i \in I\} \subseteq \mathscr{B}$, we can define recursively

$$\bigvee_{i\in I} b_i,\ \bigwedge_{i\in I} b_i \in \mathscr{B}$$

in a natural way. In this case, we denote:

$$\bigwedge B := \bigwedge_{i\in I} b_i \text{ and } \bigvee B := \bigvee_{i\in I} b_i.$$

However, for I infinite, when defining $\bigvee_{i\in I} b_i$ and $\bigwedge_{i\in I} b_i$ using the idea of the supreme and the infimum, in general, there are problems with the existence in $\mathscr{B}$: for example, it is clear that $(\mathcal{L}(\mathbb{R}), \cap, \cup, {}^{c}, \emptyset, \mathbb{R})$ is a Boolean algebra because it is a Boolean subalgebra of $\mathcal{P}(\mathbb{R})$. If $V \subseteq \mathbb{R}$ is a non-Lebesgue measurable set, then $\{\{x\} \colon x \in A\} \subseteq \mathcal{L}(\mathbb{R})$ has no supreme in $\mathcal{L}(\mathbb{R})$.

A Boolean algebra such that every infinite non-empty subset has supreme is called *complete*, and a Boolean algebra such that every countable non-empty subset has supreme is called *σ-complete*. For example, $\mathcal{P}(X)$ is complete for any set X.

1.4.1 Boolean homomorphisms

Let $\mathscr{B}, \mathscr{C}$ be Boolean algebras.

1. A *Boolean homomporhism* from $\mathscr{B}$ into $\mathscr{C}$ is a function $h \colon \mathscr{B} \to \mathscr{C}$ such that, for all $a, b \in \mathscr{B}$:
   (a) $h(a \wedge b) = h(a) \wedge h(b)$,
   (b) $h(a \vee b) = h(a) \vee h(b)$,
   (c) $h({\sim} a) = {\sim} h(a)$.
2. A *Boolean isomorphism* from $\mathscr{B}$ onto $\mathscr{C}$ is a bijective Boolean homomorphism from $\mathscr{B}$ into $\mathscr{C}$. In this case we say that $\mathscr{B}$ and $\mathscr{C}$ *are isomorphic* and we write $\mathscr{B} \cong \mathscr{C}$.

As a consequence of *Stone's representation theorem* (see [BM77, Thm. 4.1]), we can characterize any Boolean algebra using Example 1.4.3:

Theorem 1.4.7. *Every boolean algebra is isomporhic to a Boolean subalgebra of $\mathcal{P}(X)$ for some set X.*

1.4.2 Atoms

Let $\mathscr{B}$ be a Boolean algebra. An *atom of $\mathscr{B}$* is a $\leq_{\mathscr{B}}$-minimal non-zero element of $\mathscr{B}$. For example, if $x \in X$, then $\{x\}$ is an atom in $\mathcal{P}(X)$. We denote by $\mathrm{At}_{\mathscr{B}}$ the set of all atoms of $\mathscr{B}$.

We can characterize atoms in many ways:

Theorem 1.4.8. *Let $\mathscr{B}$ be a Boolean algebra and $a \in \mathscr{B}$. The following statements are equivalent:*

1. $a \in \mathrm{At}_{\mathscr{B}}$.
2. $a \neq 0$ *and for any* $b \in \mathscr{B}$, *if* $b \leq a$, *then either* $a = b$ *or* $b = 0$.
3. *For any* $b \in \mathscr{B}$ *exactly one of* $a \leq_{\mathscr{B}} b$ *or* $a \leq {\sim} b$ *holds.*
4. $a \neq 0$, *and for any finite* $B \subseteq \mathscr{B}$, *if* $a \leq \bigvee B$, *then there is some* $b \in B$ *such that* $a \leq b$.

As a consequence of (2), the atoms in $\mathscr{B}$ have an *atomic minimal property*: if $a, a' \in \mathrm{At}_{\mathscr{B}}$ and $a' \leq_{\mathscr{B}} a$, then $a = a'$.

We are particularly interested in finitely generated Boolean algebras, because we are going to be able to characterize their atoms in a very specific way, but first we must define:

Definition 1.4.9. Let $\mathscr{B}$ be a Boolean algebra and $B \subseteq \mathscr{B}$. Then,

1. For any $b \in \mathscr{B}$ and $d \in \{0, 1\}$ we define:

$$b^d := \begin{cases} b & \text{if} \quad d = 0, \\ {\sim} b & \text{if} \quad d = 1. \end{cases}$$

2. For $\sigma \in \mathrm{Fn}(B,2)$ we set:
$$a_\sigma := \bigwedge_{b \in \mathrm{dom}(\sigma)} b^{\sigma(b)}.$$

Notice that for any $\sigma \in \mathrm{Fn}(F,2)$, $a_\sigma \in \mathscr{B}$ because it is a finite Boolean combination of elements in $\mathscr{B}$. On the other hand, since by Lemma 1.4.4 an element of $\mathscr{B}$ is in the sub-algebra generated by a set B if and only if it can be written as a finite join of finite meets of elements and complements of elements from B, we get:

Lemma 1.4.10. *Let $\mathscr{B}$ be a Boolean algebra and $B \subseteq \mathscr{B}$. Then*
$$\langle B \rangle = \left\{ \bigvee_{\sigma \in C} a_\sigma \colon C \in [\mathrm{Fn}(B,2)]^{<\omega} \right\}.$$
As a consequence, if B is finite, then
$$\langle B \rangle = \left\{ \bigvee_{\sigma \in C} a_\sigma \colon C \subseteq {}^B 2 \right\}.$$

As a consequence, as we mentioned before, we can characterize the atoms of finitely generated Boolean algebras using a_σ:

Theorem 1.4.11. *Let $\mathscr{B}$ be a finitely generated Boolean algebra generated by B. Then*
$$\mathrm{At}_{\mathscr{B}} = \{a_\sigma \colon \sigma \in {}^B 2 \wedge a_\sigma \neq 0\}.$$

Proof. Let $\sigma \in {}^B 2$ be such that $a_\sigma \neq 0$ and let $b \in \mathscr{B}$ be such that $b \leq a_\sigma$. By Lemma 1.4.10, there exists some $C \subseteq {}^B 2$ such that $b = \bigvee_{\upsilon \in C} a_\upsilon$. Consider two possible cases:

1. $\sigma \in C$: in this case, $a_\sigma \leq \bigvee_{\tau \in C} a_\tau = b$. Thus, $a_\sigma = b$.
2. $\sigma \notin C$: fix $\tau \in C$. Since $\sigma \notin C$, there is some $c \in B$ such that $\sigma(c) \neq \tau(c)$. Without loss of generality, assume that $\sigma(c) = 1$ and $\tau(c) = 0$, hence, on the one hand, it is clear that $a_\tau \leq c$. On the other hand,
$$a_\tau \leq \bigvee_{\upsilon \in C} a_\upsilon = b \leq a_\sigma \leq \sim c.$$
Therefore, $a_\tau \leq c, \sim c$, hence $a_\tau = 0$. Thus, for any $\tau \in C$ we have that $a_\tau = 0$, that is, $b = 0$.

Whence follows that, for any $b \in \mathscr{B}$ below a_σ, either $b = a_\sigma$ or $b = 0$. Thus, by Theorem 1.4.8(2), $a_\sigma \in \mathrm{At}_{\mathscr{B}}$.

Reciprocally, let $a \in \mathrm{At}_{\mathscr{B}}$. By Theorem 1.4.8(3), for any $b \in \mathscr{B}$ we have that either $a \leq b$ or $a \leq \sim b$. So we can define $\sigma \colon B \to 2$ such that:
$$\sigma(b) := \begin{cases} 0, & \text{if} \quad a \leq b, \\ 1, & \text{if} \quad a \leq \sim b. \end{cases}$$

Let $b \in B$ and notice that:

$$\sigma(b) = 1 \Rightarrow a \leq \sim b = b^1 = b^{\sigma(b)} \text{ and } \sigma(b) = 0 \Rightarrow a \leq b = b^0 = b^{\sigma(b)},$$

that is, for any $b \in B$, we have that $a \leq b^{\sigma(b)}$, hence $a \leq a_\sigma$. This implies that $a_\sigma \in \mathrm{At}_{\mathscr{B}}$ because if $a_\sigma = 0$, then $a = 0$, which is a contradiction. Finally, by the atomic minimal property, we conclude that $a = a_\sigma$. $\square_{\text{Theorem 1.4.11}}$

As a consequence:

Corollary 1.4.12. *If $\mathscr{B}$ is a finitely generated Boolean algebra, then $\mathrm{At}_{\mathscr{B}}$ is finite. Furthermore, $|\mathrm{At}_{\mathscr{B}}| \leq 2^{|B|}$, where B is a generator set of $\mathscr{B}$.*

Lemma 1.4.13. *Let $\mathscr{B}$ be a Boolean algebra, $b \in \mathscr{B}$ and $A \subseteq \mathrm{At}_{\mathscr{B}}$ finite. If $b = \bigvee A$, then*

$$A = \{a \in \mathrm{At}_{\mathscr{B}} \colon a \leq b\}.$$

Proof. Notice that, if $c \in A$, then $c \leq \bigvee A = b$, so $c \leq b$. Reciprocally, let $a \in \mathrm{At}_{\mathscr{B}}$ such that $a \leq b$, hence $a \leq \bigvee A$ and, by Theorem 1.4.8(4), there exists $a' \in A$ such that $a \leq a'$. Thus, by the atomic minimal property, $a = a' \in A$. $\square_{\text{Lemma 1.4.13}}$

Theorem 1.4.14. *Let $\mathscr{B}$ be a finitely generated Boolean algebra. Then $\mathrm{At}_{\mathscr{B}}$ partitions $1_{\mathscr{B}}$ in the following sense:*

1. *If $a, a' \in \mathrm{At}_{\mathscr{B}}$ and $a \neq a'$, then $a \wedge a' = 0_{\mathscr{B}}$,*
2. *$1_{\mathscr{B}} = \bigvee \mathrm{At}_{\mathscr{B}}$.*

Furthermore, any $b \in \mathscr{B}$ is partitioned by atoms.

Proof. Let $B \subseteq \mathscr{B}$ finite such that $\langle B \rangle = \mathscr{B}$.

1. Let $a, a' \in \mathrm{At}_{\mathscr{B}}$ such that $a \neq a'$. Since in particular $a \wedge a' \leq a$, we have that either $a \wedge a' = 0_{\mathscr{B}}$ or $a \wedge a' = a$, however, $a \wedge a' = a$ implies, by the atomic minimal property, that $a = a'$. Thus $a \wedge a' = 0_{\mathscr{B}}$.
2. First, notice that $\bigvee \mathrm{At}_{\mathscr{B}}$ is defined by virtue of Corollary 1.4.12. Now, by Lemma 1.4.10, there exists $C \subseteq {}^B2$ such that $1_{\mathscr{B}} = \bigvee_{\sigma \in C} a_\sigma$. Define $A := \{a_\sigma \colon \sigma \in C\}$, hence $1_{\mathscr{B}} = \bigvee A$, and by the previous Lemma, it follows that $A = \{a \in \mathrm{At}_{\mathscr{B}} \colon a \leq 1_{\mathscr{B}}\} = \mathrm{At}_{\mathscr{B}}$. Finally, $1_{\mathscr{B}} = \bigvee \mathrm{At}_{\mathscr{B}}$.

Finally, that any $b \in \mathscr{B}$ is partitioned by atoms is clear by Lemma 1.4.10 and Theorem 1.4.11. $\square_{\text{Theorem 1.4.14}}$

1.4.3 Filters, ultrafilters and ideals

Let $\mathscr{B}$ a Boolean algebra. A *filter on $\mathscr{B}$* is a non-empty set $F \subseteq \mathscr{B}$ such that:

1. If $x, y \in F$, then $x \wedge y \in F$,
2. If $x \in F$ and $x \leq y$, then $y \in F$,
3. $0_{\mathscr{B}} \notin F$.

An *ultrafilter on $\mathscr{B}$* is a filter $F \subseteq \mathscr{B}$ such that, for any $b \in \mathscr{B}$, $b \in F$ or $\sim b \in F$.
Dual to the definition of filter, an *ideal on $\mathscr{B}$* is a non-empty set $I \subseteq \mathscr{B}$ such that:

1. If $x, y \in I$, then $x \vee y \in I$,
2. If $x \in I$ and $y \leq x$, then $y \in I$,
3. $1_{\mathscr{B}} \notin I$.

For example, if $\mathcal{X}$ is a Polish space with a measure on $\mathcal{B}(\mathcal{X})$, then $\mathcal{N}(\mathcal{X})$ is an ideal on $\mathcal{B}(\mathcal{X})$, indeed it is an σ-ideal, that is, it is closed under countable unions.
Filters and ideals on a Boolean algebra are *dual notions* in the following sense:

Lemma 1.4.15. *Let $F \subseteq \mathscr{B}$. Define $F^{\sim} := \{\sim a \colon a \in F\}$. Then F is a filter on $\mathscr{B}$ if, and only if, $F^{\sim}$ is an ideal on $\mathscr{B}$.*

Since filters are upwards closed and closed under $\wedge$, and ideals are downwards closed and closed under $\vee$, we can characterize the Boolean sub-algebra generated by a filter as follows:

Theorem 1.4.16. *If $F \subseteq \mathscr{B}$ is a filter, then $\langle F \rangle = F \cup F^{\sim}$. As a consequence, if F is an ultrafilter on $\mathscr{B}$, then $\langle F \rangle = \mathscr{B}$.*

Definition 1.4.17. Let X be a non-empty set. We say that $F \subseteq \mathcal{P}(X)$ is a *free filter* if it is a filter such that $[X]^{<\aleph_0} \subseteq F$.

Ideals will allow us to define quotients of Boolean algebras:

Definition 1.4.18. Let $\mathscr{B}$ a Boolean algebra and $B \subseteq \mathscr{B}$. We define the relation $\sim_B$ on $\mathscr{B}$ as follows:

$$a \sim_B b :\Leftrightarrow (a \wedge \sim b) \vee (b \wedge \sim a) \in B.$$

Notice that in particular, if $\mathscr{B}$ is a Boolean subalgebra of $\mathcal{P}(X)$ for some set X, then $a \sim_B b$ iff $a \triangle b \in B$, where $a \triangle b$ is the *symmetric difference* between a and b.
The relation $\sim_B$ is interesting when A is an ideal on $\mathscr{B}$:

Theorem 1.4.19. *If $I \subseteq \mathscr{B}$ is an ideal, then $\sim_I$ is an equivalence relation on $\mathscr{B}$. Furthermore, $\mathscr{B}/I := \mathscr{B}/{\sim_I}$ is a Boolean algebra with the natural operations.*

Characterizing the order of $\mathscr{B}/I$ will be useful for our definition of random forcing (Section 1.5.8).

Example 1.4.20. Let $\mathscr{B}$ be a Boolean algebra and I an ideal on $\mathscr{B}$. For $b \in B$, we denote $[b]_I :=$ $[b]_{\sim_I}$. Then we can characterize the order of $\mathscr{B}/I$ as follows:

$$\begin{aligned}[a]_I \leq [b]_I \Leftrightarrow [a]_I \wedge [b]_I = [a]_I \Leftrightarrow [a \wedge b]_I = [a]_I \Leftrightarrow (a \wedge b) \sim_I a \\ \Leftrightarrow [(a \wedge b) \wedge \sim a]_I \vee [a \wedge \sim (a \wedge b)]_I \in I \Leftrightarrow [(a \wedge \sim a) \vee (a \wedge \sim b)]_I \in I \\ \Leftrightarrow [0 \vee (a \wedge \sim b)]_I \in I \\ \Leftrightarrow a \wedge \sim b \in I.\end{aligned}$$

Thus, $[a]_I \leq_{\mathscr{B}/I} [b]_I \Leftrightarrow a \wedge \sim b \in I$.

1.5 Forcing and iterated forcing

In this section we develop the fundamental ideas of forcing and its iterations without going into technical details. We rely mainly on [Kun11], [Kun80] and [Mej20].

Remark 1.5.1. It will be common throughout this thesis to use set as models of ZFC. Although this is not precise, since it contradicts the *Gödel's second incompleteness theorem*, this is justified in the sense that, thanks to the *reflection theorems* (see [Kun11, Sec. II.5]), we know that, in ZFC, there are arbitrarily large finite fragments of ZFC. So, when we referring to N as a model of ZFC, it should be understood that N is model of a large enough finite fragment of ZFC.

1.5.1 Forcing notions

A *forcing notion* is a non-empty pre-ordered set $\mathbb{P} := (\mathbb{P}, \leq)$, that is, $\mathbb{P} \neq \emptyset$ is a set and $\leq$ is a pre-order relation on $\mathbb{P}$. If $p, q \in \mathbb{P}$ and $p \leq q$, we say that p *extends* q. Also, elements in $\mathbb{P}$ are usually called *conditions*. In general, we use letters as $\mathbb{P}, \mathbb{Q}, \mathbb{R}$, etc. to denote forcing notions.

Definition 1.5.2. Let $\mathbb{P}$ be a forcing notion and $p, q \in \mathbb{P}$. We say that:

1. p, q are *compatible* in $\mathbb{P}$, denoted by "$p \parallel_{\mathbb{P}} q$" if, and only if, there exists some $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$.

2. p, q are *incompatible* in $\mathbb{P}$, denoted by "$p \perp_{\mathbb{P}} q$" if, and only if, they are not compatible.

3. $A \subseteq \mathbb{P}$ is an an *antichain in* $\mathbb{P}$ if, and only if, for any $p, q \in \mathbb{P}$ such that $p \neq q$, $p \perp_{\mathbb{P}} q$.

4. $\mathbb{P}$ has the *countable chain condition* in $\mathbb{P}$, abbreviated as "ccc" if, and only if, in $\mathbb{P}$, every antichain is countable.

We omit the subscript "$\mathbb{P}$" in $\parallel_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ when the context is clear.

For example, if $\mathscr{B}$ is a Boolean algebra, then $(\mathscr{B}^+, \leq_{\mathscr{B}})$ is a forcing notion[2]. In this case, for $a, b \in \mathscr{B}^+,\ a \parallel_{\mathscr{B}^+} b \Leftrightarrow a \wedge b \neq 0$.

[2] Actually, $(\mathscr{B}, \leq_{\mathscr{B}})$ is also a forcing notion, however, forcing notions with least element are not interesting for us for reasons that we will see later.

We say that $p \in \mathbb{P}$ is a *atom* if, for any $q, r \leq p$, we have that $q \parallel_{\mathbb{P}} r$. If $\mathbb{P}$ has no atoms, we say that it is an *atomless* forcing notion.

Definition 1.5.3. Let $(\mathbb{P}, \leq)$ a forcing notion. The *separative order induced by* $\leq$ *on* $\mathbb{P}$, denoted by $\leq^\bullet$, is defined by: $p \leq^\bullet q$ if, and only if, for any $r \leq p$, $r \parallel q$.

Notice that, if $p \leq q$, then $p \leq^\bullet q$, however, the converse is not true in general. When the converse holds, we say that $\mathbb{P}$ is a *separative* forcing notion. For example, the Boolean algebras are separative. A converse attempt at forcing notions, in general, can be the following lemma:

Lemma 1.5.4. *Let* $\mathbb{P}$ *be a forcing notion,* $p \in \mathbb{P}$ *and* $\langle p_i \colon i < n\rangle \subseteq \mathbb{P}$. *If for every* $i < n$, $p \leq^\bullet p_i$, *then there exists some* $q \leq p$ *such that, for any* $i < n$, $q \leq p_i$.

Definition 1.5.5. Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$ and $D \subseteq \mathbb{P}$. We say that:

1. D is *pre-dense in* $\mathbb{P}$ if, and only if, for any $p \in \mathbb{P}$, there exists some $q \in E$ such that $q \parallel p$.
2. D is *dense in* $\mathbb{P}$ if, and only if, for any $p \in \mathbb{P}$, there is some $q \in D$ such that $q \leq p$.
3. D is *dense below* p if, and only if, for any $q \leq p$ there exists some $q' \in D$ such that $q' \leq q$.

A *filter* on $\mathbb{P}$ is a non-empty upwards closed set $G \subseteq \mathbb{P}$, such that every pair of elements in G are compatible in G. Also, if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$, we say that G is a $\mathbb{P}$*-generic filter over* $\mathcal{D}$ if it is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for any $D \in \mathcal{D}$.

When $\mathcal{D}$ is countable, we can prove the existence of $\mathbb{P}$-generic filters over $\mathcal{D}$:

Lemma 1.5.6. *Let* $\mathbb{P}$ *be a forcing notion and* $\mathcal{D}$ *a non-empty countable collection of dense subsets of* $\mathbb{P}$. *Then, for any* $p \in \mathbb{P}$, *there exists a* $\mathbb{P}$*-generic filter* G *over* $\mathcal{D}$ *such that* $p \in \mathbb{P}$.

1.5.2 The generic extension

For this section we fix a countable transitive model M of ZFC and if $\mathbb{P}$ is a forcing notion, we abbreviate "$(\mathbb{P}, \leq) \in \mathrm{M}$" as "$\mathbb{P} \in \mathrm{M}$".

Let $\mathbb{P}$ be a forcing notion such that $\mathbb{P} \in \mathrm{M}$. We say that $G \subseteq \mathbb{P}$ is a *generic filter over* M if, and only if, G is $\mathbb{P}$-generic over $\{D \in \mathrm{M} \colon D \subseteq \mathbb{P} \text{ is dense}\}$.

Now, we define the class of $\mathbb{P}$-*names*, denoted by $\mathrm{V}^{\mathbb{P}}$, by induction as follows: $\tau \in \mathrm{V}^{\mathbb{P}}$ if, and only if, τ is a relation and for any $(\sigma, p) \in \tau$ we have $\sigma \in \mathrm{V}^{\mathbb{P}}$ and $p \in \mathbb{P}$. Naturally if $\tau \in \mathrm{V}^{\mathbb{P}}$ we say that τ is a $\mathbb{P}$-*name*. Also $\mathrm{M}^{\mathbb{P}} \coloneqq \mathrm{V}^{\mathbb{P}} \cap \mathrm{M}$, that is, $\mathrm{M}^{\mathbb{P}} = \{\tau \in \mathrm{V}^{\mathbb{P}} \colon (\tau \text{ is a } \mathbb{P}\text{-name})^{\mathrm{M}}\}$. To each $\mathbb{P}$-name we can assign a value respect to a generic filter over M: if $\tau \in \mathrm{V}^{\mathbb{P}}$ and $G \subseteq \mathbb{P}$ is a generic filter over M then by recursion we define $\tau[G]$, the value of τ in G as follows:

$$\tau[G] \coloneqq \{\sigma[G] \colon \exists p \in G((\sigma, p) \in \tau)\}.$$

Also, $\mathrm{M}[G] \coloneqq \{\tau[G] \colon \tau \in \mathrm{M}^{\mathbb{P}}\}$ is called the *generic extension of* M *respect to* G and M is called *the ground model* in this context.

Remark 1.5.7. When dealing with forcing, unless otherwise specified, we work on some ground model M.

Example 1.5.8. Let $\mathbb{P}$ be a forcing notion and τ, σ be $\mathbb{P}$-names. If we define the following $\mathbb{P}$-names:

- $\mathrm{up}(\sigma,\tau) \coloneqq \{(\tau,p)\colon p\in\mathbb{P}\}\cup\{(\sigma,p)\colon p\in\mathbb{P}\}$.
- $\mathrm{op}(\tau,\sigma) \coloneqq \{\mathrm{up}(\mathrm{up}(\tau,\tau),\mathrm{up}(\tau,\sigma))\}$.
- $\mathrm{un}(\tau) \coloneqq \{(\pi,r)\colon r\in\mathbb{P}\wedge\exists(\sigma,p)\in\tau\,\exists q\in\mathbb{P}[(\pi,q)\in\sigma\wedge r\leq q\wedge r\leq p]\}$,

and G is $\mathbb{P}$-generic over M, then:

$$\mathrm{up}(\tau,\sigma)[G]=\{\tau[G],\sigma[G]\},\ \mathrm{op}(\tau,\sigma)[G]=(\tau[G],\sigma[G])\text{ and }\mathrm{un}(\tau)[G]=\bigcup\tau[G].$$

Notice that up, op and un are absolute notions for transitive models of ZFC.

In a natural way, we can define $\mathbb{P}$-names of the members of the ground model and for the generic set: for any $x\in\mathrm{M}$ we define

$$\check{x}\coloneqq\{(\check{z},p)\colon z\in x\wedge p\in\mathbb{P}\}\text{ and }\dot{G}_{\mathbb{P}}\coloneqq\{(\check{p},p)\colon p\in\mathbb{P}\}.$$

It is clear that those are $\mathbb{P}$-names and for any $x\in\mathrm{M}$, $\check{x}[G]=x$ and $\dot{G}_{\mathbb{P}}[G]=G$, if G is a $\mathbb{P}$-generic filter over M. As a consequence, the ground model is a subset of the generic extension and $G\in\mathrm{M}[G]$. Moreover:

Theorem 1.5.9. *Let* $\mathbb{P}\in\mathrm{M}$ *be a forcing notion and* $G\subseteq\mathbb{P}$ *a generic filter over* M. *Then:*

1. $\mathrm{M}\subseteq\mathrm{M}[G]$ *and* $G\in\mathrm{M}[G]$.
2. $\mathrm{M}\cap\mathrm{Ord}=\mathrm{M}[G]\cap\mathrm{Ord}$.
3. $\mathrm{M}[G]$ *is a countable transitive model of* ZFC.
4. *Indeed* $\mathrm{M}[G]$ *is the minimum model of* ZFC, *respect to* $\subseteq$, *containing* M *such that* $G\in\mathrm{M}$, *that it, if* N *is a model of* ZFC *such that* $\mathrm{M}\subseteq\mathrm{N}$ *and* $G\in\mathrm{N}$, *then* $\mathrm{M}[G]\subseteq N$.

Let $\mathbb{P}\in\mathrm{M}$ a forcing notion with minimun p. Notice that, $\mathbb{P}$ is a $\mathbb{P}$-generic filter over M because every pair of elements in $\mathbb{P}$ are compatible via p. As a consequence, since in general, if H and G are generic filters and $H\subseteq G$ then $G=H$, it follows that $\mathbb{P}$ has a unique $\mathbb{P}$-generic filter over M, namely, $\mathbb{P}$ itself. So for every $\mathbb{P}$-generic G over M, we have that $G\in\mathrm{M}$, and therefore, by Theorem 1.5.9(4), $\mathrm{M}[G]=\mathrm{M}$, that is, all the forcing that we can do in the generic extension is trivialized. Thus, we always work with atomless forcing notions.

1.5.3 The forcing relation

For a forcing notion $\mathbb{P}$, the *forcing language* $\mathcal{FL}_\mathbb{P}$ is the class of logical formulas formed using the binary relation "$\in$", and all names in $\mathrm{V}^\mathbb{P}$ as constant symbols. Although $\mathrm{V}^\mathbb{P}$ is a proper class, we are interested only in $\mathrm{M}^\mathbb{P}$, which is a countable set. If ψ is a sentence of $\mathcal{FL}_\mathbb{P}$ whose constant names are in $\mathrm{M}^\mathbb{P}$, then "$\mathrm{M}[G] \models \psi$" has its usual model-theoretic meaning, interpreting "$\in$" as membership and each name τ as its value $\tau[G]$. So we can define a *forcing relation*: we say that $p \Vdash^\mathrm{M}_\mathbb{P} \psi$ holds if, and only if, $\mathrm{M}[G] \models \psi$ for every filter G on $\mathbb{P}$ such that $p \in G$, and G is $\mathbb{P}$-generic over M.[3] When the context is clear we omit the subscripts "$\mathbb{P}$" and "M" in $\Vdash^\mathrm{M}_\mathbb{P}$.

The forcing relation can be defined without resorting to models, in the ground model, and with absolute notions, however, it is a careful and tedious construction (see the relation $\Vdash^*$ in [Kun11, Sec. IV.2], which interpreted in M is $\Vdash^\mathrm{M}_\mathbb{P}$ in M).

The so-called *Truth Lemma* allows us to characterize the forcing relation:

Theorem 1.5.10. *Let $\mathbb{P} \in \mathrm{M}$ be a forcing notion, $G \subseteq \mathbb{P}$ a generic filter over M and $\varphi \in \mathcal{FL}_\mathbb{P}$ with constants in $\mathrm{M}^\mathbb{P}$. Then,*

$$\mathrm{M}[G] \models \varphi \textit{ if, and only if, } \exists p \in G(p \Vdash \varphi).$$

Notation 1.5.11. If $\mathbb{P} \in \mathrm{M}$ is a forcing notion, $p \in \mathbb{P}$ and $\varphi \in \mathcal{FL}_\mathbb{P}$, we denote $\Vdash_\mathbb{P} \varphi$ if, and only if, for any $p \in \mathbb{P}$, $p \Vdash_\mathbb{P} \varphi$.

1.5.4 Embeddings and completions

We start by defining the notions of dense and complete embedding, which allow information to be transferred between forcing notions, in a similar way to isomorphisms in algebraic structures.

Definition 1.5.12. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions and $\iota \colon \mathbb{P} \to \mathbb{Q}$ a function. We say that ι is a *complete embedding* if:

1. for any $p, p' \in \mathbb{P}\, p \leq_\mathbb{P} p' \Rightarrow \iota(p) \leq_\mathbb{Q} \iota(p')$,
2. for any $p, p' \in \mathbb{P},\ p \perp_\mathbb{P} p' \Leftrightarrow \iota(p) \perp_\mathbb{Q} \iota(p')$,
3. If $A \subseteq \mathbb{P}$ is a maximal anti-chain in $\mathbb{P}$, then $\iota[A]$ is a maximal anti-chain in $\mathbb{Q}$.

Also, we say that ι is a *dense embedding* if (1) and (2) hold along with:

4. $\iota[\mathbb{Q}]$ is a dense subset of $\mathbb{P}$.

It is not difficult to prove that every dense embedding is complete.

Given forcing notions $\mathbb{P}$, $\mathbb{Q}$, a function $\iota \colon \mathbb{P} \to \mathbb{Q}$ and $q \in \mathbb{Q}$, we say that $r \in \mathbb{P}$ is a *ι-reduction of* q or simply a *reduction of* q , if for any $p \leq r$, $\iota(p) \parallel_\mathbb{Q} q$. This notion allows us to get a characterization of complete embeddings:

[3] In a similar way to the definition of "$\mathfrak{A} \models \psi$" (see [Kun12]).

Lemma 1.5.13. *Let $\mathbb{P}$, $\mathbb{Q}$ be forcing notions. Then $\iota\colon \mathbb{P} \to \mathbb{Q}$ is a complete embedding if, and only if, (1) and (2) from Definition 1.5.12 hold, and for any condition in $\mathbb{Q}$, there exists an ι-reduction in $\mathbb{P}$.*

The following example will be helpful to define the *random forcing notion*. Considering $\mathcal{B}({}^{\omega}2) \setminus \mathcal{N}$ ordered by inclusion, then:

Example 1.5.14. The map $\iota\colon \mathcal{B}({}^{\omega}2) \setminus \mathcal{N} \to \mathcal{B}({}^{\omega}2)/\mathcal{N}$ defined by $\iota(B) \coloneqq [B]_{\mathcal{N}}$ is a dense embedding. Indeed, let $A, B \in \mathcal{B}({}^{\omega}2) \setminus \mathcal{N}$.

1. Assume that $A \subseteq B$, hence $A \cap B^{\mathrm{c}} = \emptyset \in \mathcal{N}$. Therefore, by Example 1.4.20, it follows that $[A]_{\mathcal{N}} \leq [B]_{\mathcal{N}}$.
2. If A, B are incompatible in $\mathcal{B}({}^{\omega}2) \setminus \mathcal{N}$, then $A \cap B \in \mathcal{N}$. Therefore, $[A]_{\mathcal{N}} \wedge [B]_{\mathcal{N}} = [A \cap B]_{\mathcal{N}} = [\emptyset]_{\mathcal{N}}$, that is, $[A]_{\mathcal{N}}$ and $[B]_{\mathcal{N}}$ are incompatible in $\mathcal{B}({}^{\omega}2)/\mathcal{N}$.
3. Let $[B]_{\mathcal{N}} \in \mathcal{B}({}^{\omega}2)/\mathcal{N}$. Since we consider $\mathcal{B}({}^{\omega}2)/\mathcal{N}$ as a forcing notion, $[\emptyset]_{\mathcal{N}} \notin \mathcal{B}({}^{\omega}2)/\mathcal{N}$, hence $[B]_{\mathcal{N}} \neq [\emptyset]_{\mathcal{N}}$, that is, by Definition 1.4.18, $B = B \triangle \emptyset \notin \mathcal{N}$. Therefore, $B \in \mathcal{B}({}^{\omega}2) \setminus \mathcal{N}$ and $\iota(B) = [B]_{\mathcal{N}}$. Thus, $\operatorname{ran}(\iota)$ is dense.

Thus ι is a dense embedding.

To simplify writing, we use the following notation:

Notation 1.5.15. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions. We write "$\mathbb{P} \lessdot \mathbb{Q}$" when $\mathbb{P} \subseteq \mathbb{Q}$ and the identity map is a complete embedding.

It is clear that,

Lemma 1.5.16. *Let $\mathbb{P}, \mathbb{Q}$ be forcing notions. If $Q \subseteq \mathbb{P}$ is dense subposet and $\mathbb{Q} \lessdot Q$, then $\mathbb{Q} \lessdot \mathbb{P}$.*

Given a complete embedding $\iota\colon \mathbb{P} \to \mathbb{Q}$, we can naturally induce a correspondence between the class names:

Definition 1.5.17. If $\iota\colon \mathbb{P} \to \mathbb{Q}$ is a complete embedding, we define $\iota^*\colon \mathrm{V}^{\mathbb{P}} \to \mathrm{V}^{\mathbb{Q}}$ recursively, as follows:

$$\iota^*(\tau) \coloneqq \{(\iota^*(\sigma), \iota(p))\colon (\sigma, p) \in \tau\}.$$

A complete embedding $\iota\colon \mathbb{P} \to \mathbb{Q}$ establishes a correspondence between the generic filters of $\mathbb{P}$ and the generic filters of $\mathbb{Q}$:

Theorem 1.5.18. *Assume that $\iota\colon \mathbb{P} \to \mathbb{Q}$ is a complete embedding and H is $\mathbb{Q}$-generic filter over M. Then:*

1. *$G \coloneqq \iota^{-1}[H]$ is $\mathbb{P}$-generic over M and $\mathrm{M}[G] \subseteq \mathrm{M}[H]$. Even more, if ι is a dense embedding in M, then $\mathrm{M}[G] = \mathrm{M}[H]$.*
2. *If $\tau \in \mathrm{V}^{\mathbb{P}}$ and $\sigma \coloneqq \iota^*(\tau)$, then $\sigma[H] = \tau[G]$.*

3. *If* $p \in \mathbb{P}, \tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$ *and* $\varphi(x_0, \ldots, x_{n-1})$ *is an absolute formula between transitive models of* ZFC, *then*

$$p \Vdash \text{“}\varphi(\tau_0, \ldots \tau_{n-1})\text{”} \Leftrightarrow \iota(p) \Vdash \text{“}\varphi(\iota^*(\tau_0), \ldots, \iota^*(\tau_{n-1}))\text{”}.$$

4. *If* ι *is a dense embedding, then (c) is valid without the absoluteness requirement of* φ.

5. *If* ι *is a dense embedding and* G' *is* $\mathbb{P}$*-generic over* M, *then*

$$H' := \iota\langle G' \rangle := \{r \in \mathbb{Q} \colon \exists q \in \iota[G](q \leq r)\}$$

is $\mathbb{Q}$*-generic over* M *and* $\mathrm{M}[H'] = \mathrm{M}[G']$.

We can define the notion of *complete embedding* with respect to models: if $\mathbb{Q} \in \mathrm{M}$, $\mathbb{R} \in \mathrm{N}$ are forcing notions, $f \in \mathrm{N}$ is a function and $\mathrm{M} \subseteq \mathrm{N}$, we say that $f \colon \mathbb{Q} \to \mathbb{R}$ is a *complete embedding respect to* M, if (1) and (2) from Definition 1.5.12 hold and, for any maximal anti-chain $A \subseteq \mathbb{P}$ such that $A \in \mathrm{M}$, we have that $f[A]$ is a maximal anti-chain in $\mathbb{R}$. So we get similar results to (1) and (3) from Theorem 1.5.18 with respect to models:

Lemma 1.5.19. *Let* $f \colon \mathbb{Q} \to \mathbb{R}$ *be a complete embedding with respect to* M, $\mathbb{Q} \in \mathrm{M}$ *and* $\mathbb{R}, f \in \mathrm{N}$. *If* $H \subseteq \mathbb{R}$ *is* $\mathbb{R}$*-generic over* N, *then* $G := f^{-1}[H]$ *is* $\mathbb{Q}$*-generic over* M *and* $\mathrm{M}[G] \subseteq \mathrm{N}[H]$.

Lemma 1.5.20. *Assume that* $\mathrm{M} \subseteq \mathrm{N}$ *are transitive models of* ZFC, $\mathbb{Q} \in \mathrm{M}$ *and* $\mathbb{R} \in \mathrm{N}$ *are forcing notions, and let* $\iota \in \mathrm{N}$ *such that,* $\iota \colon \mathbb{Q} \to \mathbb{R}$ *is complete embedding with respect to* M. *If* $p \in \mathbb{Q}$, $\tau_0, \ldots, \tau_{n-1} \in \mathrm{M}$ *are* $\mathbb{P}$*-names and* $\varphi(x_0, \ldots, x_{n-1})$ *is a formula upwards-absolute for transitive models of* ZFC, *then* $p \Vdash^{\mathrm{M}}_{\mathbb{Q}}$ *“*$\varphi(\tau_0, \ldots, \tau_{n-1})$*” implies* $\iota(p) \Vdash^{\mathrm{N}}_{\mathbb{R}}$ *“*$\varphi(\iota(\tau_0), \ldots, \iota(\tau_{n-1}))$*”. The converse is true when* φ *is absolute for transitive models of* ZFC.

Forcing completions

Given a forcing notion $\mathbb{P}$, we say that $(\mathscr{B}_{\mathbb{P}}, \iota_{\mathbb{P}})$ is a *forcing completion* of $\mathbb{P}$, if $\mathscr{B}_{\mathbb{P}}$ is a complete Boolean algebra and $\iota_{\mathbb{P}} \colon \mathbb{P} \to \mathscr{B}^+$ is a dense embedding.

Theorem 1.5.21. *Every forcing notion has a forcing completion and it is unique up to isomorphism.*

Using completions, we can define a notion of forcing equivalence:

Definition 1.5.22. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions. We say that $\mathbb{P}, \mathbb{Q}$ are *forcing equivalent*, if its forcing completions are Boolean isomorphic. In this case we write $\mathbb{P} \equiv \mathbb{Q}$.

It can be shown that dense embedding induces isomorphisms in the completions:

Theorem 1.5.23. *Let* $\mathbb{P}, \mathbb{Q}$ *be notions forcing and* $(\mathscr{B}_{\mathbb{P}}, \iota_{\mathbb{P}})$, $(\mathscr{B}_{\mathbb{Q}}, \iota_{\mathbb{Q}})$ *its forcing completions, respectively. Consider the* completion diagram of $\mathbb{P}$ and $\mathbb{Q}$ *defined in Figure 1.5. Then:*

1. *If* $\iota \colon \mathbb{P} \to \mathbb{Q}$ *is a complete embedding, then there is a unique complete embedding* $f \colon \mathscr{B}^+_{\mathbb{P}} \to \mathscr{B}^+_{\mathbb{Q}}$, *such that the diagram in Figure 1.5 commutes.*

2. *If* $\iota \colon \mathbb{P} \to \mathbb{Q}$ *is a dense embedding, then there is a unique isomorphism* $f \colon \mathscr{B}^+_{\mathbb{P}} \to \mathscr{B}^+_{\mathbb{Q}}$, *such that the diagram in Figure 1.5 commutes.*

$$\begin{tikzcd} \mathbb{P} \arrow[r, "\iota"] \arrow[d, "\iota_{\mathbb{P}}"'] & \mathbb{Q} \arrow[d, "\iota_{\mathbb{Q}}"] \\ \mathscr{B}^+_{\mathbb{P}} \arrow[r, "f"] & \mathscr{B}^+_{\mathbb{Q}} \end{tikzcd}$$

Figure 1.5: Completion diagram of $\mathbb{P}$ and $\mathbb{Q}$.

As a consequence, the existence of a dense embedding is enough to have forcing equivalence:

Corollary 1.5.24. *If $\mathbb{P}, \mathbb{Q}$ are forcing notions and $\iota\colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then $\mathbb{P} \equiv \mathbb{Q}$.*

As a consequence, by virtue of Example 1.5.14, we have that $\mathcal{B}({}^{\omega}2) \setminus \mathcal{N}$ and $\mathcal{B}({}^{\omega}2)/\mathcal{N}$ are forcing equivalent.

1.5.5 Linkedness properties

Linkedness properties are combinatorial properties of forcing notions or subsets of forcing notions. Since the combinatorial structure of forcing notions affects what they can force, it is very useful to study these properties. Below we define those that will appear throughout this thesis:

Definition 1.5.25. Let $\mathbb{P}$ be a forcing notion, $Q \subseteq \mathbb{P}$ and μ an infinite cardinal.

1. For $2 \leq n < \omega$, we say that Q *is n-linked*, if for every subset $F \subseteq Q$ such that $|F| \leq n$, there exists some $q \in \mathbb{P}$ such that, for any $p \in F$, $q \leq p$. When $n = 2$, we just say that Q is *linked*.

2. $Q \subseteq \mathbb{P}$ is *centered* if it is n-linked for any $2 \leq n < \omega$.

3. $\mathbb{P}$ is *μ-linked* if there exists a sequence $\langle Q_\alpha \colon \alpha < \mu \rangle$ of linked subsets of $\mathbb{P}$ whose union is $\mathbb{P}$. When $\mu = \aleph_0$, we just say that $\mathbb{P}$ is *σ-linked*. Also, if each Q_α is m-linked, we say that $\mathbb{P}$ is *μ-m-linked*.

4. $\mathbb{P}$ is *μ-centered*, if it is a union of μ-many centered subsets. When $\mu = \aleph_0$, we just say *σ-centered*.

5. We say that $\mathbb{P}$ is μ-cc, if every antichain in $\mathbb{P}$ has size $< \mu$. Notice that, ccc (see Definition 1.5.2(4)) corresponds to $\aleph_1$-cc.

6. $\mathbb{P}$ is *κ-Knaster* if, for any $Q \in [\mathbb{P}]^\kappa$, there exists some linked subset $Q' \in [Q]^\kappa$. For $\kappa = \aleph_1$ we just say *Knaster*.

7. $\mathbb{P}$ is *κ-m-Knaster* if, for any $Q \in [\mathbb{P}]^\kappa$, there exists some m-linked subset $Q' \in [Q]^\kappa$. Notice that κ-2-Knaster is just κ-Knaster.

For example, if $G \subseteq \mathbb{P}$ is a generic filter, then it is centered.

For Boolean algebras as forcing notions, we can characterize the notion of σ-centered as follows:

Lemma 1.5.26. *A Boolean algebra $\mathscr{B}$ is σ-*centered *if, and only if, there exists a sequence of ultrafilters $\langle F_n \colon n < \omega\rangle$ such that $\mathscr{B}^+ = \bigcup_{n<\omega} F_n$.*

Below we present some relationships between the notions that we have just defined:

Theorem 1.5.27. *Let $\mathbb{P}$ be a forcing notion. Then,*

1. *If $\mathbb{P}$ is κ-Knaster, then it is κ-*cc.
2. *If $\mathbb{P}$ is μ-m-linked, then it is μ^+-m-Knaster*
3. *If $\mathbb{P}$ is μ-m-linked, then it is μ^+-*cc.
4. *If $\mathbb{P}$ is σ-centered, then it is* ccc.

Fréchet-linkedness

Diego Mejía in [Mej19] introduced a new and more sophisticated linkedness property:

Definition 1.5.28. Let $\mathbb{P}$ be a forcing notion and μ an infinite cardinal.

1. We say that a set $Q \subseteq \mathbb{P}$ is *Fréchet-linked in* $\mathbb{P}$, abbreviated Fr-*linked* if, for any sequence $\bar{p} = \langle p_n \colon n < \omega\rangle \in Q^\omega$, there exists some $q \in \mathbb{P}$ such that $q \Vdash_{\mathbb{P}}$ "$|\{n < \omega \colon p_n \in \dot{G}\}| \geq \aleph_0$".
2. We say that $\mathbb{P}$ is μ-Frechet-linked, abbreviated by μ-Fr-linked, if $\bigcup_{\alpha<\mu} Q_\alpha$ is dense in $\mathbb{P}$, for some sequence $\langle Q_\alpha \colon \alpha < \mu\rangle$ of Fr-linked subsets of $\mathbb{P}$.
3. We say that $\mathbb{P}$ is κ-Fr-*Knaster* if, for any $Q \in [\mathbb{P}]^\kappa$, there is some Fr-linked $Q' \in [Q]^\kappa$.

A *dominating real over* M is a real number $d \in {}^\omega\omega$ such that, for any $x \in \omega^\omega \cap \mathrm{M}$, $x \leq^* d$. The Fr-linkedness notion arises implicitly in *Arnold Miller*'s proof (see [Mil81]) that the *eventually different forcing notion* $\mathbb{E}$ does not add dominating reals, and it turned out to be very useful because it implies preservation properties (see [Mej19, Sec. 3]).

For example, every forcing notion $\mathbb{P}$ is $|\mathbb{P}|$-Fr-linked (see Corollary 4.2.11 and Example 4.2.14).

Below, we present some properties of Fr-linked forcing notions that will be important for us:

Lemma 1.5.29. *Let $\mathbb{P}$ a forcing notion. Then*

1. *No* Fr-*linked subset of $\mathbb{P}$ can contain infinite antichains.*
2. μ-Fr-*linked forcing notions are μ^+-*cc. *Moreover, they are μ^+-Knaster.*

For the reason of the definition, it is natural to expect that:

Theorem 1.5.30. *If $\mathbb{P}$ is σ-*Fr-*linked, then $\mathbb{P}$ does not add dominating reals.*

As a consequence, if $\mathbb{P}$ adds dominating reals, then it is not σ-Fr-linked.

One of the main objectives of this thesis is to introduce a new linkedness property, that we call μ-FAM-*linked* (see Definition 4.2.8), which will be related with μ-Fr-linked.

1.5.6 Δ-systems

In this section we present a very usefull combinatorial result: the Δ-*system lemma*. Our main reference here is [Kun80].

Definition 1.5.31. A family of sets $\mathcal{A}$ forms a Δ-*system with root* Δ if, $X \cap Y = \Delta$ whenever $X, Y \in \mathcal{A}$ with $X \neq Y$.

As long as $\mathcal{A}$ is a family of finite sets and $|\mathcal{A}| = \kappa$ is regular, we can guarantee the existence of Δ-systems, because the regularity of κ allows us to perform the appropriate reductions to build a root. Moreover,

Theorem 1.5.32. *Let κ be an uncountable regular cardinal and let $\mathcal{A} = \langle A_\gamma \colon \gamma < \kappa\rangle$ be a family of finite sets, such that $(\bigcup \mathcal{A}, \lhd)$ is a well-order. Then, there are $E \in [\kappa]^\kappa$, $n^* < \omega$ and $r^* \subseteq n^*$, such that:*

1. *$\mathcal{B} := \{A_\xi \colon \xi \in E\}$ forms a Δ-system with root Δ,*
2. *for any $\xi \in E$, $\mathrm{dom}(A_\xi) = \{a_{n,\gamma} \colon n < n^*\}$ is arranged in $\lhd$-increasing order,*
3. *$a_{n,\xi} \in \Delta \Leftrightarrow n \in r^*$, for any $\xi \in E$,*
4. *for any $n \in n^* \setminus r^*$ and $\xi, \zeta \in E$, if $\xi < \zeta$ then $a_{n,\xi} < a_{n\,\zeta}$.*

Although this is not the usual way of stating the Δ-*system lemma* (see, for example, [Kun11, Lem. III.2.6]), as it is suited to our particular needs, the proof is the same as the one presented by *Kenneth Kunen* in [Kun80, Thm. 1.6].

1.5.7 Nice names

Let $\mathbb{P}$ be a forcing notion. For a function h such that $\mathrm{dom}(h)$ is an anti-chain in $\mathbb{P}$ and for any $p \in \mathbb{P}$, $h(p) \in \mathrm{V}^{\mathbb{P}}$, we denote:

$$\mathrm{an}(h) := \mathrm{un}(\{(h(p), p) \colon p \in \mathrm{dom}(h)\}),$$

hence, for any $p \in \mathrm{dom}(h)$, we have that $p \Vdash$“$\mathrm{an}(h) = \check{h(p)}$”.
Now, fix two sets B and C.

1. We say that $\dot{x}$ is *nice* $\mathbb{P}$-*name of a member of* C if $\dot{x} = \mathrm{an}(h)$ for some function h into $\mathrm{dom}(\check{C}) = \{\check{y} \colon y \in C\}$ such that $\mathrm{dom}(h) \subseteq \mathbb{P}$ is a maximal anti-chain.
2. If H is a function from B into $\mathrm{M}^{\mathbb{P}}$, denote:
$$\mathrm{fn}(H) := \{(\mathrm{op}(\check{x}, H(x)), p) \colon x \in B \wedge p \in \mathbb{P}\}.$$
3. Say that $\dot{f}$ is a nice $\mathbb{P}$-*name of a function from* B *into* C if $\dot{f} = \mathrm{fn}(H)$ for some function $H \colon B \to \mathrm{nice}(C)$. Also denote $\mathrm{ncf}(B, C) := \mathrm{ncf}_{\mathbb{P}}(B, C)$ the collection of all nice names of functions from B into C.

The following results justify the name of "nice names":

Lemma 1.5.33. *Let $B, C \in \mathrm{M}$. Then:*

1. *If $\dot{x} \in \mathrm{M}$ is a nice name of a member of C, then $\Vdash$"$\dot{x} \in \check{C}$".*
2. *If $\dot{f} \in \mathrm{M}$ is a nice name of a function from B into C, then $\Vdash$"$\dot{f} \colon \check{B} \to \check{C}$".*

Theorem 1.5.34. *In M, let B and C be sets, $p \in \mathbb{P}$ and $\sigma \in \mathrm{M}^{\mathbb{P}}$. Then:*

1. *If $p \Vdash$"$\sigma \in \check{C}$", then there exists some nice $\mathbb{P}$-name $\dot{x}$ of a member of C such that $p \Vdash$"$\sigma = \dot{x}$".*
2. *If $p \Vdash$"$\sigma \colon \check{B} \to \check{C}$", then there exists some nice $\mathbb{P}$-name $\dot{f}$ of a function from B into C such that $p \Vdash$"$\sigma = \dot{f}$".*

The following result allows us to estimate the cardinal of $\mathrm{nice}_{\mathbb{P}}(C)$ and $\mathrm{ncf}_{\mathbb{P}}(B, C)$. In general when we use this result we refer to it using phrases like "*by counting nice names*" (see, for example Construction 5.5.1).

Theorem 1.5.35. *Let B and C be sets, and let κ be an infinite cardinal. If $\mathbb{P}$ has the κ-cc, then:*

1. $|\mathrm{nice}_{\mathbb{P}}(C)| \leq |\mathbb{P}|^{<\kappa} \cdot |C|^{<\kappa}$.
2. $|\mathrm{ncf}_{\mathbb{P}}(B, C)| \leq (|\mathbb{P}|^{<\kappa} \cdot |C|^{<\kappa})^{|B|}$.

To define forcing iterations, we need the existence of a set with the following properties:

Lemma 1.5.36. *Let $\mathbb{P}$ be a forcing notion and $\dot{Q}$ a $\mathbb{P}$-name of a non-empty set. Then there exists a set of $\mathbb{P}$-names $\langle \dot{Q} \rangle_{\mathbb{P}}$ satisfying:*

1. *for any $q \in \langle \dot{Q} \rangle_{\mathbb{P}}$, $\Vdash$ "$\dot{q} \in \dot{Q}$",*
2. *if τ is a $\mathbb{P}$-name and $\Vdash$ "$\tau \in \dot{Q}$", then there is some $\dot{q} \in \langle \dot{Q} \rangle_{\mathbb{P}}$ such that $\Vdash$ "$\tau = \dot{q}$",*
3. *for any $\dot{q}$, $\dot{q}' \in \langle \dot{Q} \rangle_{\mathbb{P}}$, $\dot{q} = \dot{q}'$ if, and only if, $\Vdash$ "$\dot{q} = \dot{q}'$".*

Furthermore, if κ and μ are cardinals, $\mathbb{P}$ is κ-cc and $\Vdash_{\mathbb{P}}$ "$|\dot{Q}| \leq \mu$", then $|\langle \dot{Q} \rangle_{\mathbb{P}}| \leq \left|[\mathbb{P}]^{<\kappa}\right| \cdot \mu^{<\kappa}$.

1.5.8 Some forcing notions

This section deals with defining the forcing notions that we are going to use throughout the thesis. Fundamentally there are four forcing notions: *random forcing*, *Cohen forcing*, *Hechler forcing* and $\tilde{\mathbb{E}}$.

We start with random forcing, the most important forcing notion in this thesis:

Random forcing: $\mathbb{B}$

Random forcing is the poset $\mathcal{B}({}^\omega 2) \setminus \mathcal{N}$ ordered with $\subseteq$, however, in terms of forcing, we have other alternatives to define it:

Definition 1.5.37. $\mathbb{B} := \{T \subseteq {}^{<\omega}2 \colon T$ is a tree and $\mathrm{Leb}([T]) > 0\}$ ordered with $\subseteq$.

Let us show that the two forcing notions above are equivalent forcing:

Lemma 1.5.38. *The forcing notion $\mathbb{B}$ is forcing equivalent with $\mathcal{B}({}^\omega 2) \setminus \mathcal{N}$.*

Proof. By Corollary 1.5.24, it is enough to prove that the natural function $\iota \colon \mathbb{B} \to \mathcal{B}({}^\omega 2) \setminus \mathcal{N}$ defined by $\iota(T) := [T]$ is a dense embedding. Indeed, it is clear that ι is well-defined and the preservation of the order $\subseteq$ is easy. For the preservation of incompatibilities, assume that $T, T' \in \mathbb{B}$ and that $[T] \cap [T']$ has positive measure. Consider the tree $T'' := T \cap T'$. Then $[T''] = [T] \cap [T']$, so $T'' \in \mathbb{B}$ and T'' is a subset of both T and T', hence $T \parallel_{\mathbb{B}} T'$. Density follows because, for any Borel $B \subseteq {}^\omega 2$,

$$\mathrm{Leb}(B) = \sup\{\mathrm{Leb}(C) \colon C \subseteq B \text{ closed in } {}^\omega 2\}. \qquad \Box_{\text{Lemma 1.5.38}}$$

As a consequence, by Example 1.5.14, we can consider random forcing as $\mathbb{B}$. Also, since for example Example 1.5.14, $\mathcal{B}({}^\omega 2)/\mathcal{N}$ is forcing equivalent to random forcing, it follows that we have three different ways of presenting the random forcing notion: $\mathcal{B}({}^\omega 2) \setminus \mathcal{N}$, $\mathcal{B}({}^\omega 2)/\mathcal{N}$ and $\mathbb{B}$. In general, when we refer to *random forcing*, we will be referring to $\mathbb{B}$, except in the proof of Theorem 4.2.19, but there it will be made explicit who we are working with.

Finally, it is not difficult to show that, if G is a $\mathbb{B}$-generic filter, then $\bigcap_{T \in G}[T]$ is a singleton, which allows us to define the notion of *random real*:

Definition 1.5.39. If G is a $\mathbb{B}$-generic filter, the *random real added by G* is defined as the unique $r \in \bigcap_{T \in G}[T]$, and its $\mathbb{B}$-name is denoted by $\dot{r}$.

Finally, regarding combinatorial properties, $\mathbb{B}$ is σ-linked.

Cohen forcing: $\mathbb{C}$

Although the usual definition of *Cohen forcing* is $\mathbb{C} := \mathrm{Fn}(\omega, 2)$, ordered by the inverse inclusion, that is, $f \leq_{\mathrm{Fn}(\omega,2)} g$ if, and only if, $g \subseteq f$, we can characterize it in a general way:

Theorem 1.5.40. *Any atomless countable forcing notion is forcing equivalent to $\mathbb{C}$.*

If G is a $\mathbb{C}$-generic filter set, the *Cohen real added by* $\mathbb{C}$ is $c = \bigcup G$, and its $\mathbb{C}$-name is denoted by $\dot{c}$. So $\mathbb{C}$ adds a single Cohen real. More generally, if λ if an infinite cardinal, we define $\mathbb{C}_\lambda := \mathrm{Fn}(\lambda \times \omega, 2)$, which is a variant of Cohen forcing adding a λ-sequence of Cohen reals: if G is a $\mathbb{C}_\lambda$-generic set and $g := \bigcup G$, which is a function from $\lambda \times \omega$ into 2, the *λ-sequence of Cohen reals added by* $\mathbb{C}_\lambda$ is $\langle c_\gamma \colon \gamma < \lambda \rangle$, where $c_\gamma(n) := g(\gamma, n)$, for any $\gamma < \lambda$ and $n < \omega$.

We can characterize $\mathbb{C}_\lambda$, in forcing terms, as a finite support product:

Theorem 1.5.41. *Let λ be an infinite cardinal. Then $\mathbb{C}_\lambda \equiv \prod^{<\aleph_0}_{\alpha<\lambda} \mathbb{Q}_\alpha$, where $\mathbb{Q}_\alpha := \mathbb{C}$ for any $\alpha < \lambda$.*

Regarding combinatorial properties, it is clear that Cohen forcing has the ccc because it is countable, however, it satisfies a much stronger property: it is σ-centered.

Hechler forcing: $\mathbb{D}$

Hechler forcing, denoted by $\mathbb{D}$, is the canonical forcing notion to add dominanting reals. Its set of conditions is $\omega^{<\omega} \times \omega^{\omega}$, ordered by:

$$(t, y) \leq (s, x) :\Leftrightarrow s \subseteq t,\ x \leq y \text{ and } \forall i \in |t| \setminus |s|\,(x(i) \leq t(i)).$$

If G is $\mathbb{D}$-generic filter over M, then we denote $d := \bigcup\{s \in {}^{<\omega}\omega \colon \exists\, x \in \omega^{\omega} \cap \mathrm{M}((s, x) \in G)\}$, which is called the *Hechler real over* M, and it can be shown that it is a dominating real over M. Regarding combinatorial properties, Hechler forcing is σ-centered.

In Chapter 5 (see Construction 5.5.1), we are going to force $\mathfrak{b} \geq \kappa$ for some cardinal κ, and for this, we will use *partial Hechler forcing*, that is, $\mathbb{D}$ restricted to a suitable model N of ZFC. So, we need that such a restriction does not affect some fundamental properties of Hechler forcing:

Theorem 1.5.42. *Let κ be an infinite cardinal and N a transitive model of* ZFC *of size $< \kappa$. Then, $\mathbb{D}^N$ is σ-centered, it adds a dominating real over N, and $|\mathbb{D}^N| < \kappa$.*

The forcing notion $\tilde{\mathbb{E}}$

Although $\tilde{\mathbb{E}}$ was first introduced by *Haim Horowitz* and *Saharon Shelah* (see [HS16]), the definition below is based on [KST19, Def. 1.12].

Definition 1.5.43. By induction on the height $h \geq 0$, we define a countable tree $\tilde{\mathcal{T}} \subseteq {}^{<\omega}\omega$, functions $\varrho, \pi, a, M \colon \omega \to \omega$ and a map $\mu_h \colon M(h) + 1 \to \mathbb{R}$ as follows:

1. $\mathcal{L}_0(\tilde{\mathcal{T}}) := \{\langle\rangle\}$, that is, the unique element of height 0 is $\langle\rangle$, and:
   - $\varrho(0) := 2$,
   - $\pi(0) := 2$,
   - $a(0) := 4$,
   - $M(0) := 16$,
   - $\mu_0(n) := \log_4\left(\frac{16}{16-n}\right)$ for $n < 16$,
   - $\mu_0(16) := \infty$.
2. Assume we have defined $\mathcal{L}_h(\tilde{\mathcal{T}})$ for $h < \omega$. For any $\rho \in \mathcal{L}_h(\tilde{\mathcal{T}})$, define

   $$\mathrm{succ}_\rho(\mathcal{T}) := \{\rho^\frown\langle\ell\rangle \colon \ell \in M(h)\}$$

   and

   $$\mathcal{L}_{h+1}(\tilde{\mathcal{T}}) := \bigcup_{\rho \in \mathcal{L}_h(\tilde{\mathcal{T}})} \mathrm{succ}_\rho(\tilde{\mathcal{T}}).$$

   Now,
   - $\varrho(h+1) := \max\{|\mathcal{L}_{h+1}(\tilde{\mathcal{T}})|,\ h+3\}$,
   - $\pi(h+1) := [(h+2)^2 \varrho(h+1)^{h+2}]^{\varrho(h+1)^{h+1}}$,
   - $a(h+1) := \pi(h+1)^{h+3}$,
   - $M(h+1) := a(h+1)^2$,

- $\mu_{h+1}(n) \coloneqq \begin{cases} \log_{a(h+1)}\left(\frac{M(h+1)}{M(h+1)-n}\right), & \text{if} \quad 0 \leq n < M(h+1), \\ \infty, & \text{if} \qquad n = M(h+1). \end{cases}$
- If $\rho \in \mathcal{L}_{h+1}(\tilde{\mathcal{T}})$ and $A \subseteq \operatorname{succ}_\rho$, then we set $\mu_\rho(A) \coloneqq \mu_{n+1}(|A|)$.

Finally, define

$$\tilde{\mathcal{T}} \coloneqq \bigcup_{h<\omega} \mathcal{L}_h(\tilde{\mathcal{T}}),$$

and if $p \subseteq \tilde{\mathcal{T}}$ is a tree and $\rho \in p$, $\mu_\rho(p) \coloneqq \mu_\rho(\operatorname{succ}_\rho(p))$.

So we have defined a countable tree $\tilde{\mathcal{T}} \subseteq {}^\omega\omega$ and, for each node $\rho \in \tilde{\mathcal{T}}$ a norm μ_ρ has been defined on the subsets of succ_ρ. We can intuitively think of the norm μ_ρ as a way of measuring how many immediate successors has ρ. So, if μ_ρ is big, we have more possibilities to extend subtrees in node ρ.

From the definition, it is clear that:

Lemma 1.5.44. *For each $\rho \in \tilde{\mathcal{T}}$, $|\operatorname{succ}_\rho(\tilde{\mathcal{T}})| = M(h)$, $\mu_\rho(\emptyset) = 0$, $\mu_\rho(\operatorname{succ}_\rho(\tilde{\mathcal{T}})) = \infty$ and, if $A \subseteq \operatorname{succ}_\rho(\tilde{\mathcal{T}})$, then $|A| = |\operatorname{succ}_\rho(\tilde{\mathcal{T}})|(1 - a(h)^{-\mu_\rho(A)})$.*

We can now define the forcing $\tilde{\mathbb{E}}$:

Definition 1.5.45. Assume that $\tilde{\mathcal{T}}$, ϱ, π, a, M and μ_ρ are as in Definition 1.5.43. The forcing $\tilde{\mathbb{E}}$ is the set

$$\tilde{\mathbb{E}} \coloneqq \left\{ p \subseteq \tilde{\mathcal{T}} \colon p \text{ is a tree and } \forall \rho \in p \left(\rho \geq \operatorname{trunk}(p) \Rightarrow \mu_\rho(p) \geq 1 + \frac{1}{\lg(\operatorname{trunk}(p))} \right) \right\},$$

endowed with $\subseteq$.

One of the important components of $\tilde{\mathbb{E}}$, apart from the trunk function, is the loss function:

Definition 1.5.46. Let $p \subseteq \tilde{\mathcal{T}}$ be a tree. If there is $2 < m < \omega$ such that:

1. $\lg(\operatorname{trunk}(p)) > 3m$,
2. for any $\rho \in p$, $\lg(\rho) \geq \lg(\operatorname{trunk}(p))$ entails $\mu_\rho(p) \geq 1 + \frac{1}{m}$,

we say define $\operatorname{loss}(p) \coloneqq \frac{1}{m}$, where m is the maximal of such m.

So loss is a function from a subset of $\tilde{\mathbb{E}}$ to $[0,1]_\mathbb{Q}$, moreover:

Theorem 1.5.47.

1. *$\operatorname{dom}(\operatorname{loss})$ is a dense subset of $\tilde{\mathbb{E}}$.*
2. *For any $p \in \operatorname{dom}(\operatorname{loss})$, $\frac{\operatorname{Leb}([p])}{\operatorname{Leb}([\operatorname{trunk}(p)])} \geq 1 - \frac{\operatorname{loss}(p)}{2}$.*
3. *There exists some Boolean subalgebra $\mathscr{B}$ of $\mathcal{B}({}^\omega 2)/\mathcal{N}$, such that $\tilde{\mathbb{E}}$ is forcing equivalent to $\mathscr{B}$. More specifically, $\iota \colon \tilde{\mathbb{E}} \to \mathscr{B}$, defined by $\iota(p) \coloneqq [[p]]_\mathcal{N}$, is a dense embedding.*

1.5.9 Iterated forcing: finite support iterations

Let $\mathbb{P}$ be a forcing notion such that $\mathbb{P} \in \mathrm{M}$. We say that $\dot{\mathbb{Q}} = (\dot{\mathbb{Q}}, \leq_{\dot{\mathbb{Q}}})$ is a $\mathbb{P}$-*name of a forcing notion* if $\Vdash_{\mathbb{P}}$"$\dot{\mathbb{Q}}$ is a forcing notion". In this case, we define

$$\mathbb{P} \circledast \dot{\mathbb{Q}} := \{(p, \dot{q}) \in \mathbb{P} \times \mathrm{V}^{\mathbb{P}} \colon\ p \Vdash_{\mathbb{P}} \text{“}\dot{q} \in \dot{\mathbb{Q}}\text{”}\}$$

with the order relation $(p', \dot{q}') \leq_{\mathbb{P} \circledast \dot{\mathbb{Q}}} (p, \dot{q}) :\Leftrightarrow p' \leq_{\mathbb{P}} p$ and $p' \Vdash_{\mathbb{P}}$ "$\dot{q}' \leq_{\dot{\mathbb{Q}}} \dot{q}$". Subscripts are omitted when clear from the context. Notice that, although in general, $\mathbb{P} \circledast \dot{\mathbb{Q}}$ is a proper class, there are dense subsets of it, for example, $\{(p, \dot{q}) \in \mathbb{P} \circledast \dot{\mathbb{Q}} \colon \dot{q} \in \mathrm{dom}\dot{\mathbb{Q}}\}$.[4].

Now, define $\mathbb{P} * \dot{\mathbb{Q}}$ as any dense subset of $\mathbb{P} \circledast \dot{\mathbb{Q}}$ and call it the *two-step iteration of* $\mathbb{P}$ *and* $\dot{\mathbb{Q}}$. It is clear that, in forcing terms, it is well defined because any two dense subsets of $\mathbb{P} \circledast \dot{\mathbb{Q}}$ are forcing equivalent. It is an *iteration* in the following sense: if G is a $\mathbb{P}$ generic filter over M and H is a $\mathbb{Q}$-generic filter over $\mathrm{M}[G]$, we define

$$G * H := \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} \colon p \in G \wedge \dot{q}[G] \in H\}.$$

This establish a bi-univocal correspondence between

$$\{(G, H) \colon G \text{ is } \mathbb{P}\text{-generic over } \mathrm{M} \text{ and } H \text{ is } \dot{\mathbb{Q}}\text{-generic over } \mathrm{M}[G]\} \text{ and,}$$

$$\{K \colon\ K \text{ is } (\mathbb{P} * \dot{\mathbb{Q}})\text{-generic over } \mathrm{M}\},$$

such that $\mathrm{M}[G * H] = \mathrm{M}[G][H]$. Graphically, we get an iteration as in Figure 1.6.

$$\mathrm{M} \xrightarrow{\ \mathbb{P}\ } \mathrm{M}[G] \xrightarrow{\ \dot{\mathbb{Q}}\ } \mathrm{M}[G][H] = \mathrm{M}[G * H]$$

Figure 1.6: An example of two-step iteration

Inductively, we can perform n-step iterations for any $n < \omega$ and we can generalize this for ordinals α, however we must be careful in the definition of limit steps. We are only interested in a certain type of iterations called *finite support iterations*:

Definition 1.5.48. Let $\pi \in \mathrm{Ord}$. We say that $\mathbb{P}_\pi = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi \rangle$ defined by transfinite induction on $\alpha \leq \pi$ is a *finite support iteration* if it satisfies:

1. $\mathbb{P}_0 := \{0\}$,
2. For any $\alpha \leq \pi$, if $p \in \mathbb{P}_\alpha$ then p is a function with $\mathrm{dom}(p) \in [\alpha]^{<\omega}$,
3. For $\alpha < \pi$, $\dot{\mathbb{Q}}_\alpha$ is $\mathbb{P}_\alpha$-name of a forcing notion and

$$\mathbb{P}_{\alpha+1} = \{p \colon p \restriction \alpha \in \mathbb{P}_\alpha \wedge \alpha \in \mathrm{dom}(p) \Rightarrow p(\alpha) \in \langle \dot{\mathbb{Q}}_\alpha \rangle_{\mathbb{P}_\alpha}\},$$

ordered by $q \leq_{\alpha+1} p :\Leftrightarrow q \restriction \alpha \leq_\alpha p \restriction \alpha$ and $q \restriction \alpha \Vdash_\alpha$ "$q(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p(\alpha)$",

[4]Indeed, this is *Kenneth Kunen*'s definition of $\mathbb{P} \circledast \dot{\mathbb{Q}}$ (see [Kun11, Sec. V.3.3])

4. $\alpha < \pi \Rightarrow \mathbb{P}_\alpha \subseteq \mathbb{P}_{\alpha+1}$,

5. For $\gamma \leq \pi$ limit, $\mathbb{P}_\gamma \coloneqq \operatorname{limdir}_{\alpha<\pi}\mathbb{P}_\alpha \coloneqq \bigcup_{\alpha<\gamma}\mathbb{P}_\alpha$, ordered by:

$$q \leq_\gamma p :\Leftrightarrow \forall\alpha < \gamma(q\restriction\alpha \leq_\alpha p\restriction\alpha).$$

Finite support iterations are usually constructed by induction on $\alpha \leq \pi$ where, when reaching $\mathbb{P}_\alpha$, we choose $\dot{\mathbb{Q}}_\alpha$ and get $\mathbb{P}_{\alpha+1}$. Notice that for any $\alpha < \pi$, $\mathbb{P}_{\alpha+1} \equiv \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

Now, by induction we can prove that for $\alpha \leq \beta \leq \pi$, $\mathbb{P}_\alpha \subsetdot \mathbb{P}_\beta$ and therefore, whenever G is $\mathbb{P}_\pi$-generic filter over M, $G_\alpha \coloneqq G \cap \mathbb{P}_\alpha$ is a $\mathbb{P}_\alpha$-generic filter over M. In this case, we denote $\mathrm{M}_\alpha \coloneqq \mathrm{M}[G_\alpha]$ and $G(\alpha) \coloneqq \{p(\alpha)[G_\alpha] \colon p \in G_{\alpha+1} \wedge \alpha \in \operatorname{dom}(p)\}$. Thereby, we get an increasing sequence $\langle M_\alpha \colon \alpha \leq \pi\rangle$ of models of ZFC.

Remark 1.5.49. We sometimes abuse the notation to denote by $\mathrm{M}^{\mathbb{P}_\alpha}$ the generic extension $\mathrm{M}_\alpha[G_\alpha]$, where G is a $\mathbb{P}_\pi$ generic filter over M.

Now, we state some fundamental results about finite support iterations, which we will use throughout the thesis.

We start with a characterization of the conditions at each step of the iteration:

Lemma 1.5.50. *Assume that $\mathbb{P}_\pi = \langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi\rangle$ is a finite support iteration. Then, for any $\alpha \leq \pi$, $p \in \mathbb{P}_\alpha$ if, and only if, p is a function with $\operatorname{dom}(p) \in [\alpha]^{<\aleph_0}$ and for any $\xi \in \operatorname{dom}(p)$, $p(\xi) \in \langle\dot{\mathbb{Q}}_\xi\rangle_{\mathbb{P}_\xi}$.*

Now, a characterization for a condition to belong to a generic set:

Lemma 1.5.51. *Let $\mathbb{P} = \langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi\rangle$ a finite support iteration. If G is $\mathbb{P}_\pi$-generic over M and $p \in \mathbb{P}_\pi$, then*

$$p \in G \Leftrightarrow \forall\alpha \in \operatorname{dom}(p)\left(p(\alpha)[G_\alpha] \in G(\alpha)\right).$$

Our definition in the successor step of the finite support iterations allows us to estimate the cardinality of $\mathbb{P}_\alpha$ at each step of the iteration:

Lemma 1.5.52. *Let $\langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi\rangle$ be a finite support iteration and λ be an infinite cardinal. Assume that $|\pi| \leq \lambda = \lambda^{\aleph_0}$ and, for any $\alpha < \pi$, $|\mathbb{P}_\alpha| \leq \lambda$ implies that $\mathbb{P}_\alpha$ forces $|\dot{\mathbb{Q}}_\alpha| \leq \lambda$. Then, for any $\alpha \leq \pi$, $|\mathbb{P}_\alpha| \leq \lambda$.*

As a consequence of Theorem 1.5.41, we can characterize finite support iterations, using Cohen forcing:

Theorem 1.5.53. *Let λ be an infinite cardinal. Consider $\mathbb{P}_\lambda = \langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \lambda\rangle$, a finite support iteration, where for any $\alpha < \lambda$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name of $\mathbb{C}$. Then, $\mathbb{C}_\lambda \equiv \mathbb{P}_\lambda$.*

Thanks to the nice name's structure (see Subsection 1.5.7), we have:

Lemma 1.5.54. *Let $\mathbb{P}_\pi = \langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi\rangle$ a finite support iteration and κ a regular cardinal. If $\mathbb{P}_\pi$ is κ-cc and $\operatorname{cf}(\pi) \geq \kappa$, then for any nice $\mathbb{P}_\pi$-name $\dot{x}$, there exists some $\xi < \pi$, such that $\dot{x}$ is a $\mathbb{P}_\xi$-name.*

Finally, two important and useful result about finite supports iterations is that it preserves the θ-cc condition and does not add new reals in the steps of cofinality $\geq \theta$:

Theorem 1.5.55. *Let θ be a regular uncountable cardinal. If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi \rangle$ is a finite support iteration of θ-cc forcing notions, that is, for $\alpha < \pi$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is $\theta\text{-cc}$", then $\mathbb{P}_\pi$ is $\theta\text{-cc}$. Also, if $\mathrm{cf}(\pi) \geq \theta$, then ${}^\omega\omega \cap \mathrm{M}_\pi = {}^\omega\omega \cap \cup_{\alpha<\pi} \mathrm{M}_\alpha$.*

As a consequence, finite support iterations of ccc forcing notions is ccc and it does not add new reals in steps of uncountable cofinality.

CHAPTER 2

Probability trees

The new always happens against the overwhelming odds of statistical laws and their probability, which for all practical, everyday purposes amounts to certainty; the new therefore always appears in the guise of a miracle.

Hannah Arendt[1]

Initially, this chapter was going to be a small part of the preliminaries, where we were going to define the elementary notions of probability theory. However, as we progressed in the proofs of the most important theorems, we became aware of the need to apply results related to probability trees in the context of forcing theory. Unfortunately, the bibliography on this subject turned out to be insufficient, and therefore, we had to structure a complete chapter dedicated to defining and studying the necessary properties of probability trees. Moreover, it was necessary to introduce a notion of relative expected value in such trees (see Definition 2.3.5).

In the first two sections we present the basic definitions and results of probability theory as developed, for example, in [Ros98] and [Chu74]. In the third section, we define the notion of probability tree and study some of its properties. In contrast to the first two sections, all definitions and results presented in the third section are our own.

2.1 Elementary probability notions

We say that $\Omega := (\Omega, \mathcal{A}, \mathrm{Pr})$ is a *probability space* if Ω is a non-empty set, $\mathcal{A}$ is a σ-algebra on Ω and $\mathrm{Pr}\colon \mathcal{A} \to [0,1]$ is a measure such that $\mathrm{Pr}(\Omega) = 1$. In this case, we say that Pr is a *probability measure on* Ω. Elements in $\mathcal{A}$ are called *events* and, if $E, F \in \mathcal{A}$, then $\mathrm{Pr}(E)$ is called the *probability of success of* E. Also, if $\mathrm{Pr}(E \cap F) := \mathrm{Pr}(E, F) = \mathrm{Pr}(E) \cdot \mathrm{Pr}(F)$, we say that E and F are *independent events*.

[1] See [Are18].

Since probability spaces are in particular measure spaces, they inherit their basic properties:

Lemma 2.1.1. *Let* $(\Omega, \mathcal{A}, \mathrm{Pr})$ *a probability space and* $E, F \in \mathcal{A}$. *Then,*

1. *If* $\mathrm{Pr}(E) > 0$, *then* $E \neq \emptyset$,
2. *If* $E \subseteq F$, *then* $\mathrm{Pr}(E) \leq \mathrm{Pr}(F)$,
3. $\mathrm{Pr}(E \cup F) = \mathrm{Pr}(E) + \mathrm{Pr}(F) - \mathrm{Pr}(E \cap F)$,
4. $\mathrm{Pr}(E^{\mathrm{c}}) = 1 - \mathrm{Pr}(E)$.

In the practice, all the probability spaces that we are going to consider in this work are finite. So, the following lemma will be very useful to provide any finite set with a probability space structure:

Lemma 2.1.2. *Let* Ω *be a finite set and* $\mathrm{Pr}\colon \Omega \to [0,1]$ *a function such that* $\sum_{o\in\Omega} \mathrm{Pr}(o) = 1$. *Then there exists a probability function* $\mathrm{Pr}_\Omega \colon \mathcal{P}(\Omega) \to [0,1]$ *such that,* $(\Omega, \mathcal{P}(\Omega), \mathrm{Pr}_\Omega)$ *is a probability space and, for any* $o \in \Omega$, $\mathrm{Pr}_\Omega(\{o\}) = \mathrm{Pr}(o)$.

***Proof*.** Since Ω is finite, we can write it as $\Omega = \{o_n \colon n < n^*\}$ for some $n^* < \omega$. For any $A \subseteq \Omega$, define $n_A \coloneqq \{n < n^* \colon o_n \in A\}$ and consider $\mathrm{Pr}_\Omega \colon \mathcal{P}(\Omega) \to [0,1]$ such that for any $A \subseteq \Omega$,

$$\mathrm{Pr}_\Omega(A) \coloneqq \sum_{n \in n_A} \mathrm{Pr}(o_n).$$

Since $A, B \in \mathcal{P}(\Omega)$ and $A \cap B = \emptyset$ imply $n_A \cap n_B = \emptyset$, it is clear that Pr_Ω is a measure on $\mathcal{P}(\Omega)$, and as $n_\Omega = n^*$, it follows that $\mathrm{Pr}_\Omega(\Omega) = \sum_{o\in\Omega} \mathrm{Pr}(o) = 1$. Thus, $(\Omega, \mathcal{P}(\Omega), \mathrm{Pr}_\Omega)$ is a probability space, and it is clear that, for any $o \in \Omega$, $\mathrm{Pr}_\Omega(\{o\}) = \mathrm{Pr}(o)$. $\square_{\text{Lemma 2.1.2}}$

It will be useful not to make a distinction between Pr and Pr_Ω in the Lemma above. That is, abusing the notation, it is enough to define the probability in the elements of a finite set to define a probability space on it.

Notation 2.1.3. For simplicity of notation, if $o \in \Omega$ and $\mathrm{Pr}\colon \mathcal{P}(\Omega) \to [0,1]$ then, for any $o \in \Omega$, we denote $\mathrm{Pr}(o) \coloneqq \mathrm{Pr}(\{o\})$. This then justifies us not making any distinction between the functions Pr and Pr_Ω in Lemma 2.1.2, that is, when we want to define a probability space over a finite set Ω, we are going to define a function $\mathrm{Pr}\colon \Omega \to [0,1]$ that satisfies the conditions of Lemma 2.1.2 and denote Pr_Ω by Pr.

2.2 Random variables

Fix, for the rest of this subsection, a probability space $(\Omega, \mathcal{A}, \mathrm{Pr})$. We introduce the notion of *random variable* on Ω:

Definition 2.2.1. We say that a function $X\colon \Omega \to \mathbb{R}$ is a *random variable on* Ω if, for any $a \in \mathbb{R}$, $\{o \in \Omega \colon X(o) \leq a\} \in \mathcal{A}$. Also, we say that X is a *discrete random variable* if its range is countable.

For instance, the constant functions are easily random variables. Notice that, in measure theory terms, a random variable is simply an $\mathcal{A}$-measurable function. All the random variables that we are going to consider in this work are discrete, so when we refer to a random variable it will be understood that it is discrete.

We say that a *trial* is an experiment where there are only two possible outcomes, one with probability p and the other with probability $1-p$. Intuitively, discrete random variables correspond to values from experiments in which the number of times an event has occurred is counted. For example:

Definition 2.2.2. Let X be a random variable on Ω. Then,

1. If X counts the probability of success in a single trial with probability p, then we say that X has a *Bernoulli distribution with parameter* p, and we denote it as $X \sim \mathrm{Bernoulli}(p)$.
2. If X counts the number of successes in a sequence of n independent trials, each with probability of success p, we say that X has *binomial distribution with parameters* n, p, and we denote it as $X \sim \mathrm{Binomial}(n, p)$.

Since random variables are $\mathcal{A}$-measurable functions, we have that:

Lemma 2.2.3. *If X, Y are random variables on Ω, and $r \in \mathbb{R}$ then rX, $X \cdot Y$, $|X|$ and $X + Y$, are random variables.*

By the definition of a random variable, for any $r \in \mathbb{R}$, $\Pr(\{o \in \Omega \colon X(o) = a\})$, $\Pr(\{o \in \Omega \colon X(o) \leq a\})$ and $\Pr(\{o \in \Omega \colon X(o) \geq a\})$ are defined. So, to simplify the writing, we use the following notation:

- $\Pr[X = a] := \Pr(\{o \in \Omega \colon X(o) = a\})$,
- $\Pr[X \leq a] := \Pr(\{o \in \Omega \colon X(o) \leq a\})$.
- $\Pr[X \geq a] := \Pr(\{o \in \Omega \colon X(o) \geq a\})$.

The definition of independent events can be naturally extended to random variables:

Definition 2.2.4. Let X, Y be random variables on Ω. We say that X and Y are *independent* if, for any $r, s \in \mathbb{R}$, $\Pr[X = r, Y = s] = \Pr[X = r] \cdot \Pr[Y = s]$.

Also, Definition 2.2.4 can be extended naturally to finite sequences of random variables.

Next we define a function that measures the probability that a random variable is equal to a given value:

Definition 2.2.5. Let X a random variable on Ω. We define the *mass probability function*, or simply *probability function of* X, as the function $p_X \colon \mathbb{R} \to \mathbb{R}$ such that for any $r \in \mathbb{R}$,

$$p_X(r) := \Pr[X = r].$$

Recall that, for $n, k < \omega$,

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

For instance, if $X \sim \mathrm{Binomial}(n, p)$, then

$$p_X(r) := \binom{n}{r} p^r (1-p)^{n-r},$$

when $r \in [0,1] \cap \mathbb{Z}$, and if $X \sim \mathrm{Bernoulli}(p)$, then $p_X(r) = p^r(1-p)^{1-r}$, for any $r \in \{0, 1\}$. Also, it is clear that $\sum_{r \in \mathrm{ran}(X)} p_X(r) = 1$ for any random variable X.

Example 2.2.6. Let $n < \omega$ and $p \in [0,1]$. We define $\Omega_n := \{i < \omega \colon i \leq n\}$, $\mathcal{A}_n := \mathcal{P}(\Omega_n)$ and $\mathrm{Pr}_n \colon \mathcal{A}_n \to \mathbb{R}$ such that, for any $i \leq n$,

$$\mathrm{Pr}(i) := \binom{n}{i} p^i (1-p)^{n-i}.$$

It is cleat that $(\Omega_n,\ \mathcal{A}_n,\ \mathrm{Pr}_n)$ is a probability space. Then, the identity function $\mathrm{B}_{n,p} \colon \Omega_n \to \mathbb{R}$ is a random variable.

Now, we introduce the expected value, the variance and the covariance:

Definition 2.2.7. Let X, Y be random variables on Ω. Then:

1. $\mathrm{E}[X] := \sum_{r \in \mathrm{ran}(X)} r\, p_X(r)$ is called the *expected value* of X.
2. $\mathrm{Cov}[X, Y] := \mathrm{E}[XY] - \mathrm{E}[X] \cdot \mathrm{E}[Y]$ is called the *covariance* of X and Y.
3. $\mathrm{Var}[X] := \mathrm{Cov}[X, X]$ is called the *variance* of X.

The expected value is an attempt to choose a value that "best" represents all random variable values. The variance is a way of measuring how dispersed the values of the random variable are around the expected value, which can be intuited from the following property:

$$\mathrm{Var}[X] = 0 \Leftrightarrow \mathrm{Pr}[X = \mathrm{E}[X]] = 1.$$

The covariance is a value that indicates the degree of variation of two random variables with respect to their expected values. It is a data that determines if there is a dependency between both variables. So if the variables are independent, their covariance must be zero, however, the converse is not always true.

It is easy to calculate the expected value and the variance when X has a Bernoulli or Binomial distribution:

Example 2.2.8. Let X a random variable. Then,

1. If $X \sim \mathrm{Bernoulli}(p)$, then $\mathrm{E}[X] = p$ and $\mathrm{Var}[X] = p(1-p)$
2. If $X \sim \mathrm{Binomial}(n, p)$, then $\mathrm{E}[X] = np$ and $\mathrm{Var}[X] = np(1-p)$.

By adding finite independent random variables with Bernoulli distribution and identically distributed, a random variable with binomial distribution is obtained:

Lemma 2.2.9. *Let* $\langle X_n : n < n^* \rangle$ *be a finite sequence of independent random variables on* Ω*. If for any* $n < n^*$, $X_n \sim \mathrm{Bernoulli}(p)$, *then* $\sum_{n<n^*} X_n \sim \mathrm{Binomial}(n^*, p)$.

Next, we present the basic properties of the expected value that we use in this work:

Theorem 2.2.10. *Let* X, Y *be a random variables on* Ω *and* $r, s \in \mathbb{R}$*. Then:*

1. $\mathrm{E}[r] = r$.
2. *If* $X \leq Y$, *then* $\mathrm{E}[X] \leq \mathrm{E}[Y]$.
3. *If* $r \leq X \leq s$, *then* $r \leq \mathrm{E}[X] \leq s$.
4. $\mathrm{E}[rX + s] = r\mathrm{E}[X] + s$.
5. *If* $\langle X_n : n < n^* \rangle$ *is a sequence of random variables on* Ω, *then*
$$\mathrm{E}\left[\sum_{n<n^*} X_i\right] = \sum_{n<n^*} \mathrm{E}[X_n].$$
6. $|\mathrm{E}[X]| \leq \mathrm{E}[|X|]$.

Now, we review some elementary properties of the variance and covariance:

Theorem 2.2.11. *Let* X *and* Y *be a random variables on* Ω *such that* $|\mathrm{E}[X]| < \infty$, *and* $r \in \mathbb{R}$*. Then,*

1. $\mathrm{Cov}[X, Y]$ *is a bilinear function.*
2. $\mathrm{Cov}[X, r] = \mathrm{Cov}[r, Y] = 0$.
3. *If* $\langle X_n : n < n^* \rangle$ *is a sequence of random variables on* Ω, *then*
$$\mathrm{Var}\left[\sum_{i<n} a_i\, X_i\right] = \sum_{i<n} a_i^2\, \mathrm{Var}[X_i] + \sum_{i,j \leq n,\, i \neq j} a_i a_j \mathrm{Cov}[X_i, X_j].$$

As a consequence of (1) and (2), we have that $\mathrm{Var}[X] \geq 0$ and $\mathrm{Var}[X + r] = \mathrm{Var}[X]$.
Finally, we state a result that will be fundamental in several parts of this work: the one known as *Chebyshev's inequality* .

Theorem 2.2.12. *Let* X *be a random variable on* Ω *with* $|\mathrm{E}[X]| < \infty$, *then*
$$\forall \varepsilon > 0 \left(\Pr[\,|X - \mathrm{E}[X]| \geq \varepsilon] \leq \frac{\mathrm{Var}[X]}{\varepsilon^2} \right).$$

2.3 Probability trees

Although, broadly speaking, only three probability trees appear throughout this work, they appear precisely in, perhaps, the two most important proofs (see the proof of Lemma 4.2.16 and Main Lemma 4.3.17). So, in this subsection we define what we mean by a probability tree and consider some necessary results for us. In particular, we are going to define a relative expected value in probability trees that will help us simplify the proof of Lemma 4.2.16, a proof that is still quite long and technical.

Definition 2.3.1. Let Z be a non-empty set. We say that $\mathcal{T}$ is a *probability tree on* ${}^{<\omega}Z$ if it is a well pruned tree[2] on ${}^{<\omega}Z$ and, for any $\rho \in \mathcal{T} \setminus \max(\mathcal{T})$ there exists a σ-algebra $\mathcal{A}_\rho$ with $[\mathrm{succ}_\rho]^{<\aleph_0} \subseteq \mathcal{A}_\rho$, and a function $\mathrm{Pr}^{\mathcal{T}}_\rho$ such that $(\mathrm{succ}_\rho,\, \mathcal{A}_\rho,\, \mathrm{Pr}^{\mathcal{T}}_\rho)$ is a probability space.

For example, let $\mathcal{T}$ be the tree on ${}^{<\omega}Z$ for some set Z as in Figure 2.1 such that $z_i^j \in Z$ for any $i \in \{0, 1\}$ and $j \in \{0, 1, 2, 3\}$.

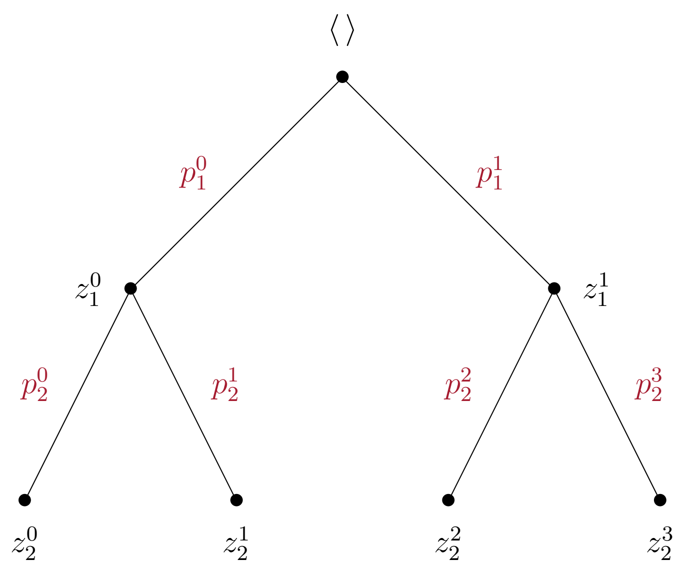

Figure 2.1: Example of a probability tree

If we define

- $\mathrm{Pr}_{\langle\rangle}(\langle z_0^i \rangle) := p_0^i$, for $i \in \{0, 1\}$,
- $\mathrm{Pr}_{\langle z_1^0 \rangle}(\langle z_1^0,\, z_2^j \rangle) := p_2^j$, for $i \in \{0, 1\}$,
- $\mathrm{Pr}_{\langle z_1^1 \rangle}(\langle z_1^1,\, z_2^j \rangle) := p_2^j$, for $j \in \{2, 3\}$,

then $\mathcal{T}$ is a probability tree if, and only if:

$$p_1^0 + p_1^1 = 1,\ p_2^0 + p_2^1 = 1 \ \text{ and } \ p_2^2 + p_2^3 = 1.$$

[2]See Definition 1.2.2

Notice that, in that case, it satisfies the following:

$$p_1^0\, p_2^0 + p_1^0\, p_2^1 + p_1^1\, p_2^2 + p_1^1\, p_2^3 = p_1^0\,(p_2^0 + p_2^1) + p_1^1\,(p_2^2 + p_2^3) = p_1^0 + p_1^1 = 1,$$

that is, if for any $\rho = \langle z_j^k, z_l^m \rangle \in \mathcal{L}_2(\mathcal{T})$ we define $\mathrm{Pr}_2(\rho) := p_j^k\, p_l^m$, then $(\mathcal{L}_2(\mathcal{T}),\ \mathcal{P}(\mathcal{L}_2(\mathcal{T})),\ \mathrm{Pr}_2)$ is a probability space. The same happens trivially at level 1. So, $\mathcal{T}$ induces a probability space on each of its levels, which illustrates a general result:

Theorem 2.3.2. *Every probability tree $\mathcal{T}$ with finite levels induces a probability space in each of its levels.*

Proof. Let $\mathcal{T}$ be a probability tree on $Z^{<\omega}$ such that, for each $h < \mathrm{ht}(\mathcal{T})$, $\mathcal{L}_h(\mathcal{T})$ is finite. For any $\rho \in \mathcal{T} \backslash \max(\mathcal{T})$, fix a σ-algebra $\mathcal{A}_\rho$ on succ_ρ containing all its singletons such that $(\mathrm{succ}_\rho,\ \mathcal{A}_\rho,\ \mathrm{Pr}_\rho)$ is a probability space.

For any $h < \mathrm{ht}(\mathcal{T})$ and $\rho \in \mathcal{L}_h(\mathcal{T})$ we define:

$$\mathrm{Pr}_h(\rho) := \prod_{0 \leq i \leq h-1} \mathrm{Pr}_{\rho \restriction i}(\rho \restriction (i+1)). \tag{2.3.1}$$

Since $\mathcal{L}_h(\mathcal{T})$ is finite, to prove that $(\mathcal{L}_h(\mathcal{T}), \mathcal{P}(\mathcal{L}_h(\mathcal{T})), \mathrm{Pr}_h)$ is a probability space, it is enough, by Lemma 2.1.2, to show that:

$$\sum_{\rho \in \mathcal{L}_h(\mathcal{T})} \mathrm{Pr}_h(\rho) = 1.$$

For this, we apply induction on $h < \mathrm{ht}(\mathcal{T})$. If $h = 0$, then

$$\sum_{\rho \in \mathcal{L}_0(\mathcal{T})} \mathrm{Pr}_0(\rho) = \mathrm{Pr}_0(\langle\,\rangle) = \prod_{0 \leq i \leq -1} \mathrm{Pr}_{\rho \restriction i}(\rho \restriction (i+1)) = 1,$$

where the last equality holds because empty products are equal to 1.

Now, suppose that $h + 1 < \mathrm{ht}(\mathcal{T})$ and

$$\sum_{\rho \in \mathcal{L}_h(\mathcal{T})} \mathrm{Pr}_h(\rho) = 1. \tag{2.3.2}$$

Since each level of $\mathcal{T}$ is finite, for any $\rho \in \mathcal{T} \setminus \max(\mathcal{T})$, there are $N_\rho < \omega$ and a sequence $\langle z_n^\rho \colon n < N_\rho \rangle \subseteq Z$ such that

$$\mathrm{succ}_\rho(\mathcal{T}) = \{\rho^\frown \langle z_n^\rho \rangle \colon n < N_\rho\}.$$

Also, since for any $\rho \in \mathcal{T}$, succ_ρ is a probability space, we have that:

$$\sum_{\eta \in \mathrm{succ}_\rho} \mathrm{Pr}_\rho(\eta) = 1 \tag{2.3.3}$$

As a consequence, using Equation 2.3.2 and Equation 2.3.3, we get:

$$\begin{aligned}
\sum_{\rho\in\mathcal{L}_{h+1}(\mathcal{T})} \Pr_{h+1}(\rho) &= \sum_{\rho\in\mathcal{L}_h(\mathcal{T})}\left(\sum_{n<N_\rho} \Pr_{h+1}(\rho^\frown\langle z_n^\rho\rangle)\right)\\
&= \sum_{\rho\in\mathcal{L}_h(\mathcal{T})}\left(\sum_{n<N_\rho}\left(\prod_{0\le i\le h} \Pr_{\rho^\frown\langle z_n^\rho\rangle\restriction i}(\rho^\frown\langle z_n^\rho\rangle\restriction(i+1))\right)\right)\\
&= \sum_{\rho\in\mathcal{L}_h(\mathcal{T})}\left(\sum_{n<N_\rho}\left(\Pr_\rho(\rho^\frown\langle z_n^\rho\rangle\restriction(h+1))\prod_{0\le i\le h-1} \Pr_{\rho^\frown\langle z_n^\rho\rangle\restriction i}(\rho^\frown\langle z_n^\rho\rangle\restriction(i+1))\right)\right)\\
&= \sum_{\rho\in\mathcal{L}_h(\mathcal{T})}\left(\sum_{n<N_\rho}\left(\Pr_\rho(\rho^\frown\langle z_n^\rho\rangle)\prod_{0\le i\le h-1} \Pr_{\rho\restriction i}(\rho\restriction(i+1))\right)\right)\\
&= \sum_{\rho\in\mathcal{L}_h(\mathcal{T})}\left(\sum_{\eta\in\mathrm{succ}_\rho} \Pr_\rho(\eta)\Pr_h(\rho)\right)\\
&= \sum_{\rho\in\mathcal{L}_h(\mathcal{T})}\left(\Pr_h(\rho)\sum_{\eta\in\mathrm{succ}_\rho} \Pr_\rho(\eta)\right)\\
&= \sum_{\rho\in\mathcal{L}_h(\mathcal{T})} \Pr_h(\rho)\\
&= 1.
\end{aligned}$$

Thus, $(\mathcal{L}_{h+1}(\mathcal{T}),\ \mathcal{P}(\mathcal{L}_{h+1}(\mathcal{T})),\ \Pr_{h+1})$ is a probability space. $\square_{\text{Theorem 2.3.2}}$

2.3.1 Relative expected value in probability trees

The proof that random forcing is σ-FAM-linked (see Lemma 4.2.16) is quite long and technical. To simplify it a bit, we decided to define a relative expected value in probability trees. In order to state this relative expected value, we need some additional notions of trees:

Definition 2.3.3. Let Z be a non-empty set, $\mathcal{T}$ a tree on ${}^{<\omega}Z$ and $\rho\in\mathcal{T}$. We say that $\mathcal{S}\subseteq\mathcal{T}_{\geq\rho}$ is a tree on $\mathcal{T}_{\geq\rho}$ when, for any $\eta,\nu\in\mathcal{T}_{\geq\rho}$, if $\eta\subseteq\nu$ and $\nu\in\mathcal{S}$, then $\eta\in\mathcal{S}$.

It is clear that, if $\mathcal{S}$ is a tree on $\mathcal{T}_{\geq\rho}$, then it is isomorphic to some tree on ${}^{<\omega}Z$ (see Figure 2.2). So, from now on, without loss of generality, we can apply to them the results we have shown for trees. In particular, we will be interested in the case when $\mathcal{S}$ is $\mathcal{T}_{\geq\rho}$ itself.
Now, from Definition 2.3.3 it is immediate that:

Lemma 2.3.4. *Let Z be a non-empty set, $\mathcal{T}$ a tree on ${}^{<\omega}Z$, $\rho\in\mathcal{T}$ and $\mathcal{S}$ a tree on $\mathcal{T}_{\geq\rho}$. Then:*

1. *If $\mathcal{T}$ is a probability tree, then $\mathcal{S}$ is too.*

2. *If* $0 < n < \omega$, *and* $\rho \in \mathcal{L}_h(\mathcal{T})$, *then* $\mathcal{L}_n(\mathcal{S}) \subseteq \mathcal{L}_{h+n}(\mathcal{T})$.

We can now define the *relative expected value* in probability trees:

Definition 2.3.5. Let $\mathcal{T}$ be a probability tree on ${}^{<\omega}Z$, $h < \omega$ and $\rho \in \mathcal{L}_h(\mathcal{T})$. Let $0 < n < \omega$ and X be a random variable on $\mathcal{L}_{h+n}(\mathcal{T})$. Then, we define:

$$\mathrm{E}_{\mathcal{L}_{h+n}(\mathcal{T})}[X : \eta \restriction h = \rho] := \mathrm{E}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}[X \restriction \mathcal{L}_n(\mathcal{T}_{\geq\rho})],$$

and call it the *relative expected value of X respect to ρ in $\mathcal{L}_{h+n}(\mathcal{T})$.*
When the context is clear, we simply write $E_{h+n}[X : \eta \restriction h = \rho]$ or even $\mathrm{E}[X : \eta \restriction h = \rho]$ instead of $\mathrm{E}_{\mathcal{L}_{h+n}(\mathcal{T})}[X : \eta \restriction h = \rho]$.

Although the definition does not depend on η but on its level, we decided to use that notation because of its intuitive meaning.
Notice that the relative expected value is well defined because, on the one hand, by Lemma 2.3.4(1) $\mathcal{T}_{\geq\rho}$ inherits the probability space structure from $\mathcal{T}$ and, on the other hand, if X is a random variable on $\mathcal{L}_{h+n}(\mathcal{T})$, then by Lemma 2.3.4(2), we have that $\mathcal{L}_n(\mathcal{T}_{\geq\rho}) \subseteq \mathcal{L}_{h+n}(\mathcal{T})$, hence $X \restriction \mathcal{L}_n(\mathcal{T}_{\geq\rho})$ is a random variable on $\mathcal{L}_n(\mathcal{T}_{\geq\rho})$, so calculating its expected value makes sense.
Since the relative expected value is defined in terms of a usual expected value, it is clear that:

Theorem 2.3.6. *Let $\mathcal{T}$ be a probability tree on ${}^{<\omega}Z$, $h < \omega$ and $\rho \in \mathcal{L}_h(\mathcal{T})$. Consider $0 < n < \omega$, two random variables X, Y on $\mathcal{L}_{h+n}(\mathcal{T})$ and $r, s \in \mathbb{R}$. Then,*

$$\mathrm{E}_{h+n}[rX + sY : \eta \restriction h = \rho] = r\mathrm{E}_{h+n}[X : \eta \restriction h = \rho] + s\mathrm{E}_{h+n}[Y : \eta \restriction h = \rho].$$

The following result allows us to decompose the probability of the successors of ρ at the level $h+n$ of $\mathcal{T}$, in terms of the probability at the level n of $\mathcal{T}_{\geq\rho}$:

Lemma 2.3.7. *Let $\mathcal{T}$ be a probability tree on ${}^{<\omega}Z$, $h < \omega$ and $\rho \in \mathcal{L}_h(\mathcal{T})$. Let $0 < n < \omega$ and $\eta \in \mathcal{L}_n(\mathcal{T}_{\geq\rho})$. Then,*

$$\mathrm{Pr}_{\mathcal{L}_{h+n}(\mathcal{T})}(\eta) = \mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}(\eta) \cdot \mathrm{Pr}_{\mathcal{L}_h(\mathcal{T})}(\rho).$$

Proof. Since by Lemma 2.3.4, $\mathcal{T}_{\geq\rho}$ is a probability tree, by Theorem 2.3.2, $\mathcal{L}_n(\mathcal{T}_{\geq\rho})$ is a probability space, that is, considering $\mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}$ makes sense. Now, if $\eta \in \mathcal{L}_k(\mathcal{T}_{\geq\rho})$, by Lemma 2.3.4(55) we have that $\eta \in \mathcal{L}_{h+k}(\mathcal{T})$, then:

$$\begin{aligned}
\mathrm{Pr}_{\mathcal{L}_{h+n}(\mathcal{T})}(\eta) &= \prod_{0\leq i\leq h+n-1} \mathrm{Pr}_{\eta\restriction i}(\eta \restriction (i+1)) \\
&= \left(\prod_{0\leq i\leq h-1} \mathrm{Pr}_{\eta\restriction i}(\eta \restriction (i+1))\right) \cdot \left(\prod_{h\leq i\leq h+n-1} \mathrm{Pr}_{\eta\restriction i}(\eta \restriction (i+1))\right) \\
&= \left(\prod_{0\leq i\leq h-1} \mathrm{Pr}_{\rho\restriction i}(\rho \restriction (i+1))\right) \cdot \left(\prod_{0\leq j\leq n-1} \mathrm{Pr}_{\eta\restriction(h+j)}(\eta \restriction (h+j+1))\right) \\
&= \mathrm{Pr}_{\mathcal{L}_h(\mathcal{T})}(\rho) \cdot \mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}(\eta).
\end{aligned}$$

Thus,

$$\mathrm{Pr}_{\mathcal{L}_{h+n}(\mathcal{T})}(\eta) = \mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}(\eta) \cdot \mathrm{Pr}_{\mathcal{L}_h(\mathcal{T})}(\rho). \qquad \square_{\text{Lemma 2.3.7}}$$

Now, we can show that, to calculate a relative expected value, we can do *intermediate steps* (see Figure 2.2), that is:

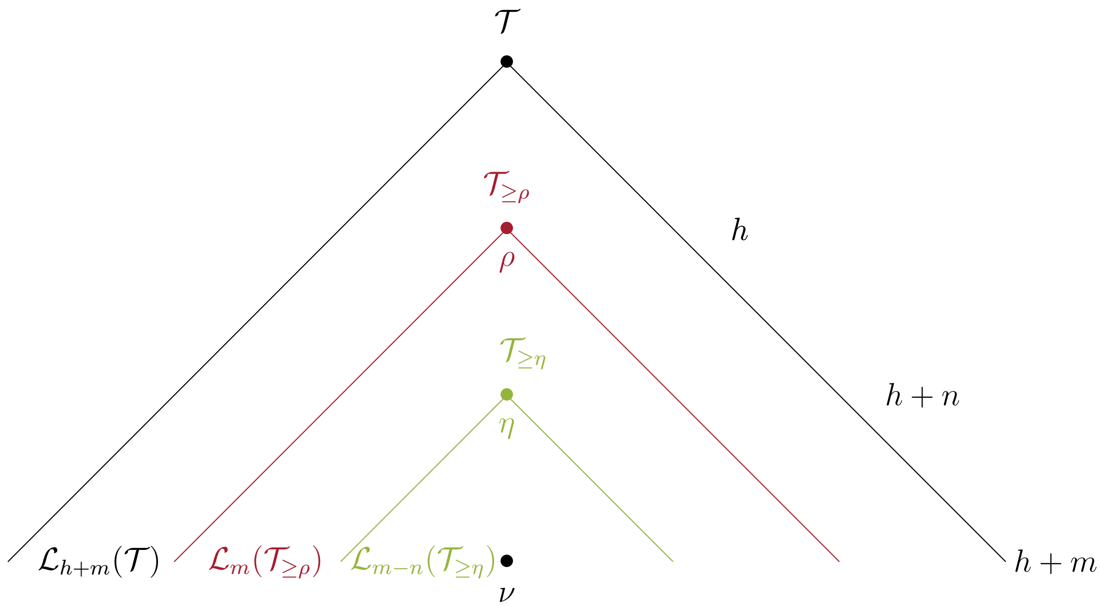

Figure 2.2: A graphic example of the situation in Theorem 2.3.8.

Theorem 2.3.8. *Let $\mathcal{T}$ be a probability tree on ${}^{<\omega}Z$ and $\rho \in \mathcal{T}$ such that $\rho \in \mathcal{L}_h(\mathcal{T})$, where $h < \omega$, and let $0 < n < m < \omega$. If X is a random variable on $\mathcal{L}_{h+m}(\mathcal{T})$, then:*

$$\mathrm{E}_{\mathcal{L}_{h+m}(\mathcal{T})}[X : \nu \restriction h = \rho] = \mathrm{E}_{\mathcal{L}_{h+n}(\mathcal{T})}[\,\mathrm{E}_{\mathcal{L}_{h+m}(\mathcal{T})}[X : \nu \restriction (h+n) = \eta] : \eta \restriction h = \rho].$$

Proof. Let $k := h+n$. By Definition 2.3.5 and the definition of expected value, we have that:

$$\begin{aligned}
\mathrm{E}_{h+n}[\,\mathrm{E}_{h+m}[X : \nu \restriction k = \eta] : \eta \restriction h = \rho] &= \sum_{\eta \in \mathcal{L}_n(\mathcal{T}_{\geq\rho})} \mathrm{E}_{h+m}[X : \nu \restriction k = \eta] \cdot \mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}(\eta) \\
&= \sum_{\eta \in \mathcal{L}_n(\mathcal{T}_{\geq\rho})} \mathrm{E}_{\mathcal{L}_m(\mathcal{T}_{\geq\eta})}[X \restriction \mathcal{L}_m(\mathcal{T}_{\geq\eta})] \cdot \mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}(\eta) \\
&= \sum_{\eta \in \mathcal{L}_n(\mathcal{T}_{\geq\rho})} \left(\sum_{\nu \in \mathcal{L}_m(\mathcal{T}_{\geq\eta})} X(\nu) \cdot \mathrm{Pr}_{\mathcal{L}_{n-m}(\mathcal{T}_{\geq\eta})}(\nu) \right) \cdot \mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}(\eta) \\
&= \sum_{\eta \in \mathcal{T}_{\geq\rho}} \left(\sum_{\nu \in \mathcal{T}_{\geq\eta}} X(\nu) \cdot \mathrm{Pr}_{\mathcal{L}_m(\mathcal{T}_{\geq\rho})}(\nu) \right) \\
&= \sum_{\nu \in \mathcal{L}_m(\mathcal{T}_{\geq\rho})} X(\nu) \cdot \mathrm{Pr}_{\mathcal{L}_m(\mathcal{T}_{\geq\rho})}(\nu) \\
&= \mathrm{E}_{\mathcal{L}_m(\mathcal{T}_{\geq\rho})}[X \restriction \mathcal{L}_m(\mathcal{T}_{\geq\rho})] \\
&= \mathrm{E}_{h+m}[X : \nu \restriction h = \rho],
\end{aligned}$$

where $\mathrm{Pr}_{\mathcal{L}_m(\mathcal{T}_{\geq\rho})}(\nu) = \mathrm{Pr}_{\mathcal{L}_{m-n}(\mathcal{T}_{\geq\eta})}(\nu) \cdot \mathrm{Pr}_{\mathcal{L}_n(\mathcal{T}_{\geq\rho})}(\eta)$ by virtue of Lemma 2.3.7. $\square_{\text{Theorem 2.3.8}}$

Finally, as a consequence, we can express the expected value of X in terms of the relative expected value:

Corollary 2.3.9. *Let $\mathcal{T}$ be a probability tree on ${}^{<\omega}Z$ and $0 < n < m < \omega$. If X is a random variable on $\mathcal{L}_m(\mathcal{T})$, then:*

$$\mathrm{E}_m[X] = \mathrm{E}_n[\,\mathrm{E}_m[X\colon \nu \restriction n = \eta]].$$

Proof. First, notice that:

$$\mathrm{E}_n[X] = \mathrm{E}_{\mathcal{L}_n(\mathcal{T}_{\geq\langle\,\rangle})}[X \restriction \mathcal{T}_{\geq\langle\,\rangle}] = \mathrm{E}_n[X\colon \nu \restriction 0 = \langle\,\rangle].$$

Now, by Theorem 2.3.8, we get:

$$\begin{aligned}\mathrm{E}_m[X] &= \mathrm{E}_m[X\colon \nu \restriction 0 = \langle\,\rangle]\\ &= \mathrm{E}_n[\,\mathrm{E}_m[X\colon \nu \restriction n = \eta]\colon \eta \restriction 0 = \langle\,\rangle]\\ &= \mathrm{E}_n[\,\mathrm{E}_m[X\colon \nu \restriction n = \eta]].\end{aligned}$$

$\square_{\text{Corollary 2.3.9}}$

2.3.2 Adding random variables with Bernoulli distribution under terrible conditions

By Lemma 2.2.9 we know that, by adding finite independent and identically distributed random variables with Bernoulli distribution, we obtain a random variable with binomial distribution. However, in one of the most important proofs of this thesis (see the proof of Main Lemma 4.3.17) we are faced with a situation in which we must compare a sum of dependent random variables, not identically distributed and worse still, that have Bernoulli distribution depending on a parameter, with some random variable with binomial distribution. The following result was obtained as a solution to this problem:

Theorem 2.3.10. *Let $n^* < \omega$ and $\mathcal{T} = {}^{\leq n^*}2$ be the complete binary tree of height $n^* + 1$ endowed with probability tree structure. Define $Y\colon \mathcal{L}_{n^*}(\mathcal{T}) \to \mathbb{R}$ such that, for any $\rho \in \mathcal{L}_{n^*}(\mathcal{T})$,*

$$Y(\rho) \coloneqq |\{n < n^*\colon \rho(n) = 0\}|.$$

Assume that there exists some $p \in [0,1]$ such that, for any $\rho \in \mathcal{T} \setminus \max(\mathcal{T})$, $p \leq p_\rho \coloneqq \Pr_\rho(\rho^\frown\langle 0\rangle)$. Then, for all $z \in \mathbb{R}$,

$$\Pr_{\mathcal{L}_{n^*(\mathcal{T})}}[Y \leq z] \leq \Pr_{\Omega_{n^*}}[\mathrm{B}_{n^*,p} \leq z].$$

Proof. For any $n < n^*$, define the random variable X_n on $\mathcal{L}_{n^*}(\mathcal{T})$ such that, for every $\rho \in \mathcal{L}_{n^*}(\mathcal{T})$,

$$X_n(\rho) \coloneqq 1 - \rho(n).$$

Thereby, clearly for any $\rho \in \mathcal{T} \setminus \max(\mathcal{T})$, $X_{\mathrm{ht}(\rho)} \restriction \mathrm{succ}_\rho \sim \mathrm{Bernoulli}(p_\rho)$ and $Y = \sum_{n<n^*} X_n$, which is the sum of random variables that we referred in the comment previous to the theorem.

For any $r \in [0,1]$ and $d \in \{0,1\}$, define:

$$I_r^d := \begin{cases} [0,r), & \text{if} \quad d = 0, \\ \\ [r,1], & \text{if} \quad d = 1. \end{cases}$$

Now, for $\rho \in \mathcal{L}_{n^*}(\mathcal{T})$, consider

$$C_\rho^\bullet := \prod_{n<n^*} I_{p_{\rho\restriction n}}^{\rho(n)},$$

and notice that, by the way we define the probability space structure on the levels (see Equation 2.3.1), it is clear that $\mathrm{Vol}(C_\rho^\bullet) = \Pr_{\mathcal{L}_{n^*}(\mathcal{T})}(\rho)$. Let us prove that $\{C_\rho^\bullet \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T})\}$ is a partition of $[0,1]^{n^*}$: it is clear that it is a collection of pairwise disjoint sets, so let $x = \langle x_n \colon n < n^* \rangle \in [0,1]^{n^*}$. Define, by recursion on $m < n^*$ a sequence $\eta_x \in \mathcal{L}_{n^*}(\mathcal{T})$, as follows:

$$\eta_x(0) := \begin{cases} 0, & \text{if} \quad x_0 \in [0,\, p_{\langle\rangle}), \\ \\ 1, & \text{if} \quad x_0 \in [p_{\langle\rangle},\, 1]. \end{cases}$$

Assume that we have constructed $\eta_x(m)$ $m+1 < n^*$. Then:

$$\eta_x(m+1) := \begin{cases} 0, & \text{if} \quad x_{m+1} \in [0,\, p_{\eta_x\restriction m+1}), \\ \\ 1, & \text{if} \quad\; x_m \in [p_{\eta_x\restriction m+1},\, 1]. \end{cases}$$

It is clear that $\eta_x \in \mathcal{L}_{n^*}(\mathcal{T})$. Now, let $n < n^*$ and consider two possible cases:

1. when $x_n \in [0, p_{\eta_x\restriction n})$: in this case, by definition of η_x, we have that $\eta_x(n) = 0$ and therefore, $x_n \in [0, p_{\eta_x\restriction n}) = I_{p_{\eta_x\restriction n}}^{\eta_x(n)}$.

2. when $x_n \in [p_{\eta_x\restriction n}, 1]$: in this case, $\eta_x(n) = 1$, hence $x_n \in [p_{\eta_x\restriction n}, 1] = I_{p_{\eta_x\restriction n}}^{\eta_x(n)}$.

Therefore, in any case, $x_n \in I_{p_{\eta_x\restriction n}}^{\eta_x}$. Thus, $x \in C_{\eta_x}^\bullet$.
Now, let $z \in \mathbb{R}$. By the definition of $C_\rho^\bullet$ and the constructions of η_x, we have that, for any $x = \langle x_n \colon n < n^* \rangle \in [0,1]^{n^*}$,

$$|\{n < n^* \colon x_n \le p_{\eta_x\restriction n}\}| \le z \Leftrightarrow \sum_{n<n^*} X_n(\eta_x) \le z,$$

whence it follows that:

$$\bigcup \left\{ C_\rho^\bullet \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} X_n(\rho) \le z \right\} = \left\{ x \in [0,1]^{n^*} \colon |\{n < n^* \colon x_n \le p_{\eta_x\restriction n}\}| \le z \right\}. \tag{2.3.4}$$

On the other hand,

$$\begin{aligned}
\Pr_{\mathcal{L}_{n^*(\mathcal{T})}}[Y \leq z] &= \sum \left\{ \Pr_{n^*(\mathcal{T})}(\rho) \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} X_n(\rho) \leq z \right\} \\
&= \sum \{ \mathrm{Vol}(C^\bullet_\rho) \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} X_n(\rho) \leq z \} \\
&= \mathrm{Vol}\left(\bigcup \{ C^\bullet_\rho \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} X_n(\rho) \leq z \} \right).
\end{aligned} \tag{2.3.5}$$

Now we are going to define a cube similar to $C^\bullet_\rho$ but, in order to be able to compare with a binomial distribution, we are going to define them with constant probability given by p: for any $\rho \in \mathcal{L}_{n^*}(\mathcal{T})$, define

$$C_\rho \coloneqq \prod_{n<n^*} I_p^{\rho(n)}.$$

In an analogous way for $C^\bullet_\rho$, we have that:

$$\bigcup \left\{ C_\rho \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} X_n(\rho) \leq z \right\} = \{ x \in [0,1]^{n^*} \colon |\{ n < n^* \colon x_n \leq p \}| \leq z \}. \tag{2.3.6}$$

Also,

$$\Pr_{\Omega_{n^*}}[\mathrm{B}_{n^*,p} \leq z] = \mathrm{Vol}\left(\bigcup \left\{ C_\rho \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} \rho(n) \leq z \right\} \right). \tag{2.3.7}$$

Since for any $\rho \in \mathcal{T}$, $p \leq p_\rho$, we have that $\{ n < n^* \colon n_n \leq p \} \subseteq \{ n < n^* \colon x_n \leq p_{\eta_x \restriction n} \}$ and therefore,

$$\{ x \in [0,1]^{n^*} \colon |\{ n < n^* \colon n_n \leq p_{\eta_x \restriction n} \}| \leq z \} \subseteq \{ x \in [0,1]^{n^*} \colon |\{ n < n^* \colon x_n \leq p \}| \leq z \} \tag{2.3.8}$$

Finally, by Equations 2.3.5, 2.3.4, 2.3.8, 2.3.6 and 2.3.7, in this order, we have get:

$$\begin{aligned}
\Pr_{\mathcal{L}_{n^*(\mathcal{T})}}[Y \leq z] &= \mathrm{Vol}\left(\bigcup \left\{ C^\bullet_\rho \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} X_n(\rho) \leq z \right\} \right) \\
&= \mathrm{Vol}(\{ x \in [0,1]^{n^*} \colon |\{ n < n^* \colon x_n \leq p_{\eta_x \restriction n} \}| \leq z \}) \\
&\leq \mathrm{Vol}(\{ x \in [0,1]^{n^*} \colon |\{ n < n^* \colon x_n \leq p \}| \leq z \}) \\
&= \mathrm{Vol}\left(\bigcup \left\{ C_\rho \colon \rho \in \mathcal{L}_{n^*}(\mathcal{T}),\ \sum_{n<n^*} \rho(n) \leq z \right\} \right) \\
&= \Pr_{\Omega_n}[\mathrm{B}_{n^*,p} \leq z].
\end{aligned}$$

$\square_{\text{Theorem 2.3.10}}$

CHAPTER 3

Finitely additive measures

We need to "let the partial randoms whisper secrets to one another", in other words, to pass information between in some way. This is done by finitely additive measures.
Saharon Shelah[1]

As mentioned before, the main goal of this thesis is to generalize the iterated forcing method using finitely additive measures, that *Saharon Shelah* developed in [She00], to prove the consistency of $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0$. Before we go into the details of the generalization, we must make a detailed study of finitely additive measures. These types of functions have been extensively studied in the context of measure theory (see, for example, [BRBR83]). However, our interest in them is focused particularly on Boolean algebras, without the need of topological or analytic structures.

In this chapter, we are dedicated to studying in detail the finitely additive measures on Boolean algebras. We present its definition as a natural way of weakening the notion of measure and studying some of its basic properties. We show that natural examples of finitely additive measures come from filters and ultrafilters and, furthermore, that there is a close connection between ultrafilters and finitely additive $\{0,1\}$-valued measures. Then we show that the set of finitely additive measures is compact and that this fact will allow us to establish extension criteria. We end the chapter by developing a detailed theory of integration with respect to finitely additive measures in Boolean algebras and showing that the integral behaves a bit similar to the integral with respect to abstract measures[2].

[1] See [She00, pg. 114].
[2] However, limit theorems are harder.

3.1 Weak measures

Without resorting to the notion of σ-algebra, we can generalize the idea of "measure" on Boolean algebras: a measure on a Boolean algebra $\mathscr{B}$ is a function $\mathbf{m}\colon \mathscr{B} \to [0, \infty]$ such that $\mathbf{m}(0_{\mathscr{B}}) = 0$ and, if $\{b_n\colon n < \omega\} \subseteq \mathscr{B}$ is such that $\bigvee_{n<\omega} b_n \in \mathscr{B}$, then

$$\mathbf{m}\left(\bigvee_{n<\omega} b_n\right) = \sum_{n<\omega} \mathbf{m}(b_n),$$

whenever for any $i, j < \omega$, if $n \neq j$, then $b_i \wedge b_j = 0_{\mathscr{B}}$. If we weaken this condition and enforce it only for finite sets, we get finitely additive measures:

Definition 3.1.1. Let $\mathscr{B}$ be a Boolean algebra. A *finitely additive measure on* $\mathscr{B}$ is a function $\Xi\colon \mathscr{B} \to [0, \infty]$ satisfying:

1. $\Xi(0_{\mathscr{B}}) = 0$,
2. $\Xi(a \vee b) = \Xi(a) + \Xi(b)$ whenever $a, b \in \mathscr{B}$ and $a \wedge b = 0_{\mathscr{B}}$.

We say that $b \in \mathscr{B}$ has Ξ-measure δ if $\Xi(b) = \delta$.

In general, we exclude the *trivial finitely additive measure*, that is, when talking about finitely additive measures, we will always assume $\Xi(1_{\mathscr{B}}) > 0$. Also, we will occasionally use the acronym "fam" or "FAM" to refer to finitely additive measures.

There are several types of finitely additive measures. In the following definition, we introduce those that will be most relevant to us:

Definition 3.1.2. Let $\mathscr{B}$ be a Boolean algebra and Ξ a finitely additive measure on $\mathscr{B}$. Then,

1. We say that Ξ is *finite*, if $\Xi(1_{\mathscr{B}}) < \infty$.
2. When $\Xi(1_{\mathscr{B}}) = 1$ we say that Ξ is *a probability finitely additive measure*.
3. If $\Xi(b) > 0$ for any $b \in \mathscr{B}^+$, we say that Ξ is *strictly positive*.

When we work with Boolean subalgebras of $\mathcal{P}(X)$ for some set X, it will be very useful to consider finitely additive measures that measure finite sets as zero. For this reason, we introduce the following notion:

Definition 3.1.3. If X is a non-empty set, $\mathscr{B}$ is a Boolean sub-algebra of $\mathcal{P}(X)$ and Ξ is a finitely additive measure on $\mathscr{B}$, we say that Ξ is a *free finitely additive measure* if, for any $x \in X$, $\{x\} \in \mathscr{B}$ and $\Xi(\{x\}) = 0$.

Notice that this implies that $[X]^{<\aleph_0} \subseteq \mathscr{B}$ and, effectively, $\Xi(F) = 0$ for any finite $F \subseteq X$.

We adopt the name "free finitely additive measure" in connection with "free filter" (see Definition 1.4.17).

Definition 3.1.4. Let $\mathscr{B}$ be a Boolean algebra and Ξ a finitely additive measure on $\mathscr{B}$. We say that $b \in \mathscr{B}$ is a Ξ-null set if $\Xi(b) = 0$. Also, we denote by $\mathcal{N}_\Xi$ the collection of Ξ-null sets.

Next, we show some elementary properties of finitely additive measures that will be useful throughout the text. We start with the *monotonicity*:

Lemma 3.1.5. *Let $\mathscr{B}$ a Boolean algebra and Ξ a finitely additive measure on $\mathscr{B}$. If $a, b \in \mathscr{B}$ and $a \leq_{\mathscr{B}} b$, then $\Xi(a) \leq \Xi(b)$.*

Proof. Assume that $a \leq_{\mathscr{B}} b$. So it is clear that $b = a \vee (b \wedge \sim a)$ and $a \wedge (b \wedge \sim a) = 0_{\mathscr{B}}$, hence $\Xi(a) \leq \Xi(a) + \Xi(b \wedge \sim a) = \Xi(b)$. Thus $\Xi(a) \leq \Xi(b)$. $\square_{\text{Lemma 3.1.5}}$

Lemma 3.1.6. *Let Ξ be a finitely additive measure on a Boolean algebra $\mathscr{B}$ and $a, b \in \mathscr{B}$. Then, $\Xi(a \vee b) + \Xi(a \wedge b) = \Xi(a) + \Xi(b)$, for any $a, b \in \mathscr{B}$.*

Proof. Let $a, b \in \mathscr{B}$. On the one hand, since $a = (a \wedge b) \vee (a \wedge \sim b)$ and $(a \wedge b) \wedge (a \wedge \sim b) = 0_{\mathscr{B}}$, we get

$$\Xi(a) = \Xi(a \wedge b) + \Xi(a \wedge \sim b) \tag{3.1.1}$$

On the other hand, as $b = (\sim a \wedge b) \vee (b \wedge a)$ and $(\sim a \wedge b) \wedge (b \wedge a) = 0_{\mathscr{B}}$, we get

$$\Xi(b) = \Xi(\sim a \wedge b) + \Xi(b \wedge a). \tag{3.1.2}$$

Finally, from Equation 3.1.1 and Equation 3.1.2, we get

$$\begin{aligned}
\Xi(a) + \Xi(b) &= \Xi(a \wedge b) + \Xi(a \wedge \sim b) + \Xi(\sim a \wedge b) + \Xi(a \wedge b) \\
&= \Xi((a \wedge b) \vee (a \wedge \sim b) \vee (\sim a \wedge b)) + \Xi(a \wedge b) \\
&= \Xi(a \vee b) + \Xi(a \wedge b).
\end{aligned}$$

$\square_{\text{Lemma 3.1.6}}$

Corollary 3.1.7. *Let $\mathscr{B}$ be a Boolean algebra, Ξ a finitely additive measure on $\mathscr{B}$ and $\langle b_i \colon i < n\rangle \subseteq \mathscr{B}$. Then, $\Xi\left(\bigvee_{i<n} b_n\right) \leq \sum_{i<n} \Xi(b_i)$. The equality holds whenever, for any $i, j < n$, if $i \neq j$, then $b_i \wedge b_j = 0_{\mathscr{B}}$.*

Proof. By an inductive argument, it is enough to prove it only for b_0, b_1. By Lemma 3.1.6, we have that $\Xi(b_0 \vee b_1) \leq \Xi(b_0 \vee b_1) + \Xi(b_0 \wedge b_1) = \Xi(b_0) + \Xi(b_1)$ and the equality holds when $b_0 \wedge b_1 = 0_{\mathscr{B}}$. $\square_{\text{Corollary 3.1.7}}$

By applying Lemma 3.1.6, we get:

Corollary 3.1.8. *Let $\mathscr{B}$ be a Boolean algebra, Ξ a finitely additive measure on $\mathscr{B}$, and $b \in \mathscr{B}$. Then $\Xi(1_{\mathscr{B}}) = \Xi(b) + \Xi(\sim b)$. As a consequence, if Ξ is of probability, then $1 = \Xi(b) + \Xi(\sim b)$.*

Next, we present an example of a probability finitely additive measure that will appear in several places throughout this work:

Example 3.1.9. Let X be a non-empty set and fix a finite non-empty set $u \in \mathcal{P}(X)$. We define $\Xi^u \colon \mathcal{P}(X) \to [0, 1]$ such that, for any $x \in \mathcal{P}(X)$, $\Xi^u(x) := \frac{|x \cap u|}{|u|}$ and we call it the *proportion finitely additive measure*. It is clear that it is a probability finitely additive measure.

Notice that, in general, Ξ^u is not strictly positive. To guarantee the existence of more interesting finitely additive measures, we must require that the Boolean algebra have more structure, for example:

Theorem 3.1.10. *Every σ-centered Boolean algebra admits a strictly positive probability finitely additive measure.*

Proof. Let $\mathscr{B}$ be a σ-centered Boolean algebra, so there exists a countable family $\{F_n : n < \omega\}$ of ultra-filters on $\mathscr{B}$ such that $\mathscr{B}^+ = \bigcup_{n<\omega} F_n$. For any $b \in \mathscr{B}$, we define $\omega_b := \{n < \omega : b \in F_n\}$. Now, we set $\Xi : \mathscr{B} \to [0,1]$ such that

$$\Xi(b) := \sum_{n \in \omega_b} \frac{1}{2^{n+1}}.$$

Let $a, b \in \mathscr{B}^+$ such that $a \wedge b = 0_{\mathscr{B}}$. It is clear that $\omega_a \cap \omega_b = \emptyset$ and $\omega_a \cup \omega_b \subseteq \omega_{a\vee b}$. Conversely, let $m \in \omega_{a\vee b}$, so $a \vee b \in F_m$. If $m \notin \omega_a$ and $m \notin \omega_b$, then $\sim a \in F_m$ and $\sim b \notin F_m$, and therefore, $\sim (a \vee b) \in F_m$, which is a contradiction because $a \vee b \in F_m$. Thus, $\omega_a \cup \omega_b = \omega_{a\vee b}$ and clearly $\omega_a \cap \omega_b = \emptyset$, hence we can calculate:

$$\Xi(a \vee b) = \sum_{n \in \omega_{a\vee b}} \frac{1}{2^{n+1}} = \sum_{n \in \omega_a} \frac{1}{2^{n+1}} + \sum_{n \in \omega_b} \frac{1}{2^{n+1}} = \Xi(a) + \Xi(b).$$

Finally, it is clear that Ξ is strictly positive and, since $1_{\mathscr{B}} \in F_n$ for all $n < \omega$, $\omega_{1_{\mathscr{B}}} = \omega$, hence $\Xi(1_{\mathscr{B}}) = 1$, that is, Ξ is a probability finitely additive measure. $\square_{\text{Theorem 3.1.10}}$

In general, to prove the existence of interesting finitely additive measures, we require the axiom of choice, that is, non-constructivist methods (see [Lau10]). In the next section, we will see that there is a very close relationship between finitely additive $\{0,1\}$-valued measures, and ultrafilters.

3.2 Connections with ultrafilters

In this section, we are going to study the connection that exists between the finitely additive measures $\{0,1\}$-valued and the ultrafilters on a Boolean algebra. This connection will be useful fundamentally for two reasons: on the one hand, natural examples of probability finitely additive measures come from filters and ultrafilters. On the other hand, there are methods of iterated forcing based on ultrafilters (see, for example, [BCM21]). In Chapter 4, we are going to define a method of iterations using finitely additive measures, so the connection between ultrafilters will allow us to establish relationships between these forcing methods.

We are going to start by showing that every filter naturally induces a finitely additive measure:

Lemma 3.2.1. *Let $\mathscr{B}$ be a Boolean algebra and F a filter on $\mathscr{B}$. Then $\Xi_F : \langle F \rangle \to \{0,1\}$ such that, for any $b \in \langle F \rangle$,*

$$\Xi_F(b) = \begin{cases} 1 & \text{if} \quad b \in F, \\ 0 & \text{if} \quad b \in F^{\sim}, \end{cases}$$

is a probability finitely additive measure. Furthermore if G is another filter on $\mathscr{B}$, then

$$F \subseteq G \Leftrightarrow \Xi_F \leq \Xi_G.$$

Proof. Notice that Ξ_F is well-defined because, by Theorem 1.4.16, $\langle F \rangle = F \cup F^{\sim}$ and those sets are disjoint. Also, since F is a filter, by Lemma 1.4.15, $F^{\sim}$ is an ideal, so $0_{\mathscr{B}} \in F^{\sim}$, hence $\Xi_F(0_{\mathscr{B}}) = 0$. To show that Ξ_F is a finitely additive measure, let $a, b \in \langle F \rangle$ such that $a \wedge b = 0_{\mathscr{B}}$. If $a \in F$, then $b \in F^{\sim}$, hence $\Xi_F(a \vee b) = 1 = 1 + 0 = \Xi_F(a) + \Xi_F(b)$. The case $a \in F^{\sim}$ and $b \in F$ is analogous. If $a, b \in F^{\sim}$, then $a \vee b \in F^{\sim}$, that is, $\Xi_F(a \vee b) = 0 = \Xi_F(a) + \Xi_F(b)$. Thus, Ξ_F is a finitely additive measure on $\langle F \rangle$.

Now, assume that $F \subseteq G$ and let $b \in \langle F \rangle$. On the one hand, if $b \in F$, then $b \in G$ and therefore, $\Xi_F(b) = 1 = \Xi_G(b)$. On the other hand, if $b \notin F$, then $\Xi_F(b) = 0 \leq \Xi_G(b)$. Thus, in any case $\Xi_F(b) \leq \Xi_G(b)$. Conversely, assume that $\Xi_F \leq \Xi_G$ and let $b \in F$, hence $1 = \Xi_F(b) \leq \Xi_G(b)$, therefore $\Xi_G(b) = 1$, that is, $b \in G$. Thus, $F \subseteq G$. $\square_{\text{Lemma 3.2.1}}$

If we choose a suitable ultrafilter, we can use Lemma 3.2.1 to construct an interesting example of a finitely additive measure that is not a measure:

Example 3.2.2. Let $\mathscr{B}$ be a Boolean sub-algebra of $\mathcal{P}(X)$, where X is a countable set, and let $F \subseteq \mathscr{B}$ be an ultrafilter on $\mathscr{B}$. If F is a free filter on $\mathscr{B}$, then Ξ_F is a finitely additive measure on $\langle F \rangle$ that is not a measure on $\langle F \rangle$. Indeed, assume that F is a free filter on $\mathscr{B}$. We know already that Ξ_F is a finitely additive measure on $\mathscr{B}$ by Theorem 1.4.16 and Lemma 3.2.1. Now, since X is countable, we can write $X = \{x_n \colon n < \omega\}$. For any $n < \omega$, define $B_n := \{x_n\}$. Notice that B_n^{c} is co-finite, and therefore, $B_n^{\mathrm{c}} \in F$, that is, $B_n \in F^{\sim}$. Thereby, $\Xi_F(B_n) = 0$ for any $n < \omega$. However, $X \in F$ because it is co-finite, hence $\Xi_F(X) = 1$. Thus,

$$\Xi_F\left(\bigcup_{n<\omega} B_n\right) = \Xi_F(X) = 1 \neq 0 = \sum_{n<\omega} \Xi_F(B_n).$$

Finally, Ξ_F is not a measure on $\langle F \rangle$.

Conversely to Lemma 3.2.1, probability finitely additive measures also induce filters:

Lemma 3.2.3. *Let $\mathscr{B}$ be a Boolean algebra, $\mathscr{C}$ be a Boolean sub-algebra of $\mathscr{B}$, and $\Xi \colon \mathscr{C} \to \{0, 1\}$ be a probability finitely additive measure. Then, $F_\Xi := \{c \in \mathscr{C} \colon \Xi(c) = 1\}$ is a filter on $\mathscr{C}$, $\langle F_\Xi \rangle = \mathscr{C}$ and $\Xi_{F_\Xi} = \Xi$, where Ξ_{F_Ξ} is as in Lemma 3.2.1 for $\mathscr{C}$. Furthermore, if $\mathscr{B} = \mathscr{C}$, then F_Ξ is an ultra-filter on $\mathscr{B}$.*

Proof. Let $c, d \in F_\Xi$, hence $\Xi(c) = \Xi(d) = 1$. Then,

1. By Lemma 3.1.6, we have that $\Xi(c \wedge d) = \Xi(c) + \Xi(d) - \Xi(c \vee d) = 2 - \Xi(c \vee d)$. If $\Xi(c \vee d) = 0$, then $\Xi(c \wedge d) = 2$, which is not possible, so $\Xi(c \vee d) = 1$ and therefore $\Xi(c \wedge d) = 1$. Thus, $c \wedge d \in F_\Xi$.

2. If $a \in \mathscr{C}$ and $c \leq a$ then, by *monotonicity* of Ξ, $\Xi(a) \geq \Xi(c) = 1$, hence $\Xi(a) = 1$. Thus, $a \in F_\Xi$.

3. Since by definition of finitely additive measure, $\Xi(0_{\mathscr{B}}) = 0$, it follows that $0_{\mathscr{B}} \notin F_\Xi$.

Thus, F_Ξ is a filter on $\mathscr{C}$.

To prove that $\langle F_\Xi \rangle = \mathscr{C}$, notice that, by Corollary 3.1.8, if $c \in \mathscr{C}$ and $\Xi(c) = 0$ then, as Ξ is a probability finitely additive measure, $\Xi({\sim} c) = \Xi(1_{\mathscr{C}}) - \Xi(c) = 1 + 0 = 1$, hence ${\sim} c \in F_\Xi$, that is, $\Xi({\sim} c) = 1$, that is, $c \in F_\Xi^{\sim}$. Thus, $\mathscr{C} = F_\Xi \cup F_\Xi^{\sim} = \langle F_\Xi \rangle$.

Now, consider the finitely additive measure Ξ_{F_Ξ} defined in Lemma 3.2.1. Since $\mathscr{C} = \langle F_\Xi \rangle$, Ξ and Ξ_{F_Ξ} have the same domain. For $c \in \mathscr{C}$, we have that $\Xi_{F_\Xi}(c) = 1 \Leftrightarrow c \in F_\Xi \Leftrightarrow \Xi(c) = 1$. Thus, $\Xi_{F_\Xi} = \Xi$.

Finally, if $\mathscr{B} = \mathscr{C}$, it is clear that F_Ξ is a ultra-filter because for any $b \in \mathscr{B}$, either $\Xi(b) = 1$ or $\Xi(b) = 0$. In the first case, $b \in F_\Xi$ and in the second one, $\Xi(\sim b) = 1$, that is, $\sim b \in F_\Xi$. $\square_{\text{Lemma 3.2.3}}$

As a consequence, if we close F_Ξ upwards in $\mathscr{B}$, we get:

Corollary 3.2.4. *Let $\mathscr{B}$ be a Boolean algebra and $\mathscr{C}$ a Boolean sub-algebra of $\mathscr{B}$. Then every probability finitely additive measure $\Xi\colon \mathscr{C} \to \{0,1\}$ induces a filter on $\mathscr{B}$, namely, $F_\Xi^{\uparrow}$.*

If we apply Lemma 3.2.3 to ultra-filters we obtain a bijection between finitely additive measures and ultrafilters, that is, the desired connection:

Theorem 3.2.5. *There is a bijection between the class of ultra-filters on $\mathscr{B}$ and the probability finitely additive measures from $\mathscr{B}$ onto $\{0,1\}$. Furthermore, if the filters are ordered by inclusion and the finitely additive measures by the usual function order, then it is an order-isomorphism.*

Proof. For any ultra-filter F on $\mathscr{B}$, define $h(F) \coloneqq \Xi_F$. By Lemma 3.2.1, Ξ_F is a probability finitely additive measure with domain $\mathscr{B}$. Notice that if F, G are ultra-filters on $\mathscr{B}$ and $\Xi_F = \Xi_G$, then

$$x \in F \Leftrightarrow \Xi_F(x) = 1 \Leftrightarrow \Xi_G(x) = 1 \Leftrightarrow x \in G,$$

that is, $F = G$ and therefore, h is one-to-one. On the other hand, if Ξ is a probability finitely additive measure on $\mathscr{B}$, then by Lemma 3.2.3, we have that F_Ξ is ultra-filter on $\mathscr{B}$ and $h(F_\Xi) = \Xi_{F_\Xi} = \Xi$, that is, h is surjective. Finally, by Lemma 3.2.1, $F \subseteq G$ if, and only if, $h(F) \leq h(G)$. Thus h is an order-isomorphism. $\square_{\text{Theorem 3.2.5}}$

As a consequence, ultra-filters are particular cases of finitely additive measures.

3.3 Compactness: the main element for extension criteria

In the first section of [She00], *Saharon Shelah* states several criteria for the extension of finitely additive measures, which will be used throughout the article and will be essential to build the iteration in that paper. Most of these criteria appear without proof or simply as “straightforward”, however, trying to prove these criteria can be difficult if not properly approached. In this section, we are going to prove that the set of finitely additive measures is compact and, in the next section, we are going to show that this is the central feature to be able to establish the desired finitely additive measure extension criteria.

Theorem 3.3.1. *Let $\mathscr{B}$ be a Boolean algebra, $\delta \in [0,\infty]$ and $Z := {}^{\mathscr{B}}[0,\delta]$ with the product topology*[3]*, which is compact*[4]*. Then, the set of finitely additive measures on $\mathscr{B}$ with $\Xi(1_{\mathscr{B}}) \leq \delta$ is closed in Z. As a consequence, it is compact in Z.*

Proof. Notice that the set $\{z \in Z\colon z(0_{\mathscr{B}}) = 0\}$ is closed in Z because we can write it as the following product of closed sets:

$$\{0\} \times \left(\prod_{a \in \mathscr{B} \setminus \{0_{\mathscr{B}},\}} [0,\delta] \right).$$

So, it remains to show that $\bigcap\{F_{a,b}\colon a,b \in \mathscr{B},\ a \wedge b = 0_{\mathscr{B}}\}$ is closed where, for any $a, b \in \mathscr{B}$ with $a \wedge b = 0_{\mathscr{B}}$, $F_{a,b} := \{z \in Z\colon\ z(a \vee b) = z(a) + z(b)\}$. Let $a, b \in \mathscr{B}$ such that $a \wedge b = 0_{\mathscr{B}}$. It is enough to show that $F := F_{a,b}$ is closed. For this, suppose that $z \in \mathrm{cl}(F)$ and consider two possible cases:

1. $z(a \vee b) < \infty$. In this case we must have $z(a), z(b) < \infty$. If this is not true, without loss of generality we can assume that $z(a) = \infty$. Consider the open set $U' := \prod_{x \in \mathscr{B}} U'_x$ where:

   - $U'_{a \vee b} := \left(z(a \vee b) - \frac{1}{2},\ z(a \vee b) + \frac{1}{2}\right) \cap [0,\infty)$,
   - $U'_a := \left(z(a \vee b) + \frac{1}{2}, \infty\right]$,
   - $U'_x := [0,\infty]$, if $x \in \mathscr{B} \setminus \{a, a \vee b\}$.

   Since U' is an open neighborhood of z, we can find $z' \in U' \cap F_{a,b}$. So, by the definition of U', we have that

   $$z'(a \vee b) = z'(a) + z'(b),\ |z'(a \vee b) - z(a \vee b)| < \frac{1}{2} \text{ and } z'(a) > z(a \vee b) + \frac{1}{2}.$$

   However, $z'(a) \leq z'(a) + z'(b) = z'(a \vee b) < z(a \vee b) + \frac{1}{2}$, which is a contradiction. Thus, $z(a), z(b) < \infty$.

   Now, let $\varepsilon > 0$ and consider the open neighborhood of z defined by $U := \prod_{x \in \mathscr{B}} U_x$, where:

   - $U_x := (z(x) - \varepsilon, z(x) + \varepsilon) \cap [0,\delta]$ when $x \in \{a, b, a \vee b\}$,
   - $U_x := [0,\delta]$ for any other $x \in \mathscr{B} \setminus \{a, b, a \vee b\}$.

   So there is some $z'' \in U \cap F$. Therefore $|z(x) - z''(x)| < \varepsilon$ for $x \in \{a, b, a \vee b\}$ and $z''(a \vee b) = z''(a) + z''(b)$. Hence

   $$|z(a \vee b) - z(a) - z(b)| = |z(a \vee b) - z''(a \vee b) + z''(a) - z(a) + z''(b) - z(b)| < 3\varepsilon.$$

   Since ε is arbitrary, $z(a \vee b) - z(a) - z(b) = 0$, so $z \in F$.

[3] Where $[0,\delta]$ has the usual topology inherited from $[-\infty, \infty]$.
[4] By *Tychonoff's theorem*.

2. $z(a \vee b) = \infty$. In this case we must have that either $z(a) = \infty$ or $z(b) = \infty$. Assume not, so $z(a), z(b) < \infty$ and consider $U' \coloneqq \prod_{x \in \mathscr{B}} U'_x$ where,

   - $U'_a \coloneqq \left(z(a) - \frac{1}{2}, z(a) + \frac{1}{2}\right) \cap [0, \infty)$,
   - $U'_b \coloneqq \left(z(b) - \frac{1}{2}, z(b) + \frac{1}{2}\right) \cap [0, \infty)$,
   - $U'_{a \vee b} \coloneqq (z(a) + z(b) + 1, \infty]$,
   - $U'_x \coloneqq [0, \infty]$, when $x \in \mathscr{B} \setminus \{a, b, a \vee b\}$.

   It is clear that U' is an open neighborhood of z and, therefore, we can find some $z' \in F \cap U'$, so $|z'(x) - z(x)| < \frac{1}{2}$ for $x \in \{a, b\}$ and $z'(a) + z'(b) = z'(a \vee b) > z(a) + z(b) + 1$. However, $z'(a) + z'(b) < z(a) + z(b) + 1$, which is a contradiction.

   Therefore $z(a) + z(b) = \infty = z(a \vee b)$, so $z \in F$.

Thus, F is closed. $\square_{\text{Theorem 3.3.1}}$

In a similar way, the set of finitely additive measures with $\Xi(1_{\mathscr{B}}) = \delta$ is closed in Z.

3.4 Compatibility and some extension criteria

As we mentioned in the previous section, in this section we are going to use compactness arguments to set extension criteria. In particular, we are going to state a theorem that is very useful in practice, called the compatibility theorem of finitely additive measures.

In the following result, we translate the compactness of finitely additive measures proved in Theorem 3.3.1, to a property of finite subsets:

Theorem 3.4.1. *Let $\mathscr{B}$ be a Boolean algebra, $0 \leq \delta \leq \infty$, K a closed subset of $[0, \delta]$, $\langle b_i \colon i \in I \rangle \subseteq \mathscr{B}$ and let $\langle K_i \colon i \in I \rangle$ be a collection of closed subsets of K. Then the following statements are equivalent.*

1. *There is a finitely additive measure Ξ on the sub-algebra generated by $\{b_i \colon i \in I\}$ such that $\operatorname{ran}(\Xi) \subseteq K$ and $\Xi(b_i) \in K_i$ for all $i \in I$.*

2. *For any finite $J \subseteq I$ and any collection $\langle G_i : i \in J \rangle$ of open subsets of K such that $K_i \subseteq G_i$ for all $i \in J$, there is some finitely additive measure Ξ on the Boolean algebra generated by $\{b_i \colon i \in J\}$ such that $\operatorname{ran}(\Xi) \subseteq K$ and $\Xi(b_i) \in G_i$ for all $i \in J$.*

Proof. The implication (1) ⇒ (2) is immediate. To prove the converse, let $\mathscr{B}'$ be the Boolean subalgebra generated by $\{b_i \colon i \in I\}$. First, fix $J \subseteq I$ finite. Since each K_i is closed, for each $i \in J$ we can find a decreasing sequence $\langle F^i_k \colon k < \omega \rangle$ of closed subsets of K such that there is an open subset G^i_k of K such that $K_i \subseteq G^i_k \subseteq F^i_k$, and $\bigcap_{k<\omega} F^i_k = K_i$. Let $\mathscr{B}_J$ be the subalgebra generated by $\{b_i : i \in J\}$. For any $k < \omega$ consider

$$C_{J,k} \coloneqq \{z \in {}^{\mathscr{B}'}K \colon z \restriction \mathscr{B}_J \text{ is a fam and } \forall i \in J(z(b_i) \in F^i_k)\}.$$

By Theorem 3.3.1, $C_{J,k}$ is a compact set and, by (2), it is a non-empty set. Moreover, since $\langle C_{J,k}\colon k<\omega\rangle$ is decreasing, it has the *finite intersection property*, so

$$C_J \coloneqq \bigcap_{k<\omega} C_{J,k} = \{z \in {}^{\mathscr{B}'}K\colon z \restriction \mathscr{B}_J \text{ is a fam and } \forall i \in J(z(b_i) \in K_i)\}$$

is a non-empty set. Also, $J \subseteq J'$ implies $C_{J'} \subseteq C_J$, so $\langle C_J\colon J \in [I]^{<\aleph_0}\rangle$ has the finite intersection property, thus it has non-empty intersection. If Ξ is in this non-empty intersection, then it is as required. $\square_{\text{Theorem 3.4.1}}$

Using Theorem 3.4.1, one can prove the *Compatibility Theorem* for finitely additive measures. We decided to omit the proof, but details can be found in [BRBR83, Ch. 3].

Theorem 3.4.2. *Let $\mathscr{B}$ be a Boolean algebra and, for $d \in \{0,1\}$, let $\mathscr{B}_d$ be a Boolean Boolean sub-algebra of $\mathscr{B}$ with a finitely additive measure $\Xi_d\colon \mathscr{B}_d \to [0,\infty)$. Then the following statements are equivalent.*

1. *There is a finitely additive measure Ξ on the Boolean sub-algebra generated by $\mathscr{B}_0 \cup \mathscr{B}_1$ extending Ξ_d for $d \in \{0,1\}$.*
2. *$\Xi_0(1_{\mathscr{B}}) = \Xi_1(1_{\mathscr{B}})$ and, for any $a \in \mathscr{B}_0$ and $a' \in \mathscr{B}_1$, if $a \leq a'$ then $\Xi_0(a) \leq \Xi_1(a')$.*
3. *For any $d, d' \in \{0,1\}$, $a \in \mathscr{B}_d$ and $a' \in \mathscr{B}_{d'}$, if $a \leq a'$ then $\Xi_d(a) \leq \Xi_{d'}(a')$.*

There are two implications that do not require too much work: the implication (1) ⇒ (2) is immediate and, to prove (2) ⇒ (3), note that (3) is clear whenever $d = d'$ or $d = 0$ and $d' = 1$, so we need to prove it when $d = 1$ and $d' = 0$. Denote $\delta \coloneqq \Xi_0(1_{\mathbb{B}}) = \Xi_1(1_{\mathbb{B}})$. If $a \in \mathbb{B}_1$, $a' \in \mathbb{B}_0$ and $a \leq a'$, then ${\sim}a' \leq {\sim}a$, so by (2) we obtain $\Xi_0({\sim}a') \leq \Xi_1({\sim}a)$, that is, $\delta - \Xi_0(a') \leq \delta - \Xi_1(a)$, so $\Xi_1(a) \leq \Xi_0(a')$, which proofs (3). However, (3) ⇒ (1) requires notions of linear algebra and tools of functional analysis[5], and is a fairly extensive proof.

Corollary 3.4.3. *Let $\mathscr{B}$ be a Boolean algebra, $\mathscr{C} \subseteq \mathscr{B}$ a sub-Boolean with a finitely additive measure $\Xi\colon \mathscr{C} \to [0,\infty)$, and let $b \in \mathscr{B}$. If $z \in [0,\infty)$ is between $\sup\{\Xi(a)\colon a \leq b,\ a \in \mathscr{C}\}$ and $\inf\{\Xi(a)\colon b \leq a,\ a \in \mathscr{C}\}$, then there is a finitely additive measure Ξ' on the Boolean algebra generated by $\mathscr{C} \cup \{b\}$, extending Ξ, such that $\Xi(b) = z$.*

***Proof*.** Note that $\{b\}$ generates the Boolean sub-algebra $\mathscr{C}' \coloneqq \{0_{\mathbb{B}}, b, {\sim}b, 1_{\mathbb{B}}\}$. By the hypothesis, $z \leq \Xi(1_{\mathscr{B}})$, so we can define the finitely additive measure $\Xi'\colon \mathscr{C}' \to [0,\infty)$ such that $\Xi'(b) \coloneqq z$ and $\Xi'(1_{\mathscr{B}}) \coloneqq \Xi(1_{\mathscr{B}})$. The result follows by Theorem 3.4.2(2) and the hypothesis on z. $\square_{\text{Corollary 3.4.3}}$

The following is one of the fundamental extension results for the development of this work:

Theorem 3.4.4. *Let Ξ_0 be a finitely additive measure on a Boolean sub-algebra of $\mathscr{B}$ and $\langle b_i\colon i \in I\rangle \subseteq \mathscr{B}$. Assume that $0 < \delta \coloneqq \Xi_0(1_{\mathscr{B}}) < \infty$ and, for every finite $J \subseteq I$ and $b \in \operatorname{dom}(\Xi_0)$, if $\Xi_0(b) > 0$ then $b \wedge \bigwedge_{i\in J} b_i \neq 0_{\mathscr{B}}$. Then, there exists a finitely additive measure Ξ on $\mathscr{B}$ extending Ξ_0 such that $\Xi(b_i) = \delta$ for every $i \in I$.*

[5] The connection with linear algebra is given because there is a bijection between finite finitely additive measures on $\mathcal{P}(N)$ and real-valued positive linear maps on $\mathbb{Q}^N$, for any $0 < N < \omega$.

***Proof*.** By the hypothesis, and using that $\Xi_0(1_{\mathscr{B}}) > 0$, we have that $\{b_i : i \in I\}$ generates a filter F on $\mathscr{B}$. Let $\mathscr{B}_0 \coloneqq \mathrm{dom}(\Xi_0)$, $\mathscr{B}_1$ be the Boolean sub-algebra generated by F and define $\Xi_1 \coloneqq \delta\,\Xi_F$, where Ξ_F is as in Lemma 3.2.1. So, $\Xi_1 : \mathscr{B}_1 \to \{0, \delta\}$ and, for any $b \in \mathscr{B}_1$, $\Xi_1(b) = \delta \Leftrightarrow b \in F$, since $b \in F \Leftrightarrow \Xi_F(b) = 1 \Leftrightarrow \Xi_1(b) = \delta$.

Now, it is enough to show that Ξ_0 and Ξ_1 satisfy Theorem 3.4.2(2). So let $a \in \mathscr{B}_0$, $b \in \mathscr{B}_1$ and assume $a \leq b$. If $b \in F$ then, by Lemma 3.1.5, $\Xi_0(a) \leq \delta = \Xi_1(b)$; otherwise $\sim b \in F$ and $\Xi_1(b) = 0$, so we must show that $\Xi_0(a) = 0$. If this is not the case and $\Xi_0(a) > 0$, then by hypothesis, we get that $a \wedge \sim b \neq 0_{\mathscr{B}}$, but since $a \leq b$ it follows that b and $\sim b$ are compatible, a contradiction. $\square_{\text{Theorem 3.4.4}}$

3.5 An integration theory with finitely additive measures

To prove that random forcing is σ-FAM-linked (see Theorem 4.2.18 and Definition 4.2.8) and to generalize the method of iterations using finitely additive measures (see Section 4.3), we will need to be able to integrate over Boolean algebras with respect to a finitely additive measure. For instance, in [She00] integrals appears defined in Definition 1.4 as $\mathrm{Av}_\Xi(\bar{a})$, where Ξ is a finitely additive measure on $\mathcal{P}(\omega)$ and $\bar{a} \in {}^{\omega}\mathbb{R}$. In this section, we do a complete development of an integration theory for finitely additive measures over Boolean algebras. This development is analogous to the usual development of the Riemann integral over $\mathbb{R}$ (see, for example, [Ros68, Ch. VI]), adjusting some details about the way we refine the partitions and also allowing to integrate over subsets in general. As a consequence, we will obtain that the Riemann integral will be a particular case of the integral with respect to finitely additive measures. We also define a notion of measurability on Boolean algebras that will be useful for certain integrability criteria. We finish by showing that the integral is absolute for transitive ZFC models. The reader is warned that, although the details in this section are developed for the fun of it, everything we prove is used later as basic properties of the integral.

For this section fix a Boolean subalgebra $\mathscr{B}$ of $\mathcal{P}(X)$ for some non-empty set X, a finitely additive measure $\Xi : \mathscr{B} \to [0, \delta]$, where δ is a non-negative real number, and a bounded function $f : X \to \mathbb{R}$. We start defining partitions and its refinements:

Definition 3.5.1. We define:

1. A *partition* of X is a sequence $\langle X_i : i \in I \rangle$ of pairwise disjoint subsets of X whose union is X. When I is finite, we say that the partition is *finite*.
2. $\mathbf{P}^\Xi$ is the set of finite partitions of X into sets in $\mathrm{dom}(\Xi) = \mathscr{B}$.
3. If $P, Q \in \mathbf{P}^\Xi$, we say that Q *is a refinement of* P, denoted by "$Q \ll P$" if every element of P can be finitely partitioned into elements of Q.
4. If $P = \langle P_n : n < n^* \rangle$ and $Q = \langle Q_m : m < m^* \rangle$ are in $\mathbf{P}^\Xi$, we define:

$$P \sqcap Q \coloneqq \bigcup \{P_n \cap Q_m : n < n^* \wedge m < m^*\}.$$

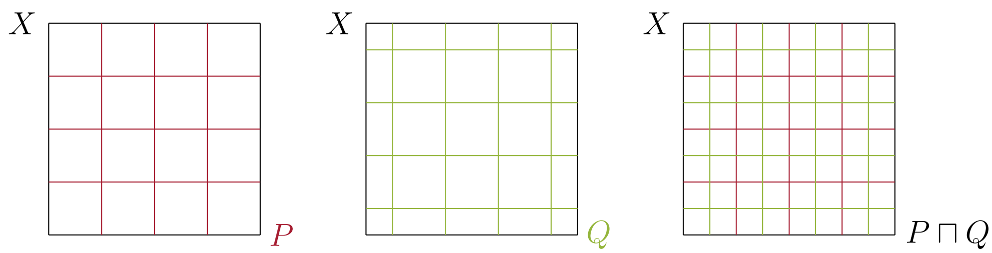

Figure 3.1: A graphic example of $P \sqcap Q$.

For example, it is clear that $\{X\} \in \mathbf{P}^{\Xi}$ and if $P \in \mathbf{P}^{\Xi}$, then $P \ll \{X\}$ and $P \ll P$. Moreover, $\ll$ is a partial order on $\mathbf{P}^{\Xi}$. Also, $P \sqcap Q$ is a common refinement of P and Q:

Lemma 3.5.2. *If $P, Q \in \mathbf{P}^{\Xi}$, then $P \sqcap Q \in \mathbf{P}^{\Xi}$ and $P \sqcap Q \ll P, Q$.*

Proof. Let $P = \langle P_n \colon n < n^* \rangle, Q = \langle Q_m \colon m < m^* \rangle \in \mathbf{P}^{\Xi}$. Notice that $P \sqcap Q \subseteq \mathscr{B}$, because it is defined as finite Boolean combinations. On the other hand, is it clear that $P \sqcap Q$ is a finite partition of X. Now, let $n < n^*$ and $x \in P_n$. Since $x \in X$ and $Q \in \mathbf{P}^{\Xi}$, there is some $m < m^*$ such that $x \in Q_m$, hence $x \in P_n \cap Q_m$, that is, $P_n = \bigcup_{m<m^*}(P_n \cap Q_m)$ and each $P_n \cap Q_m$ is in $P \sqcap Q$. Thus, $P \sqcap Q \ll P$. In an analogous way, it follows that $P \sqcap Q \ll Q$. $\square_{\text{Lemma 3.5.2}}$

Now we can define the integral with respect to Ξ:

Definition 3.5.3. We define:

1. For any $P \in \mathbf{P}^{\Xi}$,
$$\overline{\mathrm{S}}^{\Xi}(f, P) := \sum_{b \in P} \sup(f[b])\Xi(b) \text{ and } \underline{\mathrm{S}}^{\Xi}(f, P) := \sum_{b \in P} \inf(f[b])\Xi(b).$$

2. $\overline{\int_X} f d\Xi := \inf\{\overline{\mathrm{S}}(f, P) \colon P \in \mathbf{P}^{\Xi}\}$ and $\underline{\int_X} f d\Xi := \sup\{\underline{\mathrm{S}}(f, P) \colon P \in \mathbf{P}^{\Xi}\}$.

3. We say that f is Ξ-*integrable*, denoted by $f \in \mathscr{I}(\Xi)$ if, and only if, $\overline{\int_X} f d\Xi = \underline{\int_X} f d\Xi$, in which case this value is denoted by $\int_X f d\Xi$.

Naturally, when the context is clear, we omit the superscript "Ξ" in "$\overline{\mathrm{S}}^{\Xi}(f, P)$" and "$\underline{\mathrm{S}}^{\Xi}(f, P)$".

Notation 3.5.4. Later we will deal with finitely additive measures on $\mathcal{P}(\omega)$. There the functions are sequences of real numbers, and in some cases, they will be defined with respect to several parameters, for example, $\langle a_k^i \colon i, k < \omega \rangle$. So, we must establish a convention to clarify what parameter we are going to integrate with. For this, if Ξ is a finitely additive measure over $\mathcal{P}(\omega)$, we define

$$\int_\omega a_k^i \, d\Xi(k) := \int_\omega f d\Xi,$$

where $f \colon \omega \to \mathbb{R}$ and, for any $k < \omega$, $f(k) := a_k^i$, that is, we consider all parameters other than k as constants.

For example, it is clear that any constant function is Ξ-integrable. Concretely, if for all $x \in X$, $f(x) = c \in \mathbb{R}$, then $\int_X f(x)\, d\Xi = c\Xi(X)$.

Lemma 3.5.5. *Let $f\colon X \to \mathbb{R}$ be a bounded function. If $P, Q \in \mathbf{P}^{\Xi}$ and $Q \ll P$, then:*

$$\underline{\mathrm{S}}(f,P) \leq \underline{\mathrm{S}}(f,Q) \leq \overline{\mathrm{S}}(f,Q) \leq \overline{\mathrm{S}}(f,P).$$

As a consequence, $\overline{\mathrm{S}}(f,Q) - \underline{\mathrm{S}}(f,Q) \leq \overline{\mathrm{S}}(f,P) - \underline{\mathrm{S}}(f,P)$.

Proof. Let $P = \{P_i\colon i < m^*\} \in \mathbf{P}^{\Xi}$. By an inductive argument it is enough to prove the result for $Q \coloneqq \{P_i\colon 0 < i < m^*\} \cup \{A, B\}$, where $A, B \in \mathscr{B}$, $A \cap B = \emptyset$ and $A \cup B = P_0$. Notice that $\underline{\mathrm{S}}(f,Q) \leq \overline{\mathrm{S}}(f,Q)$ is clear by definition. On the one hand, since

$$\inf(f[A \cup B]) \leq \inf(f[A]), \inf(f[B]) \text{ and } \operatorname{ran}(\Xi) \subseteq [0,\delta],$$

we have that:

$$\begin{aligned}
\underline{\mathrm{S}}(f,Q) - \underline{\mathrm{S}}(f,P) &= \sum_{b\in Q} \inf(f[b])\Xi(b) - \sum_{b\in P} \inf(f[b])\Xi(b)\\
&= \inf(f[A])\Xi(A) + \inf(f[B])\Xi(B) - \inf(f[P_0])\Xi(P_0)\\
&= \inf(f[A])\Xi(A) + \inf(f[B])\Xi(B) - \inf(f[A\cup B])\Xi(A\cup B)\\
&= \Xi(A)[\inf(f[A]) - \inf(f[A\cup B])] + \Xi(B)[\inf(f[B]) - \inf(f[A\cup B])]\\
&\geq 0.
\end{aligned}$$

Thus, $\underline{\mathrm{S}}(f,Q) \geq \underline{\mathrm{S}}(f,P)$.
On the other hand, since $\sup(f[A\cup B]) \geq \sup(f[A]), \sup(f[B])$, we get:

$$\begin{aligned}
\overline{\mathrm{S}}(f,P) - \overline{\mathrm{S}}(f,Q) &= \sum_{b\in P} \sup(f[b])\Xi(b) - \sum_{b\in Q} \sup(f[b])\Xi(b)\\
&= \sup(f[P_0])\Xi(P_0) - \sup(f[A])\Xi(A) - \sup(f[B])\Xi(B)\\
&= \Xi(A\cup B)[\sup(f[A\cup B])] - \sup(f[A])\Xi(A) - \sup(f[B])\Xi(B)\\
&= \Xi(A)[\sup(f[A\cup B]) - \sup(f[A])] + \Xi(B)[\sup(f[A\cup B]) - \sup(f[B])]\\
&\geq 0.
\end{aligned}$$

Thus, $\overline{\mathrm{S}}(f,P) \geq \overline{\mathrm{S}}(f,Q)$. $\square_{\text{Lemma 3.5.5}}$

Corollary 3.5.6. *If $P, Q \in \mathbf{P}^{\Xi}$, then $\underline{\mathrm{S}}(f,P) \leq \overline{\mathrm{S}}(f,Q)$.*

Proof. Since by Lemma 3.5.2 $P \sqcap Q \ll P, Q$ we can use the previous result to get

$$\underline{\mathrm{S}}(f,P) \leq \underline{\mathrm{S}}(f,P\cup Q) \leq \overline{\mathrm{S}}(f,P\cup Q) \leq \overline{\mathrm{S}}(f,Q).$$

$\square_{\text{Corollary 3.5.6}}$

Corollary 3.5.7. $\underline{\int_X} f\, d\,\Xi \leq \overline{\int_X} f\, d\,\Xi$.

Now, we prove what we call the *Criterion of Ξ-Integrability*:

Theorem 3.5.8. *f is Ξ-integrable if, an only if, for all $\varepsilon > 0$, there exists a partition $P \in \mathbf{P}^{\Xi}$ such that $\overline{\mathrm{S}}(f,P) - \underline{\mathrm{S}}(f,P) < \varepsilon$.*

Proof. On the one hand, assume that $f \in I(\Xi)$ and let $\varepsilon > 0$. By basic properties of $\sup$ and $\inf$, there are $P, Q \in \mathbf{P}^{\Xi}$ such that:

$$\int_X f d\Xi - \frac{\varepsilon}{2} < \underline{\mathrm{S}}(f,P) \text{ and } \overline{\mathrm{S}}(f,Q) < \int_X f d\Xi + \frac{\varepsilon}{2}.$$

Consider $R \coloneqq P \sqcap Q$. By Lemma 3.5.2, $R \in \mathbf{P}^{\Xi}$ and it is a common refinement of P and Q. So, by virtue of Lemma 3.5.5,

$$\underline{\mathrm{S}}(f,P) \leq \underline{\mathrm{S}}(f,R) \text{ and } \overline{\mathrm{S}}(f,Q) \leq \overline{\mathrm{S}}(f,R).$$

Therefore,

$$\int_X f - \frac{\varepsilon}{2} < \underline{\mathrm{S}}(f,R) \text{ and } \overline{\mathrm{S}}(f,R) < \int_X f d\Xi + \frac{\varepsilon}{2}.$$

Thus, $\overline{\mathrm{S}}(f,R) - \underline{\mathrm{S}}(f,R) < \varepsilon$.

On the other hand, let $P \in \mathbf{P}^{\Xi}$ such that $\overline{\mathrm{S}}(f,P) - \underline{\mathrm{S}}(f,P) < \varepsilon$. Hence, by the definition of $\overline{\int}$ and $\underline{\int}$, we have that:

$$\overline{\int_X} f d\Xi \leq \overline{\mathrm{S}}(f,P) < \underline{\mathrm{S}}(f,P) + \varepsilon \leq \underline{\int_X} f d\Xi + \varepsilon.$$

Since ε is arbitrary, by Corollary 3.5.7 it follows that $f \in \mathscr{I}(\Xi)$. $\square_{\text{Theorem 3.5.8}}$

Next, we generalize the idea of measurability on measure spaces to Boolean algebras:

Definition 3.5.9. A function $h \colon X \to \mathbb{R}$ is *$\mathscr{B}$-measurable* if $\{x \in X : y \leq h(x) < z\} \in \mathscr{B}$ for any $y, z \in \mathbb{R}$.

For example, it is easy to prove that, if $E \in \mathscr{B}$, then χ_E is $\mathscr{B}$-measurable. Also, $\mathscr{B}$-measurability give us a useful condition of Ξ-integrability:

Theorem 3.5.10. *If $h \colon X \to \mathbb{R}$ is a $\mathscr{B}$-measurable and bounded function, then h is Ξ-integrable,*

Proof. Let $\varepsilon > 0$. Since h is bounded, there are $a, b \in \mathbb{Q}$ such that $a < b$ and $f[X] \subseteq [a,b)$. Partition $[a,b)$ into finitely many intervals $\{[a_k,b_k) \colon k < m\}$ with rational endpoints and with length $<\frac{\varepsilon}{\Xi(X)+1}$. Consider $P \coloneqq \{h^{-1}[[a_k,b_k)] : k < m\}$, which is in $\mathbf{P}^{\Xi}$ because h is $\mathscr{B}$-measurable. Hence,

$$\begin{aligned}
\overline{\mathrm{S}}(h,P) - \underline{\mathrm{S}}(h,P) &= \sum_{b \in P} (\sup h[b] - \inf h[b]) \Xi(b) \\
&\leq \sum_{b \in P} \frac{\varepsilon}{\Xi(X)+1} \Xi(b) = \frac{\varepsilon}{\Xi(X)+1} \Xi(X) < \varepsilon.
\end{aligned}$$

Thus, by the criterion of Ξ-integrability, $h \in \mathscr{I}(\Xi)$. $\square_{\text{Theorem 3.5.10}}$

One problem with $\mathscr{B}$-measurability is that in order to make fundamental proofs like h is $\mathscr{B}$-measurable if, and only if, for any $z \in \mathbb{R}$, $\{x \in X \colon f(x) > y\} \in \mathscr{B}$ or that the sum of $\mathscr{B}$-measurables is again $\mathscr{B}$-measurable, it is required that $\mathscr{B}$ is σ-complete. For example, if h_1 and h_2 are $\mathscr{B}$-measurable, then to prove that $h_1 + h_2$ is $\mathscr{B}$-measurable we need

$$\{x \in X \colon h_1(x) + h_2(x) > y\} = \bigcup_{r \in \mathbb{Q}} (\{x \in X \colon h_1(x) > r\} \cap \{x \in X \colon h_2(x) > y - r\}) \in \mathscr{B}.$$

Although using measurability to build our theory is an option, we decided against it because, being "σ-complete" is not is absolute for transitive models of ZFC. Moreover, this integration theory do not need to rely on $\mathscr{B}$-measurability. However, using $\mathscr{B}$-measurability will be very useful when $\mathscr{B}$ is $\mathcal{P}(X)$ for some[6] X because, trivially, all functions are measurable there. For example, we can use it to prove:

Theorem 3.5.11. *Let X a non-empty set. If $u \subseteq X$ is finite and non-empty, then*

$$\int_X f d\Xi^u = \frac{1}{|u|} \sum_{k \in u} f(k),$$

where Ξ^u is as in Example 3.1.9.

Proof. Let $u \subseteq X$ be a finite non-empty set. By Example 3.1.9, Ξ^u is a finitely additive measure on $\mathcal{P}(X)$, so f is $\mathcal{P}(X)$-measurable and therefore, by Theorem 3.5.10, $f \in \mathscr{I}(\Xi^u)$. Now we will prove that $\int_X f d\Xi^u = \frac{1}{|u|} \sum\limits_{x \in u} f(x)$. It is enough to show that $\overline{\int_X} f d\Xi^u = \frac{1}{|u|} \sum\limits_{x \in u} f(x)$, that is, $\frac{1}{|u|} \sum\limits_{x \in u} f(x)$ is the least lower bound of $\{\overline{\mathrm{S}}^{\Xi^u}(f, P) \colon P \in \mathbf{P}^{\Xi^u}\}$. To see this, let $P \in \mathbf{P}^{\Xi^u}$ and we prove that $\frac{1}{|u|} \sum\limits_{k \in u} f(k) \leq \sum\limits_{b \in P} \sup(f[b]) \Xi^u(b)$. Let $P' \in \mathbf{P}$ be a refinement of P such that, for any $x \in u$, $\{x\} \in P'$. Then,

$$\begin{aligned}
\sum_{b \in P'} \sup(f[b]) \Xi^u(b) &= \sum_{b \in P' \setminus [u]^1} \sup(f[b]) \Xi^u(b) + \sum_{x \in u} \sup(f[b]) \Xi^u(b) \\
&= \sum_{b \in P' \setminus u} \sup(f[b]) \cdot 0 + \sum_{x \in u} f(x) \Xi^u(\{x\}) \\
&= \frac{1}{|u|} \sum_{x \in u} f(x).
\end{aligned}$$

Not only we showed that $\frac{1}{|u|} \sum\limits_{x \in u} f(x)$ is a lower bound, but that it is equal to $\overline{\mathrm{S}}^{\Xi}(f, P')$ for some $P' \in \mathbf{P}^{\Xi}$. Thus, $\int_X f d\Xi^u = \frac{1}{|u|} \sum_{x \in u} f(x)$. $\square_{\text{Theorem 3.5.11}}$

Now our goal is to show that the integral we have defined satisfies the fundamental properties we know of, say, the Riemann integral. We start by proving that the addition and multiplication by constants of Ξ-integrable functions are again Ξ-integrable:

[6] Without relativizing in some model of ZFC.

Theorem 3.5.12. *Let $f, g \in \mathscr{I}(\Xi)$ and $c \in \mathbb{R}$. Then $cf, f+g \in \mathscr{I}(\Xi)$ and:*

1. $\int_X (cf)\, d\Xi = c \int_X f\, d\Xi$.
2. $\int_X (f+g) d\,\Xi = \int_X f d\,\Xi + \int_X g d\Xi$.

Proof.

1. Notice that cf is bounded because cf is. Also, if $c = 0$, the result is clear.

   Let $c, \varepsilon > 0$. By basic properties of $\inf$ and $\sup$, for every $P \in \mathbf{P}^{\Xi}$ we have that:

   $$\overline{\mathrm{S}}(cf, P) = c\,\overline{\mathrm{S}}(f, P) \text{ and } \underline{\mathrm{S}}(cf, P) = c\,\underline{\mathrm{S}}(f, P).$$

   Therefore,

   $$\overline{\int_E} cf d\Xi = c\overline{\int_E} f d\Xi = c\int_X f d\Xi \;\text{ and }\; \underline{\int_E} cf d\Xi = c\underline{\int_E} f d\Xi = c\int_X f d\Xi.$$

   So we can conclude that $cf \in \mathscr{I}(\Xi)$ and

   $$\int_X (cf) d\Xi = c\int_X f d\Xi.$$

   Now suppose that $c = -1$. Again, by basic properties of $\sup$ and $\inf$, for every $P \in \mathbf{P}^{\Xi}$ we have that:

   $$\overline{\mathrm{S}}(-f, P) = -\,\underline{\mathrm{S}}(f, P) \text{ and } \underline{\mathrm{S}}(-f, P) = -\,\overline{\mathrm{S}}(f, P).$$

   Therefore,

   $$\overline{\int_X}(-f) d\Xi = -\underline{\int_X} f d\Xi = -\int_X f d\Xi \;\text{ and }\; \underline{\int_X}(-f) = -\overline{\int_X} f d\Xi = -\int_X f d\Xi.$$

   Finally, suppose that $c < 0$. So, $-c > 0$ and therefore we can apply the previous cases. In particular, we get that $cf \in \mathscr{I}(\Xi)$. Now,

   $$\int_X (cf) d\Xi = -\int_X (-cf) d\Xi = -(-c)\int_X f d\Xi = c\int_X f d\Xi.$$

2. It is clear that $f + g$ is bounded. Again, by properties of $\sup$ and $\inf$ for every $p \in \mathbf{P}^{\Xi}$ we have that:

   $$\overline{\mathrm{S}}(f+g, P) \leq \overline{\mathrm{S}}(f, P) + \overline{\mathrm{S}}(g, P) \;\text{ and }\; \underline{\mathrm{S}}(f, P) + \underline{\mathrm{S}}(g, P) \leq \underline{\mathrm{S}}(f+g, P).$$

   Let $\varepsilon > 0$. By Theorem 3.5.8, there are $P, Q \in \mathbf{P}^{\Xi}$ such that:

   $$\overline{\mathrm{S}}(f, P) - \underline{\mathrm{S}}(f, P) < \frac{\varepsilon}{2} \text{ and } \overline{\mathrm{S}}(g, Q) - \underline{\mathrm{S}}(g, Q) < \frac{\varepsilon}{2}.$$

We set $R \coloneqq P \sqcap Q$. So, by Lemma 3.5.5,

$$\overline{\mathrm{S}}(f,R) - \underline{\mathrm{S}}(f,R) < \frac{\varepsilon}{2} \text{ and } \overline{\mathrm{S}}(g,R) - \underline{\mathrm{S}}(g,R) < \frac{\varepsilon}{2}.$$

Therefore,

$$\overline{\mathrm{S}}(f+g,R) - \underline{\mathrm{S}}(f+g,R) < \varepsilon.$$

Thus, $f+g \in \mathscr{I}(\Xi)$. Finally, from the above:

$$\begin{aligned}
\int_X f d\Xi + \int_X g d\Xi - \varepsilon &= \int_X f d\Xi - \frac{\varepsilon}{2} + \int_X g d\Xi - \frac{\varepsilon}{2} \\
&< \underline{\mathrm{S}}(f,R) + \underline{\mathrm{S}}(g,R) \\
&\leq \underline{\mathrm{S}}(f+g,R) \\
&\leq \int_X (f+g) d\Xi \\
&\leq \overline{\mathrm{S}}(f+g,R) \\
&\leq \overline{\mathrm{S}}(f,R) + \overline{\mathrm{S}}(g,R) \\
&< \int_X f d\Xi + \frac{\varepsilon}{2} + \int_X g d\Xi + \frac{\varepsilon}{2} \\
&< \int_X f d\Xi + \int_X g d\Xi + \varepsilon
\end{aligned}$$

Since ε is arbitrary, it follows that

$$\int_X (f+g)\, d\Xi = \int_X f d\Xi + \int_X g\, d\Xi. \qquad \square_{\text{Theorem 3.5.12}}$$

So, inductively, we get:

Corollary 3.5.13. *Let $\{f_i \colon i < n\}$ a finite sequence of Ξ-integrable functions. Then $\sum_{i<n} f_i \in \mathscr{I}(\Xi)$ and*

$$\int_X \left(\sum_{i<n} f_i\right) d\Xi = \sum_{i<n} \left(\int_X f_i \, d\Xi\right).$$

Corollary 3.5.14. *If $f, g \colon X \to \mathbb{R}$ are Ξ-integrable functions and $f \leq g$, then $\int_X f d\Xi \leq \int_X g d\Xi$.*

Proof. For all $x \in X$, define $h(x) \coloneqq g(x) - f(x)$. So $f \geq 0$. It is clear that, for every $P \in \mathbf{P}^{\Xi}$, $\overline{\mathrm{S}}(h,P) \geq 0$. Finally, since by Theorem 3.5.12 $h \in \mathscr{I}(\Xi)$, we have that

$$\int_X g\, d\Xi - \int_X f\, d\Xi = \int_X h\, d\Xi \geq 0.$$

Thus $\int_X f\, d\Xi \leq \int_X g\, d\Xi$. $\square_{\text{Corollary 3.5.14}}$

Towards showing that the product of two Ξ-integrable functions is Ξ-integrable, we prove:

Theorem 3.5.15. *If $f \in \mathscr{I}(\Xi)$, then $f^2 \in \mathscr{I}(\Xi)$.*

***Proof*.** Since f is bounded, there is some $0 < N < \omega$ such that $|f(x)| \leq N$ for all $x \in X$. Notice that f^2 is also bounded because $0 \leq f^2 \leq N^2$. Let $\varepsilon > 0$. By the criterion of Ξ-integrability, there is a partition $P \in \mathbf{P}^\Xi$ such that $\overline{\mathrm{S}}(f, P) - \underline{\mathrm{S}}(f, P) < \frac{\varepsilon}{2N}$. For all $b \in P$ define

$$M^b := \sup\{|f(c) - f(d)|\colon c, d \in b\} \text{ and } M_2^b := \sup\{|f^2(c) - f^2(d)|\colon c, d \in b\}.$$

Therefore, by properties of $\sup$ and $\inf$ we have that

$$\overline{\mathrm{S}}(f, P) - \underline{\mathrm{S}}(f, P) = \sum_{b\in P} M^b \Xi(b) \text{ and } \overline{\mathrm{S}}(f^2, P) - \underline{\mathrm{S}}(f^2, P) = \sum_{b\in P} M_2^b \Xi(b).$$

Now, notice that,

$$|f^2(c) - f^2(d)| = |f(c) + f(d)|\,|f(c) - f(d)| \leq (|f(c)| + |f(d)|)\,|f(c) - f(d)| \leq 2N|f(c) - f(d)|$$

that is, we get that for every $b \in P$, $M_2^b \leq 2NM^b$, hence

$$\overline{\mathrm{S}}(f^2, P) - \underline{\mathrm{S}}(f^2, P) = \sum_{b\in P} M_2^b \Xi(b) \leq 2N \sum_{b\in P} M^b \Xi(b) = 2N[\overline{\mathrm{S}}(f, P) - \underline{\mathrm{S}}(f, P)] < 2N\frac{\varepsilon}{2N} = \varepsilon.$$

Thus, by the criterion of Ξ-integration, we can conclude that $f^2 \in \mathscr{I}(\Xi)$. $\square_{\text{Theorem 3.5.15}}$

Since $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$, we get:

Corollary 3.5.16. *If $f, g \in \mathscr{I}(\Xi)$, then $fg \in \mathscr{I}(\Xi)$.*

Towards defining the integral over subsets of X, we need to calculate the value $\int_X \chi_E \, d\Xi$ for $E \in \mathscr{B}$, but first, we define:

Definition 3.5.17. Let $E \in \mathscr{B}$. If $P \in \mathbf{P}^\Xi$ we define $P_E := \{E \cap A\colon A \in P\}$ and $\hat{P}_E := P \sqcap \{E, E^{\mathrm{c}}\}$. We call P_E the *partition of E induced by P*.

It is clear that, in effect, for every $P \in \mathbf{P}^\Xi$, P_E is a partition of E, and $\hat{P}_E \in \mathbf{P}^\Xi$. Also, $P_E \subseteq \hat{P}_E$.

Lemma 3.5.18. *If $E \in \mathscr{B}$, then $\chi_E \in \mathscr{I}(\Xi)$ and $\int_X \chi_E \, d\Xi = \Xi(E)$.*

***Proof*.** Let $P \in \mathbf{P}^\Xi$ and notice that:

$$\overline{\mathrm{S}}(\chi_E, \hat{P}_E) = \sum_{b\in \hat{P}_E} \sup(\chi_E[b])\Xi(b) = \sum_{b\in P_E} \Xi(b) = \Xi(E),$$

where P_E is the partition of E induced by P. Similarly, $\underline{\mathrm{S}}(\chi_E, \hat{P}_E) = \Xi(E)$. Thus by the criterion of Ξ-integrability, $\chi_E \in \mathscr{I}(\Xi)$ and $\int_X \chi_E \, d\Xi = \Xi(E)$. $\square_{\text{Lemma 3.5.18}}$

This will allow us to integrate over suitable subsets of X:

Definition 3.5.19. If $E \subseteq X$ and $\chi_E f \in \mathscr{I}(\Xi)$, we define

$$\int_E f d\Xi := \int_X \chi_E f \, d\Xi.$$

Notice that, if $E \in \mathscr{B}$ and $f \in \mathscr{I}(\Xi)$, then by Lemma 3.5.18 χ_E is Ξ-integrable and therefore, by Corollary 3.5.16, $\chi_E f$ is Ξ-integrable. In general, this is the context in which we will use Definition 3.5.19.

We have integral monotonicity over subsets:

Lemma 3.5.20. *Assume that f is non-negative, $E, F \in \mathscr{B}$ and $\chi_E f, \chi_F f \in \mathscr{I}(\Xi)$. If $E \subseteq F$, then $\int_E f d\Xi \leq \int_F f d\Xi$.*

Proof. If $E \subseteq F$, then $\chi_E f \leq \chi_F f$, so by Corollary 3.5.14 we have that:

$$\int_E f d\Xi = \int_X \chi_E f d\Xi \leq \int_X \chi_F f d\Xi = \int_F f d\Xi. \qquad \square_{\text{Lemma 3.5.20}}$$

Also, integrals over Ξ-null sets are null:

Lemma 3.5.21. *If $\Xi(E) = 0$, then $\chi_E f \in \mathscr{I}(\Xi)$ and $\int_E f d\Xi = 0$.*

Proof. Let $M < \omega$ be such that for all $x \in X$, $|f(x)| \leq M$, $P \in \mathbf{P}^{\Xi}$ and consider P_E and $\hat{P}_E$ as in Definition 3.5.17. Notice that

$$\overline{\mathrm{S}}(\chi_E f, \hat{P}_E) = \sum_{b \in \hat{P}_E} \sup(\chi_E f[b])\Xi(b) = \sum_{b \in P_E} \sup(f[b])\Xi(b) \leq \sum_{b \in P_E} M\Xi(b) = M\Xi(E) = 0.$$

Since $\underline{\mathrm{S}}(f\chi_E, \hat{P}_E) \geq 0$, we can conclude that $\chi_E f \in \mathscr{I}(\Xi)$ and

$$\int_E f d\Xi = \int_X \chi_F f d\Xi = 0. \qquad \square_{\text{Lemma 3.5.21}}$$

We also have additivity in subsets:

Theorem 3.5.22. *If $f \in \mathscr{I}(\Xi)$ and $\langle E_i \colon i < n \rangle \in \mathbf{P}^{\Xi}$, then:*

$$\int_X f d\Xi = \sum_{i<n} \left(\int_{E_i} f \, d\Xi \right).$$

Proof. Let $f \in \mathscr{I}(\Xi)$ and $\langle E_i \colon i < n \rangle \in \mathbf{P}^{\Xi}$. So, it is clear that:

$$\forall x \in X \left(f(x) = \sum_{i<n} \chi_{E_i}(x) f(x) \right),$$

and therefore, by Corollary 3.5.13 we have that:

$$\int_X f d\Xi = \int_X \left(\sum_{i<n} \chi_{E_i} f \right) d\Xi = \sum_{i<n} \left(\int_X \chi_{E_i} f \, d\Xi \right) = \sum_{i<i} \left(\int_{E_i} f \, d\Xi \right).$$

$\square_{\text{Theorem 3.5.22}}$

As expected, Ξ-null sets do not affect the value of the integral:

Corollary 3.5.23. *If $f \in \mathscr{I}(\Xi)$ and $\Xi(E) = 0$, then*

$$\int_X f d\Xi = \int_{X\setminus E} f d\Xi.$$

Proof. By Theorem 3.5.22 and Lemma 3.5.21, we have:

$$\int_X f\, d\Xi = \int_{X\setminus E} f\, d\Xi + \int_E f\, d\Xi = 0 + \int_{X\setminus E} f\, d\Xi = \int_{X\setminus E} f\, d\Xi. \qquad \square_{\text{Corollary 3.5.23}}$$

Finally, for $a < b \in \mathbb{R}$, consider $\mathscr{C}$ as the Boolean algebra generated by $\{[x, y) \cap [a, b] \colon x, y \in \mathbb{R}\}$ and $\Xi := \mathrm{Leb}_{\mathbb{R}} \restriction \mathscr{C}$. It is clear that Ξ is a finitely additive measure. Thereby, if $f \colon [a, b] \to \mathbb{R}$ is a bounded Ξ-integrable function, then

$$\int_{[a,b]} f d\Xi = \int_a^b f d\,\mathrm{Leb}_{\mathbb{R}}$$

is the usual Riemann integral, that is, Ξ-integrability is equivalent to Riemann-integrability and the properties and results that we have proved in this section correspond to properties of this integral. So, as mentioned earlier, the Riemann integration is a particular case of the integration with respect to finitely additive measures.

3.5.1 Some criteria of extension and approximation with integrals

In this subsection, we prove some extension criteria for finitely additive measures involving integrals.

First, we use Theorem 3.3.1 to prove the following lemma:

Lemma 3.5.24. *Let X a non-empty set, $\delta \in [0, \infty)$ and $r \in \mathbb{R}$. Define $Z := {}^{\mathcal{P}(X)}[0, \delta]$. Then the set*

$$A(f, r) := \left\{ \Xi \in Z \colon \Xi \textit{ is a fam, } \Xi(X) = \delta \textit{ and } \int_X f d\Xi \geq r \right\},$$

is closed in $[0, \delta]^{\mathcal{P}(X)}$.

Proof. Notice that f is $\mathscr{B}$-measurable because it is bounded (see Theorem 3.5.10). Now, fix any $\Xi \in \mathrm{cl}_Z(A(f, r))$, so by Theorem 3.3.1 Ξ is a finitely additive measure on $\mathcal{P}(X)$ with $\Xi(X) = \delta$. Let $P \in \mathbf{P}^{\Xi}$. It is enough to prove that $\sum\limits_{b \in P} \sup(f[b])\, \Xi(b) \geq r$.

Since f is bounded, there exists some $M > 0$ such that $|f(x)| < M$ for all $x \in X$. Let $\varepsilon > 0$ and $\varepsilon' := \frac{\varepsilon}{M|P|}$. For $b \in P$, set $U_b := (\Xi(b) - \varepsilon', \Xi(b) + \varepsilon') \cap [0, \delta]$. Let $U := \prod\limits_{a \in \mathcal{P}(\omega)} U_a$, where $U_a = [0, \delta]$ for each $a \notin P$. Since U is an open neighborhood of Ξ, we can pick some Ξ' such that

$|\Xi(b) - \Xi'(b)| < \varepsilon'$ for every $b \in P$ and $\sum_{b \in P} \sup(f[b])\, \Xi'(b) \geq r$. Then,

$$\begin{aligned}
\left|\sum_{b\in P} \sup(f[b])\Xi(b) - \sum_{b\in P} \sup(f[b])\Xi'(b)\right| &\leq \sum_{b\in P} |\sup(f[b])\,\Xi(b) - \sup(f[b])\,\Xi'(b)| \\
&= \sum_{b\in P} |\sup(f[b])||\,\Xi(b) - \Xi'(b)| \\
&\leq \sum_{b\in P} |\sup(f[b])|\varepsilon' \\
&\leq \sum_{b\in P} M\varepsilon' = |P|M\varepsilon' \\
&= \varepsilon.
\end{aligned}$$

Since, $\sum_{b\in P} \sup(f[b])\,\Xi'(b) \geq r$, we get

$$r - \varepsilon \leq \sum_{b\in P} \sup(f[b])\,\Xi'(b) - \varepsilon \leq \sum_{b\in P} \sup(f[b])\,\Xi(b).$$

Finally, as $\varepsilon > 0$ is arbitrary, $\sum_{b\in P} \sup(f[b])\,\Xi(b) \geq r$, which proves that $\Xi \in A(f, r)$. $\square_{\text{Lemma 3.5.24}}$

As a consequence,

Theorem 3.5.25. *Let I be an index set and for each $i \in I$, let $r_i \in [0, +\infty)$ and $f_i \colon X \to \mathbb{R}$ bounded. If Ξ_0 is a finite finitely additive measure on some Boolean sub-algebra $\mathscr{B} \subseteq \mathcal{P}(X)$, such that $\delta := \Xi_0(X)$ and, for any $P \in \mathbf{P}^{\Xi_0}$, $\varepsilon > 0$ and a finite set $J \subseteq I$, there is some finite $u \subseteq X$ such that:*

1. $\left|\Xi_0(b) - \delta\frac{|b\cap u|}{|u|}\right| \leq \varepsilon$ *for any $b \in P$, and*
2. $\frac{\delta}{|u|}\sum_{k\in u} f_i(k) \geq r_i - \varepsilon$ *for any $i \in J$,*

then there is some finitely additive measure Ξ extending Ξ_0 such that, for any $i \in I$, $\int_X f_i d\Xi \geq r_i$.

Proof. Notice that, f is $\mathscr{B}$-measurable because it is bounded (see Theorem 3.5.10). Let $Z := {}^{\mathcal{P}(X)}[0, \delta]$. For $J \subseteq I$, $P \in \mathbf{P}^{\Xi_0}$ and $\varepsilon \geq 0$ define:

$$F_{J,P,\varepsilon} := \left\{\Xi \in Z \colon \ \Xi \text{ is a fam}, \forall\, b \in P\, (|\Xi(b) - \Xi_0(b)| \leq \varepsilon) \text{ and } \forall\, i \in J \left(\int f_i d\Xi \geq r_i - \varepsilon\right)\right\},$$

hence, if u satisfies the conditions (1) and (2), then $\delta\Xi^u \in F_{J,P,\varepsilon}$. Also, using Lemma 3.5.24 and Theorem 3.3.1, $F_{J,P,\varepsilon}$ is a compact set. Notice that $\varepsilon' \leq \varepsilon$ implies that $F_{J,P,\varepsilon'} \subseteq F_{J,P,\varepsilon}$, so we get that $\{F_{J,P,\varepsilon} : \varepsilon > 0\}$ has the finite intersection property. Thus, by compactness, $F_{J,P,0}$ is non-empty and closed.

Now, if $J \subseteq J'$ and P' is a refinement of P in $\mathbf{P}^{\Xi_0}$ then $F_{J',P',0} \subseteq F_{J,P,0}$, which clearly implies that the family $\{F_{P,J,0} \colon P \in \mathbf{P}^{\Xi_0},\ J \in [I]^{<\aleph_0}\}$ has the finite intersection property, so there is some Ξ in its intersection. This Ξ is as desired. $\square_{\text{Theorem 3.5.25}}$

Finally, we show a criteria that allow us to approximate integrals:

Theorem 3.5.26. *Let Ξ be a free finitely additive measure on $\mathcal{P}(X)$ such that $\Xi(X) = \delta < \infty$, $E \subseteq X$ with $\Xi(E) > 0$ and $i^* < \omega$. For any $i < i^*$, let $f_i \colon X \to \mathbb{R}$ a bounded function. Then, for all $\varepsilon > 0$ and any finite set $F \subseteq X$, there exists a finite set $u \subseteq X \setminus F$ such that, for any $i < i^*$ we have that:*

$$\left| \frac{\delta}{|u|} \sum_{x \in u} f_i(x) - \frac{\delta}{\Xi(E)} \int_E f_i d\Xi \right| < \varepsilon.$$

Proof. First, notice that each f_i is Ξ-integrable by Theorem 3.5.10. Also, since each f_i is bounded, we can find an $M < \omega$ such that, for any $i < i^*$, $|f_i(x)| < M$ for all $x \in X$. Let $\varepsilon > 0$. Since for each $i < i^*$, f_i is Ξ-integrable, there is an induced partition P^i of E such that:

$$\forall i < i^* \left(\delta \left[\overline{\mathrm{S}}(f_i, P^i) - \underline{\mathrm{S}}(f_i, P^i) \right] < \frac{\varepsilon \Xi(E)}{2} \right).$$

Let $P := \sqcap_{i<i^*} P^i$. By Lemma 3.5.2, for each $i < i^*$, it follows that:

$$\delta[\overline{\mathrm{S}}(f_i, P) - \underline{\mathrm{S}}(f_i, P)] \leq \delta[\overline{\mathrm{S}}(f_i, P^i) - \underline{\mathrm{S}}(f_i, P^i)] < \frac{\Xi(E)\varepsilon}{2}.$$

Suppose that $P = \langle P_j \colon j < j^* \rangle$ and, without loss of generality, assume that for all $j < j^*$, $\Xi(P_j) > 0$.

Now, by density of $\mathbb{Q}$, there exists a finite sequence of rational numbers $\langle r_j \colon j < j^* \rangle$ such that, for all $j < j^*$:

- $r_j \in (0,1)_{\mathbb{Q}}$,
- $\sum_{j<j^*} r_j = 1$,
- $\delta \left| r_j - \frac{\Xi(P_j)}{\Xi(E)} \right| < \frac{\varepsilon}{2j^* M}$,
- there are $d, k_j < \omega$ such that $d, k_j > 0$ and $r_j = \frac{k_j}{d}$.

By assumption, each P_j with $j < j^*$ is infinite because Ξ is a free finitely additive measure, so there exists $u_j \subseteq P_j \setminus F$ such that $|u_j| = k_j$. Define $u := \bigcup_{j<j^*} u_j$ and notice that:

$$|u| = \sum_{j<j^*} |u_j| = \sum_{j<j^*} k_j = d \sum_{j<j^*} \frac{k_m}{d} = d \sum_{j<j^*} r_m = d.$$

Therefore, $|u| = d$ and it is clear that $u \subseteq X \setminus F$.

Let $i < i^*$. Then:

$$\begin{aligned}
\sum_{x \in u} \frac{\delta\, f_i(x)}{|u|} &= \sum_{j<j^*} \left(\sum_{x \in u_j} \delta \frac{f_i(x)}{d} \right) \leq \sum_{j<j^*} \sup(f_i[P_j]) \frac{k_j}{d} \delta \\
&\leq \sum_{j<j^*} \left[\sup(f_i[P_j]) \left(\delta \frac{\Xi(P_j)}{\Xi(E)} + \frac{\varepsilon}{2Mj^*} \right) \right] \\
&= \frac{\delta}{\Xi(E)} \sum_{j<j^*} \sup(f_i[P_j]) \Xi(P_j) + \sum_{j<j^*} \sup(f_i[P_j]) \frac{\varepsilon}{2Mj^*} \\
&\leq \frac{\delta}{\Xi(E)}\, \overline{\mathrm{S}}(f_i, P) + \sum_{j<j^*} \frac{\varepsilon}{2j^*} \\
&< \frac{1}{\Xi(E)} \left(\frac{\varepsilon \Xi(E)}{2} + \delta\, \underline{\mathrm{S}}(f_i, P) \right) + \frac{\varepsilon j^*}{2j^*} \\
&\leq \frac{1}{\Xi(E)} \left(\frac{\varepsilon \Xi(E)}{2} + \delta \int_E f_i d\Xi \right) + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{2} + \frac{\delta}{\Xi(E)} \int_E f_i d\Xi + \frac{\varepsilon}{2} \\
&= \frac{\delta}{\Xi(E)} \int_E f_i d\Xi + \varepsilon.
\end{aligned}$$

On the other hand, notice that if $j < j^*$ then

$$\sum_{x \in u_j} \frac{\delta f_i(x)}{|u|} \geq \sum_{x \in u_j} \frac{\delta \inf(f_i[P_j])}{|u|} = \delta \inf(f_i[P_j]) \frac{|u_j|}{|u|} = \delta \inf(f_i[P_j]) \frac{k_j}{|u|}.$$

Therefore, the other inequality is very similar:

$$\begin{aligned}
\sum_{x \in u} \frac{\delta f_i(x)}{|u|} &= \sum_{j<j^*} \left(\sum_{x \in u_j} \frac{\delta f_i(x)}{|u|} \right) \geq \sum_{j<j^*} \delta \inf(f_i[P_j]) \frac{k_j}{d} \\
&\geq \sum_{j<j^*} \left[\inf(f_i[P_j]) \left(\delta \frac{\Xi(P_j)}{\Xi(E)} + \frac{\varepsilon}{2Mj^*} \right) \right] > \frac{\delta}{\Xi(E)} \int_E f_i d\Xi - \varepsilon.
\end{aligned}$$

Finally, we get:

$$\left| \frac{\delta}{|u|} \sum_{x \in u} f_i(x) - \frac{\delta}{\Xi(E)} \int_E f_i d\Xi \right| < \varepsilon. \qquad \square_{\text{Theorem 3.5.26}}$$

Notice that this is an approximation result in the following sense: by Theorem 3.5.11, if u is the finite set from conclusion of Theorem 3.5.26, then $\int_X f_i d\Xi^u = \frac{1}{|u|} \sum_{x \in u} f_i(x)$, so this allows us to approximate $\frac{\delta}{\Xi(E)} \int_E f_i d\Xi$ using the integral with respect to $\delta\, \Xi^u$.

3.5.2 Integrating over models: the integral absoluteness

In this section, we are going to show that the integral we defined before is absolute for transitive models of ZFC. We must first see how the integrability and the integrals are related when we integrate with respect to embedded finitely additive measures:

Theorem 3.5.27. *Let $\mathscr{B}_0$, $\mathscr{B}_1 \subseteq \mathcal{P}(X)$ be Boolean algebras such that $\mathscr{B}_0 \subseteq \mathscr{B}_1$, and Ξ_0, Ξ_1 be finitely additive measures on $\mathscr{B}_0$, $\mathscr{B}_1$ respectively, such that $\Xi_0 \subseteq \Xi_1$. Let $f\colon X \to \mathbb{R}$ a bounded function. Then*

$$f \in \mathscr{I}(\Xi_0) \Rightarrow f \in \mathscr{I}(\Xi_1),$$

in which case

$$\int_X f \, d\Xi_0 = \int_X f d\,\Xi_1.$$

The converse is true when $\mathscr{B}_0 = \mathscr{B}_1$.

Proof. For any $P \in \mathbf{P}^{\Xi_0}$ we have that $\overline{\mathrm{S}}^{\Xi_0}(f,P) = \overline{\mathrm{S}}^{\Xi_1}(f,P)$ and $\underline{\mathrm{S}}^{\Xi_0}(f,P) = \underline{\mathrm{S}}^{\Xi_1}(f,P)$, so

$$\{\overline{\mathrm{S}}^{\Xi_0}(f,P)\colon P \in \mathbf{P}^{\Xi_0}\} \subseteq \{\overline{\mathrm{S}}^{\Xi_1}(f,P)\colon P \in \mathbf{P}^{\Xi_1}\} \tag{3.5.1}$$

and

$$\{\underline{\mathrm{S}}^{\Xi_0}(f,P)\colon P \in \mathbf{P}^{\Xi_0}\} \subseteq \{\underline{\mathrm{S}}^{\Xi_1}(f,P)\colon P \in \mathbf{P}^{\Xi_1}\}. \tag{3.5.2}$$

Therefore,

$$\overline{\int_X} f d\,\Xi_1 = \inf\{\overline{\mathrm{S}}^{\Xi_1}(f,P)\colon P \in \mathbf{P}^{\Xi_1}\} \leq \inf\{\overline{\mathrm{S}}^{\Xi_0}(f,P)\colon P \in \mathbf{P}^{\Xi_0}\} = \overline{\int_X} f \, d\Xi_0,$$

and

$$\underline{\int_X} f \, d\Xi_0 = \sup\{\underline{\mathrm{S}}^{\Xi_0}(f,P)\colon P \in \mathbf{P}^{\Xi_0}\} \leq \sup\{\underline{\mathrm{S}}^{\Xi_1}(f,P)\colon P \in \mathbf{P}^{\Xi_1}\} = \underline{\int_X} f d\,\Xi_1.$$

Whence, it follows that:

$$\underline{\int_X} f d\,\Xi_0 \leq \underline{\int_X} f \, d\Xi_1 \leq \overline{\int_X} f \, d\Xi_1 \leq \overline{\int_X} f d\,\Xi_0. \tag{3.5.3}$$

Thus, if f is Ξ_0-integrable, then f is Ξ_1-integrable, and it is clear that their values coincide.
To see the converse, notice that if $\mathscr{B}_0 = \mathscr{B}_1$, then $\mathbf{P}^{\Xi_0} = \mathbf{P}^{\Xi_1}$ and therefore, inclusions in Equation 3.5.1 and Equation 3.5.2 are really equalities, so from Equation 3.5.3 we get:

$$\underline{\int_X} f d\,\Xi_0 = \underline{\int_X} f \, d\Xi_1 = \overline{\int_X} f \, d\Xi_1 = \overline{\int_X} f d\,\Xi_0.$$

Thus, $f \in \mathscr{I}(\Xi_0) \Leftrightarrow f \in \mathscr{I}(\Xi_1)$, and it is clear that both integrals coincide. $\square_{\text{Theorem 3.5.27}}$

Lemma 3.5.28. *Let* M, N *be transitive models of* ZFC *such that* $\mathrm{M} \subseteq \mathrm{N}$. *If* $X \in M$ *and* $X \subseteq \mathbb{R}^{\mathrm{M}}$, *then* $\sup^{M}(X) = \sup^{\mathrm{N}}(X)$.

Proof. Let $X \in \mathrm{M}$ such that $X \subseteq \mathbb{R}^{\mathrm{M}}$. It is clear that, in N, $\sup^{\mathrm{M}}(X)$ is a upper bound of X, so $\sup^{N}(X) \leq \sup^{\mathrm{M}}(X)$. Now, towards contradiction, working in N, assume that $\sup^{\mathrm{M}}(X) > \sup^{\mathrm{N}}(X)$ and let $r \in \mathbb{Q}^{\mathrm{N}}$ such that $\sup^{\mathrm{M}}(X) > r > \sup^{\mathrm{N}}(X)$.

Now, working in M, since being upper bound and rational number are absolute for transitive models, we have that $r \in \mathbb{Q}^{\mathrm{M}}$ and it is an upper bound of X, hence $\sup^{\mathrm{M}}(X) \leq r$. This implies that $r < r$ in N, a contradiction. Therefore $\sup^{\mathrm{N}}(X) = \sup^{\mathrm{M}}(X)$. $\square_{\text{Lemma 3.5.28}}$

We can already prove that the integral with respect to finitely additive measures in Boolean algebras is absolute for transitive models of ZFC:

Theorem 3.5.29. *Let* M, N *be transitive models of* ZFC *such that* $\mathrm{M} \subseteq \mathrm{N}$. *Let* $\mathscr{B}, \Xi \in \mathrm{M}$ *be such that* $\mathscr{B}$ *is a Boolean algebra in* M *and* Ξ *is a finitely additive measure on* $\mathscr{B}$ *in* M. *If* $f \colon X \to \mathbb{R}$ *is a bounded function in* M, *then*

$$(f \in \mathscr{I}(\Xi))^{\mathrm{M}} \Leftrightarrow (f \in \mathscr{I}(\Xi))^{\mathrm{N}},$$

in which case,

$$\left(\int_X f d\Xi\right)^{\mathrm{N}} = \left(\int_X f d\Xi\right)^{\mathrm{M}}.$$

Proof. By Lemma 3.5.28, we have that:

$$\{\overline{\mathrm{S}}^{\mathrm{M}}(f, P) \colon P \in (\mathbf{P}^{\Xi})^{\mathrm{M}}\} = \{\overline{\mathrm{S}}^{\mathrm{N}}(f, P) \colon P \in (\mathbf{P}^{\Xi})^{\mathrm{N}}\}$$

and

$$\{\underline{\mathrm{S}}^{\mathrm{M}}(f, P) \colon P \in (\mathbf{P}^{\Xi})^{\mathrm{M}}\} = \{\underline{\mathrm{S}}^{\mathrm{M}}(f, P) \colon P \in (\mathbf{P}^{\Xi})^{\mathrm{N}}\}.$$

So, taking $\inf$ in the first equation and $\sup$ in the second one, we get:

$$\left(\underline{\int_X} f d\,\Xi\right)^{\mathrm{M}} = \left(\underline{\int_X} f d\,\Xi\right)^{\mathrm{N}} \text{ and } \left(\overline{\int_X} f d\,\Xi\right)^{\mathrm{M}} = \left(\overline{\int_X} f d\,\Xi\right)^{\mathrm{N}},$$

which proves the theorem. $\square_{\text{Theorem 3.5.29}}$

We are going to extend finitely additive measures along forcing iterations preserving the values of the integrals, however, we will not always have the same domain, for example, if we have a finitely additive measure over $\mathcal{P}(\omega)^{\mathrm{M}}$, when iterating new reals can appear and $\mathcal{P}(\omega)^{\mathrm{M}} \subsetneq \mathcal{P}(\omega)^{\mathrm{N}}$. So we will use the following result, which is obtained simply by relativizing Theorem 3.5.27 and applying the integral absoluteness:

Corollary 3.5.30. *Let* M, N *be transitive models of* ZFC *such that* Ξ_0, $\mathscr{B}_0 \in \mathrm{M}$, $\Xi_1, \mathscr{B}_1 \in \mathrm{N}$. *Assume that* Ξ_0, Ξ_1 *are finitely additive measures on* $\mathscr{B}_0$, $\mathscr{B}_1$ *respectively, such that* $\Xi_0 \subseteq \Xi_1$ *and* $\mathscr{B}_0 \subseteq \mathscr{B}_1 \subseteq \mathcal{P}(X)$ *for some set* $X \in \mathrm{M}$. *Then:*

$$(f \in \mathscr{I}(\Xi_0))^{\mathrm{M}} \Rightarrow (f \in \mathscr{I}(\Xi_1))^{\mathrm{N}},$$

in which case,

$$\left(\int_X f d\Xi_0\right)^{\mathrm{M}} = \left(\int_X f d\Xi_1\right)^{\mathrm{N}}.$$

Finally, as we mentioned before, the Riemann integral is a special case of the integral with respect to finitely additive measures, so:

Corollary 3.5.31. *The Riemann integral is absolute for transitive models of* ZFC.

CHAPTER 4

A general theory of iterated forcing with finitely additive measures

No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.

David Hilbert[1]

As mentioned in the introduction of this thesis, *Saharon Shelah* in [She00] managed to define a finite support iteration using finitely additive measures that allowed to show that, consistently, $\mathrm{cov}(\mathcal{N})$ is singular, moreover, that it may have countable cofinality. The original iteration basically uses partial random forcing, since using *book-keeping* arguments allow to increase $\mathrm{cov}(\mathcal{N})$ and, on the other hand, the treatment with finitely additive measures allows to preserve not too large covering families of null sets, and consequently achieve an upper bound of $\mathrm{cov}(\mathcal{N})$.[2] Since the iteration depends on the choice, at each step, of certain finitely additive measures satisfying some special properties, one cannot iterate with just any forcing notion, as one needs some structure that allows one to properly extend finitely additive measures in the iteration steps. In 2019, *Jakob Kellner*, *Saharon Shelah* and *Anda Tănasie* (see [KST19]) succeeded in proving that $\tilde{\mathbb{E}}$ (see Definition 1.5.45) also works to build the iteration.

By analyzing the role of both $\mathbb{B}$ (see Definition 1.5.37) and $\tilde{\mathbb{E}}$ (see Definition 1.5.43) in the construction of the iteration, we managed to identify specific properties, which do not depend on these notions of forcing, that seemed to be the key to being able to construct the iteration. These properties led us to a new linkedness notion, which we call μ-FAM-*linked*, and which, as we show in the third section of this chapter, turns out to be the right property to be able to construct the iteration, which allowed us to generalize and obtain a general theory of iterated forcing with finitely additive measures.

[1] See [Hil27].

[2] This will be developed in detail in Chapter 5.

So, the goal of this chapter is to define and study the new notion of linkedness, and show that by iterating with this type of forcing notions, we can generalize the iteration built by *Saharon Shelah*. One of the properties we identified in $\mathbb{B}$ and $\tilde{\mathbb{E}}$ is related to the *intersection number*, so we must start by generalizing this idea to forcing notions.

4.1 The intersection number for forcing notions

The concept of *intersection number* for Boolean algebras was introduced by *John. L. Kelley*, at the end of the Fifties, as a useful notion for establishing criteria for the existence of finitely additive measures in Boolean algebras (see [Kel59]). In this section, we generalize that idea of intersection number to apply it to forcing notions, which will play a fundamental role in our generalization of the iterated forcing method with finitely additive measures, in particular, it will allow us to generalize the limit step of such iterations (see Theorem 4.2.20) using Crucial Lemma 4.1.10 below.

Definition 4.1.1. Let $\mathbb{P}$ be a forcing notion and $Q \subseteq \mathbb{P}$.

1. For a finite sequence $\bar{q} = \langle q_i \colon i < n \rangle \in \mathbb{P}^n$, we define
$$i_*^{\mathbb{P}}(\bar{q}) \coloneqq \max\{|F| \colon F \subseteq n \wedge \{q_i \colon i \in F\} \text{ has a lower bound in } \mathbb{P}\}.$$
2. $\mathrm{int}^{\mathbb{P}}(\mathrm{Q})$, the *intersection number of Q in* $\mathbb{P}$, is defined by
$$\mathrm{int}^{\mathbb{P}}(Q) \coloneqq \inf\left\{\frac{i_*^{\mathbb{P}}(\bar{q})}{n} \colon \bar{q} \in Q^n \wedge n \in \omega \setminus \{0\}\right\}.$$

Naturally, when the context is clear, we omit the superscript "$\mathbb{P}$" in $i_*^{\mathbb{P}}(\bar{q})$ and $\mathrm{int}^{\mathbb{P}}(Q)$.

The following theorem motivates the name *intersection number* and it establishes a relation with Boolean algebras.

Theorem 4.1.2. *Let $\mathbb{P}$ be a forcing notion, $\mathscr{B}$ a Boolean algebra, and $\iota \colon \mathbb{P} \to \mathscr{B}$ a complete embedding. If $n \in \omega \setminus \{0\}$ and $\bar{q} = \langle q_i \colon i < n \rangle \in \mathbb{P}^n$, then*
$$i_*(\bar{q}) = \max\left\{|I| \colon I \subseteq n \wedge \bigwedge_{i \in I} \iota(q_i) \neq 0_{\mathscr{B}}\right\}.$$

***Proof*.** Suppose that $\mathscr{B}$ is a Boolean algebra, $\iota \colon \mathbb{P} \to \mathscr{B}$ is a dense embedding, $n \in \omega \setminus \{0\}$ and define
$$m \coloneqq \max\left\{|I| \colon I \subseteq n \wedge \bigwedge_{i \in I} \iota(q_i) \neq 0_{\mathscr{B}}\right\}.$$

On the one hand, let $I \subseteq n$ be such that $b^* \coloneqq \bigwedge_{i \in I} \iota(q_i) \neq 0_{\mathscr{B}}$ and $|I| = m$. Since $b^* \neq 0_{\mathscr{B}}$ and ι is a complete embedding, there exists $r \in \mathbb{P}$ such that it is a reduction of b^*, hence it is clear that $r \leq^\bullet q_i$ for all $i \in I$. By Lemma 1.5.4, we can find a lower bound $q \in \mathbb{P}$ of $\{q_i \colon i \in I\}$ and, therefore, $|I| \leq i_*^{\mathbb{P}}(\bar{q})$. Thus $m \leq i_*^{\mathbb{P}}(\bar{q})$.

On the other hand, consider $F \subseteq n$ such that $\{q_i \colon i \in F\}$ has a lower bound in $\mathbb{P}$ and $|F| = i_*^{\mathbb{P}}(\bar{q})$. It is clear that $\bigwedge_{i \in I} \iota(q_i) \neq 0_{\mathscr{B}}$ because $\{q_i \colon i \in F\}$ has a lower bound and ι is a complete embedding. Thus $i_*^{\mathbb{P}}(\bar{q}) = |F| \leq m$. $\square_{\text{Theorem 4.1.2}}$

Since, by Theorem 1.4.7, every Boolean algebra is isomorphic to a Boolean sub-algebra of $\mathcal{P}(X)$ for some set X, Theorem 4.1.2 implies that $i_*^{\mathscr{B}}(\langle b_0, \dots, b_{n-1}\rangle)$ is the maximum number of members $b_0, \dots, b_m$ with non-empty intersection, which as mentioned above motivates the name of *intersection number*. Indeed, this was Kelley's original definition in [Kel59].

The characterization of $i_*^{\mathbb{P}}$ in Theorem 4.1.2 also establishes a relation between $\operatorname{int}^{\mathscr{B}}(\iota(Q))$ and $\operatorname{int}^{\mathbb{P}}(Q)$ for $Q \subseteq \mathbb{P}$:

Corollary 4.1.3. *Let $\mathbb{P}$ be a forcing notion, $\mathscr{B}$ a Boolean algebra, $\iota\colon \mathbb{P} \to \mathscr{B}^+$ a complete embedding and $Q \subseteq \mathbb{P}$. If $\bar{q} = \langle q_i \colon i < n\rangle \in Q^n$ for $n \in \omega \setminus \{0\}$ and $\bar{b} := \langle \iota(q_i) \colon i < n\rangle$, then $i_*^{\mathbb{P}}(\bar{q}) = i_*^{\mathscr{B}}(\bar{b})$. As a consequence,* $\operatorname{int}^{\mathbb{P}}(Q) = \operatorname{int}^{\mathscr{B}}(\iota(Q))$.

Since the completion of a complete Boolean algebra is the identity, we can apply Corollary 4.1.3 to get that the intersection number is preserved by complete embeddings for Boolean algebras:

Corollary 4.1.4. *Let $\mathscr{B}, \mathscr{C}$ be complete Boolean algebras and $f\colon \mathscr{B}^+ \to \mathscr{C}^+$ a complete embedding. If $Q \subseteq \mathscr{B}$, then* $\operatorname{int}^{\mathscr{B}}(Q) = \operatorname{int}^{\mathscr{C}}(f[Q])$.

As a consequence, we get the analogous result for forcing notions and complete embeddings:

Corollary 4.1.5. *Let $\mathbb{P}, \mathbb{Q}$ be forcing notions, $\iota\colon \mathbb{P} \to \mathbb{Q}$ a complete embedding and $Q \subseteq \mathbb{P}$. Then* $\operatorname{int}^{\mathbb{P}}(Q) = \operatorname{int}^{\mathbb{Q}}(\iota[Q])$. *As a consequence, for $R \subseteq \mathbb{Q}$,* $\operatorname{int}^{\mathbb{Q}}(R) \leq \operatorname{int}^{\mathbb{P}}(\iota^{-1}[R])$.

Proof. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions, $(\mathscr{B}_{\mathbb{P}}, \iota_{\mathbb{P}}), (\mathscr{B}_{\mathbb{Q}}, \iota_{\mathbb{Q}})$ their forcing completions, respectively, and $\iota\colon \mathbb{P} \to \mathbb{Q}$ a complete embedding. Therefore, by Theorem 1.5.23, there exists a Boolean complete embedding $f\colon \mathscr{B}_{\mathbb{P}} \to \mathscr{B}_{\mathbb{Q}}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "\iota"] \arrow[d, "\iota_{\mathbb{P}}"'] & \mathbb{Q} \arrow[d, "\iota_{\mathbb{Q}}"] \\
\mathscr{B}_{\mathbb{P}} \arrow[r, "f"] & \mathscr{B}_{\mathbb{Q}}
\end{tikzcd}$$

So $f \circ \iota_{\mathbb{P}} = \iota_{\mathbb{Q}} \circ \iota$ and, by applying Corollary 4.1.3 and Corollary 4.1.4, we get

$$\operatorname{int}^{\mathbb{P}}(Q) = \operatorname{int}^{\mathscr{B}_{\mathbb{P}}}(\iota_{\mathbb{P}}[Q]) = \operatorname{int}^{\mathscr{B}_{\mathbb{Q}}}(f[\iota_{\mathbb{P}}[Q]]) = \operatorname{int}^{\mathscr{B}_{\mathbb{Q}}}(\iota_{\mathbb{Q}}[\iota[Q]]) = \operatorname{int}^{\mathbb{Q}}(\iota[Q]),$$

which proves the Corollary.

Finally, let $R \subseteq \mathbb{Q}$. Then, $\operatorname{int}^{\mathbb{Q}}(R) \leq \operatorname{int}^{\mathbb{Q}}(\iota[\iota^{-1}[R]]) = \operatorname{int}^{\mathbb{P}}(\iota^{-1}[R])$. $\square_{\text{Corollary 4.1.5}}$

Below we present some basic properties of the intersection number for forcing notions, although some of them are not relevant to the development of this work.

Lemma 4.1.6. *Let $\mathbb{P}$ be a forcing notion, $Q \subseteq \mathbb{P}$ such that $Q \neq \emptyset$ and $n \in \omega \setminus \{0\}$. Then,*

1. *If Q is m-linked for some $m < \omega$ then, for all $\bar{q} \in Q^n$, $i_*^{\mathbb{P}}(\bar{q}) \geq \min\{m, n\}$.*
2. *If $\bar{q} \in Q^n$, then $1 \leq i_*(\bar{q}) \leq n$. As a consequence,* $\operatorname{int}(Q)$ *is a real number and it belongs to* $[0, 1]$.

3. *If Q is centered, then for all $\bar{q} \in Q^n$, $i_*(\bar{q}) = n$. As a consequence* $\mathrm{int}(Q) = 1$ *if, and only if, Q is centered.*

4. $\forall p \in \mathbb{P}(\mathrm{int}(\{p\}) = 1)$.

5. *If Q is finite, then* $\mathrm{int}(Q) \geq \frac{1}{|Q|} > 0$.

6. *Let Q be an anti-chain in $\mathbb{P}$, then:*

   (a) *If Q is finite then* $\mathrm{int}(Q) = \frac{1}{|Q|}$.

   (b) *If Q is infinite then* $\mathrm{int}(Q) = 0$.

7. *If $m \in (1, \omega)$ and* $\mathrm{int}(Q) \geq 1 - \frac{1}{m+1}$, *then Q is m-linked.*

8. *If $Q \subseteq P \subseteq \mathbb{P}$, then* $\mathrm{int}(P) \leq \mathrm{int}(Q)$.

Proof. Let $n \in \omega \setminus \{0\}$. Notice that (1) is clear by Definition 4.1.1, (2) and (3) follow from (1) and (4) is a direct consequence of (3). For the other items, we must work a little bit more:

5. Assume that Q is finite. Let $\bar{q} \in Q^n$ and $\mathrm{ran}(\bar{q}) = \{t_i : i < k\} \subseteq Q$. So $k \leq |Q|$. For each $j < k$ define $r_j := |\{i < n : q_i = t_j\}|$, that is, r_j indicates the number of times that t_j is repeated in $\bar{q}$. Notice that $\sum_{j<k} r_j = n$. Now consider $r := \max\{r_j : j < k\}$. It is clear that $n \leq rk$, $i_*(\bar{q}) \geq r$ and, therefore, we have that
$$\frac{i_*(\bar{q})}{n} \geq \frac{r}{n} \geq \frac{r}{rk} = \frac{1}{k} \geq \frac{1}{|Q|}.$$
Thus, $\mathrm{int}(Q) \geq \frac{1}{|Q|}$.

6. Let Q an anti-chain in $\mathbb{P}$.

   (a) Assume that Q is finite. So $0 < |Q| < \omega$. By the previous result it is enough to prove that $\mathrm{int}(Q) \leq \frac{1}{|Q|}$. For this, let $\bar{q} \in Q^{|Q|}$ be such that it lists all members of Q. Since Q is an anti-chain is clear that $i_*(\bar{q}) = 1$. Therefore, $\mathrm{int}(Q) \leq \frac{i_*(\bar{q})}{|Q|} = \frac{1}{|Q|}$. Thus $\mathrm{int}(Q) \leq \frac{1}{|Q|}$.

   (b) Assume that Q is infinite. It is clear that for $m \in \omega \setminus \{0\}$ and $\bar{q} \in Q^m$ without repetitions,
$$\mathrm{int}(Q) \leq \frac{i_*(\bar{q})}{m} \leq \frac{1}{m},$$
hence $\mathrm{int}(Q) \leq \frac{1}{m}$ for all $0 < m < \omega$. Thus, by (2) it follows that $\mathrm{int}(Q) = 0$.

7. Suppose that $\mathrm{int}(Q) \geq 1 - \frac{1}{m+1}$ for some $m \in (1, \omega)$. Towards contradiction assume that Q is not m-linked. So, there exists a set $A = \{a_i : i < k\} \subseteq Q$ which has no lower bound and $1 < k \leq m$. Define $\bar{q} := \langle a_i : i < k \rangle \in Q^k$. It is clear that $i_*(\bar{q}) < k$. Therefore, we have that:
$$1 - \frac{1}{m+1} \leq \mathrm{int}(Q) \leq \frac{i_*(\bar{q})}{k} \leq \frac{k-1}{k} = 1 - \frac{1}{k}.$$
This implies that $m + 1 \leq k$, which is a contradiction because $k \leq m$. Thus Q is m-linked.

8. It is clear that $Q \subseteq P$ implies

$$\left\{ \frac{i_*^{\mathbb{P}}(\bar{q})}{n} \colon \bar{q} \in Q^n \wedge n \in \omega \setminus \{0\} \right\} \subseteq \left\{ \frac{i_*^{\mathbb{P}}(\bar{p})}{n} \colon \bar{p} \in P^n \wedge n \in \omega \setminus \{0\} \right\}.$$

Thus, by basic properties of $\inf$, we get $\mathrm{int}(P) \leq \mathrm{int}(Q)$. $\square_{\text{Lemma 4.1.6}}$

Kelley proved (see [Kel59, Prop. 1]) that finitely additive measures can be used to define subsets of Boolean algebras whose intersection number is bounded below by a given value:

Theorem 4.1.7. *Let $\mathscr{B}$ be a Boolean algebra and $\Xi \colon \mathscr{B} \to [0,1]$ a finitely additive measure. Consider $\mathbb{P} := \mathscr{B} \setminus \{0_{\mathscr{B}}\}$ and $\delta \in [0,1]$. If $Q := \{p \in \mathbb{P} \colon \Xi(p) \geq \delta\}$, then $\mathrm{int}^{\mathbb{P}}(Q) \geq \delta$.*

Proof. Without loss of generality, by Theorem 1.4.7 we can assume that there exists a set X such that $\mathscr{B}$ is a a sub-Boolean algebra of $\mathcal{P}(X)$. Let $n \in \omega \setminus \{0\}$ and $\bar{q} = \langle q_i \colon i < n \rangle \in Q^n$. For each $i < n$ consider $\chi_i \colon X \to \{0,1\}$ as the characteristic function of q_i in X. Now, let $x \in X$ and $F := \{i < n \colon \chi_i(x) = 1\}$. So $|F| = \sum_{i=0}^{n-1} \chi_i(x)$, $x \in \bigcap_{i \in F} q_i$ and $\bigcap_{i \in F} q_i$ is a lower bound of $\{q_i \colon i \in F\}$ in $\mathscr{B}$, hence, by Definition 4.1.1(1), we have that:

$$\sum_{i=0}^{n-1} \chi_i(x) = |F| \leq i_*(\bar{q}).$$

Therefore, for all $x \in X$, $\sum_{i=0}^{n-1} \chi_i(x) \leq i_*(\bar{q})$. Using that $\mathrm{ran}(\Xi) \subseteq [0,1]$, since by Lemma 3.5.18 each χ_i is Ξ-integrable, we can apply the basic integral properties (see Corollary 3.5.13, Corollary 3.5.14 and Lemma 3.5.18) to get

$$i_*(\bar{q}) \geq i_*(\bar{q})\Xi(X) = \int_X i_*(\bar{q}) d\Xi \geq \int_X \left(\sum_{i=0}^{n-1} \chi_i \right) d\Xi = \sum_{i=0}^{n-1} \left(\int_X \chi_i d\Xi \right) = \sum_{i=0}^{n-1} \Xi(q_i) \geq n\delta.$$

Hence $\frac{i_*(\bar{q})}{n} \geq \delta$. Thus $\mathrm{int}(Q) \geq \delta$. $\square_{\text{Theorem 4.1.7}}$

In particular, we get such a result for random forcing in two versions (see Section 1.5.8):

Corollary 4.1.8. *Let $\mathbb{P}$ be the random forcing notion*[3]*, $C \subseteq {}^{\omega}2$ be a non-empty clopen set and $\delta \in [0,1]$. If $Q := \{p \in \mathbb{P} \colon \mathrm{Leb}_C(p) \geq \delta\}$, then $\mathrm{int}(Q) \geq \delta$.*

The following result is known as *Kelley's theorem* (see [Kel59, Thm. 2]) and constitutes one of the main results of [Kel59]. Unfortunately, for time constraints we omit the proof.

Theorem 4.1.9. *Let $\mathscr{B}$ be a Boolean algebra. If $Q \subseteq \mathscr{B}$ then there exists a probability finitely additive measure $\Xi \colon \mathscr{B} \to [0,1]$ such that $\inf\{\Xi(b) \colon b \in Q\} = \mathrm{int}(Q)$.*

As a consequence (see [Kel59, Cor. 3]), under the conditions of the previous theorem, if Q is a non-empty set, then the intersection number is the supremum of the numbers $\inf\{\Xi(b) \colon b \in Q\}$ for all measures Ξ on $\mathscr{B}$.

[3] Either $\mathbb{B}$ or $\mathcal{B}({}^{\omega}2)/\mathcal{N}$.

Let $\mathscr{B}$ a Boolean algebra, $n < \omega$, $B := \{b_i \colon i < n\} \subseteq \mathscr{B}$ and Ξ a probability finitely additive measure on $\mathscr{B}$ such that, for any $i < n$, $\Xi(b_i) \geq \delta \in [0,1]$. Consider $\mathscr{C} := \langle B \rangle$. Since $\mathscr{C}$ is finitely generated, by Corollary 3.5.13, we have that $\mathrm{At}_{\mathscr{C}} = \{b_\sigma \colon \sigma \in {}^B 2 \wedge b_\sigma \neq 0\}$. Define $\Sigma_{\mathscr{C}} := \{\sigma \in {}^B 2 \colon b_\sigma \in \mathrm{At}_{\mathscr{C}}\}$ and let $\varepsilon > 0$. We can approximate the Ξ-measure of the atoms of $\mathscr{C}$ by rational numbers as follows: there exists $f \colon \mathrm{At}_{\mathscr{C}} \to [0,1]_{\mathbb{Q}}$ such that $\sum_{\sigma \in \Sigma_{\mathscr{C}}} f(\sigma) = 1$ and, for any $i < n$, $|\Xi(b_\sigma) - f(\sigma)| < \varepsilon$, and

$$\sum \{f(\sigma) \colon \sigma \in \Sigma_{\mathscr{C}} \wedge \sigma(i) = 0\} > \delta - \varepsilon.$$

This result is not difficult to prove, in fact, the proof is similar to the one we are going to present for Crucial Lemma 4.1.10, below.

We know that random forcing can be defined as a Boolean algebra (see Example 1.5.14), and by Theorem 1.5.47, we can embed $\tilde{\mathbb{E}}$ in a random forcing Boolean sub-algebra. Furthermore, in both, we have the Lebesgue measure, so in both cases, there is a function f that allows us to approximate the measure of the atoms of some finitely generated sub-algebra, as we discussed previously. As it turns out, the lower bound $\delta - \varepsilon$ plays a fundamental role in the proof of the limit step of the iteration, both for random and for $\tilde{\mathbb{E}}$ (see [She00, Lem. 2.14] and [KST19, Lem. 2.39], respectively).

The way in which we approach the generalization of the limit step is based on being able to generalize that approximation for forcing notions. In principle we have two obstacles: on the one hand, we need a "set of atoms" and on the other, we need a finitely additive measure. The set of atoms is not a problem because, if $\mathbb{P}$ is a forcing notion, $\{p_i \colon i < n\} \in Q^n$ and $Q \subseteq \mathbb{P}$, then

$$\{\sigma \in {}^n 2 \colon \exists q \in \mathbb{P}\, \forall i < n\, [(\sigma(p_i) = 0 \Rightarrow q \leq p_i) \wedge (\sigma(p_i) = 1 \Rightarrow p_i \perp q)]\},$$

behaves, for our purposes, like codes for a set of atoms. However, the existence of the finitely additive measure is not so easy and, in fact, this is where the intersection number for forcing notions plays its stellar role because if $\mathrm{int}^{\mathbb{P}}(Q)$ is large enough, *Kelley's theorem* guarantees us the existence of a non-trivial finitely additive measure, which will play the role of Ξ.

In the following lemma, which constitutes one of the most important results of this thesis, we formalize the previous discussion for forcing notions in general.

Crucial Lemma 4.1.10. Let $\mathbb{P}$ be a forcing notion, $\delta \in [0,1]$ and $Q \subseteq \mathbb{P}$ such that $\mathrm{int}^{\mathbb{P}}(Q) \geq \delta$. Let $\varepsilon > 0$, $n < \omega$ and $\bar{p} = \langle p_i \colon i < n \rangle \in Q^n$. Define

$$\Sigma := \{\sigma \in {}^n 2 \colon \exists q \in \mathbb{P}\, \forall i < n\, [(\sigma(p_i) = 0 \Rightarrow q \leq p_i) \wedge (\sigma(p_i) = 1 \Rightarrow p_i \perp q)]\}.$$

Then, there exists a function $f \colon \Sigma \to [0,1]_{\mathbb{Q}}$ such that:

$$\forall i < n \left(\sum \{f(\sigma) \colon \sigma \in \Sigma \wedge \sigma(i) = 0\} > \delta - \varepsilon \right) \text{ and } \sum_{\sigma \in \Sigma} f(\sigma) = 1.$$

Proof. Let $\mathbb{P}$ be a forcing notion and $(\mathscr{B}, \iota)$ its forcing completion. Define $A := \iota[Q] \subseteq \mathscr{B} \setminus \{0_{\mathscr{B}}\}$. By Corollary 4.1.3, we have that $\mathrm{int}^{\mathscr{B}}(A) = \mathrm{int}^{\mathbb{P}}(Q) \geq \delta$. On the other hand, by Kelley's theorem (see Theorem 4.1.9), there exists a finitely additive measure $\Xi \colon \mathscr{B} \to [0,1]$ such that, for any $a \in A$, $\Xi(a) \geq \delta$.

Let $\sigma \in {}^{n}2$. Define $b_\sigma := \bigwedge_{i<n} \iota(p_i)^{\sigma(i)}$, $I_0 := \{i < n \colon \sigma(i) = 0\}$ and $I_1 := \{i < n \colon \sigma(i) = 1\}$. Then,

$$\begin{aligned}
b_\sigma \neq 0_{\mathscr{B}} &\Leftrightarrow \bigwedge_{i\in I_0} \iota(p_i) \wedge \bigwedge_{i\in I_1} {\sim}\iota(p_i) \neq 0_{\mathscr{B}} \\
&\Leftrightarrow \exists b \in \mathscr{B} \left(b \leq \bigwedge_{i\in I_0} \iota(p_i) \text{ and } b \leq \bigwedge_{i\in I_1} {\sim}\iota(p_i) \right) \\
&\Leftrightarrow \exists b \in \mathscr{B}\, \forall i < n\, [i \in I_0 \Rightarrow b \leq \iota(p_i)) \wedge (i \in I_1 \Rightarrow b \leq {\sim}\iota(p_i))] \\
&\Leftrightarrow \exists b \in \mathscr{B}\, \forall i < n\, [(\sigma(i) = 0 \Rightarrow b \leq \iota(p_i)) \wedge (\sigma(i) = 1 \Rightarrow b \perp \iota(p_i))] \\
&\Leftrightarrow \sigma \in \Sigma.
\end{aligned}$$

Therefore, for all $\sigma \in {}^{n}2$, $b_\sigma \neq 0_{\mathscr{B}} \Leftrightarrow \sigma \in \Sigma$.
Now we can find a sequence $\langle f(\sigma) \colon \sigma \in \Sigma \rangle$ such that, for each $\sigma \in \Sigma$,

- $f(\sigma) \in (0,1)_{\mathbb{Q}}$,
- $|\Xi(b_\sigma) - f(\sigma)| < \frac{\varepsilon}{2^n}$,
- $\sum_{\sigma\in\Sigma} f(\sigma) = 1$.

Let $i < n$. Then,

$$\begin{aligned}
\sum \{f(\sigma) \colon \sigma \in \Sigma \wedge \sigma(i) = 0\} &> \sum \left\{ \Xi(b_\sigma) - \frac{\varepsilon}{2^n} \colon \sigma \in \Sigma \wedge \sigma(i) = 0 \right\} \\
&= \Xi(\iota(p_i)) - \sum \left\{ \frac{\varepsilon}{2^n} \colon \sigma \in \Sigma \wedge \sigma(i) = 0 \right\} \\
&\geq \Xi(\iota(p_i)) - \varepsilon \\
&\geq \delta - \varepsilon,
\end{aligned}$$

which proves the result. $\square_{\text{CrucialLemma 4.1.10}}$

4.2 μ-FAM-linkedness

In this section we introduce a new linkedness notion: μ-FAM-*linkedness*, and we study some of its properties. In particular, we show that this notion is stronger than μ-Fr-linkedness (see Section 1.5.5). Finally, based on [She00] and [KST19], we show that both $\mathbb{B}$ and $\tilde{\mathbb{E}}$ are examples of μ-FAM-linked forcing notions.

Remark 4.2.1. From the beginning of this section and until the end of the thesis, all the finitely additive measures that we are going to consider are probability free finitely additive measures. So, henceforth, when we say "finitely additive measure", we mean "probability free finitely additive measure on $\mathcal{P}(\omega)$".

4.2.1 $(\Xi, \bar{I}, \varepsilon_0)$-FAM-linkednesss

Jakob Kellner, *Saharon Shelah* and *Anda Tănasie* proved that (see [KST19, Lem. 1.20]), if all elements of $Q \subseteq \tilde{\mathbb{E}}$ have the same trunk and $\mathrm{loss} \leq \varepsilon$, for a given $\varepsilon > 0$, then they can define a limit $\lim^{\Xi} \colon Q^\omega \to \tilde{\mathbb{E}}$ that they called *strong* FAM *limit for intervals* with respect to a finitely additive measure Ξ, and this limit satisfies fundamental properties, both to build the iteration using finitely additive measures with $\mathbb{E}$, and in its applications. On the other hand, if for each $T \in \mathbb{B}$ we define

$$\mathrm{loss}(T) \coloneqq \mathrm{Leb}([\mathrm{trunk}(T)]) - \mathrm{Leb}([T]),$$

we obtain functions of trunk and loss for $\mathbb{B}$. Thanks to *Saharon Shelah* (see [She00, Lem. 2.17, Lem. 2.18]), we have an analogous result for random forcing, and again, the strong FAM limit in random forcing plays a key role in the original construction of the iteration and in the application to show that $\mathrm{cov}(\mathcal{N})$ may have countable cofinality. So being able to define this limit in a general way is a fundamental step for our construction of an iterated forcing theory with finitely additive measures.

In this section, we are going to define a new notion of subsets, which manages to capture the idea of the strong FAM limit for intervals as a linkedness notion, and we study some of its fundamental properties, particularly those involved in iteration construction. It is important to mention that the following definition is based on ideas and intuitions of [KST19], in particular, it extends [KST19, Def. 1.7] and [KST19, Def. 1.10].

Definition 4.2.2. Let Ξ be a finitely additive measure, $\bar{I} = \langle I_k \colon k < \omega \rangle$ be an interval partition of ω and $\varepsilon_0 \in (0, 1)$. Let $\mathbb{P}$ be a forcing notion. We say that $Q \subseteq \mathbb{P}$ is $(\Xi, \bar{I}, \varepsilon_0)$-*linked*[4] if there is a function $\lim^{\Xi} \colon Q^\omega \to \mathbb{P}$,[5] satisfying the following condition. Given

- $i^* < \omega$ and $\bar{q}^i = \langle q^i_\ell \colon \ell < \omega \rangle \in Q^\omega$, for each $i < i^*$,
- $m^* < \omega$ and a partition $\langle B_m \colon m < m^* \rangle$ of ω,
- $\varepsilon > 0$ and $k^* < \omega$,
- $q \in \mathbb{P}$ such that, for all $i < i^*$, $q \leq \lim^{\Xi}(\bar{q}^i)$,

there are a finite set $u \subseteq \omega \setminus k^*$, and some $q' \in \mathbb{P}$ such that $q' \leq q$ and

1. $\left| \frac{|u \cap B_m|}{|u|} - \Xi(B_m) \right| < \varepsilon$, for all $m < m^*$,
2. $\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q' \leq q^i_\ell\}|}{|I_k|} \geq 1 - \varepsilon_0 - \varepsilon$, for all $i < i^*$.

When the context is clear we omit the superscript "Ξ" in "$\lim^{\Xi}$".

[4] In [KST19, Def. 1.10], this notion is called *strong*-FAM-*limit for intervals*.

[5] In [KST19], this function is called FAM-*limit*.

To simplify the writing, from now on when we say "Q is $(\Xi, \bar{I}, \varepsilon_0)$-linked", it will be understood that Ξ is a finitely additive measure, $\bar{I} = \langle I_k : k < \omega \rangle$ is an interval partition of ω and $\varepsilon_0 \in (0, 1)$.

In the context of Definition 4.2.2, consider for any $i < i^*$, $f_i : \omega \to \mathbb{R}$ such that, for each $k < \omega$, $f_i(k) := \frac{|\{\ell \in I_k : q' \leq q^i_\ell\}|}{|I_k|}$, and consider the *counting finitely additive measure* Ξ^u, as in Example 3.1.9. Then, the conclusion of Definition 4.2.2 allow us to approximate the values $\Xi(B_m)$ with $\Xi^u(B_m)$, for any $m < m^*$, where the integral of f_i with respect to Ξ^u are big. That is, by Theorem 3.5.11, we can reformulate (1) and (2) in terms of Ξ^u, as follows:

1. $|\Xi^u(B_m) - \Xi(B_m)| < \varepsilon$, for all $m < m^*$,
2. $\int_\omega f_i \, d\Xi^u \geq 1 - \varepsilon_0 - \varepsilon$, for all $i < i^*$.

Now, we present a first easy example of a $(\Xi, \bar{I}, \varepsilon_0)$-linked subset:

Example 4.2.3. Let $\mathbb{P}$ be a forcing notion and $Q := \{p\} \subseteq \mathbb{P}$. Then Q is $(\Xi, \bar{I}, \varepsilon_0)$-linked. Indeed, define $\lim : Q^\omega \to \mathbb{P}$ such that, for all $\bar{q} \in Q^\omega$, $\lim(\bar{q}) := p$. Let $i^* < \omega$, $\bar{q}^i \in Q^\omega$ for each $i < i^*$, $k^* < \omega$, $\langle B_m : m \leq m^* \rangle$ a partition of ω, $\varepsilon > 0$ and a condition $q \in \mathbb{P}$, such that for every $i < i^*$, $q \leq \lim(\bar{q}^i)$.

Let $q' := q$. For the first condition of Definition 4.2.2, we can define the finite set u as in the proof of Theorem 3.5.26, and the second one is immediate because, in this case, it does not depend on the choice of u, that is, since for all $i < i^*$ and $\ell \in I_k$, $q' = q \leq \lim(\bar{q}) = p = q^i_\ell$, we have that:

$$\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k : q' \leq q^i_\ell\}|}{|I_k|} = \frac{1}{|u|} \sum_{k \in u} \frac{|I_k|}{|I_k|} = \frac{1}{|u|} \sum_{k \in u} 1 = \frac{|u|}{|u|} = 1 \geq 1 - \varepsilon_0 - \varepsilon.$$

Finally, we conclude that for every $p \in \mathbb{P}$, $\{p\}$ is $(\Xi, \bar{I}, \varepsilon_0)$-linked.

A first interesting characteristic of $(\Xi, \bar{I}, \varepsilon_0)$-linkedness is that it is preserved under complete embeddings:

Theorem 4.2.4. *Let $\mathbb{P}, \mathbb{Q}$ be forcing notions and $\iota : \mathbb{P} \to \mathbb{Q}$ a complete embedding. If $Q \subseteq \mathbb{P}$ is $(\Xi, \bar{I}, \varepsilon_0)$-linked, then so is $\iota[Q] \subseteq \mathbb{Q}$.*

Proof. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions and $\iota : \mathbb{P} \to \mathbb{Q}$ a complete embedding. Suppose that $Q \subseteq \mathbb{P}$ is $(\Xi, \bar{I}, \varepsilon_0)$-linked and define $R := \iota[Q]$. We denote by $\lim_Q : Q^\omega \to \mathbb{P}$ the function given by Definition 4.2.2.

Notice that, for any $\bar{r} = \langle r_\ell : \ell < \omega \rangle \in R^\omega$, there exists some $\bar{q}_{\bar{r}} = \langle q_{\bar{r},\ell} : \ell < \omega \rangle \in Q^\omega$ such that, for all $\ell < \omega$, $\iota(q_{\bar{r},\ell}) = r_\ell$. So we can define $\lim_R : R^\omega \to \mathbb{Q}$ such that, for $\bar{r} = \langle r_\ell : \ell < \omega \rangle \in R^\omega$, $\lim_R(\bar{r}) := \iota(\lim_Q(\bar{q}_{\bar{r}}))$. Let $i^* < \omega$, $\bar{r}^i \in R^\omega$ for each $i < i^*$, $k^* < \omega$, $\langle B_m : m < m^* \rangle$ a finite partition of ω, $\varepsilon > 0$ and $r \in \mathbb{Q}$ be such that $r \leq \lim_R(\bar{r}^i)$ for all $i < i^*$. Let $q \in \mathbb{P}$ be a reduction of r, hence

$$\iota(q) \parallel r \leq \lim_R(\bar{r}^i) = \iota(\lim_Q(\bar{q}^i_{\bar{r}})).$$

Therefore, we can find $q' \leq q$ such that, for any $i < i^*$, $q' \leq \lim_Q(\bar{q}^i_{\bar{r}})$ and, applying that Q is $(\Xi, \bar{I}, \varepsilon_0)$-linked, we get a finite set $u \subseteq \omega \setminus k^*$ and $q'' \leq q$ such that:

1_Q. $\left|\frac{|u\cap B_m|}{|u|} - \Xi(B_m)\right| < \varepsilon$, for all $m < m^*$,

2_Q. $\frac{1}{|u|}\sum_{k\in u}\frac{|\{\ell\in I_k\colon q''\leq q^i_{\bar{r},\ell}\}|}{|I_k|} \geq 1-\varepsilon_0-\varepsilon$, for all $i<i^*$.

Find $r'\in\mathbb{Q}$ such that $r'\leq\iota(q''),\, r$. We must verify properties (1) and (2) from Definition 4.2.2 for r' and $\bar{r}^i$ with $i<i^*$. Notice that (1) is $(1)_Q$ because it does not depend on Q. On the other hand, let $k\in u$ and $\ell\in I_k$. By our definitions,

$$q''\leq_{\mathbb{P}} q^i_{\bar{r},\ell}\Rightarrow \iota(q'')\leq_{\mathbb{Q}}\iota(q^i_{\bar{r},\ell})\Rightarrow r'\leq_{\mathbb{Q}} r^i_\ell.$$

Then by $(2)_Q$ we have that:

$$\frac{1}{|u|}\sum_{k\in u}\frac{|\{\ell\in I_k\colon r'\leq_{\mathbb{Q}} r^i_\ell\}|}{|I_k|}\geq\frac{1}{|u|}\sum_{k\in u}\frac{|\{\ell\in I_k\colon q''\leq_{\mathbb{P}} q^i_{\bar{r},\ell}\}|}{|I_k|}\geq 1-\varepsilon_0-\varepsilon,$$

Finally, we conclude that R is $(\Xi,\bar{I},\varepsilon_0)$-linked. $\square_{\text{Theorem 4.2.4}}$

It is very useful in applications to have conditions that force infinitely many elements of the forcing notion to be in the generic filter, for instance, this is the defining feature of Fr-linked subsets (Definition 1.5.28). It turns out that the limit $\lim^\Xi$ of the Definition 4.2.2 not only forces infinitely many conditions into the generic filter but also forces something stronger:

Theorem 4.2.5. *Let $\mathbb{P}$ be a forcing notion. If $Q\subseteq\mathbb{P}$ is $(\Xi,\bar{I},\varepsilon_0)$-linked, there exists a $\mathbb{P}$-name $\dot{\Xi}^*$ of a probability free finitely additive measure with domain $\mathcal{P}(\omega)$ extending Ξ, such that, for any $\bar{q}\in Q^\omega$,*

$$\lim(\bar{q})\Vdash\text{“}\int_\omega\frac{|\{\ell\in I_k\colon q_\ell\in\dot{G}_{\mathbb{P}}\}|}{|I_k|}\,d\,\dot{\Xi}^*(k)\geq 1-\varepsilon_0\text{”}.$$

Proof. Let $G\subseteq\mathbb{P}$ be a generic filter over M.
Working in $\mathrm{M}[G]$, define $\mathcal{L}\coloneqq\{\bar{q}\in Q^\omega\cap\mathrm{M}\colon\lim(\bar{q})\in G\}$ and let $i^*<\omega$ and, for each $i<i^*$, $\bar{q}^i=\langle q^i_\ell\colon\ell<\omega\rangle\in\mathcal{L}$.
Now, working in the ground model, let $\varepsilon>0$, $k^*<\omega$ and a finite partition $\langle B_m\colon m<m^*\rangle$ of ω.
Define $\mathcal{D}_1$ as the set of conditions $r\in\mathbb{P}$, such that there exists a finite set $u\subseteq\omega\setminus k^*$ satisfying:

1_r. $\left|\frac{|u\cap B_m|}{|u|} - \Xi(B_m)\right| < \varepsilon$, for all $m < m^*$,

2_r. $\frac{1}{|u|}\sum_{k\in u}\frac{|\{\ell\in I_k\colon r\leq q^i_\ell\}|}{|I_k|} \geq 1-\varepsilon_0-\varepsilon$, for all $i<i^*$.

Consider $\mathcal{D}_2\coloneqq\{r\in\mathbb{P}\colon\exists i<i^*(r\perp\lim(\bar{q}^i))\}$ and $\mathcal{D}\coloneqq\mathcal{D}_1\cup\mathcal{D}_2$. Notice that $\mathcal{D}$ is dense in $\mathbb{P}$. Indeed, let $p\in\mathbb{P}$. Let us show that,

$$\exists r\in\mathcal{D}_2(r\leq p)\ \text{ or }\ \exists r\leq p\,\forall i<i^*(r\leq\lim(\bar{q})^i)\tag{4.2.1}$$

Assume that, for any $r\in\mathcal{D}_2$, $r\not\leq p$ and apply induction on $i\leq i^*-1$:

1. The base step $i = 0$: if $p \perp \lim(\bar{q}^0)$, then $p \in \mathcal{D}_2$, hence $p \nleq p$, which is a contradiction. Therefore, there exists some $r \leq p$ such that, $r \leq \lim(\bar{q}^0)$.

2. The induction step: by the induction hypothesis, there is some $r_0 \leq p$ such that, for each $j \leq i-1$, $r_0 \leq \lim(\bar{q}^j)$. If $r_0 \perp \lim(\bar{q}^i)$, then $r_0 \in \mathcal{D}_2$, hence $r_0 \nleq p$, which is a contradiction. As a consequence, we can find $r \leq r_0$ such that $r \leq \lim(\bar{q}^i)$. It is clear that r is as desired.

In the first case of Equation 4.2.1, $r \in \mathcal{D}$ and $r \leq p$, so we are done. On the other hand, in the second one, we get a condition $r \leq p$ to which we can apply Definition 4.2.2, by virtue of which there is some $r' \in \mathcal{D}_1$ such that $r' \leq r \leq p$. Finally, $\mathcal{D}$ is dense in $\mathbb{P}$.

It is clear also that $\mathcal{D} \in \mathrm{M}$, because its definition does not depend on the generic filter.

Working on the generic extension again, since $\mathcal{D} \in \mathrm{M}$ is dense, $G \cap \mathcal{D} \neq \emptyset$. Choose $q' \in \mathcal{D}$ such that $q' \in G$. Since $q' \in \mathcal{D}$, necessarily $q' \in \mathcal{D}_1$, hence there is a finite set $u \subseteq \omega \setminus k^*$ such that:

1. $\left|\frac{|u \cap B_m|}{|u|} - \Xi(B_m)\right| < \varepsilon$, for all $m < m^*$.

2. $\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q^i_\ell \in G\}|}{|I_k|} \geq \frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q' \leq q^i_\ell\}|}{|I_k|} \geq 1 - \varepsilon_0 - \varepsilon$, for all $i < i^*$.

By Theorem 3.5.25, there exists a finitely additive measure Ξ^* extending Ξ such that, for all $\bar{q} = \langle q_\ell \colon \ell < \omega \rangle \in \mathcal{L}$, we have that:

$$\int_\omega \frac{|\{\ell \in I_k \colon q_\ell \in G\}|}{|I_k|} \, d\Xi^*(k) \geq 1 - \varepsilon_0 - \varepsilon.$$

Finally, working in M, let $\dot{\Xi}^*$ a $\mathbb{P}$-name of Ξ^*. It is clear that $\dot{\Xi}^*$ is as required since, for any $\mathbb{P}$-generic G over M if $\lim(\bar{q}) \in G$, then, in $\mathrm{M}[G]$, $\bar{q} \in \mathcal{L}$ and

$$\int_\omega \frac{|\{\ell \in I_k \colon q_\ell \in G\}|}{|I_k|} \, d\,\Xi^*(k) \geq 1 - \varepsilon_0.$$

As a consequence, in the ground model,

$$\lim(\bar{q}) \Vdash \text{“} \int_\omega \frac{|\{\ell \in I_k \colon q_\ell \in \dot{G}_{\mathbb{P}}\}|}{|I_k|} d\dot{\Xi}^*(k) \geq 1 - \varepsilon_0\text{”}.$$

$\square_{\text{Theorem 4.2.5}}$

We can prove some combinatorial results about the $(\Xi, \bar{I}, \varepsilon_0)$-linked forcing notions:

Lemma 4.2.6. *If* $\lim_{k \to \infty} |I_k| = \infty$ *then no* $(\Xi, \bar{I}, \varepsilon_0)$*-linked subset of a forcing notion* $\mathbb{P}$ *can contain infinite antichains.*

Proof. Let $\mathbb{P}$ be a forcing notion and let $Q \subseteq \mathbb{P}$ be a $(\Xi, \bar{I}, \varepsilon_0)$-linked subset. Towards contradiction, suppose that $A \subseteq Q$ is an infinite antichain in $\mathbb{P}$. So we can define $\bar{q} = \langle q_\ell \colon \ell < \omega \rangle \in A^\omega \subseteq Q^\omega$ without repetitions such that for any $i, j < \omega$, if $i \neq j$ then $q_i \perp q_j$. Let $\varepsilon > 0$ such that $1 - \varepsilon_0 - \varepsilon > 0$ and $N < \omega$ such that $\frac{1}{N} < 1 - \varepsilon_0 - \varepsilon$. Since $\lim_{k \to \infty} |I_k| = \infty$, there exits some $k^* < \omega$ such that, for any $k > k^*$, we have that $|I_k| > N$. Let $q \leq \lim(\bar{q})$. By Definition 4.2.2, there are a finite set $u \subseteq \omega \setminus k^*$ and some condition $q' \leq q$ such that:

$$\frac{1}{|u|}\sum_{k\in u}\frac{|\{\ell \in I_k \colon q' \leq q_\ell\}|}{|I_k|} \geq 1 - \varepsilon_0 - \varepsilon.$$

However, notice that $k \in u \Rightarrow k > k^*$ and since $\operatorname{ran}(\bar{q}) \subseteq A$ and it is an antichain, necessarily for any $k \in u$ and $\ell < \omega$, $|\{\ell \in I_k \colon q' \leq q_\ell\}| \leq 1$. Then,

$$\begin{aligned} 1 - \varepsilon_0 - \varepsilon &\leq \frac{1}{|u|}\sum_{k\in u}\frac{|\{\ell \in I_k \colon q' \leq q_\ell\}|}{|I_k|} \leq \frac{1}{|u|}\sum_{k\in u}\frac{1}{|I_k|} \\ &< \frac{1}{|u|}\sum_{k\in u}\frac{1}{N} = \frac{1}{|u|}\frac{|u|}{N} = \frac{1}{N} < 1 - \varepsilon_0 - \varepsilon, \end{aligned}$$

which is a contradiction. Finally, if $A \subseteq Q$ is an anti-chain, then A is finite. $\square_{\text{Lemma 4.2.6}}$

As mentioned in Subsection 1.5.5, not containing infinite anti-chains is a particular property of Fr-linked subsets. The following lemma shows that this is not just a coincidence: as a consequence of Theorem 4.2.5, the notion of $(\Xi, \bar{I}, \varepsilon_0)$-linked is stronger than that of Fr-linked:

Lemma 4.2.7. *Let $\mathbb{P}$ be a forcing notion. If $Q \subseteq \mathbb{P}$ is $(\Xi, \bar{I}, \varepsilon_0)$-linked, then Q is* Fr*-linked.*

***Proof*.** Let $\bar{q} = \langle q_\ell \colon \ell < \omega\rangle \in Q^\omega$. Let us prove that $\lim(\bar{q}) \Vdash$ "$\exists^\infty_{\ell<\omega}(q_\ell \in \dot{G}_{\mathbb{P}})$". Let G be a $\mathbb{P}$-generic filter over M such that $\lim(\bar{q}) \in G$. Fix Ξ^* as in Theorem 4.2.5.
Working in the generic extension, define $f \colon \omega \to \mathbb{R}$ such that, for all $k < \omega$, $f(k) := \frac{|\{\ell\in I_k \colon q_\ell \in G\}|}{|I_k|}$. Towards contradiction, suppose that $\{\ell < \omega \colon q_\ell \in G\}$ is finite. This implies that $\Xi(D) = 0$ where $D := \{k < \omega \colon f(k) \neq 0\}$ and therefore, by Corollary 3.5.23,

$$\int_\omega f d\,\Xi^* = \int_{\omega\setminus D} f d\,\Xi^* = \int_{\omega\setminus D} 0 d\,\Xi^* = 0,$$

which contradicts Theorem 4.2.5. Thus, $\{\ell < \omega \colon q_\ell \in G\}$ is infinite, and therefore,

$$\lim(q) \Vdash \text{“}\exists^\infty_{\ell<\omega}\,(q_\ell \in \dot{G}_{\mathbb{P}})\text{”},$$

that is, Q is Fr-linked. $\square_{\text{Lemma 4.2.7}}$

Furthermore, it will be seen later (see Theorem 4.2.21) that, in fact, it is strictly stronger, that is, there are σ-FAM-linked forcing notions that are not σ-Fr-linked.

4.2.2 μ-FAM-linkedness

The iteration that *Saharon Shelah* built in [She00] has a very particular structure, for example, in each step the iteration is accompanied by a succession of finitely additive measures on $\mathcal{P}(\omega)$ that satisfy some special properties (see [She00, Def. 2.11] or Definition 4.3.13). So, in order to build an iteration of a certain length, in each step it must be possible to define the above mentioned sequence of finitely additive measures, which is not easy and in fact, the random forcing structure is heavily used to do so. *Jakob Kellner*, *Saharon Shelah* and *And Tănasie* in [KST19] proved that the key to being able to extend the iteration at the successor steps is in the notion of $(\Xi, \bar{I}, \varepsilon_0)$-linked (see [KST19, Lem. 2.25]), however, the problem remained open for the limit step, although they managed to show that with $\tilde{\mathbb{E}}$ the iteration can also be built, that is, that it can also be extended in the limit steps.

Analyzing the structure of $\tilde{\mathbb{E}}$ and $\mathbb{B}$ that allows establishing extension theorems at limit steps, we were able to abstract a property that allowed us to generalize the iteration and that led us to a new notion of linkedness: μ-FAM-linked, which extends $(\Xi, \bar{I}, \varepsilon_0)$-linked, but also includes a property that will be the key to generalizing the iteration: a condition on the intersection number. We could say, concisely, that the notion of μ-FAM-linked is an abstraction of the properties of $\mathbb{B}$ and $\tilde{\mathbb{E}}$ that allows the iteration built on [She00] works.

Using this linkedness notion, conditions for establishing extension theorems are already achieved in both the limit step and the successor step, therefore **this will allow us to define a general theory of iterated forcing with finitely additive measures.**

Definition 4.2.8. Let Ξ be a finitely additive measure on $\mathcal{P}(\omega)$, $\mathbb{P}$ be a forcing notion and μ be a cardinal. We say that $\mathbb{P}$ is μ-Ξ-*linked*, if there exists a sequence $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu \wedge \varepsilon \in (0,1)_{\mathbb{Q}} \rangle$ of subsets of $\mathbb{P}$, such that:

1. For any $\bar{I} \in \mathcal{I}_\infty$, each $Q_{\alpha,\varepsilon}$ is $(\Xi, \bar{I}, \varepsilon)$-linked, where $\mathcal{I}_\infty$ is the collection of interval partitions on ω such that $\lim_{k\to\infty} |I_k| = \infty$,
2. For every $\varepsilon \in (0,1)_{\mathbb{Q}}$, $\bigcup_{\alpha<\mu} Q_{\alpha,\varepsilon}$ is dense in $\mathbb{P}$,
3. For any $\alpha < \mu$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$, $\operatorname{int}(Q_{\alpha,\varepsilon}) \geq 1 - \varepsilon$,

Finally, we say that $\mathbb{P}$ is μ-FAM-*linked*, if there exists a sequence $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu \wedge \varepsilon \in (0,1)_{\mathbb{Q}} \rangle$ of subsets of $\mathbb{P}$, witnessing that $\mathbb{P}$ is μ-Ξ-linked for all finitely additive measures Ξ on $\mathcal{P}(\omega)$.

As usual, when $\mu = \aleph_0$, we write "σ-FAM-linked" instead of "$\aleph_0$-FAM-linked".

We know that conditions (1) and (2) are usual in this type of definition (see, for example Definition 1.5.28(2)). Although condition (3) seems to be taken out of the sleeve, it is the one that will allow us, using Crucial Lemma 4.1.10, to extend in the limit steps, and therefore, build a theory of iterated forcing with finitely additive measures.

As a consequence of some properties about $(\Xi, \bar{I}, \varepsilon_0)$-linkedness that we have already proved, we get two results about this new linkedness notion. The first is that dense embeddings preserve the μ-FAM-linked property:

Theorem 4.2.9. *Let $\mathbb{P}$, $\mathbb{Q}$ be forcing notions. If $\iota \colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding and $\mathbb{P}$ is μ-*FAM-*linked, then $\mathbb{Q}$ is μ-*FAM-*linked.*

Proof. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions and $\iota \colon \mathbb{P} \to \mathbb{Q}$ a dense embedding. Suppose that $\mathbb{P}$ is μ-FAM-linked witnessed by $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu \wedge \varepsilon \in (0,1)_{\mathbb{Q}} \rangle$. In the natural way, define for all $\alpha < \mu$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$, $R_{\alpha,\varepsilon} := \iota[Q_{\alpha,\varepsilon}]$. Let us show that $\langle R_{\alpha,\varepsilon} \colon \alpha < \mu \wedge \varepsilon \in (0,1)_{\mathbb{Q}} \rangle$ witnesses that $\mathbb{Q}$ is μ-FAM-linked by verifying the conditions of Definition 4.2.8:

1. Since each for any $\alpha < \mu$, $Q_{\alpha,\varepsilon}$ is $(\Xi, \bar{I}, \varepsilon)$-linked, by Theorem 4.2.4 so is $R_{\alpha,\varepsilon}$.

2. Fix $\varepsilon \in (0,1)_{\mathbb{Q}}$ and let $r \in \mathbb{Q}$. Since $\operatorname{ran}(\iota)$ is dense in $\mathbb{Q}$, there are $p \in \mathbb{P}$ and $r' \in \mathbb{Q}$ such that $\iota(p) = r' \leq_{\mathbb{Q}} r$. On the other hand, since $\bigcup_{\alpha<\mu} Q_{\alpha,\varepsilon}$ is dense in $\mathbb{P}$, there are an $\alpha^* < \mu$ and $p' \in Q_{\alpha^*,\varepsilon}$ such that $p' \leq_{\mathbb{P}} p$. Therefore, $\iota(p') \leq_{\mathbb{P}} \iota(p) = r' \leq r$. Thus $\bigcup_{\alpha<\mu} R_{\alpha,\varepsilon}$ is dense in $\mathbb{Q}$.

3. By Corollary 4.1.5, $\operatorname{int}^{\mathbb{Q}}(R_{\alpha,\varepsilon}) = \operatorname{int}^{\mathbb{Q}}(\iota[Q_{\alpha,\varepsilon}]) = \operatorname{int}^{\mathbb{P}}(Q_{\alpha,\varepsilon}) \geq 1 - \varepsilon$.

Thus, $\mathbb{Q}$ is also μ-FAM-linked. $\square_{\text{Theorem 4.2.9}}$

Recall that $\varphi(x)$ is a *forcing property* if, and only if, for any pair of forcing notions $\mathbb{P}$, $\mathbb{Q}$, if $\mathbb{P} \equiv \mathbb{Q}$ and $\varphi(\mathbb{P})$ holds, then $\varphi(\mathbb{Q})$ holds.

Remark 4.2.10. We have an equivalent reformulation of μ-FAM-linked: if we replace condition Definition 4.2.8(2) by "for every $\varepsilon \in (0,1)$, $\bigcup Q_{\alpha,\varepsilon} = \mathbb{P}$", since $Q \subseteq \mathbb{P}$ is $(\Xi, \bar{\imath}, \varepsilon)$-linked if, and only if, $Q^{\uparrow}$ is $(\Xi, \bar{I}, \varepsilon)$-linked, and $\operatorname{int}^{\mathbb{P}}(Q) = \operatorname{int}^{\mathbb{P}}(Q^{\uparrow})$. This equivalent definition allows to show that μ-FAM-linked is a forcing property.

Also, by Lemma 4.2.7, we get that:

Corollary 4.2.11. *Every* μ-FAM-*liked forcing notion is* μ-Fr-*linked.*

As a consequence, μ-FAM-linked forcing notions inherit the properties of μ-Fr-linkedness. For example, we can conclude by Lemma 1.5.29(3) that every μ-FAM-linked forcing notion is μ^+-c.c. However, by having more robust conditions in the definition, we can improve this result:

Theorem 4.2.12. *Every* μ-FAM-*linked forcing notion is* μ-m-*linked for all* $0 < m < \omega$.

Proof. Let $\mathbb{P}$ be a μ-FAM-linked forcing notion witnessed by $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu \wedge \varepsilon \in (0,1)_{\mathbb{Q}} \rangle$, and $m < \omega$ such that $m > 0$. Since $\frac{1}{m+1} \in (0,1)$, there exists some $\varepsilon_m \in (0,1)_{\mathbb{Q}}$ such that $\varepsilon_m < \frac{1}{m+1}$. Therefore by Definition 4.2.8(4), for all $\alpha < \mu$ we have that,

$$1 - \frac{1}{m+1} < 1 - \varepsilon_m \leq \operatorname{int}(Q_{\alpha,\varepsilon_m}).$$

Applying Lemma 4.1.6(7), we get that for all $\alpha < \mu$, Q_{α,ε_m} is m-linked. Now define for all $\alpha < \mu$, $Q^{\uparrow}_{\alpha} \coloneqq \{p \in \mathbb{P} \colon \exists q \in Q_{\alpha,\varepsilon_m}(q \leq p)\}$. It is clear that each $Q^{\uparrow}_{\alpha}$ is m-linked because Q_{α,ε_m} is. Finally, let $p \in \mathbb{P}$. By Definition 4.2.8(2), there are $\alpha_0 < \mu$ and $q \in Q_{\alpha_0,\varepsilon_m}$ such that $q \leq p$, hence $p \in Q^{\uparrow}_{\alpha_0}$. So $\bigcup_{\alpha<\mu} Q^{\uparrow}_{\alpha} = \mathbb{P}$. Thus, $\mathbb{P}$ is μ-m-linked. $\square_{\text{Theorem 4.2.12}}$

As a consequence, by applying Theorem 1.5.27, we get:

Corollary 4.2.13. *Every* μ-FAM-*linked forcing notion is* μ^+-m-*Knaster for all* $0 < m < \omega$.

As a result of Example 4.2.3 and Lemma 4.1.6(4), we get:

Example 4.2.14. Every forcing notion $\mathbb{P}$ is $|\mathbb{P}|$-FAM-linked witnessed by the sequence $\langle Q_{p,\varepsilon} \colon p \in \mathbb{P} \wedge \varepsilon \in (0,1)_{\mathbb{Q}} \rangle$, where $Q_{p,\varepsilon} \coloneqq \{p\}$ for all $p \in \mathbb{P}$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$.

In particular, since Cohen forcing is countable, it is σ-FAM-linked.

4.2.3 More sophisticated examples

Earlier, we said that the notion of μ-FAM-linkedness results from abstracting the properties of $\mathbb{B}$ and $\tilde{\mathbb{E}}$ that allows the iteration built in [She00] works. So, it is natural to expect that both $\mathbb{B}$ and $\tilde{\mathbb{E}}$ are, indeed, μ-FAM-linked for some cardinal μ. In this section we show that they really are, in fact, $\mu = \aleph_0$ in both cases.

For this section, fix a finitely additive measure Ξ on $\mathcal{P}(\omega)$ and $\bar{I} = \langle I_k \colon k < \omega \rangle \in \mathcal{I}_\infty$.

Random forcing is σ-FAM-linked

We start with random forcing. This result is due to *Saharon Shelah* (see [She00, Lem. 2.17 and Lem. 2.18]). The proof of [She00, Lem. 2.17] is perhaps the most difficult that appears in the paper, and there are steps, particularly related to probability, that are difficult to notice and follow, so here we take the trouble to present a complete and very detailed proof, even though that implied a long extension of it.

Lemma 4.2.15. *Let $\langle B_m \colon m < m^* \rangle$ be a finite partition of ω such that for any $m < m^*$, $a_m := \Xi(B_m)$. Let $r \in \mathbb{B}$, $i^* < \omega$, $\langle b_i \colon i < i^* \rangle$ a sequence of non-negative real numbers and for each $i < i^*$, $\bar{r}^i = \langle r^i_\ell \colon \ell < \omega \rangle \in \mathbb{B}^\omega$. For any $i < i^*$, $k < \omega$ and $r' \in \mathbb{B}$ such that $r' \leq r$, define:*

$$a^i_k(r') := \frac{1}{|I_k|} \sum_{\ell \in I_k} \mathrm{Leb}_{[r']}([r^i_\ell]),$$

and consider for $m \in M := \{m < m^* \colon a_m > 0\}$, $i < i^*$ and $r' \leq r$,

$$c_{i,m}(r') := \frac{1}{a_m} \int_{B_m} a^i_k(r')\, d\Xi(k).$$

Assume that:

$$\forall r' \leq r\, \forall i < i^* \left(\int_\omega a^i_k(r') d\Xi(k) \geq b_i \right) \tag{4.2.2}$$

Then, for all $\varepsilon > 0$, there are $r^* \leq r$ and a sequence of real numbers $\bar{c} = \langle c_{i,m} \colon i < i^* \wedge m < m^* \rangle$ such that, for all $i < i^*$ and $m < m^*$:

1. $0 \leq c_{i,m} \leq 1$,
2. $\sum_{m<m^*} c_{i,m} a_m \geq b_i$,
3. $D^* := \{r' \in \mathbb{B} \colon \forall i < i^*\, \forall m \in M\, (|c_{i,m}(r') - c_{i,m}| < \varepsilon)\}$ *is dense below r^* and $r^* \in D^*$.*

Proof. For any $i < i^*$ and $r' \leq r$, by Equation 4.2.2 and Theorem 3.5.22, we have that:

$$\begin{aligned}\sum_{m \in M} c_{i,m}(r')\, a_m &= \sum_{m \in M} \left(\frac{1}{a_m} \int_{B_m} a^i_k(r') d\Xi(k) \right) a_m = \sum_{m \in M} \left(\int_{B_m} a^i_k(r')\, d\Xi(k) \right) \\ &= \int_\omega a^i_k(r')\, d\Xi(k) \geq b_i,\end{aligned}$$

so we get:

$$\forall r' \leq r \, \forall i < i^* \left(\sum_{m \in M} c_{i,m}(r') \, a_m \geq b_i \right). \tag{4.2.3}$$

Let $\varepsilon > 0$ and $K < \omega$ large enough such that $\frac{1}{K} < \varepsilon$ and $\mathcal{C} \neq \emptyset$, where $\mathcal{C}$ is the set of finite sequences $\bar{c}$ such that:

- $\bar{c} = \langle c_{i,m} \colon i < i^* \wedge m < m^* \rangle$,
- $c_{i,m} \in [0,1]_{\mathbb{Q}}$,
- $K c_{i,m} \in \mathbb{Z}$,
- $\sum_{m < m^*} c_{i,m} \, a_m \geq b_i$.

It is clear that $\mathcal{C}$ is a finite set, hence there exists some $s^* < \omega$ such that $\mathcal{C} = \{\bar{c}^s \colon s < s^*\}$.
Now, **suppose** that, recursively on $s \leq s^*$, we can build a sequence $\langle r_s \colon s < s^* \rangle$ of random conditions such that:

- $r_0 \coloneqq r$ and for any $s < s^*$, $r_{s+1} \leq r_s$,
- for any $r' \leq r_{s+1}$, there are $i < i^*$ and $m \in M$ such that:

$$|c_{i,m}(r') - c^s_{i,m}| \geq \varepsilon.$$

On the other hand, suppose that $j < i^*$, $m \in M$ and

$$\left\{ \frac{\ell}{K} \colon \ell \leq K \right\} \cap [c_{j,m}(r_{s^*}), \, c_{j,m}(r_{s^*}) + \varepsilon) = \emptyset.$$

Consider two cases:

- If $c_{j,m}(r_{s^*}) + \varepsilon > 1$, then $c_{j,m}(r_{s^*}) \leq 1 = \frac{K}{K} < c_{j,m}(r_{s^*}) + \varepsilon$, which is a contradiction.
- If $c_{j,m}(r_{s^*}) + \varepsilon \leq 1$, let $\ell_0 \coloneqq \max \left\{ \ell \leq K \colon \frac{\ell}{K} \leq c_{j,m}(r_{s^*}) \right\}$. It is clear that $\ell_0 < K$. Also,

  $$\varepsilon = c_{j,n}(r_{s^*}) + \varepsilon - c_{j,m}(r_{s^*}) < \frac{\ell_0 + 1}{K} - \frac{\ell_0}{K} = \frac{1}{K} < \varepsilon,$$

  which is a contradiction.

As a consequence, for any $i < i^*$ and $m \in M$, we can find $c^*_{i,m} \in \left\{ \frac{\ell}{K} \colon \ell \leq K \right\}$ such that

$$c_{i,m}(r_{s^*}) \leq c^*_{i,m} < c_{i,m}(r_{s^*}) + \varepsilon.$$

In particular, we have that $|c^*_{i,m} - c_{i,m}(r_{s^*})| < \varepsilon$, for any $i < i^*$ and $m \in M$.

Since $r_{s^*} \leq r_0 = r$, by Equation 4.2.3, we have that:

$$b_i \leq \sum_{m \in M} c_{i,m}(r_{s^*})\, a_m \leq \sum_{m \in M} c^*_{i,m} a_m,$$

that is, $\bar{c}^* := \{c^*_{i,m} \colon i < i^* \wedge m \in M\} \in \mathcal{C}$, hence there is some $s < s^*$ such that $\bar{c}^* = \bar{c}^s$.

Now, since $r_{s^*} \leq r_{s+1}$, we have that $|c^s_{i,m} - c_{i,m}(r_{s^*})| \geq \varepsilon$ for some $i < i^*$ and $m \in M$. However, by the construction of $\bar{c}^*$,

$$|c^s_{i,m} - c_{i,m}(r_{s^*})| = |c^*_{i,m} - c_{i,m}(r_{s^*})| < \varepsilon,$$

which is a contraction. Therefore, r_{s^*} contradicts the choice of r_{s+1} and, as a consequence, we cannot reach step s^* in the induction, so we are "stuck" at some step $s < s^*$.

Let us show that D^* is dense below r_s: let $q \leq r_s$. Since r_{s+1} cannot be defined, there exists some $r' \leq q$ such that, for any $i < i^*$ and $m \in M$, $|c^s_{i,m} - c_{i,m}(r')| < \varepsilon$, that is, $r' \in D^*$. Finally, let $r^* \in D^*$ such that $r^* \leq r_s$. It is clear that r^*, $\bar{c}^s$ is as required. $\square_{\text{Lemma 4.2.15}}$

The following result is [She00, Lem. 2.17] and it is the key to prove that random forcing is σ-FAM-linked.

Lemma 4.2.16. *Let $\langle B_m \colon m < m^* \rangle$ be a finite partition of ω such that for any $m < m^*$, $a_m := \Xi(B_m)$. Let $r \in \mathbb{B}$, $i^* < \omega$, $\langle b_i \colon i < i^* \rangle$ a sequence of non-negative real numbers and for each $i < i^*$, $\bar{r}^i = \langle r^i_\ell \colon \ell < \omega \rangle \in \mathbb{B}^\omega$. Assume that*

$$\forall r' \leq r\, \forall i < i^* \left(\int_\omega a^i_k(r') d\Xi(k) \geq b_i \right), \tag{4.2.4}$$

where a^i_k are defined as in Lemma 4.2.15.

Then, for all $\varepsilon > 0$ and $k^* < \omega$, there are a finite set $u \subseteq \omega \setminus k^*$ and an $r^\oplus \leq r$ such that:

1. $\left| \frac{|u \cap B_m|}{|u|} - \Xi(B_m) \right| < \varepsilon$, *for all* $m < m^*$.

2. $\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon r^\oplus \leq r^i_\ell\}|}{|I_k|} \geq b_i - \varepsilon$, *for all* $i < i^*$.

***Proof*.** As in the previous Lemma, we define $M := \{m < m^* \colon a_m > 0\}$, hence $\sum_{m \in M} a_m = 1$ because $\langle B_m \colon m < m^* \rangle$ is a partition of ω. Now, define for $m \in M$, $i < i^*$ and $r' \in \mathbb{B}$,

$$c_{i,m}(r') := \frac{1}{a_m} \int_{B_m} a^i_k(r')\, d\,\Xi(k),$$

and fix $\varepsilon > 0$ and $k^* < \omega$. Also define, for any $r' \in \mathbb{B}$ and $i < i^*$ the map $\varrho^i_{r'} \colon \omega \to \mathbb{R}$ such that $\varrho^i_{r'}(k) = \frac{|\{\ell \in I_k \colon r' \leq r^i_\ell\}|}{|I_k|}$.

Since, for all $m \in M$, $\frac{2a_m(1-a_m)}{\varepsilon^2} \geq 0$, and $\frac{(\frac{\varepsilon}{2})^2}{2(m^*+i^*)} > 0$, there exists some $h^* < \omega$ such that h^* is even and:

$$\frac{2\, a_m(1-a_m)}{h^* \varepsilon^2} < \frac{1}{m^* + i^*} \text{ and } \frac{1}{h^*} < \frac{(\frac{\varepsilon}{2})^2}{2(m^* + i^*)} \tag{4.2.5}$$

On the other hand, since we choose h^* such that $\frac{1}{h^*} < \frac{(\frac{\varepsilon}{2})^2}{2(m^*+i^*)}$, there exists some $\varepsilon^* > 0$ such that $\varepsilon^* < \varepsilon$ and:

$$\frac{\frac{2}{h^*} + \varepsilon^*}{(\frac{\varepsilon}{2})^2} < \frac{1}{m^* + i^*} \tag{4.2.6}$$

By Lemma 4.2.15 applied to ε^*, there are $r^* \leq r$ and a sequence $\bar{c} = \langle c_{i,m} \colon i < i^* \wedge m \in M \rangle$ such that:

- $0 \leq c_{i,m} \leq 1$,
- $\sum_{m<m^*} c_{i,m} a_m \geq b_i$,
- $D^* \coloneqq \left\{ r' \in \mathbb{B} \colon \forall i < i^* \, \forall m \in M \left(|c_{i,m}(r') - c_{i,m}| < \frac{\varepsilon^*}{4} \right) \right\}$ is dense below r^* and $r^* \in D^*$.

The rest of the proof is technical and quite long, so to make it easier to understand, we divided it into several parts:

Part 1: The tree construction.

We set $\mathcal{F} \coloneqq \bigcup_{u \in [\omega]^{<\omega}} {}^{\left(i^* \times \bigcup_{k \in u} I_k\right)}2$ and $\pi_2 \colon \mathcal{F} \times \omega \to \omega$ such that $\pi_2(\sigma, k) \coloneqq k$, that is, π_2 is the second component projection.

Now, let us build, by recursion on the height h, a tree $\mathcal{T}$ on $(M \cup (\mathcal{F} \times \omega))^{<\omega}$, a function $\mathbf{r} \colon \mathcal{T} \to \mathbb{B}$ such that, for any $\rho \in \mathcal{T}$, $\mathbf{r}(\rho) \coloneqq r_\rho$ and a probability space on succ_ρ for any $\rho \in \mathcal{T}$. To illustrate the construction, see Figure 4.1. Notice that, there, $\eta = \langle m, (\sigma, k), m' \rangle$.

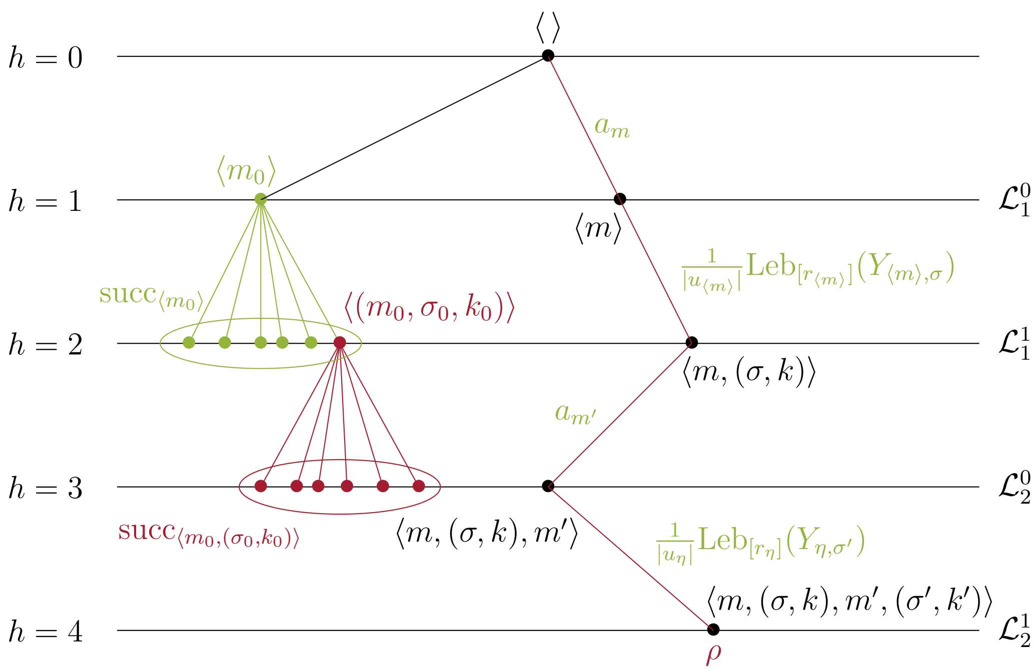

Figure 4.1: A graphic example of the early levels of $\mathcal{T}$.

In the base step, we define $\mathcal{L}_0(\mathcal{T}) \coloneqq \{\langle\rangle\}$ and $r_{\langle\rangle} \coloneqq r^*$. Now, suppose we have built the first h levels and define the level $h+1$ of $\mathcal{T}$. For this we will consider two possible cases:

1. When h is even, we define:

$$\mathcal{L}_{h+1}(\mathcal{T}) \coloneqq \mathcal{L}^0_{\frac{h+2}{2}} \coloneqq \{\rho^\frown\langle m\rangle \colon m \in M \wedge \rho \in \mathcal{L}_h(\mathcal{T})\}.$$

If $\eta \in \mathcal{L}_{h+1}(\mathcal{T})$, then there exists a unique $\rho \in \mathcal{L}_h(\mathcal{T})$ such that $\eta = \rho^\frown\langle m\rangle$. In this case, we define $r_\eta \coloneqq r_\rho$.

In order to define the probability space, let $\rho \in \mathcal{L}_h(\mathcal{T})$ and $\eta \in \mathrm{succ}_\rho$, hence, there exists $m \in M$ such that $\eta = \rho^\frown\langle m\rangle$. In this case, we define $\mathrm{Pr}_\rho(\{\eta\}) \coloneqq a_m$. Thus, since

$$\sum_{\eta \in \mathrm{succ}_\rho} \mathrm{Pr}_\rho(\eta) = \sum_{m \in M} a_m = 1,$$

it follows that $(\mathrm{succ}_\rho, \mathcal{P}(\mathrm{succ}_\rho), \mathrm{Pr}_\rho)$ is a finite probability space.

2. When h is odd, we must work hard.

Let $\rho \in \mathcal{L}_h(\mathcal{T})$. Since $h-1$ is even, by the previous case, there exists $m \in M$ such that $\rho = \nu^\frown\langle m\rangle$ where $\nu \in \mathcal{L}_{h-1}(\mathcal{T})$. We set

$$\mathcal{K} \coloneqq \{\pi_2(\rho'(i)) \colon \rho' \in \mathcal{L}_h(\mathcal{T}) \wedge 0 \leq i < h \text{ is odd}\} \cup \{k^*\}.$$

It is clear that $\mathcal{K} \subseteq \omega$ is a finite set.

By Theorem 3.5.26, there is a finite set $u_\rho \subseteq B_m$ such that $\min(u_\rho) > \max(\mathcal{K})$, and

$$\forall i < i^* \left(\left| c_{i,m}(r_\rho) - \frac{1}{|u_\rho|} \sum_{k \in u_\rho} a^i_k(r_\rho) \right| < \frac{\varepsilon^*}{4} \right) \tag{4.2.7}$$

Now, consider $\mathscr{B}_\rho$ as the sub-Boolean algebra of $\mathcal{P}([r_\rho]) \subseteq \mathcal{P}({}^\omega 2)$ generated by the sets

$$\{[r_\rho] \cap [r^i_\ell] \colon i < i^*, \ell \in I_k, k \in u_\rho\}.$$

Defining

$$\forall \sigma \in {}^{i^* \times \bigcup_{k \in u_\rho} I_k} 2 \left(Y_{\rho,\sigma} \coloneqq \bigcap_{(i,\ell) \in \mathrm{dom}(\sigma)} [r_\rho] \cap [r^i_\ell]^{\sigma(i,\ell)} \right),$$

we get, whenever $Y_{\rho,\sigma} \neq \emptyset$,

(a) $Y_{\rho,\sigma} \subseteq [r^i_\ell] \Leftrightarrow \sigma(i,\ell) = 0$,
(b) $Y_{\rho,\sigma} \cap [r^i_\ell] = \emptyset \Leftrightarrow \sigma(i,\ell) = 1$.

Also, since $\mathscr{B}_\rho$ finitely generated, by Theorem 1.4.11,

$$\mathrm{At}_{\mathscr{B}_\rho} = \{Y_{\rho,\sigma} \colon \sigma \in 2^{i^* \times \bigcup_{k \in u_\rho} I_k} \wedge Y_{\rho,\sigma} \neq \emptyset\}.$$

On the other hand, by Theorem 1.4.14, defining

$$\Sigma_\rho \coloneqq \{\sigma \in 2^{i^* \times \bigcup_{k \in u_\rho} I_k} \colon \mathrm{Leb}_{[r_\rho]}(Y_{\rho,\sigma}) > 0\},$$

it is obtained that:

$$\sum_{\sigma\in\Sigma_\rho} \mathrm{Leb}_{[r_\rho]}(Y_{\rho,\sigma}) = 1. \tag{4.2.8}$$

So, we get u_ρ and Σ_ρ for any $\rho \in \mathcal{L}_h(\mathcal{T})$, from which we can define:

$$\mathcal{L}_{h+1}(\mathcal{T}) \coloneqq \mathcal{L}^1_{\frac{h+1}{2}} \coloneqq \{\rho^\frown\langle(\sigma,k)\rangle \colon \rho\in\mathcal{L}_h(\mathcal{T}),\, \sigma\in\Sigma_\rho \wedge k\in u_\rho\}.$$

To define $\mathbf{r}$ at this level, let $\eta \in \mathcal{L}_{h+1}(\mathcal{T})$, hence there is some $\rho \in \mathcal{L}_h(T)$ such that $\eta = \rho^\frown\langle(\sigma,k)\rangle$ for some $(\sigma,k) \in \Sigma_\rho \times u_\rho$. By density, we can choose $r_\eta \in \mathbb{B}$ such that $[r_\eta] \subseteq Y_{\rho,\sigma}$, $r_\eta \leq r_\rho$ and $r_\eta \in D^*$. Therefore:

- $\sigma(i,\ell) = 0 \Rightarrow r_\eta \leq r^i_\ell$,
- $\sigma(i,\ell) = 1 \Rightarrow r_\eta \perp_{\mathbb{B}} r^i_\ell$.

In order to define the probability space, fix $\rho \in \mathcal{L}_h(\mathcal{T})$. For any $\rho^\frown\langle(\sigma,k)\rangle \in \mathrm{succ}_\rho$ we define:

$$\mathrm{Pr}_\rho(\eta^\frown\langle(\sigma,k)\rangle) \coloneqq \frac{1}{|u_\rho|}\mathrm{Leb}_{[r_\rho]}(Y_{\rho,\sigma}).$$

To prove that $(\mathrm{succ}_\rho, \mathcal{P}(\mathrm{succ}_\rho), \mathrm{Pr}_\rho)$ is a probability space, it is enough to show that

$$\sum_{\eta\in\mathrm{succ}_\rho} \mathrm{Pr}_\rho(\eta) = 1.$$

In effect, by Equation 4.2.8,

$$\begin{aligned}
\sum_{\eta\in\mathrm{succ}_\rho} \mathrm{Pr}_\rho(\eta) &= \sum_{\sigma\in\Sigma_\rho}\left(\sum_{k\in u_\rho}\mathrm{Pr}_\rho(\rho^\frown\langle(\sigma,k)\rangle)\right)\\
&= \sum_{\sigma\in\Sigma_\rho}\left(\sum_{k\in u_\rho}\frac{1}{|u_\rho|}\mathrm{Leb}_{[r_\rho]}(Y_{\rho,\sigma})\right)\\
&= \sum_{\sigma\in\Sigma_\rho}\left(\mathrm{Leb}_{[r_\rho]}(Y_{\rho,\sigma})\sum_{k\in u_\rho}\frac{1}{|u_\rho|}\right)\\
&= \sum_{\sigma\in\Sigma_\rho}\mathrm{Leb}_{[r_\rho]}(Y_{\rho,\sigma}) = 1.
\end{aligned}$$

Thus, $(\mathrm{succ}_\rho, \mathcal{P}(\mathrm{succ}_\rho), \mathrm{Pr}_\rho)$ is a finite probability space.

Finally, we define $\mathcal{T} \coloneqq \bigcup_{h<\omega}\mathcal{L}_h(\mathcal{T})$ and $\mathbf{r}\colon \mathcal{T}\to\mathbb{B}$ such that $\mathbf{r}(\rho) \coloneqq r_\rho$. It is clear by construction that $\mathcal{T}$ is a probability tree.

For simplicity, we will use the following notation: for any $h \leq h^*$ consider

$$E_h \coloneqq \{n<\omega\colon 1\leq n\leq h \text{ even}\} \text{ is and } O_h \coloneqq \{n<\omega\colon 1\leq n\leq h \text{ is odd}\}.$$

Also, if $\rho \in \mathcal{L}_h(\mathcal{T})$ for $h \geq 2$, then by construction either $\rho = \eta^\frown\langle m, (\sigma, k)\rangle$ or $\rho = \eta^\frown\langle(\sigma, k), m\rangle$. In any case, we denote:

$$m_\rho := m, \sigma_\rho := \sigma \text{ and } k_\rho := k.$$

That is, m_ρ, k_ρ, σ_ρ are the last m, k, σ that appear in ρ. For ρ of even length ≥ 2, set $u_\rho := u_{\rho\restriction h-1}$, and allow $m_\rho = m$ when $\rho = \langle m\rangle$. As a consequence, for $k \in u_\rho$ where ρ of even length and $i < i^*$,

$$\varrho^i_{r_\rho}(k) = \frac{|\{\ell \in I_k \colon r_\rho \leq r^i_\ell\}|}{|I_k|} = \frac{|\{\ell \in I_k \colon \sigma_\rho(i, \ell) = 0\}|}{|I_k|} := \varrho^i_\rho(k).$$

As a consequence, since $\mathcal{T}$ is a probability tree, by Theorem 2.3.2, it induces a probability space in each of its levels, where the probability of $\rho \in \mathcal{L}_h(\mathcal{T})$ is given by:

$$\operatorname{Pr}_h(\rho) := \prod_{n \in O_h} a_{m_{\rho\restriction n}} \cdot \prod_{n \in E_h} \frac{1}{|u_{\rho\restriction n}|} \operatorname{Leb}_{[r_{\rho\restriction n}]}(Y_{\rho\restriction n, \sigma_{\rho\restriction n}}). \tag{4.2.9}$$

Part 2: Finding a suitable $\rho \in \mathcal{L}_{h^*}(\mathcal{T})$ with high probability.

In general, as the length of the elements of $\mathcal{T}$ increases, by Equation 4.2.9, probabilities are getting smaller and smaller. What we do is try to define random variables that allow us to model what happens in both condition (1) and condition (2) of the conclusion of the lemma. For this, we divide this part into two sub-parts: in the first, we deal with condition (1), and in the second one, with condition (2).

For $\rho \in T$, define $u^*_\rho := \{k_{\rho\restriction h} \colon h \in E_{h^*}\}$.[6] It is clear by construction that $|u^*_\rho| = \frac{h^*}{2}$, $u^*_\rho \subseteq \bigcup_{m<m^*} B_m$ and $u^*_\rho \subseteq \omega \setminus k^*$.

Part 2.1: Random variables to model the first condition.

For $m \in m^* \setminus M$, $u^*_\rho \cap B_m = \emptyset$, therefore,

$$\left|\frac{|u^*_\rho \cap B_m|}{|u^*_\rho|} - a_m\right| = 0 < \varepsilon.$$

On the other hand, for $m \in M$ consider $U_m, V_m \colon \mathcal{L}_{h^*}(\mathcal{T}) \to \mathbb{R}$ such that, if $\rho \in \mathcal{L}_{h^*}(\mathcal{T})$, then:

$$V_m(\rho) := \frac{|u^*_\rho \cap B_m|}{|u^*_\rho|} \text{ and } U_m(\rho) := |\{j \in E_{h^*} \colon m_{\rho\restriction j} = m\}|.$$

Since the σ-algebra in $\mathcal{L}_{h^*}(\mathcal{T})$ is $\mathcal{P}(\mathcal{L}_{h^*}(\mathcal{T}))$, U_m and V_m are, trivially, random variables. Also, $U_m \sim \operatorname{Binomial}(\frac{h^*}{2}, a_m)$, since it counts the success after $\frac{h^*}{2}$ tries, each with probability a_m, and we can express V_m in terms of U_m: $V_m(\rho) = \frac{2U_m(\rho)}{h^*}$. So, by Theorem 2.2.10(4)

$$\operatorname{E}[V_m] = \frac{2\operatorname{E}[U_m]}{h^*} = \frac{\frac{2h^*}{2} a_m}{h^*} = a_m$$

[6] We are counting only the even numbers because the information is repeated in the construction, for example, if h is even, at the levels h and $h+1$ we have the same k_ρ for any $\rho \in \mathcal{L}_h(\mathcal{T}) \cup \mathcal{L}_{h+1}(\mathcal{T})$. So, adding over all numbers below h^* would affect the value of the probabilities.

and

$$\operatorname{Var}[V_m] = \frac{\frac{4\,h^*}{2}\, a_m\,(1-a_m)}{(h^*)^2} = \frac{2\,a_m(1-a_m)}{h^*}.$$

Thereby, by *Chebyshev inequality*, we can conclude:

$$\operatorname{Pr}_{h^*}[|V_m - a_m| \geq \varepsilon] \leq \frac{\operatorname{Var}[V_m]}{\varepsilon^2} = \frac{2\,a_m\,(1-a_m)}{h^*\,\varepsilon^2}.$$

Thus, by the choice of h^*,[7] we get:

$$\forall m \in M \left(\operatorname{Pr}_{h^*}[|V_m - a_m| \geq \varepsilon] \leq \frac{1}{m^* + i^*} \right) \tag{4.2.10}$$

In the next part, we will do the same for the second condition.

Part 2.2: Random variables to model the second condition.

Now, for $h \in E_h$ and $i < i^*$, consider $Z^i_h \colon \mathcal{L}_h(\mathcal{T}) \to \mathbb{R}$ such that $Z^i_s(\rho) := \varrho^i_\rho(k_\rho)$ for any $\rho \in \mathcal{L}_h(\mathcal{T})$. For $h \leq h^*$ even and $\eta \in \mathcal{L}_{h-1}(\mathcal{T})$, we calculate the expected value of Z^i_h "given that" $\eta\restriction_{h-1} = \rho^\frown\langle m\rangle$. For this, since $h-1$ is odd, the successors of ρ are as in Case 2 of Part 1 and we use Definition 2.3.5:

$$\begin{aligned}
\operatorname{E}[Z^i_h \colon \eta\restriction_{h-1} = \rho] &= \operatorname{E}_{\operatorname{succ}_\rho}[Z^i_h \restriction \operatorname{succ}_\rho] \\
&= \sum_{k \in u_\rho} \left(\sum_{\sigma \in \Sigma_\rho} \frac{1}{|u_\rho|} \operatorname{Leb}_{[r_\rho]}(Y_{\rho,\sigma})\, Z^i_h(\rho^\frown\langle(\sigma,k)\rangle) \right) \\
&= \sum_{k \in u_\rho} \left(\sum_{\sigma \in \Sigma_\rho} \frac{1}{|u_\rho|} \operatorname{Leb}_{[r_\rho]}(Y_{\rho,\sigma})\, \varrho^i_{\rho^\frown\langle m,(\sigma,k)\rangle}(k) \right) \\
&= \frac{1}{|u_\rho|} \sum_{k \in u_\rho} \left(\sum_{\sigma \in \sigma_\rho} \operatorname{Leb}_{[r_\rho]}(Y_{\rho,\sigma})\, \frac{|\{\ell \in I_k \colon \sigma(i,\ell) = 0\}|}{|I_k|} \right) \\
&= \frac{1}{|u_\rho|} \sum_{k \in u_\rho} \left[\frac{1}{|I_k|} \sum_{\sigma \in \Sigma_\rho} \operatorname{Leb}_{[r_\rho]}(Y_{\rho,\sigma}) \left(\sum_{\ell \in I_k,\ \sigma(i,\ell)=0} 1 \right) \right] \\
&= \frac{1}{|u_\rho|} \sum_{k \in u_\rho} \left[\frac{1}{|I_k|} \sum_{\ell \in I_k} \left(\sum_{\sigma \in \Sigma_\rho,\ \sigma(i,\ell)=0} \operatorname{Leb}_{[r_\rho]}(Y_{\rho,\sigma}) \right) \right] \\
&= \frac{1}{|u_\rho|} \sum_{k \in u_\rho} \left[\frac{1}{|I_k|} \sum_{\ell \in I_k} \left(\operatorname{Leb}_{[r_\rho]}([r^i_\ell]) \right) \right] \\
&= \frac{1}{|u_\rho|} \sum_{k \in u_\rho} a^i_k(r_\rho).
\end{aligned}$$

[7] In fact, Equation 4.2.10 was what motivated the choice of h^* in Equation 4.2.5

Therefore, by the choice of u_ρ (see Equation 4.2.7), it follows that:

$$\forall i < i^* \left(\left| \mathrm{E}[Z_h^i \colon \eta \restriction_{h-1} = \rho^\frown \langle m \rangle] - c_{i,m}(r_\rho) \right| < \frac{\varepsilon^*}{4} \right).$$

As a consequence, since $r_\rho \in D^*$, we can conclude:

$$\forall i < i^* \left(\left| \mathrm{E}[Z_h^i \colon \eta \restriction_{h-1} = \rho^\frown \langle m \rangle] - c^*_{i,m} \right| < \frac{\varepsilon^*}{2} \right) \tag{4.2.11}$$

Now we are going to calculate the expected value, "given that" we restrict it one more level. Since $h-2$ is even, the successors of $\nu \in \mathcal{L}_{h-2}(\mathcal{T})$ are as in the first case of part (1). By Theorem 2.3.8, we have that:

$$\begin{aligned}
\mathrm{E}[Z_h^i \colon \eta \restriction_{h-2} = \nu] &= \mathrm{E}[\mathrm{E}[Z_h^i \colon \eta \restriction_{h-1} = \rho] \colon \rho \restriction_{h-2} = \nu] \\
&= \mathrm{E}\left[\frac{1}{|u_\rho|} \sum_{k \in u_\rho} a_k^i(r_\nu) \ \middle| \ \rho \restriction_{h-2} = \nu \right] \\
&= \mathrm{E}_{\mathrm{succ}_\nu} \left[\frac{1}{|u_\rho|} \sum_{k \in u_\rho} a_k^i(r_\nu) \right] \\
&= \sum_{m \in M} \left(a_m \cdot \frac{1}{|u_{\nu^\frown \langle m \rangle}|} \sum_{k \in u_{\nu^\frown \langle m \rangle}} a_k^i(r_\nu) \right).
\end{aligned}$$

and therefore,

$$\begin{aligned}
\left| \mathrm{E}[Z_h^i \colon \eta \restriction_{h-2} = \nu] - \int_\omega a_k^i(r_\nu)\, d\Xi \right| &= \left| \sum_{m \in M} \left(a_m \frac{1}{|u_{\nu^\frown \langle m \rangle}|} \sum_{k \in u_{\nu^\frown \langle m \rangle}} a_k^i(r_\nu) \right) - \int_\omega a_k^i(r_\nu)\, d\Xi \right| \\
&= \left| \sum_{m \in M} \left(a_m \frac{1}{|u_{\nu^\frown \langle m \rangle}|} \sum_{k \in u_\nu} a_k^i(r_\nu) \right) - \sum_{m \in M} \left(\int_{B_m} a_k^i(r_\nu) \right) \right| \\
&= \left| \sum_{m \in M} \left(a_m \frac{1}{|u_{\nu^\frown \langle m \rangle}|} \sum_{k \in u_{\nu^\frown \langle m \rangle}} a_k^i(r_\nu) \right) - \sum_{m \in M} a_m\, c_{i,m}(r_\nu) \right| \\
&= \left| \sum_{m \in M} a_m \left(\frac{1}{|u_{\nu^\frown \langle m \rangle}|} \sum_{k \in u_{\nu^\frown \langle m \rangle}} a_k^i(r_\nu) - c_{i,m}(r_{\nu^\frown \langle m \rangle}) \right) \right| \\
&\leq \sum_{m \in M} a_m \left| \left(\frac{1}{|u_{\nu^\frown \langle m \rangle}|} \sum_{k \in u_{\nu^\frown \langle m \rangle}} a_k^i(r_\nu) - c_{i,m}(r_{\nu^\frown \langle m \rangle}) \right) \right| \\
&< \frac{\varepsilon^*}{4}.
\end{aligned}$$

Since $r_\nu \leq r$, we have that $\int_\omega a^i_k(r_\nu) \geq b_i$, hence, by Equation 4.2.11:

$$\forall i < i^* \left(\mathrm{E}[Z^i_h \colon \eta \restriction_{h-2} = \nu\,] > b_i - \frac{\varepsilon^*}{4} \right) \tag{4.2.12}$$

In a similar way, we can prove that

$$\forall i < i^* \left(\left| \mathrm{E}[Z^i_h \colon \eta \restriction_{h-2} = \rho\,] - \sum_{m \in M} a_m c_{i,m} \right| < \frac{\varepsilon^*}{2} \right) \tag{4.2.13}$$

Furthermore, we can generalize the previous equation, inductively as follows. For all $i < i^*$, $h \in E_{h^*}$, $j \in E_{h-1}$, $\eta \in \mathcal{L}_h(\mathcal{T})$, and $\rho \in \mathcal{L}_j(\mathcal{T})$, we have that

$$\left| \mathrm{E}[Z^i_h \colon \eta \restriction_j = \rho\,] - \sum_{m \in M} a_m c_{i,m} \right| < \frac{\varepsilon^*}{2} \tag{4.2.14}$$

Finally, by Corollary 2.3.9, $\mathrm{E}[Z_i] = \mathrm{E}[\mathrm{E}[Z^i_h \colon \eta \restriction_{h-2} = \rho]]$, hence we get:

$$\forall i < i^* \left(\mathrm{E}[Z^i_h] > b_i - \frac{\varepsilon^*}{4} \text{ and } \left| \mathrm{E}[Z^i_h] - \sum_{m \in M} a_m\, c_{i,m} \right| < \frac{\varepsilon^*}{2} \right) \tag{4.2.15}$$

Now, for any $i < i^*$ consider the random variable $Y_i \colon \mathcal{L}_{h^*}(\mathcal{T}) \to \mathbb{R}$ such that, for every $\rho \in \mathcal{L}_{h^*}(\mathcal{T})$,

$$Y_i(\rho) := \frac{1}{|u^*_\rho|} \sum_{k \in u^*_\rho} \frac{|\{\ell \in I_k \colon r_\rho \leq r^i_\ell\}}{|I_k|}.$$

We can express Y_i in terms of Z^i_h for $h \in E_{h^*}$:

$$Y_i = \frac{2}{h^*} \sum_{h \in E_{h^*}} Z^i_h,$$

hence, by Theorem 2.2.10(5), and Equation 4.2.15, we get:

$$\mathrm{E}[Y_i] = \mathrm{E}\left[\frac{2}{h^*} \sum_{h \in E_{h^*}} Z^i_h \right] = \frac{2}{h^*} \sum_{h \in E_{h^*}} \mathrm{E}[Z^i_h] > \frac{2}{h^*} \sum_{h \in E_{h^*}} \left(b_i - \frac{\varepsilon^*}{4} \right) = b_i - \frac{\varepsilon^*}{4}.$$

On the other hand, since $\varepsilon^* < \varepsilon$, it is clear that $\mathrm{E}[X^i_h] > b_i - \frac{\varepsilon}{4}$ and $b_i - \frac{\varepsilon}{4} < \mathrm{E}[Y_i]$, hence:

$$\Pr[Y_i \leq b_i - \varepsilon] \leq \Pr\left[Y_i \leq \mathrm{E}[Y_i] - \frac{\varepsilon}{2} \right] \leq \Pr\left[\frac{\varepsilon}{2} \leq |Y_i - \mathrm{E}[Y_i]| \right],$$

and applying *Chebyshev's inequality*, we get:

$$\forall i < i^* \left(\Pr[Y_i \leq b_i - \varepsilon] \leq \frac{\mathrm{Var}[Y_i]}{(\frac{\varepsilon}{2})^2} \right) \tag{4.2.16}$$

Part 2.2.1: Properly bound the variance of Y_i.

Since we want $\Pr[\forall i < i^*(Y_i > b_i - \varepsilon)] > 0$, we must show that $\mathrm{Var}(Y_i)$ is small enough. For this, let us start by noting that, by Theorem 2.2.11(4):

$$\mathrm{Var}(Y_i) = \frac{4}{(h^*)^2}\left(\sum_{h\in E_{h^*}} \mathrm{Var}(Z^i_h) + \sum_{h,j\in E_{h^*},\, h\neq j} \mathrm{Cov}(Z^i_j, Z^i_h)\right).$$

It is clear that, for any $i < i^*$, $0 \leq \mathrm{Var}[Z^i_h] \leq 1$, because $0 \leq Z^i_h \leq 1$, and by Theorem 2.2.10(3), $0 \leq \mathrm{E}[Z^i_h] \leq 1$ and $|Z^i_h - \mathrm{E}[Z^i_h]| \leq 1$. So, we must to bound the covariance. For this, let $j, h \in E_{h^*}$ such that $j < h$. Then, by Corollary 2.3.9, Equation 4.2.15 and Equation 4.2.13, we get:

$$\begin{aligned}
\mathrm{Cov}[Z^i_j, Z^i_h] &= \mathrm{E}[Z^i_j \cdot Z^i_h] - \mathrm{E}[Z^i_j]\cdot \mathrm{E}[Z^i_h]\\
&= \mathrm{E}_j[\,\mathrm{E}\,[Z^i_j\cdot Z^i_h \colon \nu\restriction_j =\ \eta\,]] - \mathrm{E}_j[Z^i_j \cdot \mathrm{E}\,[Z^i_h]]\\
&= \mathrm{E}_j\left[Z^i_j \cdot \left(\mathrm{E}\,[Z^i_h \colon \nu\restriction_j =\ \eta] - \mathrm{E}_h[Z^i_h]\right)\right]\\
&\leq \mathrm{E}_j\left[Z^i_j \cdot \left(\left|\,\mathrm{E}[Z^i_h \colon \nu\restriction_j =\ \eta] - \mathrm{E}_h[Z^i_h]\,\right|\right)\right]\\
&\leq \mathrm{E}_j\left[Z^i_j \cdot \left(\left|\mathrm{E}[Z^i_j \colon \nu\restriction_j =\ \eta] - \sum_{m\in M} c_{i,m}\, a_m\right| + \left|E[Z^i_h] - \sum_{m\in M} c_{i,m}\, a_m\right|\right)\right]\\
&< \mathrm{E}_j\left[Z^i_j\cdot\left(\frac{\varepsilon^*}{2} + \frac{\varepsilon^*}{2}\right)\right]\\
&= \varepsilon^* \cdot \mathrm{E}_j[Z^i_j]\\
&\leq \varepsilon^*.
\end{aligned}$$

As a consequence,

$$\mathrm{Var}[Y_i] \leq \frac{4}{(h^*)^2}\left[\frac{h^*}{2} + \frac{h^*}{2}\left(\frac{h^*}{2} - 1\right)\varepsilon^*\right] = \frac{2}{h^*} + \varepsilon^* - \frac{2}{h^*}\varepsilon^* = \frac{2}{h^*} + \left(\frac{h^*-2}{h^*}\right)\varepsilon^* < \frac{2}{h^*} + \varepsilon^*.$$

Thus, Equation 4.2.16 and the choice of h^* implies:

$$\forall i < i^*\left(\Pr[Y^i_{h^*} < b_i - \varepsilon] < \frac{1}{m^* + i^*}\right) \tag{4.2.17}$$

Part 2.3: Some ρ of high probability and the conclusion.

Consider the following events:

- $E := \{\rho \in \mathcal{L}_{h^*}(\mathcal{T}) \colon \forall m \in M\, \forall i < i^*\, (|V_{m,h^*}(\rho) - a_m| < \varepsilon \wedge Y_i(\rho) > b_i - \varepsilon)\}$,
- $F := \{\rho \in \mathcal{L}_{h^*}(\mathcal{T}) \colon \exists m \in M(|V_{m,h^*}(\rho)) - a_m| \geq \varepsilon\}$,
- $G := \{\rho \in \mathcal{L}_{h^*}(\mathcal{T}) \colon \exists i < i^*(Y_i(\rho) \leq b_i - \varepsilon\}$.

It is clear that $E = F^{\mathrm{c}} \cap G^{\mathrm{c}} = (F \cup G)^{\mathrm{c}}$ hence, $E^{\mathrm{c}} = F \cup G$. Also, by Equation 4.2.10 and Equation 4.2.17 we have that

$$\Pr(F) \leq \sum_{m \in M} \Pr[|V_{m,h^*} - a_m| \geq \varepsilon] < \sum_{m \in M} \frac{1}{m^* + i^*} \leq \frac{m^*}{m^* + i^*},$$

and

$$\Pr(G) \leq \sum_{i < i^*} \Pr[Y^i_{h^*} \leq b_i - \varepsilon] < \sum_{i < i^*} \frac{1}{m^* + i^*} = \frac{i^*}{m^* + i^*}.$$

Therefore,

$$\begin{aligned}
\Pr(E) &= 1 - \Pr(E^{\mathrm{c}}) = 1 - \Pr(F \cup G) \\
&= 1 - [\Pr(F) + \Pr(G)] - \Pr(F \cap G)] \\
&> 1 - \left(\frac{m^*}{m^* + i^*} + \frac{i^*}{m^* + i^*}\right) + \Pr(F \cap G) \\
&= 1 - \frac{m^* + i^*}{m^* + i^*} + \Pr(F \cap G) \\
&= 1 - 1 + \Pr(F \cap G) \\
&= \Pr(F \cap G) \geq 0,
\end{aligned}$$

hence $\Pr(E) > 0$. As a consequence, $E \neq \emptyset$. Let $\rho \in E$, $u := u^*_\rho$ and $r^\oplus := \mathbf{r}(\rho) = r_\rho$. Then, by the construction of T, $u \subseteq \omega \setminus k^*$ and $r^\oplus = r_\rho \leq r^* \leq r$. Also, since $\rho \in E$, for $m \in M$ and $i < i^*$, we have that:

$$\left|\frac{|u \cap B_m|}{|u|} - a_m\right| = \left|\frac{u^*_\rho \cap B_m|}{|u^*_\rho|} - a_m\right| = |V_{m,h^*}(\rho) - a_m| < \varepsilon$$

and,

$$\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon r^\oplus \leq r^i_\ell\}|}{|I_k|} = \frac{1}{|u^*_\rho|} \sum_{k \in u^*_\rho} \frac{\{\ell \in I_k \colon r_\rho \leq r^i_\ell\}}{|I_k|} = Y_i(\rho) > b_i - \varepsilon.$$

Thus, finally:

1. $\left|\frac{|u \cap B_m|}{|u|} - a_m)\right| < \varepsilon$, for all $m < m^*$.
2. $\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon r^\oplus \leq r^i_\ell\}|}{|I_k|} > 1 - \varepsilon$, for all $i < i^*$. $\square_{\text{Lemma 4.2.16}}$

The probabilistic method is a very powerful probabilistic tool that allows tackle problems mainly in discrete mathematics. It was first used by *Paul Erdös* in the context of graph theory (see [Erd47]). The author of the book *The Probabilistic Method* (see [AS16]), *Joel H. Spencer*, defines it as follows:

> Roughly speaking, the method works as follows: Trying to prove that a structure with certain desired properties exists, one defines an appropriate probability space of structures and then shows that the desired properties hold in this space with positive probability.

Notice that, the description of the probabilistic method given by *Joel Spencer*, fits very well with the way we proved Lemma 4.2.16, so we can think that it is an application of the probabilistic method to the Set Theory. In a similar way, we are going to prove the theorem that will allow us to extend at limit steps of forcing iterartion with finitely additive measures (see Main Lemma 4.3.17).

The following will allow defining the limit as in Definition 4.2.2 for random forcing.

Lemma 4.2.17. *Let $b \in [0,1]$, $r^* \in \mathbb{B}$ and $\bar{r} = \langle r_\ell \colon \ell < \omega \rangle \in \mathbb{B}^\omega$ be such that, for any $\ell < \omega$, $\mathrm{Leb}_{[r^*]}([r_\ell]) \geq b$. Then there exists $r^\otimes \in \mathbb{B}$ such that $r^\otimes \leq r^*$ and,*

$$\forall r \leq r^\otimes \left(\int_\omega a_k(r)\, d\,\Xi(k) \geq b \right),$$

where, for any $r \in \mathbb{B}$ and $k < \omega$, $a_k(r) \coloneqq \frac{1}{|I_k|} \sum_{\ell \in I_k} \mathrm{Leb}_{[r]}([r_\ell])$.

Proof. We consider the set

$$I \coloneqq \left\{ r \in \mathbb{B} \colon r \leq r^* \wedge \int_\omega a_k(r)\, d\,\Xi(k) < b \right\}.$$

Suppose that I is not dense below r^*. So there exists $r^\otimes \leq r^*$ such that, for all $r \in I$, $r \not\leq r^\otimes$, which implies that $\forall r \leq r^\otimes (r \notin I)$, that is, $r^\otimes$ is as required. So it is enough to prove that I is not dense below p^*. Towards contradiction, assume that I is dense below r^*. So we can find a maximal anti-chain $A \coloneqq \{s_i \colon i < i_*\} \subseteq I$ below r^*. Since random forcing is c.c.c., we have that $0 < i_* \leq \omega$. Also we can think of A almost like a partition in the sense that $i, j < i_*$ and $i \neq j$ imply $\mathrm{Leb}([s_i] \cap [s_j]) = 0$. As a consequence, $\mathrm{Leb}([r^*]) = \sum_{i<i_*} \mathrm{Leb}([s_i])$.
Now, for all $j < \min\{i^* + 1, \omega\}$ define $s^j \coloneqq \bigcup_{i<j} s_i$. So, by Lemma 1.2.3(1), we have that $[s^j] = \bigcup_{i<j} [s_i]$. Also expanding properly, we can express $a_k(s^j)$ in terms of $a_k(s_i)$ for $k < \omega$, $0 < j < i_*$ and $i < j$, as follows:

$$a_k(s^j) = \sum_{i<j} \mathrm{Leb}_{[s^j]}([s_i]) a_k(s_i). \tag{4.2.18}$$

Since $s_0 \in I$, $\int_\omega a_k(s_0)\, d\,\Xi(k) < b$ and $\varepsilon \coloneqq b - \int_\omega a_k(s_0)\, d\,\Xi(k) > 0$. Then, by integration properties,

$$\begin{aligned}
\int_\omega a_k(s^j)\, d\,\Xi(k) &= \int_\omega \left(\sum_{i<j} \mathrm{Leb}_{[s^j]}([s_i]) a_k(s_i) \right) d\,\Xi(k) \\
&= \sum_{i<j} \mathrm{Leb}_{[s^j]}([s_i]) \int_\omega a_k(s_i)\, d\,\Xi(k) \\
&= \mathrm{Leb}_{[s^j]}([s_0]) \int_\omega a_k(s_0)\, d\,\Xi(k) + \sum_{0<i<j} \mathrm{Leb}_{[s^j]}([s_i]) \int_\omega a_k(s_i)\, d\,\Xi(k) \\
&\leq \mathrm{Leb}_{[s^j]}([s_0])(b - \varepsilon) + \sum_{0<i<j} \mathrm{Leb}_{[s^j]}([s_i]) b \\
&= \sum_{i<j} \mathrm{Leb}_{[s^j]}([s_i]) b - \mathrm{Leb}_{[s^j]}([s_0]) \varepsilon \\
&= b - \mathrm{Leb}_{[s^j]}([s_0]) \varepsilon \\
&\leq b - \mathrm{Leb}([s_0])\, \varepsilon.
\end{aligned}$$

Hence,

$$\int_\omega a_k(s^j)\, d\,\Xi(k) \leq b - \text{Leb}([s_0])\,\varepsilon. \tag{4.2.19}$$

On the other hand, since $\lim_{j\to i^*}\text{Leb}([s^j]) = \text{Leb}([r^*])$ and $\lim_{j\to i^*}([r^*]\setminus[s^j]) = 0$, there exists $0 < j < \min\{i^*+1, \omega\}$ such that:

$$\text{Leb}_{[r^*]}([r^*]\setminus[s^j]) < \text{Leb}([s_0])\varepsilon. \tag{4.2.20}$$

Expressing $[r^*] = [s^j] \cup ([r^*]\setminus[s^j])$ and using Equations 4.2.18, 4.2.20 and 4.2.19, we get:

$$\begin{aligned}\int_\omega a_k(r^*)\, d\,\Xi(k) &= \text{Leb}_{[r^*]}([r^*]\setminus[s^j]) \int_\omega a_k([r^*]\setminus[s^j])\, d\,\Xi(k) \; + \; \text{Leb}_{[r^*]}([s^j]) \int_\omega a_k([s^j])\, d\,\Xi(k) \\ &< \text{Leb}([s_0])\varepsilon + (b - \text{Leb}([s_0])\varepsilon) \\ &= b.\end{aligned}$$

Therefore $\int_\omega a_k(r^*)\, d\,\Xi(k) < b$, so $r^* \in I$.

Finally, since for all $\ell < \omega$, $\text{Leb}_{[r^*]}([r_\ell]) \geq b$, we have that $a_k(r^*) \geq b$, therefore $\int_\omega a_k(r^*)\, d\,\Xi(k) \geq b$, that is, $r^* \notin I$, which is a contradiction. Thus, I is not dense below r^*. $\square_{\text{Lemma 4.2.17}}$

Finally, we can prove that $\mathbb{B}$ is σ-FAM-linked:

Theorem 4.2.18. *Random forcing is* σ-FAM-*linked.*

Proof. For each $t \in {}^{<\omega}2$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$, consider the set Let us prove that the sequence $\langle Q_{t,\varepsilon} \colon t \in {}^{<\omega}2 \wedge \varepsilon \in (0,1)_{\mathbb{Q}}\rangle$ witnesses that $\mathbb{B}$ is σ-FAM-linked. We must verify the three conditions in Definition 4.2.8.

1. Fix $t \in {}^{<\omega}2$, $\varepsilon_0 \in (0,1)_{\mathbb{Q}}$, a finitely additive measure Ξ on $\mathcal{P}(\omega)$ and an interval partition $\bar{I} = \langle I_k \colon k < \omega\rangle$ of ω such that $\lim_{k\to\infty}|I_k| = \infty$. We must to prove that Q_{t,ε_0} is $(\Xi, \bar{I}, \varepsilon_0)$-linked.

   Consider q^* as the full tree with trunk t, hence it is clear that it belongs to Q_{t,ε_0} because $[q^*] = [t]$. Let $\bar{q} = \langle q_\ell \colon \ell < \omega\rangle \in Q^\omega_{t,\varepsilon_0}$. Therefore, for all $\ell < \omega$, $\text{Leb}_{[t]}([q_\ell]) \geq 1 - \varepsilon_0$, that is, we are under the hypothesis of Lemma 4.2.17, by virtue of which, there exists some $q^\otimes \leq q^*$ such that, for all $q \leq q^\otimes$,

   $$\int_\omega a_k(q)\, d\Xi(k) \geq 1 - \varepsilon_0, \tag{4.2.21}$$

   where $a_k(q)$ is as in Lemma 4.2.17. Thereby, we can define $\lim_{t,\varepsilon_0} \colon Q^\omega_{t,\varepsilon_0} \to \mathbb{B}$ such that $\lim_{t,\varepsilon_0}(\bar{q}) \coloneqq q^\otimes$, which satisfies, by Equation 4.2.21, that

   $$\forall q \leq \lim_{t,\varepsilon_0}(\bar{q}) \left(\int_\omega a_k(q) d\Xi(k) \geq 1 - \varepsilon_0\right).$$

   Now let $i^* < \omega$ and $\bar{q}^i = \langle q^i_\ell \colon \ell < \omega\rangle \in Q^\omega_{t,\varepsilon_0}$ for each $i < i^*$. Let $\varepsilon > 0$, $k^* < \omega$, $\langle B_m \colon m < m^*\rangle$ a finite partition of ω and $q \in \mathbb{P}$ such that $q \leq \lim(\bar{q}^i)$ for each $i < i^*$.

Notice that, by the construction of $\lim_{t,\varepsilon_0}$, for all $i < i^*$ and $q' \leq q$ we have that $q' \leq \lim_{t,\varepsilon_0}(\bar{q}^i)$ and therefore $\int_\omega a_k(q')\, d\Xi(k) \geq 1 - \varepsilon_0$ for each $i < i^*$. So we are under the hypothesis of Lemma 4.2.16, by virtue of which, for given $\varepsilon > 0$ and $k^* < \omega$, there are a finite set $u \subseteq \omega \setminus k^*$ and $q^\oplus \leq q$ such that:

(a) $\left|\frac{|u \cap B_m|}{|u|} - \Xi(B_m)\right| < \varepsilon$, for all $m < m^*$,

(b) $\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q^\oplus \leq q^i_\ell\}|}{|I_k|} \geq 1 - \varepsilon_0 - \varepsilon$, for all $i < i^*$,

which proves that $Q_{t,\varepsilon}$ is $(\Xi, \bar{I}, \varepsilon_0)$-linked.

2. Let $\varepsilon \in (0,1)_\mathbb{Q}$ and $r \in \mathbb{B}$. So $[r] \subseteq {}^\omega 2$ and $\mathrm{Leb}([r]) > 0$. By *Lebesgue density Theorem* (see Theorem 1.2.8), there exists some $x \in [r]$ such that, $\lim_{n\to\infty} \mathrm{Leb}_{[x\restriction n]}([r]) = 1$, hence there is some $N < \omega$ such that, for all $n \geq N$, $|\mathrm{Leb}_{[x\restriction n]}([r]) - 1| < \varepsilon$. Notice that, for $t \coloneqq x \restriction (N+1) \in {}^{<\omega}2$ we have that $\mathrm{Leb}_{[t]}([r]) \geq 1 - \varepsilon$, that is, $r \in Q_{t,\varepsilon}$. Thus, $\bigcup\{Q_{t,\varepsilon} \colon t \in {}^{<\omega}2\} = \mathbb{B}$.

3. See Corollary 4.1.8.

Finally, since $\{t \in {}^{<\omega}2 \colon \mathrm{Leb}([t]) > 0\}$ is a countable set, we can conclude that random forcing is σ-FAM-linked. $\square_{\text{Theorem 4.2.18}}$

The forcing notion $\tilde{\mathbb{E}}$ is σ-FAM-linked

Now, based in [KST19, Lem. 1.20] we prove that $\tilde{\mathbb{E}}$ is also σ-FAM-linked. In this case, we do not bother to give as much detail as in the proof for random forcing, since the proof for Definition 4.2.8(1) is very well detailed in [KST19, Lem. 1.20], so, we just focus on what is new: proving the condition on the intersection numbers.

Theorem 4.2.19. *$\tilde{\mathbb{E}}$ is σ-FAM-linked.*

Proof. For $t \in \tilde{\mathcal{T}}$ and $\varepsilon \in (0,1)_\mathbb{Q}$, define

$$Q_{t,\varepsilon} \coloneqq \{p \in \mathrm{dom}(\mathrm{loss}) \colon \mathrm{trunk}(p) = t \wedge \mathrm{loss}(p) \leq \varepsilon\}.$$

Let us verify the conditions of Definition 4.2.8:

1. This result is due to *Jakob Kellner*, *Saharon Shelah* and *Anda Tănasie*, see [KST19, Lem. 1.20].

2. Let $\varepsilon \in (0,1)_\mathbb{Q}$ and $p \in \tilde{\mathbb{E}}$. By Theorem 1.5.47(1), there is some $q \in \mathrm{dom}(\mathrm{loss})$ such that $q \leq p$ and, by the definition of $\tilde{\mathbb{E}}$, we can extend $\mathrm{trunk}(q)$ long enough to find a condition $r \in \tilde{\mathbb{E}}$ such that $r \leq q$ and $\mathrm{loss}(r) \leq \varepsilon$. So, if we define $t \coloneqq \mathrm{trunk}(r)$, we have that $r \in Q_{t,\varepsilon}$ and $r \leq p$.

3. Tănasie Let $t \in \tilde{\mathcal{T}}$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$. By Theorem 1.5.47(3), $\iota\colon \tilde{\mathbb{E}} \to \mathscr{B}$ such that for all $p \in \tilde{\mathbb{E}}$, $\iota(p) \coloneqq [[p]]_{\mathcal{N}}$ is a dense embedding, where $\mathscr{B}$ is a sub-Boolean algebra of $\mathcal{B}({}^{\omega}2)/\mathcal{N}$ Also, by Theorem 1.5.47(2), for all $p \in \mathrm{dom}(\mathrm{loss})$, we have that:

$$\frac{\mathrm{Leb}([p])}{\mathrm{Leb}([\mathrm{trunk}(p)])} \geq 1 - \frac{\mathrm{loss}(p)}{2}.$$

In particular, if $p \in Q_{t,\varepsilon}$ then $\mathrm{trunk}(p) = t, \mathrm{loss}(p) \leq \varepsilon$ and therefore:

$$\forall p \in Q_{t,\varepsilon} \left(\mathrm{Leb}_{[t]}([p]) \geq 1 - \varepsilon\right) \tag{4.2.22}$$

Define $\mu\colon \mathscr{B} \to [0,1]$ such that $\mu([b]_{\mathcal{N}}) \coloneqq \mathrm{Leb}_{[t]}(b)$. Then,

$$\mu([\emptyset]_{\mathcal{N}}) = \mathrm{Leb}_{[t]}(\emptyset) = 0.$$

Also, if $[a]_{\mathcal{N}}, [b]_{\mathcal{N}} \in \mathscr{B}$ are such that $[a]_{\mathcal{N}} \wedge [b]_{\mathcal{N}} = [\emptyset]_{\mathcal{N}}$, then $[a \cap b]_{\mathcal{N}} = [\emptyset]_{\mathcal{N}}$, hence $a \cap b = (a \cap b) \triangle \emptyset$. Therefore,

$$\begin{aligned}\mu([a]_{\mathcal{N}} \vee [b]_{\mathcal{N}}) &= \mu([a \cup b]_{\mathcal{N}}) \\ &= \mathrm{Leb}_{[t]}(a) + \mathrm{Leb}_{[t]}(b) - \mathrm{Leb}_{[t]}(a \cap b) \\ &= \mathrm{Leb}_{[t]}(a) + \mathrm{Leb}_{[t]}(b) - \mathrm{Leb}_{[t]}(\emptyset) \\ &= \mathrm{Leb}_{[t]}(a) + \mathrm{Leb}_{[t]}(b) \\ &= \mu_{[t]}([a]_{\mathcal{N}}) + \mu_{[t]}([b]_{\mathcal{N}}).\end{aligned}$$

Thus, μ is a finitely additive measure.

Now, define also $Q \coloneqq \{b \in \mathscr{B}\colon \mu(b) \geq 1-\varepsilon\}$. Notice that $\iota[Q_{t,\varepsilon}] \subseteq Q$. Indeed, let $b \in \iota[Q_{t,\varepsilon}]$ and $p \in Q_{t,\varepsilon}$ such that $b = [p]_{\mathcal{N}}$. Since $t = \mathrm{trunk}(p)$, $[p] \subseteq [t]$ hence, $[p] \cap [t] = [p]$. Therefore, by Equation 4.2.12 we have that:

$$\mu(b) = \mu([[p]]_{\mathcal{N}}) = \mathrm{Leb}_{[t]}([p]) \geq 1 - \varepsilon,$$

that is, $b \in Q$, hence $\iota[Q_{t,\varepsilon}] \subseteq Q$. Finally, by Lemma 4.1.6(9), Theorem 4.1.7 and Corollary 4.1.5 we have that:

$$\mathrm{int}^{\tilde{\mathbb{E}}}(Q_{t,\varepsilon}) = \mathrm{int}^{\mathscr{B}}(\iota[Q_{t,\varepsilon}]) \geq \mathrm{int}^{\mathscr{B}}(Q) \geq 1 - \varepsilon.$$

Thus, $\mathrm{int}^{\tilde{\mathbb{E}}}(Q_{t,\varepsilon}) \geq 1 - \varepsilon$.

Finally, as $\tilde{\mathcal{T}}$ is countable, we can conclude that $\tilde{\mathbb{E}}$ is σ-FAM-linked. $\square_{\text{Theorem 4.2.19}}$

Notice that, although it was not difficult to prove the condition on the intersection numbers for random forcing or for $\tilde{\mathbb{E}}$, this is a very important point in the development of this thesis, since any condition that we demanded in the definition of μ-FAM-linked is necessarily met by $\mathbb{B}$ and $\tilde{\mathbb{E}}$.

4.2.4 Some examples of no σ-FAM-linked forcing notions

By Theorem 1.5.30, we known that no σ-Fr-linked forcing notion can add dominating reals, therefore by Corollary 4.2.11, we get:

Theorem 4.2.20. *If $\mathbb{P}$ is a forcing notion adding dominating reals, then it cannot be σ-FAM-linked.*

As a result, we particularly have that Hechler forcing is not σ-FAM-linked.

Finally, thanks to an unpublished result of *Miguel Cardona* and *Diego Mejía*, we have an example of a σ-Fr-linked forcing notion that is not σ-FAM-linked, which in particular proves that the notion of μ-FAM-linked is strictly stronger than μ-Fr-linked. It is about the *eventually different forcing notion*, denoted by $\mathbb{E}$ (see [Mil81, Sec. 5]):

Theorem 4.2.21. *$\mathbb{E}$ is σ-Fr-linked, but it is not σ-FAM-linked.*

4.3 Iterating with μ-FAM-linked forcing notions: a generalization of the iterated method using finitely additive measures

In this section we are going to generalize the finite support iteration that *Saharon Shelah* constructed in [She00] to force the consistency of $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0$. In Shelah's work, for reasons we will study in the next chapter, partial random forcing was used[8] to iterate. A natural question that arises when studying such an iteration is: what are the particular properties of random forcing that allow the iteration to work?. Moreover, is it possible to construct such an iteration with other forcing notions?, In this case, what are the properties that must be satisfied?. The problem is mainly found in the extension theorems (see Theorem 4.3.16 and Theorem 4.3.18), since the forcing notion is required to have some structure, in order to be able to construct the sequence of finitely additive measures that allow us to extend the iteration.

In this section we will show that, the properties that are needed to be able to establish extension theorems, are precisely those that define the μ-FAM-linked forcing notions, that is, we will show that the abstraction of the properties of random forcing that we made in the previous section are, indeed, the right ones to be able to generalize the iteration of [She00]. So, we will define an iteration based on [She00], but not iterating with random forcing, but with restrictions of μ-FAM-linked forcing notions.

Fix, for the rest of the section, an uncountable regular cardinal κ.

4.3.1 The iteration structure

First, we introduce the iterations with FAM-linked forcing notions:

Definition 4.3.1. We define $\mathcal{K}_0(\kappa)$ as the collection of sequences

$$\mathbb{K} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha,\, \mathbb{P}^-_\alpha,\, \vec{Q}_\alpha,\, \theta_\alpha \colon \alpha < \pi,\, \beta \leq \pi \rangle, \text{ where:}$$

[8] *Saharon Shelah* also uses some forcing notions of "small" cardinality, but his intention is to prove additional things, such as the consistency with $\mathrm{MA}_{<\kappa}$. We are going to use Hechler forcing to force $\mathfrak{b} = \kappa$.

1. $\mathbb{P}_\pi = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi \rangle$ is a finite support iteration,

2. For any $\alpha < \pi$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}^-_\alpha$-name of a θ_α-FAM-linked forcing notion, witnessed by the sequence $\vec{Q}_\alpha = \langle \dot{Q}^\alpha_{\zeta,\varepsilon} \colon \zeta < \theta_\alpha,\ \varepsilon \in (0,1)_\mathbb{Q} \rangle$, where each $\dot{Q}^\alpha_{\zeta,\varepsilon}$ is a $\mathbb{P}^-_\alpha$-name, and $\theta_\alpha < \kappa$,

3. For any $\alpha < \pi$, $\mathbb{P}^-_\alpha \subsetdot \mathbb{P}^\bullet_\alpha$, where
$$\mathbb{P}^\bullet_\alpha \coloneqq \left\{ p \in \mathbb{P}_\alpha \colon \forall \xi \in \mathrm{dom}(p) \left[p(\xi) \in \mathrm{V}^{\mathbb{P}^-_\xi} \wedge \exists \zeta < \theta_\xi \, \exists \varepsilon {\in} (0,1)_\mathbb{Q} \left(\Vdash_{\mathbb{P}^-_\xi} \text{“} p(\xi) \in \dot{Q}^\xi_{\zeta,\varepsilon} \text{”} \right) \right] \right\}.$$

We show later that $\mathbb{P}^\bullet_\beta$ is dense in $\mathbb{P}_\beta$ for all $\beta \leq \pi$.
In order to simplify the writing, we introduce the following notation:

Definition 4.3.2. Let $\mathbb{K} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha,\ \mathbb{P}^-_\alpha,\ \vec{Q}_\alpha,\ \theta_\alpha \colon \alpha < \pi,\ \beta \leq \pi \rangle \in \mathcal{K}_0(\kappa)$ and $\gamma < \pi$.

1. We say that π is the *length* of $\mathbb{K}$.

2. The *restriction of* $\mathbb{K}$ *to* γ is defined by $\mathbb{K} {\restriction} \gamma \coloneqq \langle \mathbb{P}_\beta,\ \dot{\mathbb{Q}}_\alpha,\ \mathbb{P}^-_\alpha, \vec{Q}_\alpha,\ \theta_\alpha \colon \alpha < \gamma,\ \beta \leq \gamma \rangle$.

It is clear that, for any $\mathbb{K} \in \mathcal{K}_0(\kappa)$ and any $\gamma \leq \pi$, $\mathbb{K} {\restriction} \gamma \in \mathcal{K}_0(\kappa)$.

Remark 4.3.3. In general, when we say “$\mathbb{K} \in \mathcal{K}_0(\kappa)$” it will be understood that $\mathbb{K}$ has length π and $\mathbb{K} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha,\ \mathbb{P}^-_\alpha,\ \theta_\alpha \colon \alpha < \pi,\ \beta \leq \pi \rangle$, unless otherwise specified.

Also, notice that, by definition of $\mathbb{P}^\bullet_\alpha$, it is clear that $\mathbb{P}^\bullet_\alpha \subseteq \mathbb{P}^\bullet_\beta$, whenever $\alpha < \beta \leq \pi$. It is not difficult to notice that $\mathbb{P}^\bullet_\beta$ is dense in $\mathbb{P}_\beta$, in fact, we can prove something stronger. For this, we define the following set:

Definition 4.3.4. Let $\mathbb{K} \in \mathcal{K}_0(\kappa)$. For any function $\bar{\varepsilon} \colon \omega \to (0,1)_\mathbb{Q}$, define $D^\mathbb{K}_{\bar{\varepsilon}}$ as the set of conditions $p \in \mathbb{P}^\bullet_\pi$ such that:

1. there exists $n^* < \omega$ such that $\mathrm{dom}(p) = \{\alpha_n \colon n < n^*\}$,

2. $\alpha_n < \alpha_m$ whenever $m < n < n^*$, that is, $\mathrm{dom}(p)$ is arranged in decreasing order,

3. for all $n < n^*$, there exists some $\zeta < \theta_{\alpha_n}$ such that $p {\restriction} \alpha_n \Vdash_{\alpha_n}$ “$p(\alpha_n) \in \dot{Q}^{\alpha_n}_{\zeta, \bar{\varepsilon}(n)}$”.

When the context is clear, we denote $D^\mathbb{K}_{\bar{\varepsilon}}$ simply as $D_{\bar{\varepsilon}}$.

The structure of iterations in $\mathcal{K}_0(\kappa)$ guarantees that this set is dense in the final step of the iteration:

Lemma 4.3.5. *Let $\mathbb{K} \in \mathcal{K}_0(\kappa)$. Then, for any $\alpha \leq \pi$ and any function $\bar{\varepsilon} \colon \omega \to (0,1)$, $D^{\mathbb{K} {\restriction} \alpha}_{\bar{\varepsilon}}$ is dense in $\mathbb{P}_\alpha$. In particular, $D_{\bar{\varepsilon}}$ is dense in $\mathbb{P}_\pi$.*

Proof. We apply transfinite induction on $\alpha \leq \pi$: fix $\bar{\varepsilon} \colon \omega \to (0,1)$ and assume that, for any $\beta < \alpha$, $D^{\mathbb{K} {\restriction} \beta}_{\bar{\epsilon}}$ is dense in $\mathbb{P}_\beta$ for every function $\bar{\epsilon} \colon \omega \to (0,1)$. The case $\alpha = 0$ is clear, so we only deal with the other cases:

1. α successor: suppose that $\alpha = \beta + 1$ and let $p \in \mathbb{P}_\alpha$. Without loss of generality, we can assume that $\beta \in \operatorname{dom}(p)$, so $p \restriction \beta \Vdash_\beta$"$p(\beta) \in \dot{\mathbb{Q}}_\beta$". Since by induction hypothesis, $D_{\bar{\varepsilon}}^{\mathbb{K}\restriction\beta}$ is dense in $\mathbb{P}_\beta$, we get that $\mathbb{P}_\beta^\bullet$ is dense in $\mathbb{P}_\beta$ and by Lemma 1.5.16 and Definition 4.3.1(3), $\mathbb{P}_\beta^- \lessdot \mathbb{P}_\beta$. As a consequence, we can find a $\mathbb{P}_\beta^-$-name $\dot{q}$, $\zeta < \theta_\beta$ and $r_0 \in \mathbb{P}_\beta$, such that $r_0 \leq p \restriction \beta$ and
$$r_0 \Vdash_\beta \text{“}p(\beta) = \dot{q} \in \dot{Q}^\beta_{\zeta, \bar{\varepsilon}(0)}\text{”}.$$
Consider $\bar{\epsilon} \colon \omega \to (0,1)$ such that, for any $n < \omega$, $\bar{\epsilon}(n) \coloneqq \bar{\varepsilon}(n+1)$. Since $r_0 \in \mathbb{P}_\beta$, by induction hypothesis, there is some $r_1 \in D_{\bar{\epsilon}}^{\mathbb{K}\restriction\beta}$ such that $r_1 \leq r_0$. We set $r \coloneqq r_1 \cup \{(\beta, \dot{q})\}$. It is clear that $r \leq_\alpha p$. Now we show that $r \in D_{\bar{\varepsilon}}^{\mathbb{K}\restriction\alpha}$. Since $r_1 \in D_{\bar{\epsilon}}^{\mathbb{K}\restriction\beta}$, there is $m^* < \omega$ such that $\operatorname{dom}(r_1) = \{\alpha_n \colon n < m^*\}$ and this numeration is decreasing. We define $n^* \coloneqq m^* + 1$ and
$$\gamma_n \coloneqq \begin{cases} \alpha_{n-1}, & \text{if} \quad 0 < n < n^*, \\ \\ \beta, & \text{if} \qquad n = 0, \end{cases}$$
so, $\operatorname{dom}(r) = \{\gamma_n \colon n < n^*\}$ and since $\alpha_0 < \beta$, it is a decreasing numeration. Let $0 < n < n^*$. Since $r_1 \in D_{\bar{\epsilon}}^{\mathbb{K}\restriction\beta}$, by Definition 4.3.4(3), there exists $\xi < \theta_{\alpha_{n-1}}$ such that $r_1 \restriction \alpha_{n-1} \Vdash_{\alpha_{n-1}}$"$r_1(\alpha_{n-1}) \in \dot{Q}^{\alpha_{n-1}}_{\xi, \bar{\epsilon}(n-1)}$". However, as in this case $\gamma_n = \alpha_{n-1}$ and $\bar{\epsilon}(n-1) = \bar{\varepsilon}(n)$, we get that $r \restriction \gamma_n \Vdash_{\gamma_n}$"$r(\gamma_n) \in \dot{Q}^{\gamma_n}_{\xi, \bar{\varepsilon}(n)}$. Finally, if $n = 0$, then $\gamma_0 = \beta$ and we know already that $r \restriction \beta \Vdash_\beta$"$r(\beta) \in \dot{Q}^\beta_{\zeta, \bar{\varepsilon}(0)}$". Thus, $r \in D_{\bar{\varepsilon}}^{\mathbb{K}\restriction\alpha}$.

2. α limit: let $p \in \mathbb{P}_\alpha$, hence there exists some $\beta < \alpha$ such that $p \in \mathbb{P}_\beta$ and by induction hypothesis, we can find $r \in D_{\bar{\varepsilon}}^{\mathbb{K}\restriction\beta}$ such that $r \leq p$. Notice that $D_{\bar{\varepsilon}}^{\mathbb{K}\restriction\beta} \subseteq D_{\bar{\varepsilon}}^{\mathbb{K}\restriction\alpha}$ because $\mathbb{P}_\beta^\bullet \subseteq \mathbb{P}_\alpha^\bullet$ and therefore, $r \in D_{\bar{\varepsilon}}^{\mathbb{K}\restriction\alpha}$. Finally, as $\mathbb{P}_\beta \lessdot \mathbb{P}_\alpha$, we have that $r \leq_\alpha p$. $\square_{\text{Lemma 4.3.5}}$

As a consequence, as mentioned before:

Corollary 4.3.6. *If $\mathbb{K} \in \mathcal{K}_0(\kappa)$, then, for any $\alpha \leq \pi$, $\mathbb{P}_\alpha^\bullet$ is dense in $\mathbb{P}_\alpha$. As a consequence, for any $\alpha < \pi$, $\mathbb{P}_\alpha^- \lessdot \mathbb{P}_\alpha$.*

One of the fundamental parameters in Shelah's iteration is what he calls *blueprints* (see [She00, Def. 2.9]). We are going to replace that notion with that of *guardrail* ([GMS16] and [KST19]), since we consider that this facilitates the formalization and compression of the iteration:

Definition 4.3.7. Let γ, ζ be ordinals.

1. A *half guardrail for* (γ, ζ) is a function $g \in {}^\gamma[\zeta \times (0,1)_{\mathbb{Q}}]$.

2. We say that $(g, \bar{I})$ is a *guardrail for* (γ, ζ) if g is a half guardrail for (γ, ζ) and $\bar{I} \in \mathcal{I}_\infty$.

3. If $\mathcal{G}$ is a set of guardrails for (γ, ζ), then $\mathcal{G}_0 \coloneqq \{g \in {}^\gamma[\zeta \times (0,1)_{\mathbb{Q}}] \colon \exists \bar{I} \in \mathcal{I}_\infty((g, \bar{I}) \in \mathcal{G})\}$.

4. A set $\mathcal{G}_0$ of half guardrails for (γ, ζ) is a *complete set of half guardrails* if, for any function $\sigma \colon X \to \zeta \times (0,1)_{\mathbb{Q}}$ with $X \in [\gamma]^{<\aleph_1}$, there is some $g \in \mathcal{G}_0$, such that $\sigma \subseteq g$.

5. A set $\mathcal{G}$ of guardrails for (γ, ζ) is a *complete set of guardrails for* (γ, ζ), if $\mathcal{G}_0$ is a complete set of half guardrails for (γ, ζ) and, for any $g \in \mathcal{G}_0$ and $\bar{I} \in \mathcal{I}_\infty$, $(g, \bar{I}) \in \mathcal{G}$.

6. If $\mathcal{G}$ is a set of guardrails for (π, ζ) and $\gamma \leq \pi$, we define $\mathcal{G} \restriction \gamma := \{(g \restriction \gamma, \bar{I}) \colon (g, \bar{I}) \in \mathcal{G}\}$.

Abusing the notation, we sometimes identify the guardrail $h = (g, \bar{I})$ with g, and denote $h \restriction \gamma := (g \restriction \gamma,\, \bar{I})$.

Remark 4.3.8. For any half guardrail g for (γ, ζ), there are $g_0 \colon \gamma \to \zeta$ and $g_1 \colon \gamma \to (0,1)_{\mathbb{Q}}$ such that, for any $\alpha < \pi$, $g(\alpha) = (g_0(\alpha),\, g_1(\alpha))$. In general, we use this decomposition of g.

Notice that the following result is clear by the definitions:

Lemma 4.3.9. *If $\mathcal{G}$ is a complete set of guardrails for (γ, ζ) and $\alpha \leq \gamma$, then $\mathcal{G} \restriction \alpha$ is a complete set of guardrails for (α, ζ). Likewise for a complete set of half guardrails.*

The idea is that we are going to use half guardrails g such that, for some $\alpha < \pi$ in question, $g_0(\alpha) < \theta_\alpha < \kappa$. Furthermore, since $g_1(\alpha) \in (0,1)_{\mathbb{Q}}$, the guardrail g is in charge of choosing a subset $\dot{Q}_{\zeta,\varepsilon}$ in each step of the iteration, which justifies the name "guardrail", as it gives us information about the coordinates in the steps of the iteration.

The tool we are going to use to obtain complete sets of guardrails is known as the *Engelking-Karłowicz theorem* (see [EK65]) . The proof that we are going to present below is due to *Saharon Shelah*, but improved by *Assaf Rinot* in his personal blog (see [Rin12]):

Theorem 4.3.10. *Let θ, μ and χ be infinite cardinals such that $\theta \leq \mu \leq \chi \leq 2^\mu$. Then there exists $\mathcal{G} \subseteq {}^{\chi}\mu$ such that $|\mathcal{G}| \leq \mu^{<\theta}$, and every function $f \colon X \to \mu$ with $X \in [\chi]^{<\theta}$ can be extended by a function in $\mathcal{G}$.*

Proof. Consider the set $W := \{(a, \mathcal{A}, g) \colon a \in [\mu]^{<\theta},\, \mathcal{A} \in [\mathcal{P}(a)]^{<\theta}$ and $g \in {}^{\mathcal{A}}\mu\}$. It is clear that $|W| = \mu^{<\theta}$, hence we can find a numeration $W = \{(a_i,\, \mathcal{A}_i,\, g_i) \colon i < \mu^{<\theta}\}$. Since $\chi \leq 2^\mu$, there exists some sequence $\{B_\alpha \colon \alpha < \chi\}$ of distinct subsets of μ. For any $i < \mu^{<\theta}$, define $f_i \colon \chi \to \mu$ such that, for $\alpha < \chi$,

$$f_i(\alpha) := \begin{cases} g_i(a_i \cap B_\alpha) & \text{if} \quad a_i \cap B_\alpha \in \mathcal{A}_i, \\ \\ 0, & \text{if} \quad a_i \cap B_\alpha \notin \mathcal{A}_i, \end{cases}$$

and define $\mathcal{G} := \{f_i \colon i < \mu^{<\theta}\} \subseteq {}^{\chi}\mu$, therefore $|\mathcal{G}| \leq \mu^{<\theta}$. Now, let $X \in [\chi]^{<\theta}$ and a function $f \colon X \to \mu$. For any $\alpha, \beta \in X$ such that $\alpha \neq \beta$, choose $x_{\alpha,\beta} \in B_\alpha \triangle B_\beta$ and define $a := \{x_{\alpha,\beta} \colon \alpha, \beta \in X$ and $\alpha \neq \beta\}$. Then, $|a| < \theta$ and also, for $\alpha \neq \beta$, we have that $a \cap B_\alpha \neq a \cap B_\beta$ because we chosen $x_{\alpha,\beta}$ in its symmetric difference. So if we define $\mathcal{A} := \{a \cap B_\alpha \colon \alpha \in X\}$, then $|\mathcal{A}| = |a|$ and $\mathcal{A} \in [\mathcal{P}(a)]^{<\theta}$. Also, we can define $g \colon \mathcal{A} \to \lambda$ such that $g(a \cap B_\alpha) := f(\alpha)$ for any $\alpha \in X$. Finally, choose $i < \lambda$ such that $(a, \mathcal{A}, g) = (a_i, \mathcal{A}, g_i)$. It is clear by the construction that $f \subseteq f_i$. $\square_{\text{Theorem 4.3.10}}$

In particular, for the applications that we are going to present in Chapter 5, we need the following result:

Corollary 4.3.11. *If $\aleph_0 \leq \mu \leq \chi \leq 2^\mu$ then there exists a complete set of guardrails $\mathcal{G}$ for (χ, μ), such that $|\mathcal{G}| \leq \mu^{\aleph_0}$.*

Proof. By *Engełkin-Karlowicz theorem*, there is a complete set of half guardrails $\mathcal{G}_0$ for (χ,μ), such that $|\mathcal{G}_0| \leq \mu^{<\aleph_1} = \mu^{\aleph_0}$. Therefore, it is clear that $\mathcal{G} := \{(g,\bar{I})\colon g \in \mathcal{G}_0 \wedge I \in \mathcal{I}_\infty\}$ is a complete set of guardrails for (χ,μ) and $|\mathcal{G}| \leq \mu^{\aleph_0}$. $\square_{\text{Corollary 4.3.11}}$

The following definition is a generalization of [She00, Def. 2.11(d)] and it is based on [KST19, Def. 2.33] with some modifications to adapt it to our formalism.

Definition 4.3.12. Let $\mathbb{K} \in \mathcal{K}_0(\kappa)$ and g a half guardrail for (π,κ). Let $\tau = \langle (\dot{q}_\ell,\, \alpha_\ell,\, \zeta_\ell,\, \varepsilon_\ell)\colon \ell < \omega\rangle$. We say that τ *follows* g if:

1. For any $\ell < \omega$, $\dot{q}_\ell$ is a $\mathbb{P}^-_{\alpha_\ell}$-name such that $\Vdash_{\alpha_\ell}$ "$\dot{q}_\ell \in \dot{Q}^{\alpha_\ell}_{\zeta_\ell,\varepsilon_\ell}$",
2. Either the sequence $\langle \alpha_\ell\colon \ell < \omega\rangle$ is increasing or constant,
3. The sequence $\langle \varepsilon_\ell\colon \ell < \omega\rangle$ is constant,
4. $g(\alpha_\ell) = (\zeta_\ell,\, \varepsilon_\ell)$ for all $\ell < \omega$.

We can now define the *generalized iteration with finitely additive measures*:

Definition 4.3.13. Let $\mathcal{G}$ a set. Define $\mathcal{K}_1(\kappa,\mathcal{G})$ as the collection of sequences

$$\mathbb{K} = \langle \mathbb{P}_\beta,\, \dot{\mathbb{Q}}_\alpha,\, \mathbb{P}^-_\alpha,\, \vec{Q}_\alpha,\, \theta_\alpha,\, \vec{\Xi}_\beta\colon \alpha < \pi,\ \beta \leq \pi\rangle, \text{ where:}$$

1. $\langle \mathbb{P}_\beta,\, \mathbb{Q}_\alpha,\, \mathbb{P}^-_\alpha,\, \vec{Q}_\alpha,\, \theta_\alpha\colon \alpha < \pi,\ \beta \leq \pi\rangle \in \mathcal{K}_0(\kappa)$,
2. $\mathcal{G}$ is a set of guardrails for (π,κ),
3. For any $\beta \leq \pi$, $\vec{\Xi}_\beta = \langle \dot{\Xi}^g_\beta\colon g \in \mathcal{G}_0\rangle$ and $\Vdash_{\mathbb{P}_\beta}$"$\dot{\Xi}^g_\beta$ is a finitely additive measure on $\mathcal{P}(\omega)$", for all $g \in \mathcal{G}_0$,
4. $\alpha < \beta \leq \pi \Rightarrow \Vdash_{\mathbb{P}_\beta}$" $\dot{\Xi}^g_\alpha \subseteq \dot{\Xi}^g_\beta$", for any $g \in \mathcal{G}_0$,
5. $\alpha < \pi \Rightarrow \Vdash_\alpha$"$(\Xi^g_\alpha)^- \in \mathrm{M}^{\mathbb{P}^-_\alpha}$", where $(\Xi^g_\alpha)^- := \Xi^g_\alpha \restriction (\mathcal{P}(\omega) \cap \mathrm{M}^{\mathbb{P}^-_\alpha})$, for any $g \in \mathcal{G}_0$,
6. For any $(g,\bar{I}) \in \mathcal{G}$ and $\tau = \{(\dot{q}_\ell,\, \alpha_\ell,\, \zeta_\ell,\, \varepsilon_\ell)\colon \ell < \omega\}$ following g, it is satisfied that:
   (a) If the sequence $\langle \alpha_\ell\colon \ell < \omega\rangle$ is constant with value α, then:
   $$\Vdash_{\mathbb{P}_\alpha} \text{“}\lim^{(\Xi^g_\alpha)^-}_{\ell<\omega}(\langle \dot{q}_\ell\rangle_{\ell<\omega}) \Vdash_{\dot{\mathbb{Q}}_\alpha} \text{“}\int_\omega \frac{|\{\ell \in I_k\colon \dot{q}_\ell \in \dot{G}(\alpha)\}|}{|I_k|} d\,\dot{\Xi}^g_{\alpha+1}(k) \geq 1 - g_1(\alpha)\text{””}.$$
   (b) If the sequence $\langle \alpha_\ell\colon \ell < \omega\rangle$ is increasing, then for all $\varepsilon > 0$,
   $$\Vdash_{\mathbb{P}_\pi} \text{“ }\dot{\Xi}^g_\pi\left(\left\{k < \omega\colon \frac{|\{\ell \in I_k\colon \dot{q}_\ell \in \dot{G}(\alpha_\ell)\}|}{|I_k|} \geq [1 - g_1(\alpha_0)](1-\varepsilon)\right\}\right) = 1\text{”}.$$

Condition 6(a) requires a justification: why can we take that limit? Since the sequence $\langle \alpha_\ell \colon \ell < \omega\rangle$ is constant with value α and g is a function, we have that also $\langle \zeta_\ell \colon \ell < \omega\rangle$ is constant with value, say, ζ. On the other hand, $\langle \varepsilon_\ell \colon \ell < \omega\rangle$ is constant with value, say ε, because τ follows g. As a consequence, in $\mathrm{M}^{\mathbb{P}_\alpha^-}$ we have that $\langle \dot{q}_\ell \colon \ell < \omega\rangle \in (\dot{Q}^\alpha_{\zeta,\varepsilon})^\omega$. Also, as $(\Xi^g_\alpha)^-$ is a finitely additive measure in $\mathrm{M}^{\mathbb{P}_\alpha^-}$, and $\dot{Q}^\alpha_{\zeta,\varepsilon}$ is in particular $((\dot{\Xi}^g_\alpha)^-, \bar{I}, \varepsilon)$-linked in that model, we have that $\lim^{(\Xi^g_\alpha)^-}_{Q^\alpha_{\zeta,\varepsilon}}(\langle \dot{q}_\ell \colon \ell < \omega\rangle)$ make sense.

Notation 4.3.14. Let $\mathbb{K} \in \mathcal{K}_0(\kappa)$. For any sequence $\vec{\Xi} = \langle \Xi^i_\beta \colon \beta \leq \pi \wedge i \in I\rangle$, we denote

$$\mathbb{K} \sqcup \vec{\Xi} \coloneqq \langle \mathbb{P}_\beta,\, \dot{\mathbb{Q}}_\alpha,\, \mathbb{P}^\bullet_\alpha, \vec{Q}_\alpha,\, \theta_\alpha,\, \vec{\Xi}_\beta \colon \alpha < \pi,\, \beta \leq \pi\rangle,$$

where for any $\beta \leq \pi$, $\vec{\Xi}_\beta \coloneqq \langle \dot{\Xi}^i_\beta \colon i \in I\rangle$.

Concerning the finitely additive measures, if g is a guardrail for (π, ζ) and $\alpha \leq \gamma \leq \pi$, we identify $\dot{X}^{g\restriction\gamma}_\alpha$ with $\dot{\Xi}^g_\alpha$.

In order to construct iterations in $\mathcal{K}_1(\kappa, \mathcal{G})$, we need to properly extend the parameters of the iteration at the successor and limit steps, so that the iteration still belongs to $\mathcal{K}_1(\kappa, \mathcal{G})$. In general, this is going to come down to simply getting the sequences of finitely additive measures properly extended, so that they continue to satisfy the conditions of Definition 4.3.13. In the successor steps, there is no difficulty, since we will show that we can amalgamate finitely additive measures throughout the iteration. Moreover, in the limit steps of cofinality $\geq \kappa$, we will not have problems, because in that case the iteration does not add new reals (see Theorem 1.5.55) and then taking the union of the finitely additive measures works, since the domain is preserved, and by the condition Definition 4.3.13(4), we have compatibility. However, the limit step is not only the most difficult step in the construction of *Saharon Shelah* (see [She00, Lem. 2.13]) and [She00, Lem. 2.14[], but for reasons we will see later, it is the most delicate point in our generalization. We start with the successor step:

4.3.2 Extending at successor steps

As a consequence of Theorem 3.4.2, we can amalgamate finitely additive measures in forcing generic extensions. As mentioned before, this is an essential result to construct iterations with finitely additive measures, because it will allow us to extend iterations in $\mathcal{K}_1(\kappa, \mathcal{G})$ at successor steps (see Theorem 4.3.16). The following lemma extends [She00, Clm. 1.6].

Lemma 4.3.15. *Let* $\mathrm{M} \subseteq \mathrm{N}$ *be transitive models of* ZFC, $\mathbb{P} \in \mathrm{M}$ *a forcing notion,* G *a* $\mathbb{P}$*-generic over* N, $X \in M$ *and* $\Xi_0 \in \mathrm{M}$, $\Xi_1 \in \mathrm{N}$ *finite finitely additive measures on* $\mathcal{P}(X) \cap \mathrm{M}$ *and* $\mathcal{P}(X) \cap \mathrm{N}$ *respectively, such that* Ξ_1 *extends* Ξ_0. *In* M, *let* $\dot{\Xi}^*_0$ *be a* $\mathbb{P}$*-name of a finitely additive measure on* $\mathcal{P}(X) \cap \mathrm{M}[G]$ *extending* Ξ_0. *Then, in* N, *there is a* $\mathbb{P}$*-name* $\dot{\Xi}^*_1$ *such that* $\mathbb{P}$ *forces* $\dot{\Xi}^*_1$ *is a finitely additive measure on* $\mathcal{P}(X) \cap \mathrm{N}[G]$ *extending both* Ξ_1 *and* $\dot{\Xi}^*_0$.

$$\begin{array}{ccc} \mathrm{M} & \xrightarrow{\mathbb{P}} & \mathrm{M}[G] \\ \downarrow & & \downarrow \\ \mathrm{N} & \xrightarrow{\mathbb{P}} & \mathrm{N}[G] \end{array} \qquad\qquad \begin{array}{ccc} \Xi_0 & \longrightarrow & \Xi^*_0 \\ \downarrow & & \downarrow \\ \Xi_1 & \longrightarrow & \Xi^*_1 \end{array}$$

In the diagram on the left, the inclusion relationships between the models are presented. In the diagram on the right, the extension relation between the finitely additive measures are presented.

***Proof*.** In view of Theorem 3.4.2(2) and Corollary 3.4.3, it is enough to prove that, if $b \in \mathcal{P}(X) \cap \mathrm{N}$ and $\dot{a} \in \mathrm{M}$ is a $\mathbb{P}$-name of a subset of X then, in N, $\mathbb{P}$ forces $\dot{a} \subseteq b \Rightarrow \dot{\Xi}_0^*(\dot{a}) \leq \Xi_1(b)$ (it is already clear that, in N, $\mathbb{P}$ forces $\dot{\Xi}_0^*(X) = \Xi_1(X) = \Xi_0(X)$). Notice that the identify function from $\mathbb{P}$ into $\mathbb{P}$ is a complete embedding with respect to M, so we can use Lemma 1.5.19 and Lemma 1.5.20.
So assume $p \in \mathbb{P}$ and $p \Vdash_{\mathbb{P}}^{\mathrm{N}} \dot{a} \subseteq b$. Define $a_0 := \{x \in X : \exists q \leq p \colon q \Vdash_{\mathbb{P}}^{\mathrm{M}} x \in \dot{a}\}$, which is in M. Then, $a_0 \subseteq b$ and $p \Vdash_{\mathbb{P}}^{\mathrm{M}} \dot{a} \subseteq a_0$. Therefore $\Xi_0(a_0) = \Xi_1(a_0) \leq \Xi_1(b)$ and $p \Vdash_{\mathbb{P}}^{\mathrm{M}} \dot{\Xi}_0^*(\dot{a}) \leq \dot{\Xi}_0^*(a_0) = \Xi_0(a_0)$, so $p \Vdash_{\mathbb{P}}^{\mathrm{N}} \dot{\Xi}_0^*(\dot{a}) \leq \Xi_1(b)$. $\square_{4.3.15}$

We already have everything necessary to generalize the *extension theorem* at successor steps:

Theorem 4.3.16. *Let $\mathbb{K} \in \mathcal{K}_0(\kappa)$ be of length $\pi = \gamma + 1$ and $\mathcal{G}$ a set of guardrails for (π, κ). Assume that $\mathbb{K} \upharpoonright \gamma \sqcup \langle \dot{\Xi}_\beta^g \colon g \in \mathcal{G} \upharpoonright \gamma \rangle \in \mathcal{K}_1(\kappa, \mathcal{G})$ and that for any $g \in G \upharpoonright \gamma$, $\Vdash_\gamma$ "$(\dot{\Xi}_\gamma^g)^- \in \mathrm{M}^{\mathbb{P}_\gamma^-}$". Then, there exists a sequence $\langle \dot{\Xi}_\pi^g \colon g \in G \rangle$ such that $\mathbb{K} \sqcup \langle \dot{\Xi}_\beta^g \colon \beta \leq \pi,\ g \in G \rangle \in \mathcal{K}_1(\kappa, \mathcal{G})$.*

***Proof*.** Let $(g, \bar{I}) \in \mathcal{G}$, $\zeta_\gamma := g_0(\gamma)$ and $\varepsilon_0 := g_1(\gamma)$. Since the length of $\mathbb{K}$ is $\pi = \gamma + 1$, $\dot{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma^-$-name of a θ_γ-FAM-linked forcing notion witnessed by $\langle \dot{Q}_{\zeta,\varepsilon}^\gamma \colon \zeta < \theta_\gamma \wedge \varepsilon \in (0,1)_{\mathbb{Q}} \rangle$, where each $\dot{Q}_{\zeta,\varepsilon}^\gamma$ is a $\mathbb{P}_\gamma^-$-name. By hypothesis, in $\mathrm{M}^{\mathbb{P}_\gamma^-}$, $(\dot{\Xi}_\gamma^g)^- \in \mathrm{M}^{\mathbb{P}_\gamma^-}$ and in particular $\dot{Q}_{\zeta_\gamma,\varepsilon_0}^\gamma$ is $((\dot{\Xi}_\gamma^g)^-, \bar{I}, \varepsilon_0)$-linked, so by Theorem 4.2.5 there exists a $\dot{\mathbb{Q}}_\gamma$-name $(\dot{\Xi}_\gamma^g)^*$ extending $(\dot{\Xi}_\gamma^g)^-$ such that, for any $\bar{\dot{q}} \in (\dot{Q}_{\zeta_\gamma,\varepsilon_0}^\gamma)^\omega$,

$$\Vdash_{\mathbb{P}_\gamma^-} \text{“}\lim^{(\dot{\Xi}_\gamma^g)^-}_{\dot{Q}^\gamma_{\zeta_\gamma,\varepsilon_0}} (\langle \dot{q}_\ell \rangle_{\ell<\omega}) \Vdash_{\dot{\mathbb{Q}}_\gamma} \text{“} \int_\omega \frac{|\{\ell \in I_k \colon \dot{q}_\ell \in \dot{G}_{\dot{\mathbb{Q}}_\gamma}\}}{|I_k|} d(\dot{\Xi}_\gamma^g)^*(k) \geq 1 - \varepsilon_0\text{””}.$$

Moreover, since $\mathbb{P}_\gamma^- \lessdot \mathbb{P}_\gamma$, by integral absoluteness (see Theorem 3.5.29) and Theorem 1.5.18, we have:

$$\Vdash_{\mathbb{P}_\gamma} \text{“}\lim^{(\dot{\Xi}_\gamma^g)^-}_{\dot{Q}^\gamma_{\zeta_\gamma,\varepsilon_0}} (\langle \dot{q}_\ell \rangle_{\ell<\omega}) \Vdash_{\dot{\mathbb{Q}}_\gamma} \text{“} \int_\omega \frac{|\{\ell \in I_k \colon \dot{q}_\ell \in \dot{G}_{\dot{\mathbb{Q}}_\gamma}\}}{|I_k|} d(\dot{\Xi}_\gamma^g)^*(k) \geq 1 - \varepsilon_0\text{””}. \tag{4.3.1}$$

Now, by Lemma 4.3.15 we can amalgamate as in the following diagrams:

$$\begin{array}{ccc} \mathrm{M}_\gamma^- & \xrightarrow{\dot{\mathbb{Q}}_\gamma} & \mathrm{M}_\gamma^-[G(\gamma)] \\ \downarrow & & \downarrow \\ \mathrm{M}_\gamma & \xrightarrow{\dot{\mathbb{Q}}_\gamma} & \mathrm{M}_\gamma[G(\gamma)] \end{array} \qquad\qquad \begin{array}{ccc} (\Xi_\gamma^g)^- & \longrightarrow & (\Xi_\gamma^g)^* \\ \downarrow & & \downarrow \\ \Xi_\gamma^g & \longrightarrow & \Xi_\pi^g \end{array}$$

That is, in $\mathrm{M}_\gamma[G(\gamma)] = \mathrm{M}^{\mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma} = \mathrm{M}^{\mathbb{P}_{\gamma+1}} = \mathrm{M}_\pi$, there exists a $\dot{\mathbb{Q}}_\gamma$-name $\dot{\Xi}_\pi^g$ of a finitely additive measure on $\mathcal{P}(\omega) \cap \mathrm{M}_\pi$ extending Ξ_γ^g and $(\Xi_\gamma^g)^*$. As a consequence, we get a sequence of $\dot{\mathbb{Q}}_\gamma$-names, $\langle \dot{\Xi}_\pi^g \colon g \in \mathcal{G} \rangle$. Now, we must show that $\mathbb{K} \sqcup \langle \dot{\Xi}_\beta^g \colon \beta \leq \pi,\ g \in G \rangle \in \mathcal{K}_1(\kappa, \mathcal{G})$. Notice that the first five conditions from Definition 4.3.13 are immediate. Now, we deal with condition (6):

6. Suppose that $\tau = \{(\dot{q}_\ell,\ \alpha_\ell,\ \zeta_\ell,\ \varepsilon_\ell) \colon \ell < \omega\}$ follows g for some guardrail $(g, \bar{I})$.

(a) Assume the sequence $\langle\alpha_\ell\colon \ell<\omega\rangle$ is constant with value, say, α. On the one hand, if $\alpha<\gamma$, then the result follows because $\mathbb{K}{\restriction}\gamma\in\mathcal{K}_1(\kappa,\mathcal{G}{\restriction}\gamma)$. On the other hand, if $\alpha=\gamma$, then by Corollary 3.5.30, Equation 4.3.1 and Corollary 3.5.30, we get

$$\Vdash_\gamma \text{“}\lim^{(\dot\Xi^g_\gamma)^-}_{\dot Q^\gamma_{\zeta_\gamma,\varepsilon_0}}(\langle\dot q_\ell\rangle_{\ell<\omega})\Vdash_{\dot{\mathbb{Q}}_\gamma}\text{“}\int_\omega\frac{|\{\ell\in I_k\colon \dot q_\ell\in\dot G_{\dot{\mathbb{Q}}_\gamma}\}}{|I_k|}d\dot\Xi^g_\pi(k)\geq 1-\varepsilon_0\text{””}.$$

(b) Let $\varepsilon>0$. If the sequence $\langle\alpha_\ell\colon \ell<\omega\rangle$ is increasing, then it is bounded by γ, moreover for any $\ell<\omega$, $\alpha_\ell<\gamma$. Since by hypothesis, $\mathbb{K}{\restriction}\gamma\in\mathcal{K}_1(\kappa,\mathcal{G}{\restriction}\gamma)$, by Definition 4.3.13(6)(b), we get:

$$\Vdash_{\mathbb{P}_\gamma}\text{“ }\dot\Xi^g_\gamma\left(\left\{k<\omega\colon \frac{|\{\ell\in I_k\colon \dot q_\ell\in\dot G(\alpha_\ell)\}|}{|I_k|}\geq[1-g_1(\alpha_0)](1-\varepsilon)\right\}\right)=1\text{”},$$

also, as $\gamma<\pi$, and $\mathbb{P}_\pi\Vdash$“$\Xi^g_\gamma\subseteq\Xi^g_\pi$, we can conclude

$$\Vdash_{\mathbb{P}_\pi}\text{“ }\dot\Xi^g_\pi\left(\left\{k<\omega\colon \frac{|\{\ell\in I_k\colon \dot q_\ell\in\dot G(\alpha_\ell)\}|}{|I_k|}\geq[1-g_1(\alpha_0)](1-\varepsilon)\right\}\right)=1\text{”}.$$

Notice that, in this case, we are not using the new sequence of finitely additive measures, that is, the iteration had already taken care of this condition.

Finally, $\mathbb{K}\sqcup\langle\dot\Xi^g_\beta\colon \beta\leq\pi,\ g\in G\rangle\in\mathcal{K}_1(\kappa,\mathcal{G})$. $\square_{\text{Theorem 4.3.16}}$

Notice that, the notion of being “$(\Xi,\bar I,\varepsilon_0)$-linked” is not upward absolute, and for this reason it was necessary to resort to $\mathbb{P}^-_\gamma$ to be able to extend the finitely additive measures.

4.3.3 Extending at limit steps

Now, we deal with the problem of the extension of finitely additive measures at the limit steps. When analyzing the limit step extension proof for both random forcing and $\tilde{\mathbb{E}}$ (see [She00, Lem. 2.14] and [KST19, Lem. 2.39] respectively), we notice that there are two fundamental points: having available the Lebesgue measure and the Boolean structure, particularly the atoms. These allow us to find the bounds that we mentioned in the previous discussion to Crucial Lemma 4.1.10, which allows us to build a probability tree, to reason in a similar way to that of Lemma 4.2.16's proof. The problem is that, when considering more general forcing notions, we do not necessarily have a Boolean structure or some measure, so we need other alternatives. It is here where the condition on the intersection numbers (see Definition 4.2.8(3)) plays its stellar role: on the one hand, the measure that we obtain from Theorem 4.1.9 is the one that plays the role of the Lebesgue measure, and on the other hand, the set Σ from Crucial Lemma 4.1.10 is the one that plays the role of the set of atoms. In addition, Crucial Lemma 4.1.10 allows us to obtain the appropriate bound to be able to generalize the proof of the limit step. All this makes sense with the following lemma, one of the most important of this thesis:

Main Lemma 4.3.17. *Let $\mathbb{P}_\pi=\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha\colon \alpha\leq\pi\rangle$ be a finite support iteration with π limit. Assume that,*

1. *For any* $\alpha < \pi$, $\dot{\Xi}_\alpha$ *is a* $\mathbb{P}_\alpha$*-name of a finitely additive measure and* $\bar{I} \in \mathcal{I}_\infty$.
2. $\alpha < \beta \Rightarrow \Vdash_\beta$ *"*$\dot{\Xi}_\alpha \subseteq \dot{\Xi}_\beta$*".*
3. *For any* $\alpha < \pi$, $\mathbb{P}^-_\alpha \lessdot \mathbb{P}_\alpha$, $\dot{\mathbb{Q}}_\alpha$ *is a* $\mathbb{P}^-_\alpha$*-name and* $\dot{Q}'_\alpha$ *is a* $\mathbb{P}^-_\alpha$*-name of a subset of* $\dot{\mathbb{Q}}_\alpha$ *such that* $\operatorname{int}^{\dot{\mathbb{Q}}_\alpha}(\dot{Q}'_\alpha) \geq 1 - \varepsilon'_\alpha$, *with* $\varepsilon'_\alpha \in (0,1)_{\mathbb{Q}}$.
4. *If* $\langle \beta_\ell \colon \ell < \omega\rangle$ *is increasing with* $\sup_{\ell<\omega} \beta_\ell \coloneqq \beta < \pi$ *and, for any* $\ell < \omega$, $\dot{r}_\ell$ *is a* $\mathbb{P}^-_{\beta_\ell}$*-name such that* $\Vdash_{\beta_\ell}$ *"*$\dot{r}_\ell \in \dot{Q}'_{\beta_\ell}$*",* $\varepsilon > 0$ *and* $\langle \varepsilon'_{\beta\ell} \colon \ell < \omega\rangle$ *is constant with value* ε_0, *then:*

$$\Vdash_{\mathbb{P}_\beta} \text{``}\, \dot{\Xi}_\beta \left(\left\{ k < \omega \colon \frac{|\{\ell \in I_k \colon \dot{r}_\ell \in \dot{G}(\beta_\ell)\}|}{|I_k|} \geq (1-\varepsilon_0)(1-\varepsilon) \right\} \right) = 1\text{''}.$$

Then, there is a $\mathbb{P}_\pi$*-name* $\dot{\Xi}_\pi$ *of a finitely additive measure extending* $\bigcup_{\alpha<\pi} \dot{\Xi}_\alpha$ *such that, if* $\langle \alpha_\ell \colon \ell < \omega\rangle$ *is increasing and, for any* $\ell < \omega$, $\dot{q}_\ell$ *is a* $\mathbb{P}^-_{\alpha_\ell}$*-name such that* $\Vdash_{\alpha_\ell}$ *"*$\dot{q}_\ell \in \dot{Q}'_{\alpha_\ell}$*",* $\varepsilon > 0$ *and* $\langle \varepsilon'_{\alpha_\ell} \colon \ell < \omega\rangle$ *is constant with value* ε_0, *then:*

$$\Vdash_{\mathbb{P}_\pi} \text{``}\, \dot{\Xi}_\pi \left(\left\{ k < \omega \colon \frac{|\{\ell \in I_k \colon \dot{q}_\ell \in \dot{G}(\alpha_\ell)\}|}{|I_k|} \geq (1-\varepsilon_0)(1-\varepsilon) \right\} \right) = 1\text{''}.$$

Proof. With the intention of applying Theorem 3.4.4, we define $\mathscr{I}$ as the set of pairs $(\bar{c}, \varepsilon)$ such that:

- $\bar{c} = \langle (\beta_\ell, \dot{r}_\ell) \colon \ell < \omega\rangle$,
- $\langle \beta_\ell \colon \ell < \omega\rangle$ is increasing and for any $\ell < \omega$, $\beta_\ell < \pi$,
- for any $\ell < \omega$, $\dot{r}_\ell$ is a $\mathbb{P}^-_{\beta_\ell}$-name in $\dot{Q}'_{\beta_\ell}$,
- $\langle \varepsilon_{\beta_\ell} \colon \ell < \omega\rangle$ is constant with value some ε_0.
- $\varepsilon \in (0,1)_{\mathbb{Q}}$.

For each $(\bar{c}, \varepsilon) \in \mathscr{I}$, we define:

$$A_{\bar{c},\varepsilon} \coloneqq \left\{ k < \omega \colon \frac{|\{\ell < \omega \colon \dot{r}_\ell \in \dot{G}(\beta_\ell)\}|}{|I_k|} \geq (1-\varepsilon_0)(1-\varepsilon) \right\}.$$

Pick a $\mathbb{P}_\pi$-name $\dot{\Xi}$ for $\bigcup_{\alpha<\pi} \dot{\Xi}_\alpha$. Therefore, $\dot{\Xi}$ is a $\mathbb{P}_\pi$-name and $\Vdash_{\mathbb{P}_\pi}$ "$\operatorname{dom}(\dot{\Xi}) = \bigcup_{\alpha<\pi} \mathcal{P}(\omega) \cap \mathrm{M}_\alpha$"
Let $c \subseteq \mathscr{I}$ finite, $p \in \mathbb{P}$ and $\dot{a} \in \mathrm{M}^{\mathbb{P}_\pi}$ such that $p \Vdash_{\mathbb{P}_\pi}$ "$\dot{a} \in \operatorname{dom}(\dot{\Xi})$ and $\dot{\Xi}(\dot{a}) > 0$". Our aim will be to find some $q \in \mathbb{P}_\pi$ such that $q \leq p$ and $q \Vdash_{\mathbb{P}_\pi}$"$\dot{a} \cap \bigcap_{(\bar{c},\varepsilon)\in c} A_{\bar{c},\varepsilon} \neq \emptyset$", because this will allow us to apply Theorem 3.4.4 to extend $\dot{\Xi}$ as required.
Fix $i^* < \omega$ and let $(\bar{c}^i, \varepsilon^i) \in \mathscr{I}$ for each $i < i^*$ with $\bar{c}^i = \langle (\beta^i_\ell, \dot{r}^i_\ell) \colon \ell < \omega\rangle$. Since $\varepsilon' < \varepsilon$ implies $A_{\bar{c},\varepsilon'} \subseteq A_{\bar{c},\varepsilon}$, without loss of generality we can assume that, for any $i < i^*$, $\varepsilon^i = \varepsilon = \min\{\varepsilon^i \colon i < i^*\}$. For every $i < i^*$, define $A_i \coloneqq A_{\bar{c}^i, \varepsilon}$.
By strengthening p, we can find $\gamma < \pi$ such that:

- $\dot{a}$ is a $\mathbb{P}_\gamma$-name,
- Whenever $\langle \beta^i_\ell \colon \ell < \omega \rangle$ is bounded in π, γ is a upper bound of it,
- $p \in \mathbb{P}_\gamma$.

Define $j^* \coloneqq \{i < i^* \colon \langle \beta^i_\ell \colon \ell < \omega \rangle$ is bounded in $\pi\}$, so if $i \in j^*$, then γ is a upper bound of $\langle \beta^i_\ell \colon \ell < \omega \rangle$ and therefore, by (4), we have that $\Vdash_\gamma$"$\dot{\Xi}_\gamma(A_i) = 1$", so we just have to deal with the indices in $i^* \setminus j^*$. Define $\dot{a}' \coloneqq \dot{a} \cap \bigcap_{i \in j^*} A_i$, hence $\Vdash_\gamma \dot{\Xi}_\gamma(\dot{a}') > 0$".

Recall that $\langle \varepsilon_{\beta^i_\ell} \colon \ell < \omega \rangle$ is constant, say, with value ε^i_0. Since for any $i < i^*$, $\varepsilon(1 - \varepsilon^i_0) > 0$, there exists some $\varepsilon' > 0$ such that, for all $i \in i^* \setminus j^*$,

$$\varepsilon' < \varepsilon(1 - \varepsilon^i_0). \tag{4.3.2}$$

By the choice of ε', we have that $(1 - \varepsilon^i_0 - \varepsilon')(\varepsilon^i_0 + \varepsilon') > 0$ for any $i < i^*$. As a consequence, since $\lim_{k \to \infty} |I_k| = \infty$, there exists some $k_0 < \omega$ such that, for any $i \in i^* \setminus j^*$, $k \geq k_0$, $\ell \in I_k$, $\beta^i_\ell > \gamma$ and

$$\frac{1}{|I_k|} \frac{(1 - \varepsilon^i_0 - \varepsilon')\,(\varepsilon^i_0 + \varepsilon')}{[\varepsilon\,(1 - \varepsilon^i_0) - \varepsilon']^2} < \frac{1}{i^* + 1}. \tag{4.3.3}$$

Since $p \in \mathbb{P}_\gamma$, $p \Vdash_\gamma$"$\dot{\Xi}_\gamma(\dot{a}') > 0$" and $\dot{\Xi}_\gamma$ is $\mathbb{P}^-_\gamma$ a name of a free finitely additive measure, we have that $p \Vdash_\gamma$"$\dot{a}'$ is infinite", hence there exists some $q_0 \in \mathbb{P}_\gamma$ with $q_0 \leq p$ and some $k \geq k_0$ such that $q_0 \Vdash_\gamma$"$k \in \dot{a}'$".

It is clear that $\{\beta^i_\ell \colon i \in i^* \setminus j^*,\ \ell \in I_k\}$ is finite, so there exists some $m^* < \omega$ such that the sequence $\langle \beta_m \colon m < m^* \rangle$ is an increasing enumeration of it. Notice that $\varepsilon^i_0 = \varepsilon^{i'}_0$ whenever $\beta^i_\ell = \beta^{i'}_{\ell'}$.

Now, similar to the proof of Lemma 4.2.16, we split the rest of this proof in three parts. First, we build a suitable probability tree and then we find a suitable event with high probability.

Part 1: The tree construction.

We will build a tree $\mathcal{T}$ of height $m^* + 1$, a function $\mathbf{p} \colon \mathcal{T} \to \mathbb{P}_\pi$ such that $\mathbf{p}(\rho) \coloneqq p_\rho$ for each $\rho \in \mathcal{T}$, and a probability space in succ_ρ for each $\rho \in \mathcal{T}$, by induction on the level $m \leq m^*$ as follows.

In the base step, we define $\mathcal{L}_0(\mathcal{T}) \coloneqq \{\langle\,\rangle\}$ and $p_{\langle\,\rangle} \coloneqq q_0$.

For the successor step, suppose that we have built the first m levels of $\mathcal{T}$. In order to define $\mathcal{L}_{m+1}(\mathcal{T})$, let $J_m \coloneqq \{(i, \ell) \colon i \in i^* \setminus j^*,\ \ell \in I_k \wedge \beta^i_\ell = \beta_m\}$ and let $\rho \in \mathcal{L}_m(\mathcal{T})$. Working in $\mathrm{M}^{\mathbb{P}_{\beta_m}}$, we have that, by (2), $\mathrm{int}^{\mathbb{Q}_{\beta_m}}(Q'_{\beta_m}) \geq 1 - \varepsilon'_{\beta_m}$, and by virtue of Crucial Lemma 4.1.10, for

$$\Sigma_\rho \coloneqq \{\sigma \in {}^{J_m}2 \colon \exists q \in \mathbb{Q}_{\beta_m} \forall (i, \ell) \in J_m[(\sigma(i, \ell) = 0 \Rightarrow q \leq \dot{r}^i_\ell) \wedge (\sigma(i, \ell) = 1 \Rightarrow q \perp \dot{r}^i_\ell)]\},$$

there exists some function $f_\rho \colon \Sigma_\rho \to [0, 1]_{\mathbb{Q}}$ such that $\sum_{\sigma \in \Sigma_\rho} f(\sigma) = 1$ and, for any $(i, \ell) \in J_m$, we have that:

$$\sum \{f_\rho(\sigma) \colon \sigma \in \Sigma_\rho, \wedge\, \sigma(i, \ell) = 0\} \geq 1 - \varepsilon'_{\beta_m} - \varepsilon' \tag{4.3.4}$$

For each $\sigma \in \Sigma_\rho$, choose a witness $q_\sigma \in \mathbb{Q}_{\beta_m}$ for "$\sigma \in \Sigma_\rho$".

So working the ground model again, there is some $q'_\rho \leq p_\rho$ deciding $\Sigma_\rho = \dot{\Sigma}_\rho$ and $f_\rho = \dot{f}_\rho$. We then define $\mathrm{succ}_\rho \coloneqq \{\rho^\frown \langle \sigma \rangle \colon \sigma \in \Sigma_\rho\}$ and $p_{\rho^\frown \langle \sigma \rangle} \coloneqq q'_\rho \cup \{(\beta_m, \dot{q}_\sigma)\}$, where $\dot{q}_\sigma$ is a $\mathbb{P}^-_{\beta_m}$-name for q_σ (decided by q'_ρ).

Finally, to define the probability space, notice that $(\mathrm{succ}_\rho, \mathcal{P}(\mathrm{succ}_\rho), \mathrm{Pr}_\rho)$ is a probability space, where for $\eta = \rho^\frown\langle\sigma\rangle$, we define $\mathrm{Pr}_\rho(\eta) \coloneqq f_\rho(\sigma)$.

By the construction, it is clear that if $\eta = \rho^\frown\langle\sigma\rangle$, then $p_\eta \Vdash_{\beta_{m+1}}$ "$\dot{r}^i_\ell \in \dot{G}(\beta_m) \Leftrightarrow \sigma(i,\ell) = 0$" for any $(i,\ell) \in J_m$.

For notation, for any $(i,\ell) \in (i^* \setminus j^*) \times I_k$, define $m_{i,\ell}$ as the unique $m < m^*$ such that $\beta^i_\ell = \beta_m$.

Part 2: Comparing with a random variable with binomial distribution.

For any $i < i^*$ consider $\mathcal{T}^i \coloneqq {}^{I_k}2$, that is, the complete binary tree of height $|I_k| + 1$. Without loss of generality, for ease of notation, suppose that the levels of $\mathcal{T}^i$ are indexed by I_k, that is, the root is at $\min(I_k)$ and $\max(\mathcal{T}^i)$ is at $\max(I_k) + 1 = \min(I_{k+1})$. The key to be able to conclude the proof is to define a probability space structure on $\mathcal{T}^i$ and manage to properly transfer information from $\mathcal{T}$ into $\mathcal{T}^i$. For this, we are going to define a function $\Phi^i \colon \mathcal{T} \to \mathcal{T}^i$ such that, for any $\rho \in \mathcal{T}$,

$$\Phi^i(\rho) \coloneqq \langle \rho(m_{i,\ell})(i,\ell)) \colon \ell \in I_k \wedge m_{i,\ell} < \mathrm{ht}_{\mathcal{T}}(\rho)\rangle,$$

that is, $\Phi^i(\rho)(\ell) = \rho(m_{i,\ell})(i,\ell)$ whenever $\ell \in I_k$ and $m_{i,\ell} < \mathrm{ht}_{\mathcal{T}}(\rho)$. Notice that,

1. $\Phi^i(\rho)$ has domain $\{\ell \in I_k \colon \ell < \ell^i_\rho\}$, where $\ell^i_\rho \coloneqq \min(\{\ell \in I_k \colon m_{i,\ell} \geq \mathrm{ht}_{\mathcal{T}}(\rho)\} \cup \{n^*\}\}$ and $n^* \coloneqq \max(I_k) + 1$. As a consequence $\Phi^i[\mathcal{L}_{m_{i,\ell}(\mathcal{T})}] \subseteq \mathcal{L}_\ell(\mathcal{T}^i)$ for any $\ell \in I_k$ and also $\Phi^i[\mathcal{L}_{m^*(\mathcal{T})}] \subseteq \max(\mathcal{T}^i)$.
2. If $\rho \subseteq \eta$ in $\mathcal{T}$, then $\Phi^i(\rho) \subseteq \Phi^i(\eta)$.
3. If $\eta \in \mathcal{T}$ and $m_{i,\ell} \leq \mathrm{ht}_{\mathcal{T}}(\rho)$, then $\Phi^i(\rho \restriction m_{i,\ell}) = \Phi^i(\rho) \restriction \ell$.

For any $i \in i^* \setminus j^*$ and $\ell \in I_k$, define the random variable X^i_ℓ on $\mathcal{L}_{m_{i,\ell}+1}(\mathcal{T})$ such that, for any $\eta \in \mathcal{L}_{m_{i,\ell}+1}(\mathcal{T})$,

$$X^i_\ell(\eta) \coloneqq 1 - \eta(m_{i,\ell})(i,\ell) \in \{0,1\}.$$

Now, let us deal with the probability space structure on $\mathcal{T}^i$. First, we are going to define a probability space on its levels: for any $\ell \in I_k \cup \{n^*\}$ and $s \in \mathcal{L}_\ell(\mathcal{T}^i)$, define

$$\mathrm{Pr}^i_\ell(s) \coloneqq \mathrm{Pr}_{\mathcal{L}_{m^*(\mathcal{T})}}[\forall \ell' \in I_k \cap \ell\, (1 - X^i_{\ell'} = s(\ell'))].$$

Notice that,

$$\begin{aligned}
\sum_{s \in \mathcal{L}_\ell(\mathcal{T}^i)} \mathrm{Pr}^i_\ell(s) &= \sum_{s \in \mathcal{L}_\ell(\mathcal{T}^i)} \mathrm{Pr}_{\mathcal{L}_{m^*(\mathcal{T})}}[\forall \ell' \in I_k \cap \ell\, (1 - X^i_{\ell'} = s(\ell'))] \\
&= \sum_{s \in \mathcal{L}_\ell(\mathcal{T}^i)} \mathrm{Pr}_{\mathcal{L}_{m^*(\mathcal{T})}}[\Phi^i(\rho) \restriction \ell = s] \\
&= \mathrm{Pr}_{\mathcal{L}_{m^*(\mathcal{T})}}[\exists s \in \mathcal{L}_\ell(\mathcal{T}^i)(\Phi^i(\rho) \restriction \ell = s)] \\
&= 1,
\end{aligned}$$

where the last equality is given by virtue of (1). Thus, by Lemma 2.1.2, we have that $\mathcal{L}_\ell(\mathcal{T}^i)$ is a probability space with probability function Pr^i_ℓ.

Contrary to what we did in Theorem 2.3.2, in this case we have that the levels of $\mathcal{T}^i$ induces a probability space on succ_s: for any $\ell \in I_k$ and $s \in \mathcal{L}_\ell(\mathcal{T}^i)$, we set:

$$\mathrm{Pr}^i_s(s^\frown\langle d\rangle) := \begin{cases} \frac{\mathrm{Pr}_{\mathcal{L}_{\ell+1}(\mathcal{T}^i)}(s^\frown\langle d\rangle)}{\mathrm{Pr}_{\mathcal{L}_\ell(\mathcal{T}^i)}(s)}, & \text{if} \quad \mathrm{Pr}_{\mathcal{L}_\ell(\mathcal{T}^i)}(s) \neq 0, \\ p_i, & \text{if} \quad \mathrm{Pr}_{\mathcal{L}_\ell(\mathcal{T}^i)}(s) = 0 \wedge d = 0, \\ 1 - p_i, & \text{if} \quad \mathrm{Pr}_{\mathcal{L}_\ell(\mathcal{T}^i)}(s) = 0 \wedge d = 1. \end{cases}$$

It is not difficult to verify that, indeed, $(\mathrm{succ}_s(\mathcal{T}^i), \mathcal{P}(\mathrm{succ}_s(\mathcal{T}^i)), \mathrm{Pr}^i_s)$ is a probability space. In fact, the probability on the levels of $\mathcal{T}^i$ matches with the definition in terms of the probability of successors as in Equation 2.3.1: for $\ell \in I_k \cup \{n^*\}$

$$\mathrm{Pr}_{\mathcal{L}_\ell(\mathcal{T}^i)}(s) := \prod_{\ell' \in I_k \cap \ell} \mathrm{Pr}_{s \restriction \ell'}(s \restriction (\ell' + 1)).$$

Now we must verify the hypotheses of Theorem 2.3.10: let $\ell \in I_k$ and $s \in \mathcal{L}_\ell(\mathcal{T}^i)$. To make it easier to read, we are going to divide the equations into two parts:

$$\begin{aligned} \mathrm{Pr}^i_{\ell+1}(s^\frown\langle 0\rangle) &= \mathrm{Pr}_{\mathcal{L}_{m^*(\mathcal{T})}}[\Phi(\rho) \restriction (\ell+1) = s^\frown\langle 0\rangle] \\ &= \mathrm{Pr}_{\mathcal{L}_{m^*(\mathcal{T})}}[\Phi^i(\rho) \restriction \ell = s \wedge \Phi^i(\rho)(\ell) = 0] \\ &= \mathrm{Pr}_{\mathcal{L}_{m^*(\mathcal{T})}}[\Phi^i(\rho \restriction m_{i,\ell}) = s \wedge \Phi^i(\rho)(\ell) = 0] \\ &= \mathrm{Pr}_{\mathcal{L}_{m_{i,\ell}+1}(\mathcal{T})}[\Phi^i(\rho \restriction m_{i,\ell}) = s \wedge \Phi^i(\rho)(\ell) = 0] \\ &= \sum \left\{ \mathrm{Pr}_{\mathcal{L}_{m_{i,\ell}+1}(\mathcal{T})}(\rho) \colon \Phi^i(\rho \restriction m_{i,\ell}) = s \wedge \Phi^i(\rho)(\ell) = 0 \wedge \rho \in \mathcal{L}_{m_{i,\ell}+1}(\mathcal{T}) \right\}, \end{aligned}$$

where the third equality is given by virtue of (3), and the equalities prior to the sum are given by the definition of Φ^i and because the probability only depends on $\rho \restriction m_{i,\ell} + 1$. Now, using that $\mathrm{Pr}_{\mathcal{L}_{k+1}(\mathcal{S})}(\rho^\frown\langle x\rangle) = \mathrm{Pr}_{\mathcal{L}_k(\mathcal{S})}(\rho) \cdot \mathrm{Pr}_\rho(\rho^\frown\langle x\rangle)$ holds in any probability tree $\mathcal{S}$, and making a change of $\rho = \eta^\frown\sigma$ and denoting $t := s^\frown\langle 0\rangle$, we get:

$$\begin{aligned} \mathrm{Pr}^i_{\ell+1}(t) &= \sum \left\{ \mathrm{Pr}_{\mathcal{L}_{m_{i,\ell}+1}(\mathcal{T})}(\rho) \colon \Phi^i(\rho \restriction m_{i,\ell}) = s \wedge \Phi^i(\rho)(\ell) = 0 \wedge \rho \in \mathcal{L}_{m_{i,\ell}+1}(\mathcal{T}) \right\} \\ &= \sum \left\{ \mathrm{Pr}_{\mathcal{L}_{m_{i,\ell}}(\mathcal{T})}(\eta)\, \mathrm{Pr}_\eta(\eta^\frown\langle\sigma\rangle) \colon \Phi^i(\eta) = s,\, \sigma(i,\ell) = 0,\, \eta \in \mathcal{L}_{m_{i,\ell}}(\mathcal{T}) \wedge \sigma \in \Sigma_\eta \right\} \\ &= \sum_{\substack{\eta \in \mathcal{L}_{m_{i,\ell}}(\mathcal{T}) \\ \Phi^i(\eta) = s}} \left(\sum_{\substack{\sigma \in \Sigma_\eta \\ \sigma(i,\ell) = 0}} \mathrm{Pr}_{\mathcal{L}_{m_{i,\ell}}(\mathcal{T})}(\eta) \cdot \mathrm{Pr}_\eta(\eta^\frown\langle\sigma\rangle) \right) \\ &= \sum_{\substack{\eta \in \mathcal{L}_{m_{i,\ell}}(\mathcal{T}) \\ \Phi^i(\eta) = s}} \left(\sum_{\substack{\sigma \in \Sigma_\eta \\ \sigma(i,\ell) = 0}} \mathrm{Pr}_{\mathcal{L}_{m_{i,\ell}}(\mathcal{T})}(\eta) \cdot f_\eta(\sigma) \right) \end{aligned} .$$

Now, by Equation 4.3.4, defining $p_i := 1 - \varepsilon^i_0 - \varepsilon'$, we have that:

$$\sum_{\substack{\eta\in\mathcal{L}_{m_{i,\ell}}(\mathcal{T})\\ \Phi^i(\eta)=s}}\left(\sum_{\substack{\sigma\in\Sigma_\eta\\ \sigma(i,\ell)=0}}\Pr_{\mathcal{L}_{m_{i,\ell}}(\mathcal{T})}(\eta)\cdot f_\eta(\sigma)\right)=\sum_{\substack{\eta\in\mathcal{L}_{m_{i,\ell}}(\mathcal{T})\\ \Phi^i(\eta)=s}}\left(\Pr_{\mathcal{L}_{m_{i,\ell}}(\mathcal{T})}(\eta)\cdot\sum_{\substack{\sigma\in\Sigma_\eta\\ \sigma(i,\ell)=0)}}f_\eta(\sigma)\right)$$
$$\geq\left(\sum_{\substack{\eta\in\mathcal{L}_{m_{i,\ell}}(\mathcal{T})\\ \Phi^i(\eta)=s}}\Pr_{\mathcal{L}_{m_{i,\ell}}(\mathcal{T})}(\eta)\right)\cdot p_i$$
$$=\Pr_{\mathcal{L}_{m_{i,\ell}(\mathcal{T})}}[\Phi^i(\eta)=s]\cdot p_i$$
$$=\Pr^i_\ell(s)\cdot p_i,$$

As a consequence, for any $\ell\in I_k$ and $s\in\mathcal{L}_\ell(\mathcal{T}^i)$, we have that

$$\Pr^i_{\ell+1}(s^\frown\langle 0\rangle)\geq p_i\cdot\Pr^i_\ell(s). \tag{4.3.5}$$

If $\Pr^i_\ell(s)=0$, then by definition, $p_i=\Pr^i_s(s^\frown\langle 0\rangle)$, and, on the other hand, if $\Pr^i_s(s)\neq 0$, we have that, by Equation 4.3.5, $p_i\leq\Pr^i_s(s^\frown\langle 0\rangle)$.
For each $i\in i^*\setminus j^*$ define Y_i on $\mathcal{L}_{n^*}(\mathcal{T}^i)$ such that, for any $s\in\mathcal{L}_{n^*}(\mathcal{T}^i)$,

$$Y_i(s)\coloneqq|\{\ell\in I_k\colon s(\ell)=0\}|.$$

Notice that since by Equation 4.3.5 we are under the hypothesis of Theorem 2.3.10, we have that:

$$\forall z\in\mathbb{R}\,(\Pr_{\mathcal{L}_{n^*}(\mathcal{T}^i)}[Y_i\leq z]\leq\Pr_{\Omega_{n^*}}[\mathrm{B}_{n^*,p_i}\leq z]). \tag{4.3.6}$$

Finally, using Φ^i again, since for $\rho\in\mathcal{L}_{m^*}(\mathcal{T})$, $Y_i(\Phi^i(\rho))=\sum_{\ell\in I_k}X^i_\ell(\rho)$ we can conclude that, for any $i\in i^*\setminus j^*$ and any $z\in\mathbb{R}$,

$$\Pr_{\mathcal{L}_{m^*}(\mathcal{T})}\left[\sum_{\ell\in I_k}X^i_\ell\leq z\right]=\Pr_{\mathcal{L}_{n^*}(\mathcal{T}^i)}[Y_i\leq z]. \tag{4.3.7}$$

Part 3: Find a suitable $\rho\in\mathcal{L}_{m^*}(\mathcal{T})$ with high probability.
For any $i\in i^*\setminus j^*$, $z_i\coloneqq|I_k|(1-\varepsilon^i_0)(1-\varepsilon)$ consider the following event in $\mathcal{L}_{m^*}(\mathcal{T})$:

$$E_i\coloneqq\{\rho\in\mathcal{L}_{m^*}(\mathcal{T})\colon|\{\ell\in I_k\colon\rho(m_{i,\ell})(i,\ell)=0\}|\geq z_i\}$$

and define $F_i\coloneqq E^{\mathrm{c}}_i$, hence it is clear that, for any $i\in i^*\setminus j^*$,

$$\Pr_{\mathcal{L}_{m^*}(\mathcal{T})}(F_i)=\Pr_{\mathcal{L}_{m^*}(\mathcal{T})}\left[\sum_{\ell\in I_k}X^j_\ell<z_i\right]. \tag{4.3.8}$$

Therefore, using Equation 4.3.8, Equation 4.3.7, Equation 4.3.6, Equation 4.3.3 and *Chebyshev's inequality* for $k^* := |I_k|$, we get:

$$\begin{aligned}
\Pr_{\mathcal{L}_{m^*}(\mathcal{T})}(F_i) &= \Pr_{\mathcal{L}_{m^*}(\mathcal{T})}\left[\sum_{\ell\in I_k} X^i_\ell < z_i\right] \\
&\leq \Pr_{\mathcal{L}_{n^*}(\mathcal{T}^i)}[Y_i \leq z_i] \\
&\leq \Pr_{\Omega_{k^*}}[\mathrm{B}_{k^*,p_i} \leq z_i] \\
&= \Pr_{\Omega_{k^*}}\left[\mathrm{E}[\mathrm{B}_{k^*,p_i}] - z_i \leq \mathrm{E}[\mathrm{B}_{k^*,p_i}] - \mathrm{B}_{k^*,p_i}\right] \\
&\leq \Pr_{\Omega_{k^*}}\left[|\mathrm{B}_{k^*,p_i} - \mathrm{E}[\mathrm{B}_{k^*,p_i}]| \geq \mathrm{E}[\mathrm{B}_{k^*,p_i}] - z_i\right] \\
&\leq \frac{\mathrm{Var}[\mathrm{B}_{k^*,p_i}]}{(\mathrm{E}[\mathrm{B}_{k^*,p_i}] - z_i)^2} \\
&= \frac{|I_k|\,(1-\varepsilon^i_0-\varepsilon')\,(\varepsilon^i_0+\varepsilon')}{|I_k|^2\,[\varepsilon\,(1-\varepsilon^i_0)-\varepsilon']^2} \\
&= \frac{1}{|I_k|}\frac{(1-\varepsilon^i_0-\varepsilon')\,(\varepsilon^i_0+\varepsilon')}{[\varepsilon\,(1-\varepsilon^i_0)-\varepsilon']^2} \\
&< \frac{1}{i^*+1}.
\end{aligned}$$

As a consequence, in $\mathcal{L}_{m^*}(\mathcal{T})$ we have

$$\Pr\left(\bigcup_{i\in i^*\setminus j^*} F_i\right) \leq \sum_{i\in i^*\setminus j^*} \Pr(F_i) < \sum_{i\in i^*\setminus j^*} \frac{1}{i^*+1} \leq \sum_{i<i^*} \frac{1}{i^*+1} = \frac{i^*}{i^*+1} < 1,$$

hence,

$$\Pr\left(\bigcap_{i\in i^*\setminus j^*} E_i\right) > 0,$$

and, by Lemma 2.1.1(1) $\bigcap_{i\in i^*\setminus j^*} E_i \neq \emptyset$, hence there exists some $\eta \in \bigcap_{i\in i^*\setminus j^*} E_i$. Since, for any $(i,\ell) \in (i^*\setminus j^*)\times I_k$,

$$r_\eta \Vdash_{\beta_{m^*}} \text{“}\dot{r}^i_\ell \in \dot{G}(\beta_{\beta_{m,\ell}}) \Leftrightarrow \eta(m_{i,\ell})(i,\ell) = 0\text{”},$$

it is clear that $r_\eta \Vdash_{\beta_{m^*}}$“$k \in \bigcap_{i\in i^*\setminus j^*} \dot{A}_i$”, hence , $r_\eta \Vdash_{\beta_{m^*}}$“ $k \in \bigcap_{i<i^*} \dot{A}_i$. Finally, as we mentioned at the beginning of this proof, the result follows by Theorem 3.4.4. $\square_{\text{MainLemma 4.3.17}}$

As a consequence, we can extend iterations in $\mathcal{K}_1(\kappa,\mathcal{G})$ at limit steps, that is, we can generalize the *extension theorem* at limit steps:

Theorem 4.3.18. *Let $\mathbb{K} \in \mathcal{K}_0(\kappa)$ be of length π limit and $\mathcal{G}$ a set of guardrails for (π,κ). Given a sequence $\langle \dot{\Xi}^g_\alpha \colon \alpha<\pi,\ g\in\mathcal{G}\rangle$ such that, for any $\alpha<\pi$, $\mathbb{K}\restriction\alpha \sqcup \langle\dot{\Xi}^g_\beta\colon \beta\leq\alpha,\ g\in\mathcal{G}\restriction\alpha\rangle$ is in $\mathcal{K}_1(\kappa,\mathcal{G}\restriction\alpha)$, then there exists a sequence $\langle\dot{\Xi}^g_\pi\colon g\in\mathcal{G}\rangle$ such that*

$$\mathbb{K} \sqcup \langle\dot{\Xi}^g_\alpha\colon \alpha\leq\pi,\ g\in\mathcal{G}\rangle \in \mathcal{K}_1(\kappa,\mathcal{G}).$$

***Proof*.** Fix $(g,\bar{I})\in\mathcal{G}$ and, for any $\alpha<\pi$, define $\zeta_\alpha\coloneqq g_0(\alpha)$ and $g_1(\alpha)\coloneqq\varepsilon_\alpha$. We start verifying the conditions of Main Lemma 4.3.17:

1. Let $\alpha<\pi$. Since $\mathbb{K}\restriction\alpha\sqcup\langle\dot{\Xi}^g_\beta\colon\beta\leq\alpha,\ g\in G\restriction\alpha\rangle\in\mathcal{K}_1(\kappa,\mathcal{G}\restriction\alpha)$ it follows that $\dot{\Xi}_\alpha$ is a $\mathbb{P}_\alpha$-name of a free finitely additive measure by Definition 4.3.13(3).

2. If $\alpha<\beta<\pi$, then, by Definition 4.3.13(4), we have that $\Vdash_\beta$"$\dot{\Xi}^g_\alpha\subseteq\dot{\Xi}^g_\beta$", considering that $\mathbb{K}\restriction\beta\sqcup\langle\dot{\Xi}^g_\gamma\colon\gamma\leq\beta,\ g\in G\restriction\beta\rangle\in\mathcal{K}_1(\kappa,\mathcal{G}\restriction\beta)$.

3. For any $\alpha<\pi$, define $\dot{Q}'_\alpha\coloneqq\dot{Q}^\alpha_{g(\alpha)}=\dot{Q}^\alpha_{\zeta_\alpha,\varepsilon_\alpha}$. By Definition 4.3.1(2), we have that $\dot{Q}'_\alpha$ is a $\mathbb{P}^-_\alpha$-name of a $(\Xi^g_\alpha,\bar{I},\varepsilon_\alpha)$-linked subset of $\dot{\mathbb{Q}}_\alpha$ such that $\operatorname{int}^{\dot{\mathbb{Q}}_\alpha}(\dot{Q}'_\alpha)=\operatorname{int}^{\dot{\mathbb{Q}}_\alpha}(\dot{Q}_{\zeta_\alpha,\varepsilon_\alpha})\geq1-\varepsilon_\alpha$.

4. Assume that $\langle\beta_\ell\colon\ell<\omega\rangle$ is increasing with $\sup_{\ell<\omega}\beta_\ell\coloneqq\beta<\pi$, for any $\ell<\omega$, $\dot{r}_\ell$ is a $\mathbb{P}^-_{\beta_\ell}$-name such that $\Vdash_{\beta_\ell}$"$\dot{r}_\ell\in\dot{Q}'_{\beta_\ell}$" and $\langle\varepsilon_{\beta\ell}\colon\ell<\omega\rangle$ is constant with value ε_0. Define $\tau_0\coloneqq\langle(\dot{r}_\ell,\,\beta_\ell,\,\zeta_{\beta_\ell},\,\varepsilon_{\beta_\ell})\colon\ell<\omega\rangle$. Notice that:

    - Given $\ell<\omega$, on the one hand, $\Vdash_{\beta_\ell}$"$\dot{r}_\ell\in\dot{Q}'_{\beta_\ell}$", and on the other hand, $\Vdash_{\beta_\ell}$"$\dot{Q}'_{\beta_\ell}=\dot{Q}^{\beta_\ell}_{\zeta_{\beta\ell},\varepsilon_0}$", therefore $\Vdash_{\beta_\ell}$"$\dot{r}_\ell\in\dot{Q}^{\beta_\ell}_{\zeta_{\beta_\ell},\varepsilon_0}$".
    - If $\ell<\omega$, then $g(\beta_\ell)=(g_0(\beta_\ell),g_1(\beta_\ell))=(\zeta_{\beta_\ell},\varepsilon_0)$.

    That is, τ_0 follows g and therefore, since $\beta<\pi$, $\mathbb{K}\restriction\beta\sqcup\langle\dot{\Xi}^g_\gamma\colon\gamma\leq\beta,\ g\in G\restriction\beta\rangle$ belongs to $\mathcal{K}_1(\kappa,\mathcal{G}\restriction\beta)$, hence we have that, for any $\varepsilon>0$,

    $$\Vdash_{\mathbb{P}_\beta}\text{“}\,\dot{\Xi}^g_\beta\left(\left\{k<\omega\colon\frac{|\{\ell\in I_k\colon\dot{r}_\ell\in\dot{G}(\beta_\ell)\}|}{|I_k|}\geq(1-\varepsilon_0)(1-\varepsilon)\right\}\right)=1\text{”}.$$

So indeed, we are under the hypothesis of Main Lemma 4.3.17, by virtue of which, there exists some $\mathbb{P}_\pi$-name $\dot{\Xi}^g_\pi$ of a finitely additive measure extending $\bigcup_{\alpha<\pi}\dot{\Xi}^g_\alpha$ such that, if $\langle\alpha_\ell\colon\ell<\omega\rangle$ is increasing, for any $\ell<\omega$, $\Vdash_{\alpha_\ell}$"$\dot{q}_\ell\in\dot{Q}'_{\alpha_\ell}$" (where $\dot{q}_\ell$ is a $\mathbb{P}^-_{\alpha_\ell}$-name) and $\varepsilon>0$, then:

$$\Vdash_{\mathbb{P}_\pi}\text{“}\,\dot{\Xi}^g_\pi\left(\left\{k<\omega\colon\frac{|\{\ell\in I_k\colon\dot{q}_\ell\in\dot{G}(\alpha_\ell)\}|}{|I_k|}\geq(1-\varepsilon_0)(1-\varepsilon)\right\}\right)=1\text{”}.\tag{4.3.9}$$

Thereby, we get a collection of $\mathbb{P}_\pi$-names of free finitely additive measures, $\langle\dot{\Xi}^g_\pi\colon g\in\mathcal{G}\rangle$ satisfying the condition from Equation 4.3.9.

Now, we must show that $\mathbb{K}\sqcup\langle\dot{\Xi}^g_\alpha\colon\alpha\leq\pi,\ g\in\mathcal{G}\rangle\in\mathcal{K}_1(\kappa,\mathcal{G})$. It is clear that the first five conditions from Definition 4.3.13 are immediate, so we deal with condition (6):

6. Assume that $\tau=\{(\dot{q}_\ell,\,\alpha_\ell,\,\zeta_\ell,\,\varepsilon_\ell)\colon\ell<\omega\}$ follows g for some guardrail $(g,\bar{I})\in\mathcal{G}$. Then,

    (a) If the sequence $\langle\alpha_\ell\colon\ell<\omega\rangle$ is constant with value α, then its value must be less than π, and therefore, the result is trivial, because $\mathbb{K}\restriction\alpha\sqcup\langle\dot{\Xi}^g_\beta\colon\beta\leq\alpha,\ g\in G\restriction\alpha\rangle\in\mathcal{K}_1(\kappa,\mathcal{G}\restriction\alpha)$.

(b) If the sequence $\langle \alpha_\ell \colon \ell < \omega \rangle$ is increasing, since the sequence $\langle \varepsilon_{\alpha_\ell} \colon \ell < \omega \rangle$ is also constant with value say ε_0, we have that, by Equation 4.3.9, for all $\varepsilon > 0$,

$$\Vdash_{\mathbb{P}_\pi} \text{“}\, \dot{\Xi}^g_\pi \left(\left\{ k < \omega \colon \frac{|\{\ell \in I_k \colon \dot{q}_\ell \in \dot{G}(\alpha_\ell)\}|}{|I_k|} \geq (1-\varepsilon_0)(1-\varepsilon) \right\} \right) = 1\text{”}.$$

Finally, $\mathbb{K} \sqcup \langle \dot{\Xi}^g_\alpha \colon \alpha \leq \pi,\, g \in \mathcal{G} \rangle \in \mathcal{K}_1(\kappa, \mathcal{G})$. $\square_{\text{Theorem 4.3.18}}$

4.3.4 Uniform Δ-systems

Suppose that $\mathbb{K} \in \mathcal{K}_0(\kappa)$ and $\bar{p} = \langle p_i \colon i \in I \rangle \subseteq \mathbb{P}^\bullet_\pi$, where I is an index set. Let $i \in I$ and $\xi \in \operatorname{dom}(p_i)$. So since $p_i \in \mathbb{P}^\bullet_\pi$, there are $\zeta < \theta_\xi$ and $\varepsilon \in (0,1)_\mathbb{Q}$ such that $\Vdash_\xi$“$p_i(\xi) \in \dot{Q}^\xi_{\zeta,\varepsilon}$”. If for ξ we could define a countable *suitable* $I_\xi \subseteq I$ such that for all $j \in I_\xi$, $\xi \in \operatorname{dom}(p_j)$ and $\Vdash_\xi$ “$p_j(\xi) \in \dot{Q}^\xi_{\zeta,\varepsilon}$”, where ζ and ε does no depend on $j \in I_\xi$, then we could use the limit that, by Definition 4.2.2, we have defined in $\dot{Q}^\xi_{\zeta,\varepsilon}$ to try to define a limit $\lim_\mathbb{K}(\bar{p})$ on $\mathbb{K}$. Notice that, to control the coordinates ζ, ε we can use a half guardrail g for (π, κ). By doing a complete analysis of the conditions that are required to formalize this idea, we get the notion of *g-uniform Δ-system*:

Definition 4.3.19. Let $\mathbb{K} \in \mathcal{K}_0(\kappa)$, g be a half guardrail for (π, κ) and $(L, <_L)$ a well-ordered set. We say that $\bar{p} = \langle p_l \colon l \in L \rangle \subseteq \mathbb{P}_\pi$ is a *g-uniform-Δ-system with parameters* $(\Delta, \vec{\alpha}, n^*, r^*, \varepsilon^*)$, when:

1. For any $l \in L$, $p_l \in \mathbb{P}^\bullet_\pi$.
2. $\{\operatorname{dom}(p_l) \colon l \in L\}$ forms a Δ-system with root Δ.
3. $n^* < \omega$, for any $l \in L$, $\operatorname{dom}(p_l) = \{\alpha_{n,l} \colon n < n^*\}$ and the order is increasing, that is, $n < m < n^* \Rightarrow \alpha_{n,l} < \alpha_{m,l}$.
4. $r^* \subseteq n^*$ and for any $l \in L$ and $n < n^*$, $n \in r^* \Leftrightarrow \alpha_{n,l} \in \Delta$. So whenever $n \in r^*$, the sequence $\langle \alpha_{n,l} \colon l < \omega \rangle$ is a constant with value, say, α^*_n.
5. For any $n \in n^* \setminus r^*$, the sequence $\langle \alpha_{n,l} \colon l \in L \rangle$ is increasing, that is, if $l, j \in L$ and $l <_L j$, then $\alpha_{n,l} < \alpha_{n,j}$.
6. For any $l \in L$ and any $n < n^*$, $\Vdash_{\alpha_{n,l}}$ “ $p_l(\alpha_{n,l}) \in \dot{Q}^{\alpha_{n,l}}_{g(\alpha_{n,l})}$”.
7. $\varepsilon^* \colon n^* \to (0,1)_\mathbb{Q}$ is a function such that, for all $n < n^*$, the sequence $\langle g_1(\alpha_{n,l}) \colon l \in L \rangle$ is constant with value $\varepsilon^*(n)$.

Intuitively, we can think of g-uniform Δ-systems as matrix. Indeed, without loss of generality, suppose that $\omega \subseteq L$ and consider the matrix M of dimension $n^* \times |L|$, where the l-th row of M consists of the domain elements of p_l, arranged as follows:

$$\begin{bmatrix} \alpha_{0,0} & \alpha_{1,0} & \alpha_{2,0} & \cdots & a_{n^*-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \alpha_{2,1} & \cdots & a_{n^*-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \alpha_{2,2} & \cdots & a_{n^*-1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{0,l} & \alpha_{1,l} & \alpha_{2,l} & \cdots & a_{n^*-1,l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

Condition (3) implies that each row is increasing to the right and condition (7) entails that, for all the elements of the same column, there is the same value of g_1, which can also be characterized with ε^*.

Now, by the condition (4), there must exist some columns associated with the root of the Δ-system, which are constant, for example:

$$\begin{bmatrix} \color{red}{\alpha_0^*} & \alpha_{1,0} & \color{red}{\alpha_2^*} & \cdots & a_{n^*-1,0} \\ \color{red}{\alpha_0^*} & \alpha_{1,1} & \color{red}{\alpha_2^*} & \cdots & a_{n^*-1,1} \\ \color{red}{\alpha_0^*} & \alpha_{1,2} & \color{red}{\alpha_2^*} & \cdots & a_{n^*-1,2} \\ \color{red}{\vdots} & \vdots & \color{red}{\vdots} & \ddots & \vdots \\ \color{red}{\alpha_0^*} & \alpha_{1,l} & \color{red}{\alpha_2^*} & \cdots & a_{n^*-1,l} \\ \color{red}{\vdots} & \vdots & \color{red}{\vdots} & \ddots & \vdots \end{bmatrix}$$

In this case, the red columns[9] are those that correspond to values in r^*, which in this example is $r^* = \{0, 2\}$. There the columns are constants. The black columns correspond to values in $n^* \setminus r^*$. There, the columns are increasing, by virtue of condition (5).

In general, from a sequence of conditions of regular size θ, we will be able to obtain g-uniform Δ-systems. In this case, regularity, as in the proof of the Δ-system (see Theorem 1.5.32), plays a fundamental role in order to make the necessary reductions to the conditions:

Theorem 4.3.20. *Let $\theta \geq \kappa$ be a regular cardinal and $\mathbb{K} \in \mathcal{K}_0(\kappa)$. If $\{p_\xi \colon \xi < \theta\} \subseteq \mathbb{P}^\bullet_\pi$, then there are some $E \in [\theta]^\theta$ and a half guardrail g for (π, κ), such that $\{p_\xi \colon \xi \in E\}$ forms a g-uniform Δ-system.*

Proof. Since θ is regular, $\{\mathrm{dom}(p_\xi) \colon \xi < \theta\}$ is a family of finite sets and $\bigcup_{\xi<\theta} \mathrm{dom}(p_\xi) \subseteq \mathrm{Ord}$, we can apply Theorem 1.5.32 to get $G \in [\theta]^\theta$, Δ, $n^* < \omega$ and $r^* \subseteq n^*$, such that:

(a) $\{\mathrm{dom}(p_\xi) \colon \xi \in G\}$ forms a Δ-system with root Δ.

(b) For any $\xi \in G$, $\mathrm{dom}(p_\xi) = \{\alpha_{n,\xi} \colon n < n^*\}$ is arranged in increasing order.

(c) $\alpha_{n,\xi} \in \Delta$ if, and only if, $n \in r^*$ for any $\xi \in G$.

(d) For any $n \in n^* \setminus r^*$ and $\xi, \zeta \in G$, if $\xi < \zeta$ then $\alpha_{n,\xi} < \alpha_{n\,\zeta}$.

[9] That is, the first and third columns.

By (c), we have that, for any $n \in r^*$, the sequence $\langle \alpha_{n,\xi} \colon \xi \in G \rangle$ is constant with value, say, α^*_n. Since for every $\xi \in G$, $p_\xi \in \mathbb{P}^\bullet_\xi$, for any $n < n^*$ we can find $\varepsilon_{n,\xi} \in (0,1)_\mathbb{Q}$ and $\zeta_{n,\xi} < \theta_{\alpha_{n,\xi}}$ such that

$$\Vdash_{\alpha_{n,\xi}} \text{“} p_\xi(\alpha_{n,\xi}) \in \dot{Q}^{\alpha_{n,\xi}}_{\zeta_{n,\xi},\varepsilon_{n,\xi}} \text{”} \tag{4.3.10}$$

For any $\bar{\varepsilon} = \langle \varepsilon_n \colon n < n^* \rangle \in (0,1)^{n^*}_\mathbb{Q}$, define $G_{\bar{\varepsilon}} := \{\xi \in G \colon \forall n < n^* (\varepsilon_{n,\xi} = \varepsilon_n)\}$. It is clear that $\{G_{\bar{\varepsilon}} \colon \bar{\varepsilon} \in (0,1)^{n^*}_\mathbb{Q}\}$ is a countable family of pairwise disjoint sets, whose union is G. So, since $|G| = \theta$ is regular, there exists $\bar{\varepsilon}^* \in (0,1)^{n^*}_\mathbb{Q}$ such that $|G_{\bar{\varepsilon}^*}| = \theta$. Define $F := G_{\bar{\varepsilon}^*}$ and $\varepsilon^* \colon n^* \to (0,1)_\mathbb{Q}$ such that $\varepsilon^*(n) := \varepsilon^*_n$ for each $n < n^*$. As a consequence, we have,

$$\forall n < n^* \, \forall \xi \in F (\varepsilon_{n,\xi} = \varepsilon^*(n)) \tag{4.3.11}$$

For any $\bar{\zeta} = \langle \zeta_n \colon n \in r^* \rangle \in \prod_{n \in r^*} \theta_{\alpha^*_n}$ define $F_{\bar{\zeta}} := \{\xi \in F \colon \forall n \in r^* (\zeta_{n,\xi} = \zeta_n)\}$, hence we have that $\{F_{\bar{\zeta}} \colon \bar{\zeta} \in \prod_{n \in r^*} \theta_{\alpha^*_n}\}$ is a family of pairwise disjoint sets whose union is F and $|F_{\bar{\zeta}^*}| = \theta$. Since $|F| = \theta$ is regular, $|\prod_{n \in r^*} \theta_{\alpha^*_n}| < \kappa$ and $\theta \geq \kappa$, there is some $\bar{\zeta}^*$ such that, whenever $n \in r^*$, $\zeta_{n,\xi} = \zeta^*_n$. Define $E := F_{\bar{\zeta}^*}$, hence

$$\forall n \in r^* \, \forall \xi \in E (\zeta_{n,\xi} = \zeta^*_n) \tag{4.3.12}$$

To finish the construction, we define the half guardrail for (π, κ), $g \colon \pi \to \bigcup_{\alpha<\pi} [\theta_\alpha \times (0,1)_\mathbb{Q}]$ such that, for any $\alpha < \pi$,

$$g(\alpha) := \begin{cases} (\zeta_{n,\xi}, \varepsilon_{n,\xi}) & \text{if} \quad \exists n < n^* \exists \xi \in E (\alpha = \alpha_{n,\xi}), \\ \left(2022, \frac{1}{6}\right) & \text{if} \qquad \alpha \notin \bigcup_{\xi \in E} \operatorname{dom}(p_\xi). \end{cases}$$

Let $m, n < n^*$ and $\xi, \zeta \in E$ be such that $\alpha_{n,\xi} = \alpha_{m,\zeta}$. On the one hand, if $n \notin r^*$ and $m \notin r^*$, then $n = m$ and $\xi = \zeta$, hence $\zeta_{n,\xi} = \zeta_{m,\zeta}$ and $\varepsilon_{n,\xi} = \varepsilon_{m,\zeta}$, that is, $g(\alpha_{n,\xi}) = g(\alpha_{m,\zeta})$. On the other hand, if either $n \in r^*$ or $m \in r^*$, then $\alpha_{n,\xi}, \alpha_{m,\zeta} \in \Delta$. Since $\xi, \zeta \in E$, by Equation 4.3.11 and Equation 4.3.12, we have that $\zeta_{n,\xi} = \zeta^*_n = \zeta_{m,\zeta}$ and $\varepsilon_{n,\xi} = \varepsilon^*_n = \varepsilon_{m,\zeta}$, that is, $g(\alpha_{n,\xi}) = g(\alpha_{n,\zeta})$. Notice that the others cases are immediate, so we can conclude that g is well-defined.

Finally, we show that $\langle p_\xi \colon \xi \in E \rangle$ is a g-uniform Δ-system with parameters $(\Delta, n^*, \vec{\alpha}, r^*, \varepsilon^*)$ verifying the conditions from Definition 4.3.19:

1. Since $E \subseteq \theta$, we have that $\{p_\xi \colon \xi \in E\} \subseteq \mathbb{P}^\bullet_\pi$.
2. $\{\operatorname{dom}(p_\xi) \colon \xi \in E\}$ is a Δ-system with root Δ because $\{\operatorname{dom}(p_\xi) \colon \xi \in G\}$ is and $E \subseteq G$.
3. It is clear because $E \subseteq G$.
4. Direct consequence of (c), above.
5. Direct consequence of (d), above.
6. Let $\xi \in E$ and $n < n^*$, hence $g(\alpha_{n,\xi}) = (\zeta_{n,\xi}, \varepsilon_{n,\xi})$ and by Equation 4.3.10, we have that $\Vdash_{\alpha_{n,\xi}} \text{“} p_\xi(\alpha_{n,\xi}) \in \dot{Q}^\xi_{g(\alpha_{n,\xi})} \text{”}$.

7. Let $n < n^*$. By definition of g and Equation 4.3.11, we have that

$$\langle g_1(\alpha_{n,\xi})\colon \xi \in E\rangle = \langle \varepsilon_{n,\xi}\colon \xi \in E\rangle = \langle \varepsilon^*_n\colon \xi \in E\rangle,$$

that is, $\langle g_1(\alpha_{n,\xi})\colon \xi \in E\rangle$ is constant with value $\varepsilon^*(n)$.

Finally, $\{p_\xi\colon \xi \in E\}$ is a g-uniform Δ-system. $\square_{\text{Theorem 4.3.20}}$

As far as half guardrails are concerned, uniformity *only* depends on the restriction of the half guardrail to the parameter $\vec{\alpha}$, which is countable. So, it is always possible to reduce uniform Δ-systems in $\mathbb{K}(\kappa, \mathcal{G})$, where $\mathcal{G}$ is a complete set of guardrails, to countable uniform Δ-systems. Formally,

Theorem 4.3.21. *Assume that* $\operatorname{otp}((L, \leq_L)) = \omega$, *let* $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$ *where* $\mathcal{G}$ *is a complete set of guardrails and* $\langle p_l\colon l \in L\rangle$ *a* f*-uniform* Δ*-system with parameters* $(\Delta, \vec{\alpha}, n^*, r^*, \varepsilon^*)$, *for some half guardrail* f *for* (π, κ). *Then there exists a half guardrail* $g \in \mathcal{G}_0$ *such that* $\bar{p} = \langle p_l\colon l \in L\rangle$ *is a* g*-uniform* Δ*-system with the same parameters.*

Proof. Let $X := \{\alpha_{n,l}\colon l \in L\}$. Since $\operatorname{otp}(L) = \omega$, in particular we have that X is countable. Define $\sigma := f \restriction X$. So, it is a countable partial function in $\prod_{\alpha<\pi}[\theta_\alpha \times (0,1)_{\mathbb{Q}}]$. By virtue of the completeness of $\mathcal{G}$, there exists some $g \in \mathcal{G}_0$ such that $\sigma \subseteq g$. In order to prove that $\bar{p}$ is a g-uniform Δ-system, notice that the first five conditions from Definition 4.3.19 are clear, and conditions (6), (7) follow since, by definition of X and σ, for any $l \in L$, $n < n^*$, $g(\alpha_{n,l}) = \sigma(\alpha_{n,l}) = f(\alpha_{n,l})$. $\square_{\text{Theorem 4.3.21}}$

Notice that this result is a good motivation for the definition of a complete set of guardrails (see Definition 4.3.7).

Now, we will show that our definition of g-uniform Δ-system is a good definition, in the sense that it allows us to define the desired limit that we mentioned at the beginning of this subsection (as long as the g-uniform Δ-system is countable):

Definition 4.3.22. Let $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$, $(g, \bar{I}) \in \mathcal{G}$ and $\bar{p} = \langle p_\ell\colon \ell < \omega\rangle \subseteq \mathbb{P}_\pi$ be a g-uniform Δ-system with parameters $(\Delta,\ n^*, \vec{\alpha},\ r^*,\ \varepsilon^*)$. We define the function $\lim^g_{\mathbb{K}}(\bar{p})$, such that:

1. $\operatorname{dom}(\lim^g_{\mathbb{K}}(\bar{p})) = \Delta$,
2. For any $n \in r^*$, $\lim^g_{\mathbb{K}}(\bar{p})(\alpha^*_n) := \lim^{(\Xi^g_{\alpha^*_n})^-}_{\dot{Q}_{g(\alpha^*_n)}}(\langle p_\ell(\alpha^*_\ell)\colon \ell < \omega\rangle)$.

When the context is clear, we write simply $\lim^g(\bar{p})$ to refer to $\lim^g_{\mathbb{K}}(\bar{p})$.

We must verify that $\lim^g(\bar{p})$ is well-defined:

Lemma 4.3.23. *Let* $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$, $(g, \bar{I}) \in \mathcal{G}$ *and* $\bar{p} = \langle p_\ell\colon \ell < \omega\rangle$ *be a* g*-uniform* Δ*-system with parameters* $(\Delta,\ \vec{\alpha},\ n^*,\ r^*,\ \varepsilon^*)$. *Then,* $\lim^g(\bar{p}) \in \mathbb{P}_\pi$.

Proof. By Definition 4.3.19(4), for any $n \in r^*$, the sequence $\langle \alpha_{n,\ell} \colon \ell < \omega \rangle$ is constant with value α^*_n. Since $\bar{p}$ is a g-uniform Δ-system, we have that, for any $\ell < \omega$ and $n < n^*$, $p_\ell(\alpha_{n,\ell})$ is a $\mathbb{P}^-_{\alpha^*_n}$-name. Also, since $\Vdash_{\alpha^*_n}$"$p_\ell(\alpha^*_n) \in \dot{Q}^{\alpha^*_n}_{g(\alpha^*_n)}$", and $\mathbb{P}^-_{\alpha^*_n} \lessdot \mathbb{P}_{\alpha^*_n}$, we have that $\langle p_\ell(\alpha^*_n) \colon \ell < \omega \rangle$ is a sequence of conditions of $\dot{Q}^{\alpha^*_n}_{g(\alpha^*_n)}$ in $\mathrm{M}^{\mathbb{P}^-_{\alpha^*_n}}$ and therefore, since it is $((\Xi^g_{\alpha^*_n})^-, \bar{I}, \varepsilon^*_n)$-linked, it follows that, in $\mathrm{M}^{\mathbb{P}^-_{\alpha^*_n}}$,

$$\dot{l}_n \coloneqq \lim^{(\Xi^g_{\alpha^*_n})^-}_{\dot{Q}^{\alpha^*_n}_{g(\alpha^*_n)}} (\langle p_\ell(\alpha^*_n) \colon \ell < \omega \rangle) \in \dot{\mathbb{Q}}_{\alpha^*_n}$$

is defined. Without loss of generality, by virtue of Lemma 1.5.36(2), we can assume that, for any $n \in r^*$, $\dot{l}_n \in \langle \dot{\mathbb{Q}}_{\alpha^*_n} \rangle_{\mathbb{P}^-_{\alpha^*_n}}$, because $\Vdash_{\mathbb{P}^-_{\alpha^*_n}}$"$\dot{l}_n \in \dot{\mathbb{Q}}_{\alpha^*_n}$". So, $\mathrm{dom}(\lim^g(\bar{p})) \in [\pi]^{<\omega}$ and for any $\xi \in \mathrm{dom}(\lim^g(\bar{p}))$, we have that $\lim^g(\bar{p})(\xi) \in \langle \dot{\mathbb{Q}}_\xi \rangle$. Thus, by Lemma 1.5.50 $\lim^g(\bar{p}) \in \mathbb{P}_\pi$.

$\square_{\text{Lemma 4.3.23}}$

In the following theorem, we present some properties of the limit that we have just defined. Some of them appear implicitly in the proof of [She00, Lem. 3.4]. Property (4) is particularly interesting, since, similar to Theorem 4.2.5, $\lim^g(\bar{p})$ forces "many" conditions to fall into the generic filter.

Theorem 4.3.24. *Let $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$, $(g, \bar{I}) \in \mathcal{G}$ and $\bar{p} = \langle p_\ell \colon \ell < \omega \rangle$, a g-uniform Δ-system with parameters $(\Delta, \vec{\alpha}, n^*, r^*, \varepsilon^*)$. Then, $\lim^g(\bar{p})$ satisfies the following properties:*

1. *if $n \in r^*$, then $\lim^g(\bar{p}) \Vdash_{\mathbb{P}_\pi}$ " $\int_\omega \frac{|\{\ell \in I_k \colon p_\ell(\alpha^*_n) \in \dot{G}(\alpha^*_n)\}|}{|I_k|} \, d\, \Xi^g_\pi(k) \geq 1 - \varepsilon^*(n)$",*
2. *if $n \in n^* \setminus r^*$ and $\varepsilon > 0$, then:*
   (a) $\Vdash_{\mathbb{P}_\pi}$ " $\dot{\Xi}^g_\pi \left(\left\{ k < \omega \colon \frac{|\{\ell \in I_k \colon p_\ell(\alpha_{n,\ell}) \in \dot{G}(\alpha_{n,\ell})\}|}{|I_k|} \geq [1 - \varepsilon^*(n)](1 - \varepsilon) \right\} \right) = 1$",
   (b) $\lim^g(\bar{p}) \Vdash_{\mathbb{P}_\pi}$ " $\int_\omega \frac{|\{\ell \in I_k \colon p_\ell(\alpha_{n,\ell}) \in \dot{G}(\alpha_{n,\ell})\}|}{|I_k|} \, d\, \Xi^g_\pi(k) \geq 1 - \varepsilon^*(n)$",
3. *$\lim^g(\bar{p}) \Vdash_{\mathbb{P}_\pi}$ " $\int_\omega \frac{|\{\ell \in I_k \colon p_\ell \in \dot{G}\}|}{|I_k|} \, d\, \Xi^g_\pi(k) \geq 1 - \sum_{n < n^*} \varepsilon^*(n)$",*
4. *If $0 < \varepsilon < 1 - \sum_{n < n^*} \varepsilon^*(n)$, then $\lim^g(\bar{p}) \Vdash_{\mathbb{P}_\pi}$ "$\Xi^g_\pi(\dot{A}_\varepsilon) > 0$", where*

$$\dot{A}_\varepsilon \coloneqq \left\{ k < \omega \colon \frac{|\{\ell \in I_k \colon p_\ell \in \dot{G}\}|}{|I_k|} > \varepsilon \right\}.$$

   As a consequence, $\lim^g(\bar{p}) \Vdash_{\mathbb{P}_\pi}$"$\dot{A}_\varepsilon$ is infinite".

Proof. For any $\ell < \omega$, and $n < n^*$, define $\zeta_{n,\ell} \coloneqq g_0(\alpha_{n,\ell})$, $\varepsilon_{n,\ell} \coloneqq g_1(\alpha_{n,\ell})$. Also consider

$$\tau \coloneqq \langle (p_\ell(\alpha_{n,\ell}),\, \alpha_{n,\ell},\, \zeta_\ell,\, \varepsilon_\ell) \colon \ell < \omega \rangle.$$

To simplify the notation, let $q \coloneqq \lim^g(\bar{p})$. Then,

1. Let $n \in r^*$. Notice that,

- if $\ell < \omega$, then by Definition 4.3.19(6), $\Vdash_{\alpha_{n,\ell}}$"$p_\ell(\alpha_{n,\ell}) \in \dot{Q}^{\alpha_{n,\ell}}_{g(\alpha_{n,\ell})}$" and by the definition of ζ_ℓ and ε_ℓ, we get that $\Vdash_{\alpha_{n,\ell}}$"$p_\ell(\alpha_{n,\ell}) \in \dot{Q}^{\alpha_{n,\ell}}_{\zeta_{n,\ell},\varepsilon_\ell}$",
- the sequence $\langle \alpha_{n,\ell} \colon \ell < \omega \rangle$ is constant with value α^*_n by Definition 4.3.19(7),
- the sequence $\langle \varepsilon_\ell \colon \ell < \omega \rangle$ is equals to $\langle g_1(\alpha_{n,\ell}) \colon \ell < \omega \rangle$ by the definition of ε_ℓ and $\langle g_1(\alpha_{n,\ell}) \colon \ell < \omega \rangle$ is constant with value $\varepsilon^*(n)$ by Definition 4.3.19(7),
- by definition of ζ_ℓ and ε_ℓ, we have that $g(\alpha_{n,\ell}) = (\zeta_\ell, \varepsilon_\ell)$ for any $\ell < \omega$.

As a consequence, τ follows g and therefore, as $\langle \alpha_{n,\ell} \colon \ell < \omega \rangle$ is constant, we have by Definition 4.3.13(6)(a), that:

$$\Vdash_{\mathbb{P}_{\alpha^*_n}} \text{“}\lim^{\Xi^g_{\alpha^*_n}}_{\ell<\omega}(p_\ell(\alpha^*_n)) \Vdash_{\dot{\mathbb{Q}}_{\alpha^*_n}} \text{“}\int_\omega \frac{|\{\ell \in I_k \colon p_\ell(\alpha^*_n) \in \dot{G}(\alpha^*_n)\}|}{|I_k|}\, d\dot{\Xi}^g_{\alpha^*_n+1}(k) \geq 1 - g_1(\alpha^*_n)\text{””}.$$

Finally, since $q \restriction \alpha^*_n \Vdash_{\alpha^*_n}$"$q(\alpha^*_n) \in \dot{G}(\alpha^*_n)$", and by Corollary 3.5.30, we can conclude that:

$$q \restriction \alpha^*_n + 1 \Vdash_\pi \text{“}\int_\omega \frac{|\{\ell \in I_k \colon p_\ell(\alpha^*_n) \in \dot{G}(\alpha^*_n)\}|}{|I_k|}\, d\dot{\Xi}^g_{\alpha^*_n+1}(k) \geq 1 - \varepsilon^*(n)\text{”}.$$

2. Let $n \in n^* \setminus r^*$ and $\varepsilon > 0$. Notice that, in a similar way to (1), τ follows g, but with $\langle \alpha_{n,\ell} \colon \ell < \omega \rangle$ increasing, by Definition 4.3.19(5).

   (a) Since τ follows g and $\langle \alpha_{n,\ell} \colon \ell < \omega \rangle$ is increasing, by Definition 4.3.13(6)(b), we get:

   $$\Vdash_{\mathbb{P}_\pi} \text{“}\,\dot{\Xi}^g_\pi\left(\left\{k < \omega \colon \frac{|\{\ell \in I_k : p_\ell(\alpha_{n,\ell}) \in \dot{G}(\alpha_{n,\ell})\}|}{|I_k|} \geq [1 - \varepsilon^*(n)](1 - \varepsilon)\right\}\right) = 1\text{”}.$$

   (b) Let $G \subseteq \mathbb{P}_\pi$ be a generic filter over M such that $q \in G$. Working in $\mathrm{M}[G]$, define $\upsilon_n \colon \omega \to \mathbb{R}$ such that, for any $k < \omega$,

   $$\upsilon_n(k) := \frac{|\{\ell \in I_k \colon p_\ell(\alpha_{n,\ell}) \in G(\alpha_{n,\ell})\}|}{|I_k|}$$

   and consider $P_{\varepsilon,n} := \{k < \omega \colon \upsilon_n(k) \geq [1 - \varepsilon^*(n)](1 - \varepsilon)\}$. By the previous item, $\Xi^g_\pi(P_{\varepsilon,n}) = 1$ and therefore, $\Xi^g_\pi(\omega \setminus P_{\varepsilon,n}) = 0$. Now, by basic integral properties, for any $n < n^*$, we have that:

   $$\begin{aligned} \int_\omega \upsilon_n \, d\,\Xi^g_\pi &= \int_{\omega \setminus P_{\varepsilon,n}} \upsilon_n \, d\,\Xi^g_\pi + \int_{P_{\varepsilon,n}} \upsilon_n \, d\,\Xi^g_\pi(k) \\ &= \int_{P_{\varepsilon,n}} \upsilon_n \, d\,\Xi^g_\pi \\ &\geq \int_{P_{\varepsilon,n}} [1 - \varepsilon^*(n)](1 - \varepsilon)\, d\,\Xi^g_\pi \\ &= [1 - \varepsilon^*(n)](1 - \varepsilon)\, \Xi^g_\pi(P_{\varepsilon,n}) \\ &= [1 - \varepsilon^*(n)](1 - \varepsilon). \end{aligned}$$

Since ε is arbitrary, we can conclude

$$\int_\omega \frac{|\{\ell \in I_k \colon p_\ell(\alpha_{n,\ell}) \in G(\alpha_{n,\ell})\}|}{|I_k|}\, d\Xi(k)^g_\pi \geq 1 - \varepsilon^*_n.$$

Thus, in the ground model,

$$q \Vdash_\pi \text{``} \int_\omega \frac{|\{\ell \in I_k \colon p_\ell(\alpha_{n,\ell}) \in G(\alpha_{n,\ell})\}|}{|I_k|}\, d\Xi(k)^g_\pi \geq 1 - \varepsilon^*_n\text{''}.$$

3. Let $G \subseteq \mathbb{P}_\pi$ be a generic filter over M such that $q \in G$. Working in $\mathrm{M}[G]$, define the functions $\varrho, \varrho_n, \upsilon \colon \omega \to \mathbb{R}$ for any $n < n^*$ such that, for $k < \omega$,

$$\varrho(k) \coloneqq \frac{|\{\ell \in I_k \colon p_\ell \notin G\}|}{|I_k|},\ \ \upsilon(k) \coloneqq \frac{|\{\ell \in I_k \colon p_\ell \in G\}|}{|I_k|} \ \text{ and}$$

$$\varrho_n(k) \coloneqq \frac{|\{\ell \in I_k \colon p_\ell(\alpha_{n,\ell}) \notin G(\alpha_{n,\ell})\}|}{|I_k|}.$$

Notice that for any $k < \omega$, $\varrho(k) + \upsilon(k) = 1$ and therefore, $\upsilon(k) = 1 - \varrho(k)$.

Now, by Lemma 1.5.51, if $p \in \mathbb{P}_\pi$, then $p \in G \Leftrightarrow \forall \alpha \in \mathrm{dom}(p)\, (p(\alpha) \in G(\alpha))$, so we can relate ϱ in with ϱ_n as follows:

$$\begin{aligned}\varrho(k) &= \frac{|\{\ell \in I_k \colon \exists n < n^*(p_\ell(\alpha_{n,\ell}) \notin G(\alpha_{n,\ell}))\}|}{|I_k|}\\ &\leq \sum_{n<n^*} \frac{|\{\ell \in I_k \colon p_\ell(\alpha_{n,\ell}) \notin G(\alpha_{n,\ell})\}|}{|I_k|}\\ &= \sum_{n<n^*} \varrho_n(k).\end{aligned}$$

Also, by items (1) and (2)(b), it is clear that $\int_\omega \varrho_n d\Xi^g_\pi \leq \varepsilon^*(n)$ for any $n < n^*$ and therefore, by basic integral properties,

$$\int_\omega \varrho\, d\Xi^g_\pi \leq \sum_{n<n^*} \left(\int_\omega \varrho_n\, d\Xi^g_\pi \right) \leq \sum_{n<n^*} \varepsilon^*(n),$$

which implies that

$$\int_\omega \upsilon\, d\Xi^g_\pi = \int_\omega (1 - \varrho)\, d\Xi^g_\pi = 1 - \int_\omega \varrho\, d\Xi^g_\pi \geq 1 - \sum_{n<n^*} \varepsilon^*(n).$$

Finally, in the ground model,

$$q \Vdash_\pi \text{``} \int_\omega \frac{|\{\ell \in I_k \colon p_\ell \in \dot{G}_{\mathbb{P}_\pi}\}|}{|I_k|} d\Xi_\pi(k) \geq 1 - \sum_{n<n^*} \varepsilon^*(n)\text{''}.$$

4. Let $G \subseteq \mathbb{P}_\pi$ be a generic filter over M such that $q \in G$. Working in $\mathrm{M}[G]$, consider $\upsilon\colon \omega \to \mathbb{R}$ as in the previous item. Let $0 < \varepsilon < 1 - \sum_{n<n^*} \varepsilon^*(n)$. Thereby, $A_\varepsilon = \{k < \omega\colon \upsilon(k) > \varepsilon\}$. By (3) and integral properties, we have:

$$\begin{aligned}
\varepsilon &< 1 - \sum_{n<n^*} \varepsilon^*(n) \\
&\leq \int_\omega \upsilon \, d\, \Xi^g_\pi \\
&= \int_{\omega \setminus A_\varepsilon} \upsilon \, d\Xi^g_\pi + \int_{A_\varepsilon} \upsilon \, d\, \Xi^g_\pi \\
&\leq \int_{\omega \setminus A_\varepsilon} \varepsilon^* d\, \Xi^g_\pi + \int_{A_\varepsilon} \upsilon \, d\, \Xi^g_\pi \\
&= \varepsilon \, \Xi^g_\pi(\omega \setminus A_\varepsilon) + \int_{A_\varepsilon} \upsilon \, d\, \Xi^g_\pi \\
&= \varepsilon - \varepsilon^* \, \Xi^g_\pi(A_\varepsilon) + \int_{A_\varepsilon} \upsilon \, d\, \Xi^g_\pi,
\end{aligned}$$

whence it follows that,

$$\varepsilon \Xi^g_\pi(A_\varepsilon) < \int_{A_\varepsilon} \upsilon \, d\, \Xi^g_\pi.$$

So, if $\Xi^g_\pi(A_\varepsilon) = 0$, then by Lemma 3.5.21, $\varepsilon \, \Xi^h_\pi(A_\varepsilon) < \int_{A_\varepsilon} \varrho \, d\, \Xi^g_\pi = 0$, which is not possible. Thus, $\Xi^g_\pi(A_\varepsilon) > 0$. Finally, in the ground model, $q \Vdash_{\mathbb{P}_\pi}$ "$\Xi^g_\pi(\dot{A}_\varepsilon) > 0$", and since $\dot{\Xi}^g_\pi$ is a name of a free finitely additive measure, it is clear that $q \Vdash_\pi$"$\dot{A}_\varepsilon$ is infinite" $\square_{\text{Theorem 4.3.24}}$

As a consequence, we have that, if $\mathcal{G}$ is a complete set of guardrails for (π, κ), then any iteration in $\mathcal{K}_1(\kappa, \mathcal{G})$ is κ-Fr-Knaster:

Theorem 4.3.25. *Let $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$. If $\mathcal{G}$ is a complete set of guardrails for (π, κ) and $\theta \geq \kappa$ is regular, then the final step $\mathbb{P}_\pi$ is θ-Fr-Knaster.*

Proof. Let $\{p_\xi\colon \xi < \theta\} \subseteq \mathbb{P}_\pi$, $\bar{\varepsilon}\colon \omega \to (0,1)$ such that $\sum_{n<\omega} \bar{\varepsilon}(n) < 1$ and fix $\varepsilon > 0$ such that:

$$\varepsilon < 1 - \sum_{n<\omega} \bar{\varepsilon}(n). \tag{4.3.13}$$

By Lemma 4.3.5, for any $\xi < \kappa$, there exists some $q_\xi \in D_{\bar{\varepsilon}}$ such that $q_\xi \leq p_\xi$. Also, since $q_\xi \in D_{\bar{\varepsilon}}$, there exists $n^*_\xi < \omega$ and $\zeta_{\xi,n}$ for all $n < n^*_\xi$, such that

$$q_\xi \restriction \gamma_{n,\xi} \Vdash_{\gamma_{n,\xi}} \text{“} q_\xi(\gamma_{n,\xi}) \in \dot{Q}^{\gamma_{n,\xi}}_{\zeta_{\xi,n}, \bar{\varepsilon}(n)} \text{”},$$

where $\mathrm{dom}(q_\xi) = \{\gamma_{n,\xi}\colon n < n^*_\xi\}$ is arranged in decreasing order. To get a uniform Δ-system, we must reorganize the domains: for any $\xi < \kappa$ and $n < n^*_\xi$, let $\alpha_{n,\xi} \coloneqq \gamma_{n^*_\xi - n - 1}$, hence we have that $\mathrm{dom}(q_\xi) = \{\alpha_{n,\xi}\colon n < n^*_\xi\}$ is arranged in increasing order because, if $n < j < n^*_\xi$, then $n^*_\xi - j - 1 < n^*_\xi - n - 1$ and therefore $\alpha_{n,\xi} < \gamma_{n^*_\xi - n - 1} < \gamma_{n^*_\xi - j - 1} = \alpha_{j,\xi}$.

Using those parameters and the regularity of θ, we can proceed as in the proof of Theorem 4.3.20 to get f, a half guardrail for (π, κ), and $E \in [\theta]^\theta$ such that $\{q_\xi \colon \xi \in E\}$ forms a f-uniform Δ-system with parameters $(\Delta, n^*, \vec{\alpha}, r^*, \varepsilon^*)$, where for any $\xi \in E$, $n^*_\xi = n^*$ and

$$\forall n < n^*(\varepsilon^*(n) = \bar{\varepsilon}(n^* - n - 1)). \tag{4.3.14}$$

Now, we show that $\{p_\xi \colon \xi \in E\}$ is Fr-linked in $\mathbb{P}_\pi$. For this, let $\bar{p} \coloneqq \langle p_{\xi_\ell} \colon \ell < \omega\rangle$ with each $\xi_\ell \in E$ and consider $\langle \beta_\ell \colon \ell < \omega\rangle \subseteq \{\xi_\ell \colon \ell < \omega\}$ with order type ω. Define $\bar{q} \coloneqq \langle q_{\beta_\ell} \colon \ell < \omega\rangle$. It is clear that it is a countable f-uniform Δ-system with parameters $(\Delta,\, n^*,\, \vec{\alpha},\, r^*, \varepsilon^*)$ an therefore, by virtue of the completeness of $\mathcal{G}$, by Theorem 4.3.21, we can find $g \in \mathcal{G}_0$ such that $\bar{q}$ is an g-uniform Δ-system with the same parameters. Let $\bar{I} \in \mathcal{I}_\infty$, hence $(g, \bar{I}) \in \mathcal{G}$.

For any $k < \omega$, let $\varrho(k) \coloneqq \frac{|\ell \in I_k \colon q_{\beta_\ell} \in \dot{G}|}{|I_k|}$ and $\dot{A}_\varepsilon \coloneqq \{k < \omega \colon \varrho(k) > \varepsilon\}$. By Equation 4.3.13 and Equation 4.3.14, we have that:

$$0 < \varepsilon < 1 - \sum_{n<\omega} \bar{\varepsilon}(n) \leq 1 - \sum_{n<n^*} \bar{\varepsilon}(n^* - n - 1) = 1 - \sum_{n<n^*} \varepsilon^*(n).$$

Also, since $(g, \bar{I}) \in \mathcal{G}$, by Theorem 4.3.24(4), we get that $\lim^g(\bar{q}) \Vdash_\pi$“$\dot{A}_\varepsilon$ is infinite”. Finally, since for any $\ell < \omega$, $q_{\beta_\ell} \leq p_{\xi_\ell}$ and $\langle p_{\beta_\ell} \colon \ell < \omega\rangle \subseteq \{p_{\xi_\ell} \colon \ell < \omega\}$, it follows that

$$\lim(\bar{q}) \Vdash_\pi \text{“}\exists^\infty_{\ell<\omega}(p_{\xi_\ell} \in \dot{G})\text{”}.$$

As a consequence, $\{p_\xi \colon \xi \in E\}$ is Fr-linked. $\square_{\text{Theorem 4.3.25}}$

CHAPTER 5

Applications: $\mathrm{cov}(\mathcal{N})$ may have countable cofinality

To sort out possible theorems – after throwing away all relations which do not hold, you no longer have a heap of questions which clearly are all independent, the trash is thrown away and in what remains you find some grains of gold.

Saharon Shelah[1]

In this chapter, we are going to study some applications of the theory that we built in the previous chapter. In particular, based on [She00], we define an iteration in $\mathcal{K}_1$ that will allow us to prove the consistency of $\mathrm{cov}(\mathcal{N})$ with countable cofinality, and obtain some separations of Cichoń's diagram with $\mathrm{cov}(\mathcal{N})$ singular. To contextualize the problem, we are going to start by making a brief summary of the cofinalities of the cardinals in Cichoń's diagram.

5.1 Context: cofinalities in Cichoń's diagram

Cofinalities of the cardinal invariants in Cichoń's diagram, and of cardinal invariants in general, were extensively studied in the late 1980s, mainly by *Tomek Bartoszyński*, *Jörg Brendle*, *David Framlin*, *Haim Judah*, *Arnold W. Miller* and *Saharon Shelah* (see, for instance, [Bre91], [BIS89] and [Mil82]). Some results about these cofinalities are not difficult, for example, it is well known that $\mathfrak{b}$ is uncountable regular. Furthermore, it is known that $\mathrm{cf}(\mathfrak{d}) \geq \mathfrak{b}$. So, $\mathfrak{d}$ has uncountable cofinality. On the other hand, in general, for any ideal $\mathcal{I}$ containing the singletons, we have that $\mathrm{add}(\mathcal{I})$ is regular and, since $\mathrm{add}(\mathcal{N})$ and $\mathrm{add}(\mathcal{M}) \geq \aleph_1$, they have uncountable cofinality. Finally, by *König's theorem* (see [Kun12, Thm. I.13.13]), $\mathrm{cf}(\mathfrak{c}) > \aleph_0$. In conclusion, basically from the definitions, it follows that $\mathfrak{b}$, $\mathfrak{d}$, $\mathrm{add}(\mathcal{M}), \mathrm{add}(\mathcal{N})$, $\aleph_1$ and $\mathfrak{c}$ have uncountable cofinality.

[1] In response to the question: "What good does it do you to know all those independence results?" by *L. Harrington* (see [She93]).

For $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{N})$, $\mathrm{cof}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{N})$ we need a little more work:

Lemma 5.1.1. *Let X be a non-empty set and $\mathcal{I}$ an ideal on X containing all its singletons. Then* $\mathrm{add}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{non}(\mathcal{I}))$.

Proof. Let $Y \subseteq X$ such that $Y \notin \mathcal{I}$ and $Y = \bigcup_{\alpha<\mathrm{cf}(\mathrm{non}(\mathcal{I}))} Y_\alpha$, where for any $\alpha < \mathrm{cf}(\mathrm{non}(\mathcal{I}))$, we have that $|Y_\alpha| < \mathrm{non}(\mathcal{I})$. Therefore, $Y_\alpha \in \mathcal{I}$. As a consequence, if $\mathrm{cf}(\mathrm{non}(\mathcal{I})) < \mathrm{add}(\mathcal{I})$, then $Y \in \mathcal{I}$, which is a contradiction. Thus, $\mathrm{add}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{non}(\mathcal{I}))$. $\square_{\text{Lemma 5.1.1}}$

As a consequence, using that $\mathrm{add}(\mathcal{N})$ and $\mathrm{add}(\mathcal{M})$ are uncountable, we get:

Corollary 5.1.2. $\mathrm{non}(\mathcal{M})$ *and* $\mathrm{non}(\mathcal{N})$ *have uncountable cofinality.*

In a similar way for $\mathrm{cof}(\mathcal{I})$:

Lemma 5.1.3. *Let X be a non-empty set and $\mathcal{I}$ an ideal on X containing all its singletons. Then* $\mathrm{add}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{cof}(\mathcal{I}))$.

Proof. Let $\mathcal{F} \subseteq \mathcal{I}$ such that $\mathcal{F} = \bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha \subseteq \mathcal{I}$, $|\mathcal{F}_\alpha| < \mathrm{cof}(\mathcal{I})$ and $\kappa < \mathrm{add}(\mathcal{I})$. Since $|\mathcal{F}_\alpha| < \mathrm{cof}(\mathcal{I})$, it is not cofinal in $\mathcal{I}$, hence we can find $X_\alpha \in \mathcal{I}$ such that $X_\alpha \nsubseteq F$ for all $F \in \mathcal{F}_\alpha$ So $\bigcup_{\alpha<\kappa} X_\alpha \in \mathcal{I}$ and, for any $F \in \mathcal{F}$, $\bigcup_{\alpha<\kappa} X_\alpha \nsubseteq F$. Thus $\mathrm{add}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{cof}(\mathcal{I}))$. $\square_{\text{Lemma 5.1.3}}$

Corollary 5.1.4. $\mathrm{cof}(\mathcal{M})$ *and* $\mathrm{cof}(\mathcal{N})$ *have uncountable cofinality.*

However, proving that $\mathrm{cov}(\mathcal{M})$ has uncountable cofinality requires much more work. This result is due to *Tomek Bartoszyński* and *Jaim Judah*, who in 1988 ([BJ95, Thm. 5.1.3]) proved the following theorem:

Theorem 5.1.5. $\mathrm{cf}(\mathrm{cov}(\mathcal{M})) \geq \mathrm{add}(\mathcal{N})$.

So in 1989, it was already known that all the cardinals in Cichón's diagram, with the exception of $\mathrm{cov}(\mathcal{N})$, have uncountable cofinality. So, expecting $\mathrm{cov}(\mathcal{N})$ to also have countable cofinality was only natural. *Tomek Bartoszyński* himself made some attempts to find a proof, for instance:

Theorem 5.1.6. *If* $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}$, *then* $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) > \aleph_0$.

Although $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) > \aleph_0$ was naturally expected, the problem was left open:

Main Problem 5.1.7. May $\mathrm{cov}(\mathcal{N})$ have countable cofinality?

Historical Remark 5.1.8. Everything seems to indicate that the first person who raised the problem of the cofinality of $\mathrm{cov}(\mathcal{N})$ was *David Fremlin* before the year 1979. Although we were unable to find an exact reference since the references pointed to a seminar at the *Pierre and Marie Curie* university in Paris in 1984, we were able to find two references that support it: on the one hand, in [Bar88, pg. 9], Tomek Bartoszyński refers to Theorem 2.2, which corresponds to our Theorem 5.1.6, as:

> "*a partial solution to **a question of*** **David Fremlin** *about the cofinality of* $\mathrm{cov}(\mathcal{N})$".

Even more conclusively, at the end of [Mil81][2], there appears a section of open problems, where the problem (4), attributed to *David Fremlin*, states, using our current notation, the following:

> "*Show that the least* κ ($\mathrm{cov}(\mathcal{N})$) *such that* ${}^{\omega}2$ *can be covered by* κ *many measure zero sets cannot have countable cofinality*".

So *David Fremlin* not only raised the question but thought that $\mathrm{cov}(\mathcal{N})$ could not have countable cofinality.

If contrary to *Fremlin*'s predictions, one wants to try to prove the consistency of $\mathrm{cov}(\mathcal{N})$ with countable cofinality, it may be natural to try to attack the problem using partial random forcing because this allows us to increment $\mathrm{cov}(\mathcal{N})$ using *book-keeping* arguments (see Section 5.5). However, *Saharon Shelah*, *Haim Judah* (see [SJ93]), and *Janusz Pawlikowski* (see [Paw92]) built some examples of partial random forcing that added dominant reals, which is not good because to force $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0$, by Theorem 5.1.6, we must not increment $\mathfrak{b}$ too much. One possible solution to this was to try to find ways to iterate with partial random forcing without adding new dominating reals. For example, *Jörg Brendle* and *Haim Judah* at [BJ93] did some studies on this. They thought that considering finite combinations of random forcing it would permit to construct an iteration to force $\mathrm{cov}(\mathcal{N})$ with countable cofinality. Finally, it was *Saharon Shelah*, in the year 2000, who finally solved the problem, proving the consistency of $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0$ using a method, very well known to us by now, of iterated forcing using finitely additive measures. According to Historical Remark 5.1.8, Main Problem 5.1.7 was open for almost 20 years.

One question remains: what is the relationship between iterations using finitely additive measures and not increasing $\mathfrak{b}$? A possible answer is that, according to Theorem 4.3.25, if $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$ for some cardinal κ and some complete set of guardrails $\mathcal{G}$, then $\mathbb{P}_\pi := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi \rangle$ is κ-Fr-Knaster and by [BCM21, Thm. 3.12], the κ-Fr-Knaster forcing notions preserves $\mathfrak{b} \leq \kappa$.

5.2 Coding null sets

Since we are going to use random forcing to force $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0$, and we know that random reals can be characterized in terms of null-set codifications, we are going to introduce a particular coding a base of null sets. But first, we need:

Definition 5.2.1. Let Ω be the set of sequences $\bar{a} \colon \omega \to \omega \times [{}^{<\omega}2]^{<\omega}$ such that, for any $j < \omega$, $\bar{a}(j) = (n_j, a_j)$ satisfying:

1. $\langle n_j \colon j < \omega \rangle$ is increasing,
2. for any $j < \omega$, $a_j \subseteq {}^{n_j}2$ and $\frac{|a_j|}{2^{n_j}} > 1 - \frac{1}{7^j}$.

Now, we introduce our coding of null sets:

Definition 5.2.2. For $\bar{a} \in \Omega$ and $m < \omega$ define:

[2] Submitted in 1979.

1. $\mathrm{Tree}_m(\bar{a}) := \bigcap_{n_j > m} \{x \in 2^\omega \colon x \restriction n_j \in a_j\}$,

2. $N[\bar{a}] := 2^\omega \setminus \bigcup_{m<\omega} \mathrm{Tree}_m(\bar{a})$.

It is, indeed, a coding in the following sense:

Theorem 5.2.3. *$A \in \mathcal{N}$ if, and only if, there exists some $\bar{a} \in \Omega$ such that $A \subseteq N[\bar{a}]$.*

Proof. On the hand hand, assume $\bar{a} \in \Omega$ and $A \subseteq N[\bar{a}]$. For any $j < \omega$, we can write the set $\{x \in {}^\omega 2 \colon x \restriction n_j \in a_j\} = \bigcup_{t \in a_j} [t]$ as a disjoint union, and it is clearly Lebesgue measurable in ${}^\omega 2$, hence

$$\mathrm{Leb}(\{x \in {}^\omega 2 \colon x \restriction n_j \in a_j\}) = \sum_{t \in a_j} \mathrm{Leb}([t]) = \sum_{t \in a_j} \frac{1}{2^{n_j}} = \frac{|a_j|}{2^{n_j}}.$$

So, for any $m < \omega$, we get

$$\begin{aligned}
\mathrm{Leb}({}^\omega 2 \setminus \mathrm{Tree}_m(\bar{a})) &= \mathrm{Leb}\left(\bigcup_{n_j > m} \{x \in {}^\omega 2 \colon x \restriction n_j \in a_j\}^{\mathrm{c}}\right) \\
&\leq \sum_{n_j \geq m} \mathrm{Leb}(\{x \in {}^\omega 2 \colon x \restriction n_j \in a_j\}^{\mathrm{c}}) \\
&= \sum_{n_j > m} \left(1 - \frac{|a_j|}{2^{n_j}}\right) \\
&\leq \sum_{n_j > m} \frac{1}{7^j}.
\end{aligned}$$

Now, it is clear by Definition 5.2.2(2), that if $n < m$, then $\mathrm{Tree}_n(\bar{a}) \subseteq \mathrm{Tree}_m(\bar{a})$, hence we have that $\mathrm{Tree}_m(\bar{a})^{\mathrm{c}} \subseteq \mathrm{Tree}_n(\bar{a})^{\mathrm{c}}$. Thus

$$\begin{aligned}
\mathrm{Leb}(N[\bar{a}]) &= \mathrm{Leb}\left({}^\omega 2 \setminus \bigcup_{m<\omega} \mathrm{Tree}_m(\bar{a})\right) \\
&= \mathrm{Leb}\left(\bigcap_{m<\omega} ({}^\omega 2 \setminus \mathrm{Tree}_m(\bar{a}))\right) \\
&= \lim_{m \to \infty} (\mathrm{Leb}({}^\omega 2 \setminus \mathrm{Tree}_m(\bar{a}))) \\
&\leq \lim_{m \to \infty} \sum_{n_j > m} \frac{1}{7^j} \\
&= 0.
\end{aligned}$$

Finally, $\mathrm{Leb}(N[\bar{a}]) = 0$. Since $A \subseteq N[\bar{a}]$, $\mathrm{Leb}(A) \leq \mathrm{Leb}(N[\bar{a}])$, hence $\mathrm{Leb}(A) = 0$.

On the other hand, assume that $A \in \mathcal{N}$ and consider, for any $n < \omega$, $\varepsilon_n := \frac{1}{n+1} \cdot \frac{1}{7^{n+1}}$. Since A is null, for any $n < \omega$, there exists a sequence $\langle t_{n,k} \colon k < \omega \rangle$ such that $A \subseteq \bigcup_{k<\omega} [t_{n,k}]$ and

$\sum_{k<\omega} \mathrm{Leb}([t_{n,k}]) < \varepsilon_n$. So, using that this sum is convergent, we can find an increasing sequence $\langle k_i \colon i < \omega \rangle$ such that $k_0 = 0$ and, for any $n < \omega$,

$$\sum_{k \geq 0} \mathrm{Leb}([t_{n,k}]]) + \sum_{i<n} \left(\sum_{k \geq k_n} \mathrm{Leb}([t_{i,k}]) \right) < \frac{1}{7^{n+1}}. \tag{5.2.1}$$

Now, for $n < \omega$, we define

$$C_n := \left(\bigcup_{k<k_{n+1}} [t_{n,k}] \right) \cup \left[\bigcup_{i<n} \left(\bigcup_{k \in [k_n, k_{n+1}) \cap \omega} [t_{i,k}] \right) \right].$$

So, by Equation 5.2.1, for any $n < \omega$, we have that:

$$\mathrm{Leb}(C_n) < \frac{1}{7^{n+1}} \tag{5.2.2}$$

Also, by the construction, it follows that:

$$A \subseteq \bigcap_{m<\omega} \left(\bigcup_{n \geq m} C_n \right) = \{x \in {}^{\omega}2 \colon |\{n < \omega \colon x \in C_n\}| = \aleph_0\} \tag{5.2.3}$$

On the other hand, notice that:

$$\begin{aligned}
\mathrm{Leb}\left(\bigcap_{m<\omega} \left(\bigcup_{n \geq m} C_n \right) \right) &= \lim_{m \to \infty} \left(\bigcup_{n \geq m} C_n \right) \\
&\leq \lim_{m \to \infty} \left(\sum_{n \geq m} \mathrm{Leb}(C_n) \right) \\
&\leq \lim_{m \to \infty} \left(\sum_{n \geq m} \frac{1}{7^{n+1}} \right) \\
&= 0.
\end{aligned}$$

So far, we have used the usual ideas of this type of proof, following [BJ95, Thm. 2.3.10]. Now, we must relate this to the particular way in which we are coding.

Since $\{[t] \colon t \in {}^{<\omega}2\}$ is a countable basis of clopens for ${}^{\omega}2$, for any $j < \omega$, we can write C_j as a finite union of clopens: we can find $n_j < \omega$ and $b_j \subseteq 2^{n_j}$ such that, $C_j = \bigcup_{t \in b_j} [t]$. Without loss of generality, we can assume that $\langle n_j \colon j < \omega \rangle$ is increasing. Define, for any $j < \omega$, $a_j := 2^{n_j} \setminus b_j$ and $\bar{a} := \langle (n_j, a_j) \colon j < \omega \rangle$. Notice that,

$$\frac{|b_j|}{2^{n_j}} = \sum_{t \in b_j} \frac{1}{2^{n_j}} = \sum_{t \in b_j} \mathrm{Leb}([t]) = \mathrm{Leb}(C_j) < \frac{1}{7^{j+1}},$$

hence, $\frac{|a_j|}{2^{n_j}} > 1 - \frac{1}{7^{n_j}}$, that is, $\bar{a} \in \Omega$ and it is clear that $N[\bar{a}] = \bigcap_{n<\omega} \bigcup_{n \geq m} C_n$, whence the result follows. $\square_{\text{Theorem 5.2.3}}$

We can translate this coding into a Tukey relation with appropriate relational systems. For this, we introduce:

Definition 5.2.4. Let us define the following relations systems:

1. $\mathcal{S} := \langle \Omega, \mathcal{P}, \sqsubset \rangle$ where $\mathcal{P} := \{T \subseteq {}^{<\omega}2 \colon T \text{ is a perfect tree}\}$ and
$$\bar{a} \sqsubset T :\Leftrightarrow \exists m < \omega([T] \subseteq \mathrm{Tree}_m(\bar{a})).$$

2. $\mathcal{CN} := \langle \Omega, {}^{\omega}2, \lhd \rangle$, where $\bar{a} \lhd x :\Leftrightarrow x \notin N[\bar{a}]$.

Notice that, in $\mathcal{CN}$, $a \lhd x$ if, and only if, there exists some $m < \omega$, such that $x \in \mathrm{Tree}_m(\bar{a})$.
In this way, the coding can be understood according to the following Tukey equivalence:

Theorem 5.2.5. $\mathcal{CN} \equiv_{\mathrm{T}} \mathcal{C}_{\mathcal{N}}^{\perp}$.

***Proof*.** On the one hand, define $\psi_- \colon \Omega \to \mathcal{N}$ such that for any $\bar{a} \in \Omega$, $\psi_-(\bar{a}) := N[\bar{a}]$ and $\psi_+ \colon {}^{\omega}2 \to {}^{\omega}2$ as the identity function. Let $\bar{a} \in \Omega$ and $x \in {}^{\omega}2$. Then,

$$\psi_-(\bar{a}) \not\ni x \Leftrightarrow N[\bar{a}] \not\ni x \Leftrightarrow \bar{a} \lhd x \Leftrightarrow \bar{a} \lhd \psi_+(x).$$

Thus, $\mathcal{CN} \leq_{\mathrm{T}} \mathcal{C}_{\mathcal{N}}^{\perp}$.
On the other hand, by Theorem 5.2.3, for any $A \in \mathcal{N}$, there exists some $\bar{a}_A \in \Omega$, such that $A \subseteq N[\bar{a}_A]$. Now, define $\psi_- \colon \mathcal{N} \to \Omega$ such that, for any $A \in \mathcal{N}$, $\psi_-(A) := \bar{a}_A$ and $\psi_+ \colon {}^{\omega}2 \to {}^{\omega}2$ as the identity function. Let $A \in \mathcal{N}$ and $x \in {}^{\omega}2$. Then,

$$\psi_-(A) \lhd x \Leftrightarrow \bar{a}_A \lhd x \Leftrightarrow x \notin N[\bar{a}_A] \Rightarrow A \not\ni x \Leftrightarrow A \not\ni \psi_+(x).$$

Thus, $\mathcal{C}_{\mathcal{N}}^{\perp} \preceq_{\mathrm{T}} \mathcal{CN}$. Finally, $\mathcal{CN} \equiv_{\mathrm{T}} \mathcal{C}_{\mathcal{N}}^{\perp}$. $\square_{\text{Theorem 5.2.5}}$

As we mentioned before, random reals can be characterized in terms of their behavior with respect to null sets, however, we are only going to prove the implication that we will need in the iteration:

Theorem 5.2.6. *If r is a random real over* M *then, for any* $\bar{a} \in \Omega \cap \mathrm{M}$, $r \notin N[\bar{a}]$.

***Proof*.** Working in M, let $\bar{a} \in \Omega$ and $T_0 \in \mathbb{B}$, hence $\mathrm{Leb}([T_0] \setminus N[\bar{a}]) > 0$ and therefore, there exists $T \in \mathbb{B}$ such that:

$$[T]^{\mathrm{M}} \subseteq [T_0]^{\mathrm{M}} \setminus N[\bar{a}]^{\mathrm{M}} \tag{5.2.4}$$

It is enough to show that $T \Vdash_{\mathbb{B}}$"$\dot{r} \notin N[\bar{a}]$". So, let G a $\mathbb{B}$-generic filter over M, such that $T \in G$. Working in $\mathrm{M}[G]$, we have that, by Equation 5.2.4, $[T]^{\mathrm{M}[G]} \subseteq [T_0]^{\mathrm{M}[G]} \setminus N[\bar{a}]^{\mathrm{M}[G]}$, because it are Σ^1_1-properties, hence we can apply Theorem 1.2.10. As a consequence, since $T \in G$, $r \in [T]^{\mathrm{M}[G]}$ and therefore, $r \notin N[\bar{a}]^{\mathrm{M}[G]}$. $\square_{\text{Theorem 5.2.6}}$

In fact, random reals are those reals that manage to evade all the nulls that are coded in the ground model.
Finally, for convenience, we use the following form of Cohen forcing:

Convention 5.2.7. The *Cohen forcing* $\mathbb{C}$ is the collection of finite sequences $\langle (n_\ell, a_\ell) \colon \ell < k \rangle$, where:

- $\langle n_\ell \colon \ell < k \rangle$ is an increasing sequence of natural numbers,
- for any $\ell < k$, $a_\ell \subseteq {}^{n_\ell}2$ and $\frac{|a_\ell|}{2^{n_\ell}} > 1 - \frac{1}{7^\ell}$.

Notice that this convention is justified by Theorem 1.5.40.

5.3 Controlling $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$: preservation of strongly unbounded families

In this section, we are going to prove several results that will allow us to control $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$. Thanks to the structure with finitely additive measures of the iterations defined in the previous chapter, we are going to be able to preserve some strongly unbounded families, which will allow us to control the cardinals in question. We start by proving a necessary lemma about tree absoluteness:

Lemma 5.3.1. *In* M, *let* $\mathbb{P}$ *be a forcing notion and* $\langle \mathcal{T}_i \colon i \in I \rangle$ *a sequence of well-pruned trees on* ${}^{<\omega}\omega$. *Let* G *be a* $\mathbb{P}$ *generic filter over* M *and assume that* $\bigcap_{i \in I} [\mathcal{T}_i]^{\mathrm{M}[G]}$ *is countable. Then*

$$\bigcap_{i \in I} [\mathcal{T}_i]^{\mathrm{M}[G]} = \bigcap_{i \in I} [\mathcal{T}_i]^{\mathrm{M}}.$$

Proof. Since for any $i \in I$, $[\mathcal{T}_i]^{\mathrm{M}} \subseteq [T_i]^{\mathrm{M}[G]}$, it is clear that $\bigcap_{i \in I} [\mathcal{T}_i]^{\mathrm{M}}$ is countable, so there exists an enumeration $\langle x_n \colon n < w \rangle \in \mathrm{M}$ where $w \leq \omega$. We show that there is no $x \in \bigcap_{i \in I} [\mathcal{T}_i]^{\mathrm{M}[G]}$ such that, for any $n < w$, $x \neq x_n$. Towards contradiction, assume the contrary, hence, there are $p \in G$ and a name $\dot{x} \in \mathrm{M}^{\mathbb{P}}$ such that:

$$M \models \text{“}\dot{x} \in \bigcap_{i \in I} [\mathcal{T}_i] \wedge \forall n < w (\dot{x} \neq x_n)\text{”}.$$

Now, work in M. By induction on $k < \omega$, define a decreasing sequence $\langle p_k \colon k < \omega \rangle$ in $\mathbb{P}$ with $p_0 \leq p$, and $y \in {}^{\omega}\omega$ such that $p_k \Vdash$ “$\dot{x}(k) = y(k)$” and, whenever $k < w$, there exists $\ell_k < \omega$ such that, $p_k \Vdash$ “$\dot{x}(\ell_k) = y(\ell_k) \neq x_k(\ell_k)$”. Then $p_k \Vdash$ “$y \restriction (k+1) = \dot{x} \restriction (k+1) \in \mathcal{T}_i$” for all $i \in I$, so $y \in \bigcap_{i \in I} [\mathcal{T}_i]^{\mathrm{M}} = \{x_n \colon n < w\}$. However, for $n < w$, $p_n \Vdash$ “$y(\ell_n) \neq x_n(\ell_n)$”, hence $y \neq x_n$, which is a contradiction. $\square_{\text{Lemma 5.3.1}}$

The following reformulation of [She00, Lem. 2.7] gives us conditions to control $\mathrm{cov}(\mathcal{N})$ in finite support iterations in general:

Theorem 5.3.2. *Let* κ, λ *be uncountable cardinals such that* κ *es regular and* $\lambda \geq \kappa$. *Consider a finite support iteration* $\mathbb{P}_\pi = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \pi \rangle$ *of* κ-cc *forcing notions. Assume that*

1. $\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ *is strongly* λ-$\mathcal{CN}$*-unbounded,*

2. $\Vdash_{\mathbb{P}_\pi}$ *"$\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ is strongly κ-$\mathcal{S}$-unbounded"*[3]*,*

3. *for any $\alpha < \pi$, $\Vdash_\alpha$" $\dot{\mathbb{Q}}_\alpha$ contains a dense subset of size $< \lambda$".*

Then, $\Vdash_{\mathbb{P}_\pi}$"$\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ is strongly λ-$\mathcal{CN}$-unbounded". As a consequence, $\Vdash_\pi$"$\mathrm{cov}(\mathcal{N}) \leq \lambda$".

Proof. By induction on $\xi \leq \pi$, we show that $\Vdash_{\mathbb{P}_\xi}$"$\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ is strongly λ-$\mathcal{CN}$-unbounded". So we consider three cases:

1. The initial step $\xi = 0$: in this case $\Vdash_{\mathbb{P}_0}$"$\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ is strongly λ-$\mathcal{CN}$-unbounded" by condition (1).

2. The successor step $\xi + 1$: first, work in $\mathrm{M}^{\mathbb{P}_\xi}$. By the induction hypothesis, we have that $\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ is strongly λ-$\mathcal{CN}$-unbounded. So we must to show that $\dot{\mathbb{Q}}_\xi$ still forces this. Towards contradiction, suppose that there are $q \in \mathbb{Q}_\xi$ and a $\mathbb{Q}_\xi$-name $\dot{x}$ of a real number in ${}^\omega 2$ such that,
$$q \Vdash_{\mathbb{Q}_\xi} \text{"}|\{\alpha < \lambda : \dot{x} \notin N[\bar{a}^\alpha]\}| \geq \lambda\text{"}. \tag{5.3.1}$$
Therefore, by transfinite recursion on $\zeta < \lambda$, we can build two sequences $\langle \alpha_\zeta \colon \zeta < \lambda \rangle$ and $\langle q_{\alpha_\zeta} \colon \zeta < \lambda \rangle$ such that, for any $\zeta < \lambda$,

   - $\alpha_\zeta < \lambda$,
   - $q_{\alpha_\zeta} \leq q$
   - $\alpha_\zeta \notin \{\alpha_\varepsilon \colon \varepsilon < \zeta\}$,
   - $q_{\alpha_\zeta} \Vdash_{\mathbb{Q}_\xi}$"$\dot{x} \notin N[\bar{a}^{\alpha_\zeta}]$".

   As a consequence, defining $C \coloneqq \{\alpha_\zeta \colon \zeta < \lambda\}$, we have that $C \in [\lambda]^\lambda$ and for each $\alpha \in C$, $q_\alpha \Vdash_{\mathbb{Q}_\xi}$"$\dot{x} \notin N[\bar{a}^\alpha]$", hence for any $\alpha \in C$, there exists $m_\alpha < \omega$ and some $q'_\alpha \leq q_\alpha$ such that $q'_\alpha \Vdash_{\mathbb{Q}_\xi}$"$\dot{x} \in \mathrm{Tree}_{m_\alpha}(\bar{a}^\alpha)$".

   Since by condition (3) $\mathbb{Q}_\xi$ contains a dense subset of size $<\lambda$, we can find $C_0 \in [C]^\kappa$ and $q' \in \mathbb{Q}_\xi$ such that, for any $\alpha \in C_0$, $q' \leq q'_\alpha$. On the other hand, as κ is a regular cardinal, we can make a reduction to find some $m < \omega$, and $C_1 \subseteq C_0$ with $|C_1| = \kappa$ such that, for each $\alpha \in C_1$, $\alpha_{m_\alpha} = \alpha$. Therefore, for any $\alpha \in C_1$, we have that $q' \Vdash_{\mathbb{Q}_\xi}$"$\dot{x} \in \mathrm{Tree}_m(\bar{a}^\alpha)$", and therefore
$$q' \Vdash_{\mathbb{Q}_\xi} \text{"}\dot{x} \in \bigcap_{\alpha \in C_1} \mathrm{Tree}_m(\bar{a}^\alpha)\text{"}.$$

   Now, working in $\mathrm{M}^{\mathbb{P}_\xi}[G]$ where G is a $\mathbb{Q}_\xi$-generic over $\mathrm{M}^{\mathbb{P}_\alpha}$ containing q', we have that, by condition (2), $F_1 \coloneqq \bigcap_{\alpha \in C_1} \mathrm{Tree}_m(\bar{a}^\alpha)$ is a closed subset of ${}^\omega 2$ and, since for all $T \in \mathcal{P}$, $|\{\alpha < \lambda : \exists m < \omega([T] \subseteq \mathrm{Tree}_m(\bar{a}^\alpha))\}| < \kappa$, necessarily F_1 does not contains a perfect set. Therefore, by *Cantor-Bendixson theorem* (see Theorem 1.2.5) it is a countable set. Hence, by Lemma 5.3.1, we have that $F_1 \in \mathrm{M}^{\mathbb{P}_\xi}$. Thus $x \coloneqq \dot{x}[G] \in \mathrm{M}^{\mathbb{P}_\xi}$. However, by induction hypothesis, for any $y \in \mathrm{M}^{\mathbb{P}_\xi}$, $|\{\alpha < \lambda \colon y \notin N[\bar{a}^\alpha]\}| < \lambda$. In particular, $|\{\alpha < \lambda : x \notin N[\bar{a}^\alpha]\}| < \lambda$, which contradicts 5.3.1.

[3]This is condition $(**)_{\mathbb{P}}$ in [She00, Lem. 2.7].

3. The limit step: let γ a limit ordinal and consider two cases:

   (a) $\mathrm{cf}(\gamma) < \kappa$: towards contradiction, assume that there are $p \in \mathbb{P}_\gamma$ and a $\mathbb{P}_\gamma$-name of a real number $\dot{x}$ in ${}^\omega 2$, such that:

   $$p \Vdash_\gamma \text{"}|\{\alpha < \lambda \colon \dot{x} \notin N[\bar{a}^\alpha]\}| \geq \lambda\text{"}. \tag{5.3.2}$$

   Like in successor step, we can find $C \in [\lambda]^\lambda$ and $\{p_\alpha \colon \alpha \in C\} \subseteq \mathbb{P}_\gamma$ such that, for any $\alpha \in C$, $p_\alpha \leq p$ and $p_\alpha \Vdash_\gamma$ "$\dot{x} \notin N[\bar{a}^\alpha]$". Since $\mathrm{cf}(\gamma) < \kappa$, there exists a set $L \subseteq \gamma$ cofinal in γ with $|L| < \kappa$.

   So for each $\alpha \in C$, there exists some $\xi_\alpha \in L$, such that $p_\alpha \in \mathbb{P}_{\xi_\alpha}$. On the other hand, since $|C| = \lambda \geq \kappa > |L|$, we can find a set $C_0 \subseteq C$ with $|C_0| = \kappa$ and some $\xi \in L$ such that, for any $\alpha \in C_0$, $p_\alpha \in \mathbb{P}_\xi$, hence $\xi < \gamma$. Again, like in the successor step, there are a set $C_1 \subseteq C_0$ with $|C_1| = \kappa$ and $m < \omega$, such that, for any $\alpha \in C_1$, $p_\alpha \Vdash$ "$\dot{x} \in \mathrm{Tree}_m(\bar{a}^\alpha)$".

   On the other hand, notice that:

   $$|\{\alpha \in C_1 \colon p_\alpha \nVdash_\xi \text{"}|\{\beta \in C_1 \colon p_\beta \in \dot{G}_\xi\}| = \kappa\}| < \kappa. \tag{5.3.3}$$

   Indeed, towards contradiction, assume that there exists $D \subseteq C_1$, such that $|D| = \kappa$ and for any $\alpha \in D$, $p_\alpha \nVdash_\xi$ "$|\{\beta \in C_1 \colon p_\beta \in \dot{G}_\xi\}| = \kappa$", hence for any $\alpha \in D$, we can find a condition $q_\alpha \in \mathbb{P}_\gamma$, such that $q_\alpha \leq p_\alpha$ and $q_\alpha \Vdash_\xi$ "$|\{\beta \in C_1 \colon p_\beta \in \dot{G}_\xi\}| < \kappa$. Therefore, $|\{\beta \in C_1 \colon q_\alpha \parallel p_b\}| < \kappa$. Thereby, by transfinite recursion we can construct an increasing sequence $\{\alpha_\zeta \colon \zeta < \kappa\} \subseteq D$, such that $\{q_{\alpha_\zeta} \colon \zeta < \kappa\}$ is an antichain, which is a contradiction because by Theorem 1.5.55 $\mathbb{P}_\xi$ is κ-cc.

   As a consequence of Equation 5.3.3, there exists an $\alpha \in C_1$ such that

   $$p_\alpha \Vdash_\xi \text{"}|\{\beta \in C_1 \colon p_\beta \in \dot{G}_\xi\}| = \kappa\text{"}.$$

   Let G a $\mathbb{P}_\gamma$-generic filter over M such that $p_\alpha \in G$. Working in $\mathrm{M}[\mathbb{P}_\xi \cap G]$, let

   $$C_2 \coloneqq \{\beta \in C_1 \colon p_\beta \in \mathbb{P}_\xi \cap G\},$$

   so $|C_2| = \kappa$. Therefore, by condition (2), $F_2 \coloneqq \bigcap_{\beta \in C_2} \mathrm{Tree}_m(\bar{a}^\beta)^{\mathrm{M}[G]}$ is a countable closed subset of ${}^\omega 2$.

   Finally, working in $\mathrm{M}[G]$, we have that $F_2 \subseteq \mathrm{M}^{\mathbb{P}_\xi}$ and $x \coloneqq \dot{x}[G] \in F_2$. However, by induction hypothesis, since $\xi < \gamma$, it follows that, for each $x \in {}^\omega 2 \cap \mathrm{M}^{\mathbb{P}_\xi}$, $|\{\alpha < \lambda \colon x \notin N[\bar{a}^\alpha]\}| < \lambda$, which contradicts Equation 5.3.2.

   (b) $\mathrm{cf}(\gamma) \geq \kappa$: let $\dot{x}$ a nice $\mathbb{P}_\gamma$-name of a real number in ${}^\omega 2$. Since by Lemma 1.5.54 $\mathbb{P}_\gamma$ is κ-cc and $\mathrm{cf}(\gamma) \geq \kappa$, there exists $\xi < \gamma$ such that $\dot{x}$ is a $\mathbb{P}_\xi$-name. Therefore, by induction hypothesis, $\Vdash_{\mathbb{P}_\xi}$ "$|\{\alpha < \lambda \colon \dot{x} \notin N[\bar{a}^\alpha]\}| < \lambda$". Thus, $\mathbb{P}_\gamma$ forces the same.

Thus, $\Vdash_{\mathbb{P}_\pi}$"$\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ is strongly λ-$\mathcal{CN}$-unbounded". Therefore, by Theorem 1.3.22 we have that $\mathcal{C}_{[\lambda]^{<\lambda}} \leq_{\mathrm{T}} \mathcal{CN}$ and by Theorem 5.2.5, we conclude that $\Vdash_{\mathbb{P}}$"$\mathcal{C}_{[\lambda]^{<\lambda}} \preceq_{\mathrm{T}} \mathcal{C}_{\mathcal{N}}^{\perp}$". Finally, working in M_π, by Example 1.3.14(2), Lemma 1.3.18(4) and Lemma 1.3.16, in this order, we get:

$$\mathrm{cov}(\mathcal{N}) = \mathfrak{d}(\mathcal{C}_{\mathcal{N}}) = \mathfrak{b}(\mathcal{C}_{\mathcal{N}}^{\perp}) \leq \mathfrak{b}(\mathcal{C}_{[\lambda]^{<\lambda}}) = \mathrm{non}(\mathcal{C}_{[\lambda]^{<\lambda}}) \leq \lambda.$$

Thus, $\Vdash_{\mathbb{P}_\pi}$ "$\mathrm{cov}(\mathcal{N}) = \lambda$". $\square_{\text{Theorem 5.3.2}}$

When iterating with partial random forcing, the conditions (1) and (3) of Theorem 5.3.2 are relatively easy to handle: (3) will be obvious, and we can handle (1) by taking the ground model after adding λ-many Cohen reals. So, it only remains to establish conditions for (2). This is where the iteration structure, using finitely additive measures, plays its part. In particular, we are going to use the properties of the limit function defined in Definition 4.3.22:

Theorem 5.3.3. *Let $\kappa \leq \lambda$ be uncountable cardinals such that κ is regular. Assume that $\mathbb{K}$ is in $\mathcal{K}_1(\kappa, \mathcal{G})$ with length π, where $\mathcal{G}$ is a complete set of guardrails for (π, κ), $\pi \geq \lambda$ and, for $\alpha < \lambda$, $\dot{\mathbb{Q}}_\alpha := \mathbb{C}$ and $Q_{s,\varepsilon} := \{s\}$ for any $s \in \mathbb{C}$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$. Then,*

$$\Vdash_\pi \text{“}\langle \bar{a}^\alpha \colon \alpha < \lambda\rangle \textit{ is strongly } \kappa\text{-}\mathcal{S}\textit{-unbounded”},$$

where each $\bar{a}^\alpha$ is the Cohen real added by $\dot{\mathbb{Q}}_\alpha$ at step α of the iteration.

Proof. Towards contradiction, assume that there are a condition $p \in \mathbb{P}_\pi$, a $\mathbb{P}_\pi$-name $\dot{T}$ of a perfect tree on ${}^{<\omega}2$ and $m < \omega$, such that $p \Vdash_\pi$“$|\dot{E}| \geq \kappa$”, where $\dot{E} := \{\alpha < \lambda \colon [\dot{T}] \subseteq \mathrm{Tree}_m(\bar{a}^\alpha)\}$. To get a contradiction, we are going to build a suitable guardrail for (π, κ). So we split the rest of the proof into two parts: get the guardrail and obtain a contradiction.

Part 1: Build a guardrail $(g, \bar{I}) \in \mathcal{G}$ and a suitable g-uniform Δ-system.
Let $\bar{\varepsilon} \colon \omega \to (0,1)$ such that $\sum_{n<\omega} \bar{\varepsilon}(n) < 1$ and fix $\varepsilon > 0$ such that

$$\varepsilon < 1 - \sum_{n<\omega} \bar{\varepsilon}(n) \tag{5.3.4}$$

By transfinite recursion, we can build sequences $\{p_\xi \colon \xi < \kappa\} \subseteq D_{\bar{\varepsilon}}$ and $\{\alpha_\xi \colon \xi < \kappa\} \subseteq \lambda$, such that:

- for any $\xi < \kappa$, $p_\xi \leq p$,
- $\langle \alpha_\xi \colon \xi < \kappa\rangle$ has no repetitions,
- if $\xi < \kappa$, then $\alpha_\xi \in \mathrm{dom}(p_\xi)$,
- for any $\xi < \kappa$, $p_\xi \Vdash_\pi$“$\alpha_\xi \in \dot{E}$”.

Since each $p_\xi \in D_{\bar{\varepsilon}}$, there exists $n^*_\xi < \omega$ and $\zeta_{\xi,n}$ for all $n < n^*_\xi$, such that

$$p_\xi \restriction \gamma_{n,\xi} \Vdash_{\gamma_{n,\xi}} \text{“}p_\xi(\gamma_{n,\xi}) \in \dot{Q}^{\gamma_{n,\xi}}_{\zeta_{\xi,n},\bar{\varepsilon}(n)}\text{”},$$

where $\mathrm{dom}(p_\xi) = \{\gamma_{n,\xi} \colon n < n^*_\xi\}$ is arranged in decreasing order. To get a uniform Δ-system, we must reorganize the domains: for any $\xi < \kappa$ and $n < n^*_\xi$, let $\alpha_{n,\xi} := \gamma_{n^*_\xi - n - 1}$, hence $\mathrm{dom}(p_\xi) = \{\alpha_{n,\xi} \colon n < n^*_\xi\}$ is arranged in increasing order because, if $n < j < n^*_\xi$, then $n^*_\xi - j - 1 < n^*_\xi - n - 1$ and therefore $\alpha_{n,\xi} < \gamma_{n^*_\xi - n - 1} < \gamma_{n^*_\xi - j - 1} = \alpha_{j,\xi}$.
Using those parameters and the regularity of κ, we can proceed as in the proof of Theorem 4.3.20 to get f, a half guardrail for (π, κ), and $E \in [\kappa]^\kappa$ such that:

1. $\{p_\xi \colon \xi \in E\}$ forms a f-uniform Δ-system with parameters $(\Delta, n^*, \vec{\alpha}, r^*, \varepsilon^*)$, where for any $\xi \in E$, $n^*_\xi = n^*$ and
$$\forall n < n^*(\varepsilon^*(n) = \bar{\varepsilon}(n^* - n - 1)), \tag{5.3.5}$$
2. for any $\xi \in E$, $|p_\xi(\alpha_\xi)| > \max\{m, 10\}$,
3. there exists some $s^* \in \mathbb{C}$ such that, for any $\xi \in E$, $p_\xi(\alpha_\xi) = s^*$ and $s^* = \langle (n_\ell, a_\ell) \colon \ell < m^* \rangle$, hence $\lg(s^*) = m^*$,
4. there is some $c^* < n^*$ such that, for any $\xi \in E$, $\alpha_\xi = \alpha_{c^*, \xi}$, that is, all α_ξ's comes from the same column of the Δ-system.

Notice that, $c^* \notin r^*$ and $m^* > m, 10$.

Now, we construct the interval partition $\bar{I}$: define $j^* := 3n_{m^*-1} + 1$ and, for any $k < \omega$, let $j_k := j^* + k!!$. Consider a sequence of natural numbers $\langle s_k \colon k < \omega \rangle$ such that $s_0 = 0$ and $s_{k+1} - s_k = |[{}^{j_k}2]^{2^{j_k}(1-8^{-m^*})}|$,[4] that is, it is the number of subsets of ${}^{j_k}2$ of size $2^{j_k(1-8^{-m^*})}$, so

$$s_{k+1} - s_k = \binom{2^{j_k}}{2^{j_k}(1 - 8^{-m^*})} = \binom{2^{j_k}}{2^{j_k}8^{-m^*}},$$

and define, for any $k < \omega$, $I_k := [s_k, s_{k+1})$. It is clear that $2^{j_k}8^{-m^*} \notin \{0, 2^{j_k}\}$ and therefore, since $0 < k < n$ implies $n \geq \binom{n}{k}$, we have that $\lim_{k\to\infty} |I_k| = \infty$ and it is an interval partition of ω, hence $\bar{I} := \langle I_k \colon k < \omega \rangle \in \mathcal{I}_\infty$. On the other hand, by the choice of $\langle s_k \colon k < \omega \rangle$, we can enumerate the subsets of ${}^{j_k}2$ whose size is $2^{j_k}(1 - 8^{-m^*})$ as $\{a_\ell \colon \ell \in I_k\}$.

Let $\langle \beta_\ell \colon \ell < \omega \rangle \subseteq E$ be of order type ω. For $k < \omega$ and $\ell \in I_k$, we define a condition $p'_{\beta_\ell} \in \mathbb{P}_\pi$ such that $p'_{\beta_\ell} \leq p_{\beta_\ell}$, $\mathrm{dom}(p'_{\beta_\ell}) = \mathrm{dom}(p_{\beta_\ell})$ and,

$$p'_{\beta_\ell}(\gamma) := \begin{cases} p_{\beta_\ell}(\gamma), & \text{if} \quad \gamma \neq \beta_\ell, \\ \\ s^{*\frown}\langle (j_k, a_\ell) \rangle, & \text{if} \quad \gamma = \beta_\ell. \end{cases}$$

Since $\langle I_k \colon k < \omega \rangle$ is a partition of ω, we really defined p'_{β_ℓ} for any $\ell < \omega$. Notice that,

$$\frac{|a_\ell|}{2^{j_k}} = 1 - 8^{-m^*} > 1 - \frac{1}{7^{m^*}}$$

and therefore, each p'_{β_ℓ} is well-defined.

Notice that, $\bar{p}' := \langle p'_{\beta_\ell} \colon \ell < \omega \rangle$ stills forms a h-uniform Δ-system for some half guardrail h for (π, κ). On the other hand, as $\mathcal{G}$ is a complete set of guardrails for (π, κ), by virtue of Theorem 4.3.21, there exists a half guardrail $g \in \mathcal{G}_0$ such that $\bar{p}'$ is a g-uniform Δ-system with parameters $(\Delta,\ n^*,\ \vec{\alpha},\ r^*,\ \varepsilon^*)$. As a consequence, we have that $(g, \bar{I}) \in \mathcal{G}$.

Part 2: get a contradiction.

[4] $2^{j_k}8^{-m^*}$ is an integer because it equals to $2^{j_k - 3m^*}$ and we have that $j_k > 3m^*$, so the definition of $\langle s_k \colon k < \omega \rangle$ makes sense.

For any $k<\omega$, let $\varrho(k):=\frac{|\ell\in I_k\colon p'_{\beta_\ell}\in\dot{G}|}{|I_k|}$ and $\dot{A}_\varepsilon:=\{k<\omega\colon\varrho(k)>\varepsilon\}$. By Equation 5.3.4 and Equation 5.3.5, we have that:

$$0<\varepsilon<1-\sum_{n<\omega}\bar{\varepsilon}(n)\leq 1-\sum_{n<n^*}\bar{\varepsilon}(n^*-n-1)=1-\sum_{n<n^*}\varepsilon^*(n).$$

Also, as $(g,\bar{I})\in\mathcal{G}$, by Theorem 4.3.24(4), we get that $\lim^g(\bar{p}')\Vdash_\pi$"$\dot{A}_\varepsilon$ is infinite".

Let G be a $\mathbb{P}_\pi$-generic over M with $\lim^g(\bar{p}')\in G$. Working in $\mathrm{M}[G]$, we have that $A:=A_\varepsilon[G]$ is infinite.

Define, for any $k<\omega$, $b_k:=\{\ell\in I_k\colon p'_{\beta_\ell}\in G\}$, hence $\varrho(k)=\frac{|b_k|}{|I_k|}$. Notice that

$$k\in A\Leftrightarrow\varrho(k)>\varepsilon\Leftrightarrow\frac{|b_k|}{|I_k|}>\varepsilon\Leftrightarrow|b_k|>\varepsilon\cdot|I_k|.$$

Now, if $k\in A$, then $\mathcal{L}_{j_k}(T)\subseteq\bigcap_{\ell\in b_k}a_\ell$. Indeed, let $k\in A$ and $\ell\in b_k$. By definition of p'_{β_ℓ}, $p'_{\beta_\ell}(\beta_\ell)=s^*{}^\frown\langle(j_k,a_\ell)\rangle\in G(\beta_\ell)$. Since $\beta_\ell\in E$, it follows that

$$[T]\subseteq\mathrm{Tree}_m(\bar{a}^{\beta_\ell})=\bigcap_{n_j^{\beta_\ell}>m}\{x\in{}^\omega 2\colon x\restriction n_j^{\beta_\ell}\in a_j^{\beta_\ell}\}.$$

On the other hand, since $\lg(s^*)=m^*$, we get

$$\bar{a}^{\beta_\ell}=\langle(n_j^{\beta_\ell},a_j^{\beta_\ell})\colon j<\omega\rangle\supseteq s^*{}^\frown\langle(j_k,a_\ell)\rangle,$$

where $a_j^{\beta_\ell}\subseteq{}^{n_j^{\beta_\ell}}2$, for any $j<\omega$. Therefore,

$$T\subseteq\{x\in{}^\omega 2\colon x\restriction n_{m^*}^{\beta_\ell}\in a_{m^*}^{\beta_\ell}\}=\{x\in{}^\omega 2\colon x\restriction j_k\in a_\ell\}.$$

Thus, $\mathcal{L}_{j_k}(T)\subseteq a_\ell$.

As a consequence, we get:

$$\begin{aligned}|b_k|&=|\{a_\ell\colon\ell\in b_k\}|\\&\leq|\{a_\ell\colon\ell\in I_k,\,\mathcal{L}_{j_k}(T)\subseteq a_\ell\}|\\&\leq\{a\subseteq{}^{j_k}2\colon\mathcal{L}_{j_k}(T)\subseteq a,\,|a|=2^{j_k}(1-8^{-m^*})\}\\&=|\{a\subseteq{}^{j_k}2\setminus(\mathcal{L}_{j_k}(T))\colon|a|=2^{j_k}\cdot 8^{-m^*}\}|\\&=\binom{2^{j_k}-|\mathcal{L}_{j_k}(T)|}{2^{j_k}\cdot 8^{-m^*}}\end{aligned}$$

Consider the function $\mathbf{m}\colon\omega\to\omega$ such that, for any $k<\omega$,

$$\mathbf{m}(k):=\mathbf{m}_k:=\min\{|\mathcal{L}_{j_k}(T)|,\,2^{j_k}\cdot 8^{-m^*}\}.$$

If $k \in A$, then

$$\begin{aligned}
\varepsilon < \frac{|b_k|}{|I_k|} \leq \frac{\binom{2^{j_k}-|\mathcal{L}_{j_k}(T)|}{2^{j_k}\cdot 8^{-m^*}}}{\binom{2^{j_k}}{2^{j_k}\cdot 8^{-m^*}}} \leq \frac{\binom{2^{j_k}-\mathbf{m}_k}{2^{j_k}\cdot 8^{-m^*}}}{\binom{2^{j_k}}{2^{j_k}\cdot 8^{-m^*}}} \\
= \frac{\prod_{i<\mathbf{m}_k}(2^{j_k} - 2^{j_k}\cdot 8^{-m^*} - i)}{\prod_{i<\mathbf{m}_k}(2^{j_k}-i)} \\
= \prod_{i<\mathbf{m}_k}\left(1 - \frac{2^{j_k}\cdot 8^{-m^*}}{2^{j_k}-i}\right) \\
\leq \prod_{i<\mathbf{m}_k}\left(1 - \frac{2^{j_k}\cdot 8^{-m^*}}{2^{j_k}}\right) \\
= (1-8^{-m^*})^{\mathbf{m}_k}.
\end{aligned}$$

As a consequence, we get that, for any $k < \omega$,

$$\mathbf{m}_k < \frac{\log(\frac{1}{\varepsilon})}{\log\left(\frac{1}{1-8^{m^*}}\right)},$$

that is, $\mathbf{m}$ is a bounded function.

Now, we can write $A = A_0 \cup A_1$, where the union is disjoint,

$$A_0 \coloneqq \{k \in A \colon \mathbf{m}_k = |\mathcal{L}_{j_k}(T)|\} \text{ and } A_1 \coloneqq \{k \in A \colon \mathbf{m}_k = {}^{j_k}2\cdot 8^{-m^*}\}.$$

Since $\lim_{k\to\infty} 2^{j_k}\cdot 8^{-m^*} = \infty$, A_1 is finite, because otherwise $\mathbf{m}$ cannot be bounded. Therefore, as A is infinite, we have that A_0 is infinite, that is, there are infinitely many values $k < \omega$ for which $|\mathcal{L}_{j_k}(T)|$ is bounded, which contradicts Lemma 1.2.4 because T is a perfect tree. $\square_{\text{Theorem 5.3.3}}$

Now, we prove a similar result that will allow us to control $\mathfrak{b}$. Again, here the structure of the iteration using finitely additive measures plays a fundamental role.

Theorem 5.3.4. *Let $\kappa \leq \lambda$ be uncountable cardinals such that κ is regular. Assume that $\mathbb{K}$ is in $\mathcal{K}_1(\kappa, \mathcal{G})$ has length π, where $\mathcal{G}$ is a complete set of guardrails for $\mathbb{K}$, $\pi \geq \lambda$, for $\alpha < \lambda$, $\dot{\mathbb{Q}}_\alpha \coloneqq \mathbb{C}$ and $Q_{s,\varepsilon} \coloneqq \{s\}$ for all $s \in \mathbb{C}$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$. Then,*

$$\Vdash_\pi \text{“}\langle \bar{n}^\alpha \colon \alpha < \lambda\rangle \textit{ is strongly } \kappa\text{-}{}^\omega\omega\textit{-unbounded},$$

where each $\bar{a}^\alpha = \langle (n^\alpha_\ell, a^\alpha_\ell) \colon \ell < \omega$ is the Cohen real added by $\dot{\mathbb{Q}}_\alpha$ at the step α of the iteration, and $\bar{n}^\alpha \coloneqq \langle n^\alpha_\ell \colon \ell < \omega\rangle$. As a consequence, $\Vdash_\pi$“$\mathfrak{b} \leq \kappa$”.

Proof. The first part of this proof, is a *déjà vu* of the first part of the proof of the previous theorem: towards contradiction, assume that there are a nice $\mathbb{P}_\pi$-name $\dot{x}$ of a real number in ${}^\omega\omega$ and a condition $p \in \mathbb{P}_\pi$ such that $p \Vdash_\pi$“$|\{\alpha < \lambda \colon \bar{n}^\alpha \leq^* \dot{x}\}| \geq \kappa$”. To get a contradiction, we are going to build a suitable guardrail for (π, κ).

Let $\bar{\varepsilon} \colon \omega \to (0,1)$ such that $\sum_{n<\omega} \bar{\varepsilon}(n) < 1$ and fix $\varepsilon > 0$ such that

$$\varepsilon < 1 - \sum_{n<\omega} \bar{\varepsilon}(n) \tag{5.3.6}$$

By the regularity of κ transfinite recursion, we can build sequences $\{p_\xi\colon \xi < \kappa\} \subseteq D_{\bar{\varepsilon}}$ and $\{\alpha_\xi\colon \xi < \kappa\} \subseteq \pi$ and $m < \omega$ such that:

- for any $\xi < \kappa$, $p_\xi \leq p$,
- $\langle \alpha_\xi\colon \xi < \kappa\rangle$ has no repetitions,
- if $\xi < \kappa$, then $\alpha_\xi \in \mathrm{dom}(p_\xi)$,
- for any $\xi < \kappa$, $p_\xi \Vdash_\pi$"$\forall j \geq m\, (n_j^\alpha \leq \dot{x}(j))$".

Since each $p_\xi \in D_{\bar{\varepsilon}}$, there exists $n_\xi^* < \omega$ and $\zeta_{\xi,n}$ for all $n < n_\xi^*$, such that

$$p_\xi \restriction \gamma_{n,\xi} \Vdash_{\gamma_{n,\xi}} \text{“}p_\xi(\gamma_{n,\xi}) \in \dot{Q}^{\gamma_{n,\xi}}_{\zeta_{\xi,n},\bar{\varepsilon}(n)}\text{”},$$

where $\mathrm{dom}(p_\xi) = \{\gamma_{n,\xi}\colon n < n_\xi^*\}$ is arranged in decreasing order. To get a uniform Δ-system, we must reorganize the domains: for any $\xi < \kappa$ and $n < n_\xi^*$, let $\alpha_{n,\xi} \coloneqq \gamma_{n_\xi^*-n-1}$, hence $\mathrm{dom}(p_\xi) = \{\alpha_{n,\xi}\colon n < n_\xi^*\}$ is arranged in increasing order because, if $n < j < n_\xi^*$, then $n_\xi^* - j - 1 < n_\xi^* - n - 1$ and therefore, $\alpha_{n,\xi} < \gamma_{n_\xi^*-n-1} < \gamma_{n_\xi^*-j-1} = \alpha_{j,\xi}$.

Using those parameters and the regularity of κ, we can proceed as in the proof of Theorem 4.3.20 to get f, a half guardrail for (π, κ), and $E \in [\kappa]^\kappa$ such that:

1. $\{p_\xi\colon \xi \in E\}$ forms a f-uniform Δ-system with parameters $(\Delta, n^*, \vec{\alpha}, r^*, \varepsilon^*)$, where for any $\xi \in E$, $n_\xi^* = n^*$ and
$$\forall n < n^*(\varepsilon^*(n) = \bar{\varepsilon}(n^* - n - 1)). \tag{5.3.7}$$
2. for any $\xi \in E$, $|p_\xi(\alpha_\xi)| > m$,
3. there exists some $t \in {}^{<\omega}\omega$ such that, for any $\xi \in E$, $p_\xi(\alpha_\xi) = t$, where $\lg(t) = m^*$,
4. there is some $c^* < n^*$ such that, for any $\xi \in E$, $\alpha_\xi = \alpha_{c^*,\xi}$, that is, all α_ξ's comes from the same column of the Δ-system.

Notice that $c^* \notin r^*$ and $m^* > m$.

Choose an increasing sequence of natural numbers $\langle k_\ell\colon \ell < \omega\rangle$ such that, for any $\ell < \omega$, we have that $\ell < k_\ell$, $2^{k_\ell} > 7^{m^*}$ and $k_\ell > n_{m^*-1}$. Also, consider a sequence $\langle a_\ell\colon \ell < \omega\rangle$ such that, for any $\ell < \omega$, $a_\ell \subseteq {}^{k_\ell}2$ and,

$$\frac{|a_\ell|}{2^{k_\ell}} > 1 - \frac{1}{7^{m^*}}. \tag{5.3.8}$$

Let $\langle \beta_\ell\colon \ell < \omega\rangle \subseteq E$ be of order type ω. For any $\ell < \omega$, we define a condition $p'_{\beta_\ell} \in \mathbb{P}_\pi$ such that $p'_{\beta_\ell} \leq p_{\beta_\ell}$, $\mathrm{dom}(p'_{\beta_\ell}) = \mathrm{dom}(p_{\beta_\ell})$ and,

$$p'_{\beta_\ell}(\gamma) \coloneqq \begin{cases} p_{\beta_\ell}(\gamma), & \text{if} \quad \gamma \neq \beta_\ell, \\ t^\frown\langle (k_\ell, a_\ell)\rangle, & \text{if} \quad \gamma = \beta_\ell. \end{cases}$$

This is well-defined by virtue of Equation 5.3.8. Notice that, we can find a half guardrail h for (π, κ), such that $\bar{p}' := \langle p'_{\beta_\ell} : \ell < \omega \rangle$ is a h-uniform Δ-system with parameters $(\Delta, n^*, \vec{\alpha}, r^*, \varepsilon^*)$. On the other hand, as $\mathcal{G}$ is a complete set of guardrails for (π, χ), by virtue of Theorem 4.3.21, there exists a half guardrail $g \in \mathcal{G}_0$ such that $\bar{p}'$ is a g-uniform Δ-system with parameters $(\Delta,\ n^*,\ \vec{\alpha},\ r^*,\ \varepsilon^*)$. Now, pick any $\bar{I} \in \mathcal{I}_\infty$, hence $(g, \bar{I}) \in \mathcal{G}$.

For any $k < \omega$, let $\varrho(k) := \frac{|\ell \in I_k : p'_{\beta_\ell} \in \dot{G}|}{|I_k|}$ and $\dot{A}_\varepsilon := \{k < \omega \colon \varrho(k) > \varepsilon\}$. By Equation 5.3.6 and Equation 5.3.7, we have that:

$$0 < \varepsilon < 1 - \sum_{n<\omega} \bar{\varepsilon}(n) \leq 1 - \sum_{n<n^*} \bar{\varepsilon}(n^* - n - 1) = 1 - \sum_{n<n^*} \varepsilon^*(n).$$

Also, as $(g, \bar{I}) \in \mathcal{G}$, by Theorem 4.3.24(4), we get that $\lim^g(\bar{p}') \Vdash_\pi$“$\dot{A}_\varepsilon$ is infinite”.

Let G be a $\mathbb{P}_\pi$-generic over M with $\lim^g(\bar{p}) \in G$. Working in $\mathrm{M}[G]$, we have that $A := A_\varepsilon[G]$ is infinite.

Now, let $\ell < \omega$ such that $p'_{\beta_\ell} \in G$, hence $p'_{\beta_\ell}(\beta_\ell) \in G(\beta)$. Assume that $\langle n_j^{\beta_\ell}, a_j^{\beta_\ell} \colon j < \omega \rangle$ is the Cohen real added by the iteration at the step β_ℓ, hence we have that $n_{m^*}^{\beta_\ell} = p'_{\beta_\ell}(\beta_\ell) = k_\ell$. On the other hand, we know that, for any $j \geq m^*$, $\leq x(j)$. Thereby, we have that $\ell < k_\ell = n_{m^*}^{\beta_\ell} \leq x(m^*)$, hence $\ell < x(m^*)$. However, since A is infinite, it follows that $\{\ell < \omega \colon \ell < x(m^*)\}$ is also infinite, which is a contradiction.

Since $\langle \bar{n}^\alpha \colon \alpha < \kappa \rangle$ is strongly κ-${}^\omega\omega$-unbounded, by Theorem 1.3.22, we have that $\mathcal{C}_{[\lambda]^{<\kappa}} \preceq_{\mathrm{T}} {}^\omega\omega$. Finally, using $\lambda \geq \kappa$, by Lemma 1.3.16, it follows that:

$$\mathfrak{b} = \mathfrak{b}({}^\omega\omega) \leq \mathfrak{b}(\mathcal{C}_{[\lambda]^{<\kappa}}) = \mathfrak{b}([\lambda]^{<\kappa}) = \mathrm{non}([\lambda]^{<\kappa}) = \kappa. \qquad \square_{\text{Theorem 5.3.4}}$$

5.4 The last parameter of the iteration

We have already said that, to deal with $\mathrm{cov}(\mathcal{N})$, we are going to iterate with restricted random forcing. On the other hand, to deal with $\mathfrak{b}$, we are going to use small Hechler forcing, which although it is not σ-FAM-linked, we know by Example 4.2.14 that it is μ -FAM-linked for some cardinal $\mu < \kappa$. Also, for technical reasons, the first part of the iteration will be with Cohen forcing, which we know, again from Example 4.2.14, to be σ -FAM-linked. We also said that the existence of the complete set of guardrails is given by virtue of the *Engelking-Karłowizc theorem* (see Theorem 4.3.10 and Corollary 4.3.11). Furthermore, the existence of finitely additive measures is guaranteed by the extension theorems (see Theorem 4.3.18 and Theorem 4.3.16). So, we only have to define one parameter to be able to build the iteration: the forcing notions $\mathbb{P}^-_\alpha$. For this, there are at least two alternatives: on the one hand, we could use the *history of conditions* in iterations (see [GKMS21, Def. 4.7]). However, this would involve introducing its definition, calculating some cardinalities, and showing that it satisfies what we need. So, we lean towards using elementary substructures:

Theorem 5.4.1. *Let* $\mathbb{P}$ *be a* ccc forcing notion, $Q \subseteq \mathbb{P}$ and $\langle \dot{F}_i \colon i \in I \rangle$ a sequence of $\mathbb{P}$-names such that, for any $i \in I$, $\Vdash_\mathbb{P}$“$\dot{F}_i \colon \mathcal{P}(\omega) \to \mathbb{R}$”. Then, there exists another forcing notion $\mathbb{P}^-$ such that:

1. $\mathbb{P}^- \lessdot \mathbb{P}$,

2. $Q \subseteq \mathbb{P}^-$,

3. $|\mathbb{P}^-| \leq \max\{2, |Q|, |I|\}^{\aleph_0}$,

4. For any $i \in I$, $\Vdash_{\mathbb{P}}$“$\dot{F}_i \restriction \mathcal{P}(\omega) \cap \mathrm{M}^{\mathbb{P}^-} \in \mathrm{M}^{\mathbb{P}^-}$.

***Proof*.** For any $\dot{x} \in \mathrm{nice}_{\mathbb{P}}(\mathcal{P}(\omega))$ and any $i \in I$, define $E_i(\dot{x})$ as a nice $\mathbb{P}$-name of $F_i(\dot{x})$, which is possible by virtue of Theorem 1.5.34. For a large enough regular cardinal χ, we can use a generalization of [Kun11, Lem. III.8.4] to find a model $N \prec \mathcal{H}(\chi)$ such that ${}^{\omega}N \subseteq N$, $\mathbb{P}$, $Q \in N$, $Q \cup \{\dot{F}_i \colon i \in I\} \subseteq N$, N is closed under E_i for any $i \in I$ and $|N| \leq \{2, |Q|, |I|\}^{\aleph_0}$. We define $\mathbb{P}^- \coloneqq \mathbb{P} \cap N$ endowed with $\leq_{\mathbb{P}} \restriction \mathbb{P}^-$. We show that it works:

1. It is clear that, for any $p, p' \in \mathbb{P}^-$, if $p \leq_{\mathbb{P}^-} p'$, then $p \leq_{\mathbb{P}} p'$ and if $p \perp_{\mathbb{P}^-} p'$, then $p \perp_{\mathbb{P}} p'$. Therefore, by Definition 1.5.12 to prove that $\mathbb{P}^-$ is a complete subposet of $\mathbb{P}$, it is enough to show that any maximal antichain in $\mathbb{P}^-$ is a maximal antichain in $\mathbb{P}$. Let $A \subseteq \mathbb{P}^-$ a maximal antichain, hence it is clear that $A \subseteq \mathbb{P}$ is an antichain in $\mathbb{P}$, and therefore, since $\mathbb{P}$ is ccc, it follows that $|A| \leq \aleph_0$. As a consequence, $A \in N$, because $A \subseteq N$ and N is closed under countable sequences. Now, we have that:

$$\begin{aligned} A \text{ is maximal in } \mathbb{P}^- &\Leftrightarrow \forall p \in \mathbb{P} \cap N\, \exists q \in A (q \parallel_{\mathbb{P}^-} p) \\ &\Leftrightarrow N \models \text{“}\forall p \in \mathbb{P}\, \exists q \in A (q \parallel_{\mathbb{P}} p) \\ &\Leftrightarrow \mathcal{H}(\chi) \models \text{“}\forall p \in \mathbb{P}\, \exists q \in A (q \parallel_{\mathbb{P}} p) \\ &\Leftrightarrow \forall p \in \mathbb{P}\, \exists q \in A (q \parallel_{\mathbb{P}} p) \\ &\Leftrightarrow A \text{ is maximal in } \mathbb{P}. \end{aligned}$$

   Thus, A is a maximal antichain in $\mathbb{P}$, therefore $\mathbb{P}^- \lessdot \mathbb{P}$.

2. Since $Q \subseteq \mathbb{P}$ and $Q \subseteq N$, we have that $Q \subseteq \mathbb{P}^-$.

3. $|\mathbb{P}^-| = |\mathbb{P} \cap N| \leq \max\{2, Q, |I|\}^{\aleph_0}$.

4. For any $i \in I$, consider the $\mathbb{P}$-name

$$\tau_i \coloneqq \{(\mathrm{op}(\dot{x}, E_i(\dot{x})), p) \colon p \in \mathbb{P}^- \wedge \dot{x} \in N \cap \mathrm{nice}_{\mathbb{P}}(\mathcal{P}(\omega))\}.$$

   Notice that, if $\dot{x} \in N$, then $E_i(\dot{x}) \in N$ and, since all their components are in N, we have that $\dot{x}$ and $E_i(\dot{x})$ are $\mathbb{P}^-$-names. As a consequence, for any $i \in I$, τ_i is also a $\mathbb{P}^-$-name, and it is clear that $\Vdash_{\mathbb{P}}$“$\tau_i = F_i \restriction \mathcal{P}(\omega) \cap \mathrm{M}^{\mathbb{P}^-}$”, that is, $\Vdash_{\mathbb{P}}$“$F_i \restriction \mathcal{P}(\omega) \cap \mathrm{M}^{\mathbb{P}^-} \in \mathrm{M}^{\mathbb{P}^-}$”. $\square_{\text{Theorem 5.4.1}}$

5.5 Increasing $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$: the book-keeping idea

In this section we will construct the iteration that will allow us to force the consistency of $\mathrm{cov}(\mathcal{N})$ with countable cofinality, and in the next section (see Theorem 5.6.1) we will prove that the this iteration is indeed adequate.

Assume that κ, λ and χ are cardinals such that $\kappa < \lambda < \chi^{<\lambda} = \chi \leq 2^\kappa$, κ is regular and λ is $\aleph_1$-inaccessible. Our goal is to prove the consistency of $\operatorname{cov}(\mathcal{N}) = \lambda$ and $\mathfrak{b} = \kappa$. So, we are going to build an iteration $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$ of length χ, where $\mathcal{G}$ is a complete set of guardrails for (χ, κ).

The idea to build the iteration is the following. To satisfy the hypotheses of Theorem 5.3.3, Theorem 5.3.4 and obtain $\operatorname{cov}(\mathcal{N}) \leq \lambda$ and $\mathfrak{b} \leq \kappa$, we are going to add λ-many Cohen reals, so in the first λ steps of the iteration we are going to use Cohen forcing. Then we split the iteration into two parts: on the one hand, to deal with $\operatorname{cov}(\mathcal{N}) \geq \lambda$ we are going to iterate, using *book-keeping* arguments, with restricted random forcing and, on the other hand, to deal with $\mathfrak{b} \geq \kappa$ we are going to iterate, again using *book-keeping* arguments, with restricted Hechler forcing. We can build the iteration in $\mathcal{K}_1(\kappa, \mathcal{G})$ thanks to the extension theorems Theorem 4.3.16 and Theorem 4.3.18. It is important to note that in our hypotheses about the cardinals we are not requiring that λ is regular, so this way of iterating is a very powerful method that will allow us not only to prove the consistency of $\operatorname{cov}(\mathcal{N})$ with countable cofinality, but also, to obtain separations in Cichoń's diagram with $\operatorname{cov}(\mathcal{N})$ singular.

Construction 5.5.1. Let κ, λ and χ be uncountable cardinals such that λ is $\aleph_1$-inaccessible, κ is regular, and $\kappa < \lambda < \chi^{<\lambda} = \chi \leq 2^\kappa$. By Corollary 4.3.11, there exists a complete set of guardrails $\mathcal{G}$ for (χ, κ), such that $|\mathcal{G}| \leq \kappa^{\aleph_0}$. Let B, S be sets such that, $\chi \setminus \lambda = B \cup S$, $B \cap S = \emptyset$ and $\operatorname{otp}(B) = \operatorname{otp}(S) = \chi$. Fix bijections $\mathbf{b}_S \colon S \to \chi \times \chi$ and $\mathbf{b}_B \colon B \to \chi \times \chi$ such that $\mathbf{b}_S(\xi) = (\alpha, \beta)$ implies $\alpha \leq \xi$, (likewise for $\mathbf{b}_B$), which we call *book-keeping functions*. By transfinite recursion on χ, we are going to define $\mathbb{K} \in \mathcal{K}_1(\kappa, \mathcal{G})$ of length χ, as follows:

1. The initial step: $\mathbb{P}_0 \coloneqq \{\emptyset\}$.
2. The successor step: assume that we have constructed

$$\mathbb{K}_\xi = \langle \mathbb{P}_\beta,\, \dot{\mathbb{Q}}_\alpha,\, \mathbb{P}^-_\alpha,\, \vec{Q}_\alpha,\, \theta_\alpha,\, \vec{\Xi}_\beta \colon \alpha < \xi,\, \beta \leq \xi \rangle \in \mathcal{K}_1(\kappa, \mathcal{G} \restriction \xi),$$

and consider three cases:

(a) $\xi < \lambda$: define $\dot{\mathbb{Q}}_\xi \coloneqq \mathbb{C}$, $\mathbb{P}_{\xi+1} \coloneqq \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, $\mathbb{P}^-_\xi \coloneqq \mathbb{P}_\xi$, $\theta_\xi \coloneqq \aleph_0$ and, for any $s \in \mathbb{C}$ and $\varepsilon \in (0,1)_\mathbb{Q}$, $\dot{Q}^\xi_{s,\varepsilon} \coloneqq \{s\}$. So it is clear that $\mathbb{K} = \langle \mathbb{P}_\alpha,\, \dot{\mathbb{Q}}_\alpha,\, \mathbb{P}^-_\alpha,\, \vec{Q}_\alpha,\, \theta_\alpha \colon \alpha < \xi + 1 \rangle \in \mathcal{K}_0(\kappa)$ and by Theorem 4.3.16, we can find a sequence of names of finitely additive measures $\vec{\Xi}_{\xi+1} \coloneqq \langle \dot{\Xi}^g_{\xi+1} \colon g \in \mathcal{G} \rangle$, such that:

$$\mathbb{K}_{\xi+1} \coloneqq \langle \mathbb{P}_\beta,\, \dot{\mathbb{Q}}_\alpha,\, \mathbb{P}^-_\alpha,\, \vec{Q}_\alpha,\, \theta_\alpha,\, \vec{\Xi}_\beta \colon \alpha < \xi + 1,\, \beta \leq \xi + 1 \rangle \in \mathcal{K}_1(\kappa, \mathcal{G} \restriction \xi + 1).$$

(b) $\xi \in S$: By Lemma 1.5.52 and induction hypothesis, we know that $|\mathbb{P}_\xi| \leq \chi$, hence by counting nice names of reals numbers, we have that $\Vdash_\xi$"$\mathfrak{c} \leq \chi$". On the other hand, as $\chi^{<\kappa} = \chi$, it follows that $\Vdash_\xi$"$|[{}^\omega\omega]^{<\kappa}| \leq \chi$". By Lemma 1.5.36, we can enumerate $\langle \dot{S}_\xi \rangle_{\mathbb{P}_\xi} = \langle \dot{E}_{\xi,\beta} \colon \beta < \chi \rangle$, where $\dot{S}_\xi$ is a $\mathbb{P}_\xi$-name of $[{}^\omega\omega]^{<\kappa}$. We can assume that we have this numeration for any $\gamma \leq \xi$. Now, define $\dot{F}_\xi \coloneqq \dot{E}_{\mathbf{b}_S(\xi)}$ and $\dot{N}_\xi$ as a $\mathbb{P}_\xi$-name of a transitive model of ZFC of size some $\theta_\zeta < \kappa$ (decided in the ground model) such that, $\Vdash_\xi$ "$\dot{F}_\xi \subseteq \dot{N}_\xi$". Finally, we define $\dot{\mathbb{Q}}_\alpha \coloneqq \mathbb{D}^{\dot{N}_\alpha}$, $\mathbb{P}_{\xi+1} \coloneqq \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, $\mathbb{P}^-_\xi \coloneqq \mathbb{P}_\xi$, for any $f \in \mathbb{D}^{\dot{N}_\xi}$ and $\varepsilon \in (0,1)_\mathbb{Q}$, $\dot{Q}^\xi_{(s,f),\varepsilon} \coloneqq \{(s,f)\}$ and $\vec{\Xi}_\xi \coloneqq \langle \dot{\Xi}^g_\xi \colon g \in \mathcal{G} \rangle$ as in the conclusion of Theorem 4.3.16. As a consequence, we have that,

$$\mathbb{K}_{\xi+1} \coloneqq \langle \mathbb{P}_\beta,\, \dot{\mathbb{Q}}_\alpha,\, \mathbb{P}^-_\alpha,\, \vec{Q}_\alpha,\, \theta_\alpha,\, \vec{\Xi}_\beta \colon \alpha < \xi + 1,\, \beta \leq \xi + 1 \rangle \in \mathcal{K}_1(\kappa, \mathcal{G} \restriction \xi + 1).$$

(c) $\xi \in B$: By Lemma 1.5.52 and the induction hypothesis, we know that $|\mathbb{P}_\xi| \leq \chi$ and, by counting nice names of real numbers, we have that, $\Vdash_\xi$"$\mathfrak{c} \leq \chi = \chi^{<\lambda}$"[5]. So we can enumerate all collections of size $< \lambda$ of nice $\mathbb{P}_\xi$-names of members of Ω as $\{F_{\xi,\beta} \colon \beta < \chi\}$. Let $F := F_{\mathfrak{b}_B(\xi)}$. Notice that, as all elements of F are nice $\mathbb{P}_\xi$-names, if $\dot{a} \in F$, then it depends on some countable sequence of maximal antichains $\langle A^{\dot{a}}_k \colon k < \omega\rangle$ in $\mathbb{P}_\gamma$ (see Subsection 1.5.7), hence $|Q| < \lambda$, where

$$Q := \bigcup_{\dot{a}\in F} \left(\bigcup_{k<\omega} A^{\dot{a}}_k \right).$$

Thereby, we can apply Theorem 5.4.1 to find a forcing notion $\mathbb{P}^-_\xi \subsetdot \mathbb{P}_\xi$ containing Q such that, $|\mathbb{P}^-_\xi| \leq \max\{2, |Q|, |\mathcal{G} \restriction \xi|\}^{\aleph_0}$ and, for any $g \in \mathcal{G} \restriction \xi$, $\Vdash_\xi$"$(\dot{\Xi}^g_\xi)^- \in \mathrm{M}^{\mathbb{P}^-_\xi}$". Notice that, $|\mathbb{P}^-_\xi| < \lambda$ because $|\mathcal{G} \restriction \xi| < \kappa^{\aleph_0} < \lambda$ and $|Q| < \lambda$, and λ is $\aleph_1$-inaccessible.

Finally, let $\dot{\mathbb{Q}}_\xi := \mathbb{B}^{\mathrm{M}^{\mathbb{P}^-_\xi}}$, $\theta_\xi := \aleph_0$, $\mathbb{P}_{\xi+1} := \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, for any $t \in {}^{<\omega}2$ and $\varepsilon \in (0,1)_\mathbb{Q}$, $\dot{Q}^\xi_{t,\varepsilon}$ is as in the proof of Theorem 4.2.18, and $\vec{\Xi}_\xi := \langle \dot{\Xi}^g_\xi \colon g \in \mathcal{G}\rangle$ is as in the conclusion of Theorem 4.3.16. As a consequence, we have that,

$$\mathbb{K}_{\xi+1} := \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha, \mathbb{P}^-_\alpha, \vec{Q}_\alpha, \theta_\alpha, \vec{\Xi}_\beta \colon \alpha < \xi+1, \beta \leq \xi+1\rangle \in \mathcal{K}_1(\kappa, \mathcal{G} \restriction \xi+1)$$

3. The limit step limit: assume ξ is a limit ordinal and that we have defined $\mathbb{K}_\gamma$ for any $\gamma < \xi$. Notice that, for define $\mathbb{K}_\xi$ it is enough to define $\mathbb{P}_\xi$ and the sequence of names of finitely additive measures. As usual in finite support iterations, define $\mathbb{P}_\xi := \operatorname{limdir}_{\gamma<\xi}\mathbb{P}_\gamma$. On the other hand, the existence of finitely additive measures sequences is given by virtue of the induction hypothesis and Theorem 4.3.18. As a consequence, we have that:

$$\mathbb{K}_\xi := \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha, \mathbb{P}^-_\alpha, \vec{Q}_\alpha, \theta_\alpha, \vec{\Xi}_\beta \colon \alpha < \xi, \beta \leq \xi\rangle \in \mathcal{K}_1(\kappa, \mathcal{G} \restriction \xi)$$

Finally, define $\mathbb{K} := \mathbb{K}_\chi$.

$\mathbb{K}$ is our candidate to force $\mathfrak{b} = \kappa$ and $\operatorname{cov}(\mathcal{N}) = \lambda$, under the hypothesis of Construction 5.5.1: Notice that, by the construction of $\mathbb{K}$, since we can partition $\chi \setminus \lambda$ in different ways and use other notions of forcing, there are many alternatives to define the iteration to force different things. For example in the next section, we mention how we can build iterations to force, for instance, MA_κ or $\operatorname{add}(\mathcal{N}) = \theta$ for some regular cardinal $\theta \leq \kappa$.

5.6 Consistency of $\operatorname{cov}(\mathcal{N})$ with countable cofinality

Now, all that remains is to put together the puzzle: we are going to prove that $\mathbb{K}$ from Construction 5.5.1, allows forcing $\operatorname{cov}(\mathcal{N}) = \lambda$ and $\mathfrak{b} = \kappa$, without any assumption on the cofinality of λ.

[5]For this reason, we need the condition $\chi = \chi^{<\lambda}$ as a hypothesis.

Theorem 5.6.1. *Let* κ, λ *and* χ *be uncountable cardinals such that,* λ *is* $\aleph_1$*-inaccessible,* κ *is regular, and* $\kappa < \lambda < \chi^{<\lambda} = \chi \leq 2^\kappa$*. Then, there exists a* κ-Fr-*Knaster and* ccc *forcing notion* $\mathbb{P}$ *such that* $\Vdash_{\mathbb{P}}$ "$\mathrm{cov}(\mathcal{N}) = \lambda$ *and* $\mathfrak{b} = \kappa$".

Proof. Consider $\mathbb{K}$ as in Construction 5.5.1 and let $\mathbb{P} := \mathbb{P}_\chi := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \chi \rangle$. $\mathbb{P}$ is ccc because we are iterating with ccc forcing notions, Also, $\mathbb{P}$ is κ-Fr-Knaster by Theorem 4.3.25. Now, we prove that $\mathbb{P}$ forces $\mathrm{cov}(\mathcal{N}) = \lambda$ and $\mathfrak{b} = \kappa$:

1. $\mathfrak{b} = \kappa$:

   (a) $\mathfrak{b} \geq \kappa$: By Lemma 1.3.15, we must to prove that, in M_χ, any $F \subseteq {}^\omega\omega$ of size $< \kappa$ is bounded and, by Theorem 1.5.42, it is enough to prove that any $F \in [{}^\omega\omega]^{<\kappa}$ is a subset of N_α for some $\alpha < \kappa$. For this, working in M_χ, assume that $F \subseteq {}^\omega\omega$ has size $< \kappa$. Since $\mathbb{P}_\lambda$ is ccc and it is clear that $\chi^{<\kappa} = \chi$, we have that, there exists some $\xi < \lambda$, such that $F \in \mathrm{M}_\xi$. As a consequence, by the iteration construction, $F = E_{\xi,\beta}$ for some $\beta < \lambda$, hence $F = F_\alpha$, where $\alpha := \mathbf{b}_S^{-1}(\xi, \beta) \geq \xi$. So we are done.

   (b) $\mathfrak{b} \leq \kappa$: since $\kappa \leq \lambda$, and by the definition of $\mathbb{K}$, by Theorem 5.3.4 we have that, in M_λ, $\mathfrak{b} \leq \kappa$.

2. $\mathrm{cov}(\mathcal{N}) = \lambda$:

   (a) $\mathrm{cov}(\mathcal{N}) \geq \lambda$: by Lemma 1.3.4, it is enough to show that no family of nice $\mathbb{P}_\chi$-names of members of Ω of size $< \lambda$ cover ${}^\omega 2$. So, consider F as one of such families. For any $\dot{\bar{a}} \in F$, we can find $\alpha_{\dot{\bar{a}}} < \chi$ such that $\dot{\bar{a}}$ is a $\mathbb{P}_{\alpha_{\dot{\bar{a}}}}$-name. Since $\chi^{<\lambda} = \chi$, we have that $\mathrm{cf}(\chi) \geq \lambda$, hence there exists some $\xi < \chi$ such that, for any $\dot{\bar{a}} \in F$, $\alpha_{\dot{\bar{a}}} < \xi$, hence any $\dot{\bar{a}} \in F$ is a $\mathbb{P}_\xi$-name. Therefore, by the iteration construction, we can find some $\beta < \chi$, such that $F = F_{\xi,\beta}$. Let $\alpha \in B$ such that $\mathbf{b}_B(\alpha) = (\xi, \beta)$, hence $F_{\mathbf{b}_B(\alpha)}$ is a collection of $\mathbb{P}^-_\alpha$-names. Consider $\dot{r}_\alpha$ as the random real added by $\mathbb{P}^-_\alpha$ at the step α of the iteration, then by Theorem 5.2.6, we have that, for any $\dot{\bar{a}} \in F$, $\Vdash_{\mathbb{P}^-_\alpha}$ "$\dot{r}_\alpha \notin N[\dot{\bar{a}}]$", that is, F does not cover ${}^\omega 2 \cap \mathrm{M}^{\mathbb{P}^-_\alpha}$, and as $\mathbb{P}^-_\alpha \lessdot \mathbb{P}_\alpha$, it follows that, it does nos cover ${}^\omega 2 \cap \mathrm{M}^{\mathbb{P}_\alpha}$. As a consequence, $\Vdash_{\mathbb{P}}$"$\mathrm{cov}(\mathcal{N}) \geq \lambda$".

   (b) $\mathrm{cov}(\mathcal{N}) \leq \lambda$: it is enough to prove that, $\Vdash_\lambda$ "$\mathrm{cov}(\mathcal{N}) \leq \lambda$". For this, we must verify the hypothesis of Theorem 5.3.2.

      1. By Theorem 1.5.53, we know that $\mathbb{P}_\lambda \cong \prod^{-\aleph_0}_{\alpha<\lambda} \mathbb{C}_\alpha$, where for any $\alpha < \lambda$, $\mathbb{C}_\alpha := \mathbb{C}$ adding a Cohen real $\bar{a}^\alpha$. Now, let $\dot{x}$ a nice $\mathbb{P}_\lambda$-name of a real number in ${}^\omega 2$. So there exists a countable set $C \subseteq \lambda$ such that, $\dot{x}$ is a $\prod^{<\aleph_0}_{\alpha\in C} \mathbb{C}_\alpha$-name. Therefore, for any $\alpha \in \lambda \setminus C$, $\Vdash_\lambda$ "$\dot{x} \in N[\bar{a}^\alpha]$", hence $\Vdash_\lambda$ "$\{\alpha < \lambda \colon \dot{x} \notin N[\bar{a}^\alpha]\} \subseteq C$". As a consequence, $\Vdash_\lambda$ "$\langle \bar{a}^\alpha \colon \alpha < \lambda \rangle$ is strongly λ-$\mathcal{CN}$-unbounded".
      2. Is a consequence of Construction 5.5.1 and Theorem 5.3.3.
      3. Clear.

So we are under the hypothesis of Theorem 5.3.2, by virtue of which, $\Vdash_\lambda$"$\mathrm{cov}(\mathcal{N}) \leq \lambda$". $\square_{\text{Theorem 5.6.1}}$

Therefore, we can find models of ZFC with $\mathrm{cov}(\mathcal{N})$ singular. For instance, we can use Cohen forcing iterations to build a model where $\kappa = \aleph_1$, $\lambda = \aleph_\omega$, $\chi = \mathfrak{c}$ and, for any $n < \omega$, $2^{\aleph_n} = \chi$, and use Theorem 5.6.1 to get a wonderful result:

Corollary 5.6.2. $\mathrm{Con}(\mathrm{ZFC}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \aleph_0)$.

5.7 Effects in Cichoń's diagram: some separations with $\mathrm{cov}(\mathcal{N})$ singular

In this section, we are going to present some effects on Cichoń's diagram by iterating using finitely additive measures. We will omit the proofs because some of the equalities require *Preservation theory* (see [CM19]), which is beyond the scope of this thesis.

We start with the separation that we get from Theorem 5.6.1:

Theorem 5.7.1. *Let κ, λ and χ be uncountable cardinals such that λ is $\aleph_1$-inaccessible, κ is regular, and $\kappa < \lambda < \chi^{<\lambda} = \chi \leq 2^\kappa$. Then, there exists a κ-Fr-Knaster ccc forcing notion that forces $\mathrm{add}(\mathcal{N}) = \aleph_1$, $\mathrm{add}(\mathcal{M}) = \mathfrak{b} = \kappa$, $\mathrm{cov}(\mathcal{N}) = \lambda$, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \chi$, and $\mathrm{non}(\mathcal{M}) \in \{\lambda, \lambda^+\}$ where, if $\mathrm{cf}(\lambda) < \kappa$ then $\mathrm{non}(\mathcal{M}) = \lambda^+$, as illustrated in Figure 5.1.*

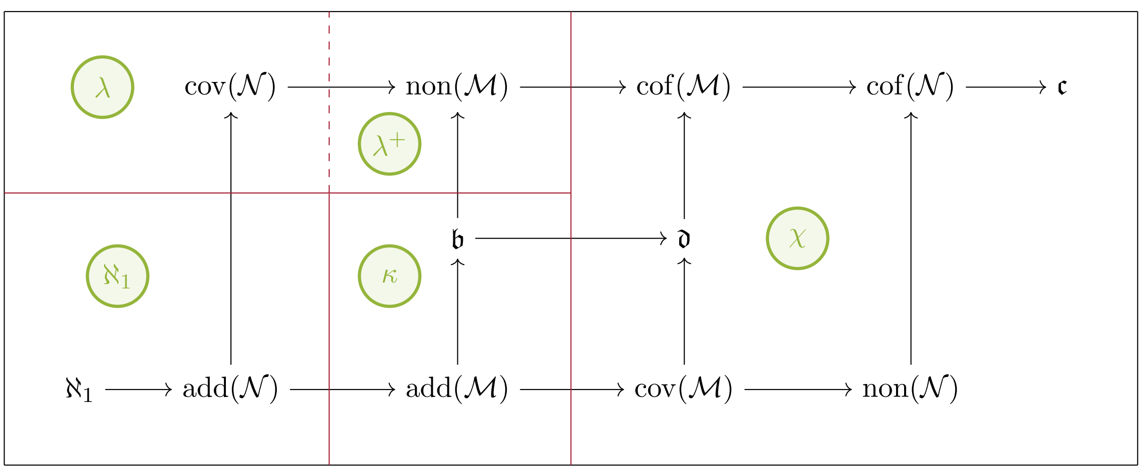

Figure 5.1: A separation of Cichoń's diagram with $\mathrm{cov}(\mathcal{N})$ possibly singular, using Hechler and random forcing.

If in Construction 5.5.1, instead of using partial Hechler forcing on S, we use ccc forcing notions of size $< \kappa$, which is in fact the original way *Saharon Shelah* constructs the iteration (see [She00, Def. 2.2(F)(α)]) and with a similar book-keeping argument, we obtain that, in M_χ, MA_κ holds. Furthermore, it is known that MA_κ entails $\mathrm{add}(\mathcal{N}) \geq \kappa$ and therefore, we obtain the following separation:

Theorem 5.7.2. *Let κ, λ and χ be uncountable cardinals such that λ is $\aleph_1$-inaccessible, κ is regular, and $\kappa < \lambda < \chi^{<\lambda} = \chi \leq 2^\kappa$. Then, there exists a κ-Fr-Knaster ccc forcing notion that forces: $\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathfrak{b} = \kappa$, $\mathrm{cov}(\mathcal{N}) = \lambda$, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \chi$, and $\mathrm{non}(\mathcal{M}) \in \{\lambda, \lambda^+\}$ where, if $\mathrm{cf}(\lambda) < \kappa$, then $\mathrm{non}(\mathcal{M}) = \lambda^+$, as illustrated in Figure 5.2.*

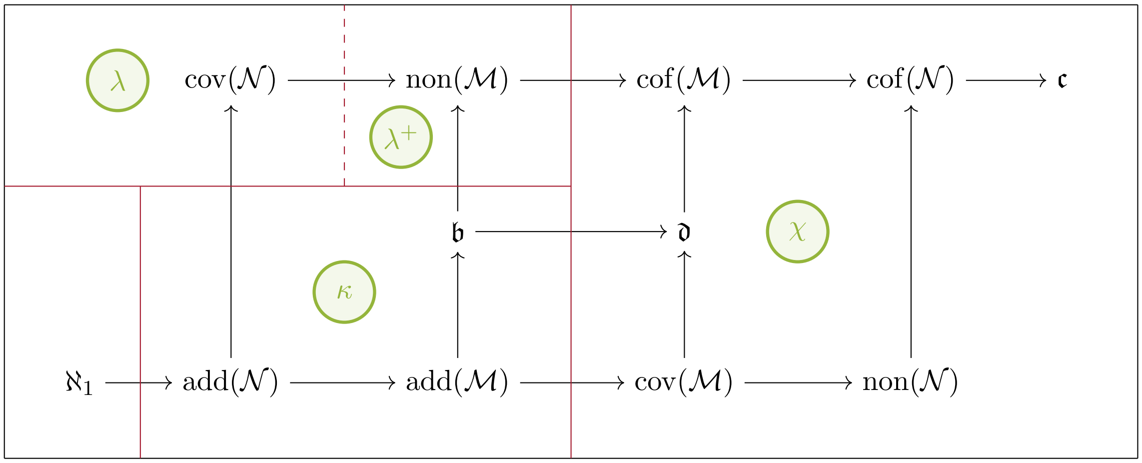

Figure 5.2: A separation of Cichoń's diagram with $\mathrm{cov}(\mathcal{N})$ possibly singular, where MA_κ holds.

If in Construction 5.5.1 we partition χ as $\chi = \lambda \cup B \cup S \cup L$ and, on L, we use $\mathbb{LOC}^N$, that is *localization forcing* (see [Tru88]) for some transitive model N of ZFC such that $|N| < \theta$, then we can deal with $\mathrm{add}(\mathcal{N})$ to get the following separation:

Theorem 5.7.3. *Let* θ, κ, λ *and* χ *be uncountable cardinals such that,* κ *and* θ *are regular,* λ *is* $\aleph_1$*-inaccessible, and* $\theta \leq \kappa < \lambda < \chi^{<\lambda} = \chi \leq 2^\kappa$*. Then, there exists a* κ-Fr-*Knaster* ccc *forcing notion that forces:* $\mathrm{add}(\mathcal{N}) = \theta$, $\mathrm{add}(\mathcal{M}) = \mathfrak{b} = \kappa$, $\mathrm{cov}(\mathcal{N}) = \lambda$, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \chi$, *and* $\mathrm{non}(\mathcal{M}) \in \{\lambda, \lambda^+\}$ *where, if* $\mathrm{cf}(\lambda) < \kappa$, *then* $\mathrm{non}(\mathcal{M}) = \lambda^+$, *as illustrated in Figure 5.3.*

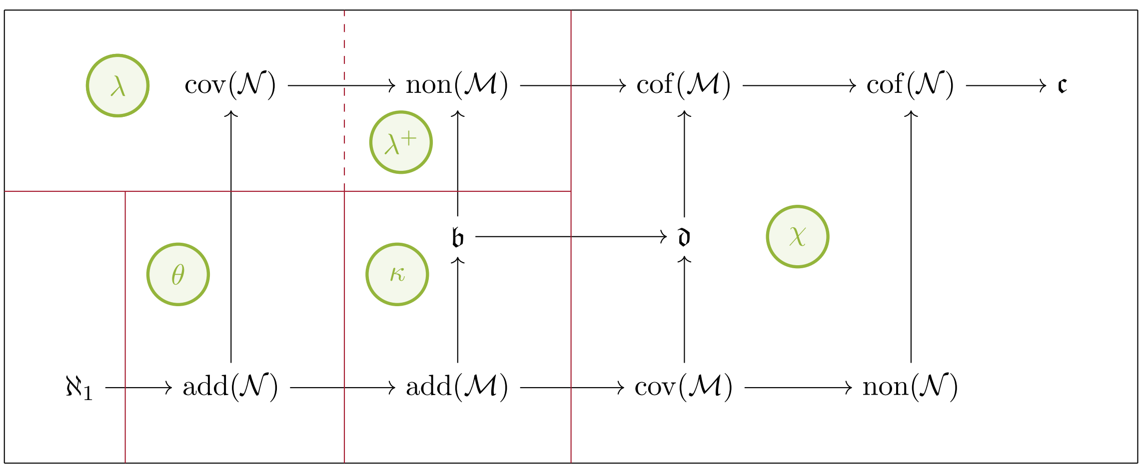

Figure 5.3: A separation of Cichoń's diagram with $\mathrm{cov}(\mathcal{N})$ possibly singular, using Hechler and localization forcing.

5.7.1 A new constellation of Cichoń's diagram: separating the left hand side allowing $\mathrm{cov}(\mathcal{N})$ singular

Finally, if in Theorem 5.7.3, we choose λ singular such that $\mathrm{cf}(\lambda) < \kappa$, we get one of the main results of this thesis: a new constellation of Cichón's diagram separating the left side with $\mathrm{cov}(\mathcal{N})$ possibly singular:

Corollary 5.7.4. *Let θ, κ, λ and χ be uncountable cardinals such that, κ and θ are regular, λ is singular $\aleph_1$-inaccessible with $\mathrm{cf}(\lambda) < \kappa$, and $\theta \leq \kappa < \lambda < \chi^{<\lambda} = \chi \leq 2^\kappa$. Then, there exists a κ-Fr-Knaster ccc forcing notion that forces: $\mathrm{add}(\mathcal{N}) = \theta$, $\mathrm{add}(\mathcal{M}) = \mathfrak{b} = \kappa$, $\mathrm{cov}(\mathcal{N}) = \lambda$, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \chi$, and $\mathrm{non}(\mathcal{M}) = \lambda^+$, as illustrated in Figure 5.4.*

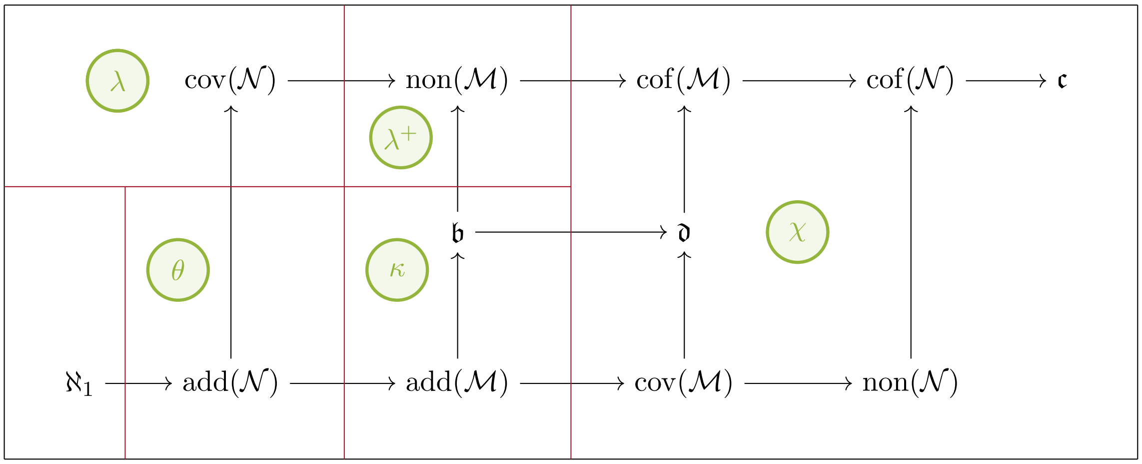

Figure 5.4: A separation of the left side of Cichoń's diagram with $\mathrm{cov}(\mathcal{N})$ singular.

Managing to obtain a left-hand side separation with singular $\mathrm{cov}(\mathcal{N})$, opens up a branch of interesting and so far unexplored questions about the possibility of forcing singular cardinals in Cichoń's diagram. In the next Chapter, we will present some of these questions as open problems (see Section 6.2).

CHAPTER 6

Open problems and future work

If the doors of perception were cleansed, everything would appear to man as it is, infinite. For man has closed himself up, till he sees all things through narrow chinks of his cavern .

William Blake[1]

In this chapter, we present some problems that were left open throughout the development of the thesis and that we consider relevant in future applications of the theory constructed in Chapter 4. We also present some problems that are not directly related to this work, but which, by obtaining results with singular cardinal invariants, are natural.

We begin by presenting the possibility of defining a general framework to force $\mathrm{cov}(\mathcal{N})$ singular, which would greatly facilitate the way we present Chapter 5.

6.1 A general framework for $\mathrm{cov}(\mathcal{N})$ singular

The following definition generalizes, as a notion of linkedness, the properties of Theorem 4.3.24 satisfied by $\lim^g$ from Definition 4.3.22:

Definition 6.1.1. Let $\mathbb{P}$ a forcing notion and $Q \subseteq \mathbb{P}$.

1. We say that Q is pt-*linked in* $\mathbb{P}$ if there exists some $\varepsilon \in (0,1)$ such that, for any $\bar{p} \in Q^\omega$ and any $\bar{I} \in \mathcal{I}_\infty$, there is some $q \in \mathbb{P}$ such that $q \Vdash$ "$\dot{A}_\varepsilon$ is infinite", where

$$\dot{A}_\varepsilon := \left\{ k < \omega \colon \frac{|\{\ell \in I_k \colon p_\ell \in \dot{G}_\mathbb{P}\}|}{|I_k|} > \varepsilon \right\}.$$

[1] See [Bla68].

2. $\mathbb{P}$ is μ-pt-*linked*, if there exists a sequence $\langle Q_\alpha \colon \alpha < \mu\rangle$ such that, for any $\alpha < \mu$, Q_α is a pt-linked subset of $\mathbb{P}$ and $\mathbb{P} = \bigcup_{\alpha<\mu} Q_\alpha$.

3. We say that $\mathbb{P}$ is κ-pt-*Knaster*, if for any $A \in [\mathbb{P}]^\kappa$, there exists some $Q \in [A]^\kappa$ such that Q is pt-linked in $\mathbb{P}$.

Notice that, if $\mathcal{G}$ is complete for (π, κ), then any iteration in $\mathcal{K}_1(\kappa, \mathcal{G})$ is pt-Fr-Knaster and, it is not difficult to prove that κ-pt-Knaster implies κ-Fr-linked.
So, a question that arises from the results proved in Chapter 5 is the following:

Open Problem 6.1.2. Let κ be an uncountable regular cardinal. Assume that $\mathbb{P}$ is κ-pt-Knaster. If $F \subseteq \Omega$ is strongly κ-$\mathcal{S}$-unbounded and $|F| \geq \kappa$, then $\Vdash_{\mathbb{P}}$"F is strongly κ-$\mathcal{S}$-unbounded".

6.2 Separations of Cichoń's diagram with $\mathrm{cov}(\mathcal{N})$ singular

Now, motivated by the results obtained in Section 5.7, we present some open questions related to Cichoń's diagram with some singular values:

Open Problem 6.2.1. It is possible to force Cichoń's maximum with $\mathrm{cov}(\mathcal{N})$ singular?

Open Problem 6.2.2. In the context of Theorem 5.7.3, when λ is singular and $\kappa \leq \mathrm{cf}(\lambda)$, is it possible to decide whether $\mathrm{non}(\mathcal{M}) = \lambda$ or $\mathrm{non}(\mathcal{M}) = \lambda^+$?

More generally:

Open Problem 6.2.3. Under the assumptions of Theorem 5.7.3, if $\mu \in [\lambda^+, \chi]$ is a regular cardinal, can we build a finite support iteration to force the separation given in Figure 6.1?

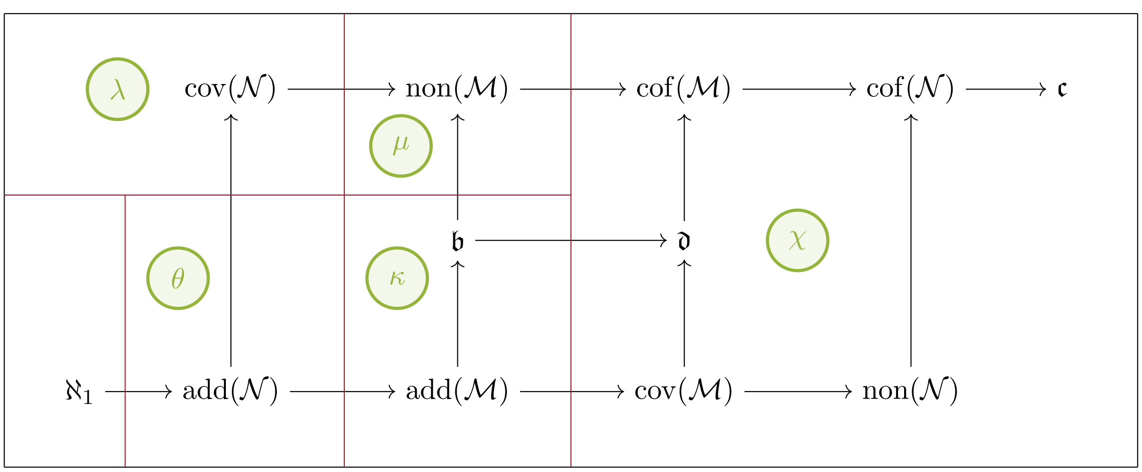

Figure 6.1: A separation of the left side of Cichoń's diagram with $\mathrm{cov}(\mathcal{N})$ possible singular.

Open Problem 6.2.4. Is it possible to separate the left side of the diagram with $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{M})$ singular?

Notice that, Open Problem 6.2.4 is equivalent to finding a finite support method to force $\mathrm{non}(\mathcal{M})$ singular, which is unknown.

Thanks to unpublished work by *Martin Goldstern*, *Jakob Kellner*, *Diego Mejía* and *Saharon Shelah*, it is known that, using large cardinals, one can force the right hand side of Cichoń's diagram with singular values, so:

Open Problem 6.2.5. Is it possible, without using large cardinals, to force a separation of the right hand side of Cichoń's diagram with singular values?

Finally, in [BCM21], two-dimensional iterations with ultrafilters were considered to separate the left side of the diagram and additionally some cardinals on the right side. Since in Section 3.2, we saw that there are connections between finitely additive measures and ultrafilters, the following question arises:

Open Problem 6.2.6. Is it possible to construct a theory of two-dimensional iterated forcing with finitely additive measures?

6.3 Future work

We hope that the presentation and development of the new formalization of the iterated forcing with finitely additive measures method will facilitate its understanding and encourage future research in applications of the method. While we know that some of the open problems raised in the previous section are very difficult and it is very ambitious to tackle some of them, we intend that these problems will be the basis of a future doctoral thesis.

Index

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