

Fixed and coincidence point theorems on partial metric spaces with an application

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ABSTRACT. The aim of this paper is to investigate some fixed and coincidence point theorems in complete, orbitally complete and (T, f) -orbitally complete partial metric spaces under the generalized contractive type conditions of mappings. Moreover, our results generalize and extend the several obtained results in the literature. Additionally some non-trivial examples are demonstrated, and an application has discussed to integral equations.

1. INTRODUCTION

Fixed point theory is an interdisciplinary branch of mathematical sciences which can be applied in several areas of mathematics and other fields, viz., game theory, mathematical economics, optimization problems, approximation theory, initial and boundary value problems in ordinary and partial differential equations, integral equations, variational inequalities, and many others. The most fundamental result in fixed point theory that influenced several researchers was due to the Polish mathematician Stefan Banach [9] in 1922, and this result is popularly known as the Banach contraction principle (BCP).

Because new discoveries of spaces and their properties are always interesting to researchers in mathematics, so many researchers have attempted to generalize the metric space structure by weakening the properties of the metric. Among them, Matthews [22] introduced the concept of the partial metric space which is one of the most important generalization of the metric space. Interestingly, Matthews explored applications of partial metric spaces in the field of computer science, especially in the study of denotational semantics of programming languages and algorithms. In fact, partial metrics

2020 *Mathematics Subject Classification*. Primary: 47H10, 54H25.

Key words and phrases. Fixed point, coincidence point, orbital continuity, orbital completeness, partial metric and Hausdorff metric.

Full paper. Received 26 January 2023, accepted 27 September 2023, available online 20 October 2023 .

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play an important role to construct models in the theory of computation and domain theory of computer science (see also [16]). Moreover, Matthews [22] also extended the BCP in the setting of partial metric spaces. Thereafter, in the view of several applications, this result has been generalized in various ways in partial metric spaces, see [1–6, 8, 10, 13, 14, 20, 24, 25, 28–30] and references therein.

Throughout the further discussion, \mathbb{N} and \mathbb{R}^+ will denote the set of positive integers and set of non-negative reals, respectively. We now recall some important definitions and lemmas as follow.

Definition 1 ([22]). A partial metric on a non-empty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$, such that for all $x, y, z \in X$:

- (p₁) $p(x, x) = p(y, y) = p(x, y) \iff x = y$;
- (p₂) $p(x, x) \leq p(x, y)$;
- (p₃) $p(x, y) = p(y, x)$;
- (p₄) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space.

Remark 1. Let (X, p) be a partial metric space. Then $p(x, x) = 0$ implies $x = y$, but on the contrary $p(x, x)$ need not be zero.

Remark 2. Let (X, p) be a partial metric space. Then open p -ball at $x \in X$ with radius $\epsilon > 0$ is defined by $B_p(x, \epsilon) = \{y \in X : p(x, y) < \epsilon + p(x, x)\}$. Also, the family of open p -balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$ is a base for the topology τ_p on X , and with respect to this topology, the space (X, p) is a T_0 space.

Remark 3. Let (X, p) be a partial metric space. Then the functions $p^s, p^t : X \times X \rightarrow \mathbb{R}^+$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ and $p^t(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$ are metrics on X .

Definition 2 ([3, 19, 22, 23]). Let (X, p) be a partial metric space. Then,

- (i) a sequence $\{x_n\}$ in X is convergent to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x)$;
- (ii) a sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and is finite;
- (iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$;
- (iv) a mapping $T : X \rightarrow X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x_0, \delta)) \subseteq B_p(Tx_0, \epsilon)$.

Lemma 1 ([3, 19, 22, 23]). Let (X, p) be a partial metric space. Then,

- (i) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in (X, p^s) ;

(ii) (X, p) is complete if and only if (X, p^s) is complete. Furthermore,
 $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

Lemma 2 ([17]). Assume $x_n \rightarrow x$ as $n \rightarrow \infty$ in a partial metric space (X, p) such that $p(x, x) = 0$. Then $\lim_{n \rightarrow \infty} p(x_n, y) = p(x, y)$ for every $y \in X$.

Recently, Kumar et al. [21] have proved a fixed point result on a complete metric space (X, d) for a mapping $T : X \rightarrow X$ satisfying a non-expansive type condition, such that for all $x, y \in X$:

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta [d(x, Tx) + d(y, Ty)] \\ &\quad + \gamma [d(x, Ty) + d(y, Tx)] + \delta [M(x, y) + h m(x, y)], \end{aligned}$$

where $\alpha \geq 0$; $\beta, \gamma, \delta > 0$; $0 < h < 1$; $\alpha + 2\beta + 2\gamma + 2\delta = 1$; $M(x, y) = \max\{d(x, Ty), d(y, Tx)\}$ and $m(x, y) = \min\{d(x, Ty), d(y, Tx)\}$.

In this paper, we will present some fixed and coincidence point theorems for generalized type contractions in partial metric spaces. The approach is based on fixed point results obtained by Kumar et al. [21] in the settings of the partial metric under similar type of contractive conditions. Our results are the extensions of Banach's contraction, Kannan's contraction, Chatterjea's contraction and Ćirić's contraction of metric spaces to partial metric spaces. Moreover, we show that our results are also true in orbitally complete and (T, f) -orbitally complete partial metric spaces, which generalize and extend the conclusions obtained in the literature. In addition, we have given some non-trivial examples to demonstrate our results, and applications have obtained to integral equations.

2. MAIN RESULTS

Theorem 1. Let (X, p) be a complete partial metric space and let $T : X \rightarrow X$ be a mapping satisfying

$$\begin{aligned} (1) \quad p(Tx, Ty) &\leq \alpha p(x, y) + \beta [p(x, Tx) + p(y, Ty)] \\ &\quad + \gamma [p(x, Ty) + p(y, Tx)] \\ &\quad + \delta [M_p(x, y) + h m_p(x, y)], \end{aligned}$$

for all $x, y \in X$, where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$, and

$$\begin{aligned} M_p(x, y) &= \max\{p(x, Ty), p(y, Tx)\}, \\ m_p(x, y) &= \min\{p(x, Ty), p(y, Tx)\}. \end{aligned}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary. Now, we define a sequence $\{x_n \in X : n \in \mathbb{N} \cup \{0\}\}$ such that $x_{n+1} = Tx_n = T^n x_0$. Then, by (1), we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq \alpha p(x_{n-1}, x_n) + \beta [p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \\ &\quad + \gamma [p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \\ &\quad + \delta [M_p(x_{n-1}, x_n) + hm_p(x_{n-1}, x_n)], \end{aligned}$$

which implies

$$\begin{aligned} (2) \quad p(x_n, x_{n+1}) &\leq \delta [\max\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\}] \\ &\quad + h \min\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\}] \\ &\quad + (\alpha + \beta + \gamma)p(x_{n-1}, x_n) + (\beta + \gamma)p(x_n, x_{n+1}). \end{aligned}$$

If $p(x_n, x_{n+1}) > p(x_{n-1}, x_n)$ for some n , then (2) implies

$$\begin{aligned} p(x_n, x_{n+1}) &< \delta [\max\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\}] \\ &\quad + h \min\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\}] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(x_n, x_{n+1}). \end{aligned}$$

Now either $p(x_{n-1}, x_{n+1}) \geq p(x_n, x_n)$ or $p(x_{n-1}, x_{n+1}) \leq p(x_n, x_n)$ in above inequality. But, in both cases, we get $p(x_n, x_{n+1}) < p(x_n, x_{n+1})$, which is not true. Hence, $p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n)$ for all n . Also, by (2), we have

$$\begin{aligned} (3) \quad p(x_n, x_{n+1}) &\leq \delta [\max\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\}] \\ &\quad + h \min\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\}] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(x_{n-1}, x_n). \end{aligned}$$

We consider the following two cases:

Case 1: If $\max\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\} = p(x_{n-1}, x_{n+1})$, then (3) implies

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \delta [2p(x_{n-1}, x_n) - (1-h)p(x_n, x_n)] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(x_{n-1}, x_n) \\ &\leq (\alpha + 2\beta + 2\gamma + 2\delta)p(x_{n-1}, x_n), \quad \text{since } h < 1. \end{aligned}$$

Case 2: If $\max\{p(x_{n-1}, x_{n+1}), p(x_n, x_n)\} = p(x_n, x_n)$, then (3) implies

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \delta [p(x_n, x_n) + hp(x_{n-1}, x_{n+1})] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(x_{n-1}, x_n) \\ &\leq \delta(1+h)p(x_n, x_n) + (\alpha + 2\beta + 2\gamma)p(x_{n-1}, x_n) \\ &\leq \delta(1+h)p(x_{n-1}, x_n) + (\alpha + 2\beta + 2\gamma)p(x_{n-1}, x_n) \quad (\text{by } (p_2)) \\ &< (\alpha + 2\beta + 2\gamma + 2\delta)p(x_{n-1}, x_n), \quad \text{since } h < 1. \end{aligned}$$

Thus, in both cases, we conclude that $p(x_n, x_{n+1}) \leq kp(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, where $k = (\alpha + 2\beta + 2\gamma + 2\delta) < 1$.

We now show that $\{x_n\}$ is a Cauchy sequence in X . Let $m, n > 0$ with $m > n$, then by (p_4) , we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots + p(x_{n+m-1}, x_m) \\ &\quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \dots - p(x_{n+m-1}, x_{n+m-1}) \\ &\leq k^n p(x_0, x_1) + k^{n+1} p(x_0, x_1) + \dots + k^{n+m-1} p(x_0, x_1) \\ &\leq k^n \left(\frac{1 - k^{m-1}}{1 - k} \right) p(x_0, x_1). \end{aligned}$$

Making $n, m \rightarrow \infty$, we get $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in X , so by Lemma 1, it is also a Cauchy sequence in the metric space (X, p^s) . Thus the sequence $\{x_n\}$ converges to $x \in X$ because (X, p) is complete. Moreover,

$$(4) \quad \lim_{n, m \rightarrow \infty} p(x_n, x_m) = \lim_{n \rightarrow \infty} p(x_n, x) = p(x, x) = 0.$$

Further, we show that x is a fixed point of T . Also, for every n , we have

$$p(x, Tx) \leq p(x, Tx_n) + p(Tx_n, Tx) - p(Tx_n, Tx_n).$$

Taking $n \rightarrow \infty$ in this inequality, we get $p(x, Tx) \leq p(Tx, Tx)$. Hence, by (p_2) , we get

$$(5) \quad p(x, Tx) = p(Tx, Tx).$$

Again from (1), we have

$$\begin{aligned} p(x, Tx) &\leq p(x, Tx_n) + p(Tx_n, Tx) - p(Tx_n, Tx_n) \\ &\leq p(x, x_{n+1}) + \alpha p(x_n, x) + \beta [p(x_n, Tx_n) + p(x, Tx)] \\ &\quad + \gamma [p(x_n, Tx) + p(x, Tx_n)] + \delta [M_p(x_n, x) + h m_p(x_n, x)] \\ &\leq p(x, x_{n+1}) + \alpha p(x_n, x) \\ &\quad + \beta [p(x_n, x) + p(x, x_{n+1}) - p(x, x) + p(x, Tx)] \\ &\quad + \gamma [p(x_n, x) + p(x, Tx) - p(x, x) + p(x, x_{n+1})] \\ &\quad + \delta [M_p(x_n, x) + h m_p(x_n, x)] \\ &\leq (1 + \beta + \gamma) p(x, x_{n+1}) + (\alpha + \beta + \gamma) p(x_n, x) + (\beta + \gamma) p(x, Tx) \\ &\quad + \delta [\max\{p(x_n, Tx), p(x, Tx_n)\} + h \min\{p(x_n, Tx), p(x, Tx_n)\}]. \end{aligned}$$

Again we consider the following cases:

Case 1: If $\max\{p(x_n, Tx), p(x, Tx_n)\} = p(x_n, Tx)$, then

$$p(x, Tx) \leq \left(\frac{1 + \beta + \gamma + h\delta}{1 - \beta - \gamma - \delta} \right) p(x, x_{n+1}) + \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta} \right) p(x_n, x).$$

Case 2: If $\max\{p(x_n, Tx), p(x, Tx_n)\} = p(x, Tx_n)$, then

$$p(x, Tx) \leq \left(\frac{1 + \beta + \gamma + \delta}{1 - \beta - \gamma - h\delta} \right) p(x, x_{n+1}) + \left(\frac{\alpha + \beta + \gamma + h\delta}{1 - \beta - \gamma - h\delta} \right) p(x_n, x).$$

Thus, taking $n \rightarrow \infty$ in both cases, we get

$$(6) \quad p(x, Tx) = 0.$$

From (4), (5) and (6), we get

$$(7) \quad p(Tx, Tx) = p(x, Tx) = p(x, x) = 0.$$

Hence, by (p_1) and (7), we get $x = Tx$, i.e., x is a fixed point of T .

To show the uniqueness, we suppose that y is another fixed point of T . Then, by (1), we get

$$\begin{aligned} p(x, y) &= p(Tx, Ty) \\ &\leq \alpha p(x, y) + \beta[p(x, Tx) + p(y, Ty)] + \gamma[p(x, Ty) + p(y, Tx)] \\ &\quad + \delta[\max\{p(x, Ty), p(y, Tx)\} + h \min\{p(x, Ty), p(y, Tx)\}] \\ &\leq [\alpha + 2\gamma + (1 + h)\delta]p(x, y), \end{aligned}$$

which implies $p(x, y) = 0$, that is, $x = y$. This completes the theorem. \square

We now consider the following example in the support of Theorem 1.

Example 1. Let $X = [0, +\infty)$. Then (X, p) , where $p(x, y) = \max\{x, y\}$, is a complete partial metric space. Define $T : X \rightarrow X$ by

$$T(t) = \begin{cases} \frac{t}{4}, & \text{if } 0 \leq t < 1; \\ \frac{t}{3+t}, & \text{if } t \geq 1. \end{cases}$$

Then T satisfies all the conditions of the Theorem 1 for $\alpha = \frac{1}{4}$, $\beta, \gamma, \delta \in (0, \frac{1}{8})$ and $0 < h < 1$. Moreover, $0 \in X$ is the only fixed point of T .

Next we prove another result for a pair of mappings as follows.

Theorem 2. Let (X, p) be a complete partial metric space and let $T, f : X \rightarrow X$ be mappings satisfying

$$(8) \quad \begin{aligned} p(Tx, Ty) &\leq \alpha p(fx, fy) + \beta[p(fx, Tx) + p(fy, Ty)] \\ &\quad + \gamma[p(fx, Ty) + p(fy, Tx)] \\ &\quad + \delta[M_p^f(x, y) + h m_p^f(x, y)], \end{aligned}$$

for all $x, y \in X$, where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$, with $\alpha + 2\beta + 2\gamma + 2\delta < 1$, and

$$\begin{aligned} M_p^f(x, y) &= \max\{p(fx, Ty), p(fy, Tx)\}, \\ m_p^f(x, y) &= \min\{p(fx, Ty), p(fy, Tx)\}. \end{aligned}$$

If $T(X) \subset f(X)$ and $f(X)$ is a complete subspace of X , then T and f have a unique coincidence point.

Proof. Let $x_0 \in X$ be an arbitrary. Since $T(X) \subset f(X)$, we define a sequence $\{fx_n : n \in \mathbb{N} \cup \{0\}\}$ such that $fx_{n+1} = Tx_n$. Then, by (8), we have

$$\begin{aligned} p(fx_{n+1}, fx_{n+2}) &\leq \alpha p(fx_n, fx_{n+1}) + \beta[p(fx_n, Tx_n) + p(fx_{n+1}, Tx_{n+1})] \\ &\quad + \gamma[p(fx_n, Tx_{n+1}) + p(fx_{n+1}, Tx_n)] \\ &\quad + \delta[M_p^f(x_n, x_{n+1}) + h m_p^f(x_n, x_{n+1})], \end{aligned}$$

which implies

$$\begin{aligned} (9) \quad p(fx_{n+1}, fx_{n+2}) &\leq \delta[\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\} \\ &\quad + h \min\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\}] \\ &\quad + (\alpha + \beta + \gamma)p(fx_n, fx_{n+1}) \\ &\quad + (\beta + \gamma)p(fx_{n+1}, fx_{n+2}). \end{aligned}$$

If $p(fx_{n+1}, fx_{n+2}) > p(fx_n, fx_{n+1})$ for some n , then (9) implies

$$\begin{aligned} p(fx_{n+1}, fx_{n+2}) &\leq \delta[\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\} \\ &\quad + h \min\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\}] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(fx_{n+1}, fx_{n+2}). \end{aligned}$$

Now, in above inequality, either $p(fx_n, fx_{n+2}) \geq p(fx_{n+1}, fx_{n+1})$ or $p(fx_n, fx_{n+2}) \leq p(fx_{n+1}, fx_{n+1})$. But, in both cases, we get $p(fx_{n+1}, fx_{n+2}) < p(fx_{n+1}, fx_{n+2})$, which is not true.

Hence, $p(fx_{n+1}, fx_{n+2}) \leq p(fx_n, fx_{n+1})$ for all n . Also, by (9), we have

$$\begin{aligned} (10) \quad p(fx_{n+1}, fx_{n+2}) &\leq \delta[\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\} \\ &\quad + h \min\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\}] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(fx_n, fx_{n+1}). \end{aligned}$$

We consider the following two cases:

Case 1: If $\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\} = p(fx_n, fx_{n+2})$, then (10) implies

$$\begin{aligned} p(fx_{n+1}, fx_{n+2}) &\leq \delta[2p(fx_n, fx_{n+1}) - (1 - h)p(fx_{n+1}, fx_{n+1})] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(fx_n, fx_{n+1}) \\ &\leq (\alpha + 2\beta + 2\gamma + 2\delta)p(fx_n, fx_{n+1}), \quad \text{since } h < 1. \end{aligned}$$

Case 2: If $\max\{p(fx_n, fx_{n+2}), p(fx_{n+1}, fx_{n+1})\} = p(fx_{n+1}, fx_{n+1})$, then (10) implies

$$\begin{aligned} p(fx_{n+1}, fx_{n+2}) &\leq \delta[p(fx_{n+1}, fx_{n+1}) + hp(fx_n, fx_{n+2})] \\ &\quad + (\alpha + 2\beta + 2\gamma)p(fx_n, fx_{n+1}) \\ &\leq \delta[(1 + h)p(fx_n, fx_{n+1})] + (\alpha + 2\beta + 2\gamma)p(fx_n, fx_{n+1}) \\ &\leq (\alpha + 2\beta + 2\gamma + 2\delta)p(fx_n, fx_{n+1}), \quad \text{since } h < 1. \end{aligned}$$

Thus, in both cases, we conclude that $p(fx_{n+1}, fx_{n+2}) \leq kp(fx_n, fx_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$, where $k = (\alpha + 2\beta + 2\gamma + 2\delta) < 1$.

We now show that $\{fx_n\}$ is a Cauchy sequence in X . Let $m, n > 0$ with $m > n$, then by (p_4) , we have

$$\begin{aligned} p(fx_n, fx_m) &\leq p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2}) + \cdots + p(fx_{n+m-1}, fx_m) \\ &\quad - p(fx_{n+1}, fx_{n+1}) - \cdots - p(fx_{n+m-1}, fx_{n+m-1}) \\ &\leq k^n p(fx_0, fx_1) + k^{n+1} p(fx_0, fx_1) + \cdots + k^{n+m-1} p(fx_0, fx_1) \\ &\leq k^n \left(\frac{1 - k^{m-1}}{1 - k} \right) p(fx_0, fx_1). \end{aligned}$$

Making $n, m \rightarrow \infty$, we get $\lim_{n, m \rightarrow \infty} p(fx_n, fx_m) = 0$. Hence $\{fx_n\}$ is Cauchy sequence in X , so by Lemma 1, it is also a Cauchy sequence in the metric space (X, p^s) . Thus the sequence $\{fx_n\}$ will converge to $x \in X$ because (X, p) is complete. Moreover,

$$(11) \quad \lim_{n, m \rightarrow \infty} p(fx_n, fx_m) = \lim_{n \rightarrow \infty} p(fx_n, fx) = p(fx, fx) = 0.$$

Further, we show that x is a coincidence point of f and T . Also, for every n , we have

$$p(fx, Tx) \leq p(fx, Tx_n) + p(Tx_n, Tx) - p(Tx_n, Tx_n).$$

Taking $n \rightarrow \infty$ in this inequality, we get $p(fx, Tx) \leq p(Tx, Tx)$. Hence, by (p_2) , we get

$$(12) \quad p(fx, Tx) = p(Tx, Tx).$$

Again from (8), we have

$$\begin{aligned} p(fx, Tx) &\leq p(fx, Tx_n) + p(Tx_n, Tx) - p(Tx_n, Tx_n) \\ &\leq p(fx, fx_{n+1}) + \alpha p(fx_n, fx) + \beta [p(fx_n, Tx_n) + p(fx, Tx)] \\ &\quad + \gamma [p(fx_n, Tx) + p(fx, Tx_n)] + \delta [M_p^f(x_n, x) + h m_p^f(x_n, x)]. \\ &\leq p(fx, fx_{n+1}) + \alpha p(fx_n, fx) \\ &\quad + \beta [p(fx_n, fx) + p(fx, fx_{n+1}) - p(fx, fx) + p(fx, Tx)] \\ &\quad + \gamma [p(fx_n, fx) + p(fx, Tx) - p(fx, fx) + p(fx, fx_{n+1})] \\ &\quad + \delta [M_p^f(x_n, x) + h m_p^f(x_n, x)] \\ &\leq (1 + \beta + \gamma) p(fx, fx_{n+1}) + (\alpha + \beta + \gamma) p(fx_n, fx) \\ &\quad + (\beta + \gamma) p(fx, Tx) + \delta [\max\{p(fx_n, Tx), p(fx, Tx_n)\}] \\ &\quad + h \min\{p(fx_n, Tx), p(fx, Tx_n)\}. \end{aligned}$$

Again we consider the following cases:

Case 1: If $\max\{p(fx_n, Tx), p(fx, Tx_n)\} = p(fx_n, Tx)$, then

$$p(fx, Tx) \leq \left(\frac{1 + \beta + \gamma + h\delta}{1 - \beta - \gamma - \delta} \right) p(fx, fx_{n+1}) + \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta} \right) p(fx_n, fx).$$

Case 2: If $\max\{p(fx_n, Tx), p(fx, Tx_n)\} = p(fx, Tx_n)$, then

$$p(fx, Tx) \leq \left(\frac{1 + \beta + \gamma + \delta}{1 - \beta - \gamma - h\delta}\right)p(fx, fx_{n+1}) + \left(\frac{\alpha + \beta + \gamma + h\delta}{1 - \beta - \gamma - h\delta}\right)p(fx_n, fx).$$

Thus, taking $n \rightarrow \infty$ in both the cases, we get

$$(13) \quad p(fx, Tx) \leq 0 \quad \Rightarrow \quad p(fx, Tx) = 0.$$

Using equations (11), (12) and (13), we get

$$(14) \quad p(Tx, Tx) = p(fx, Tx) = p(fx, fx) = 0.$$

Hence, by (p_1) and (14), we get $fx = Tx$, i.e., x is a coincidence point of f and T .

To show the uniqueness, we suppose that y is another coincidence point of f and T , i.e., $fy = Ty$, $fx = Tx$. Then, from (8), we get

$$\begin{aligned} p(Tx, Ty) &\leq \alpha p(fx, fy) + \beta[p(fx, Tx) + p(fy, Ty)] \\ &\quad + \gamma[p(fx, Ty) + p(fy, Tx)] + [\max\{p(fx, Ty), p(fy, Tx)\}] \\ &\quad + h \min\{p(fx, Ty), d(fy, Tx)\} \\ \Rightarrow p(Tx, Ty) &\leq (\alpha + 2\gamma + (1 + h)\delta)d(Tx, Ty), \end{aligned}$$

which implies $Tx = Ty$, and so $fx = fy$. This completes the proof. \square

We further provide the following example in the support of Theorem 2.

Example 2. Let $X = [0, \frac{1}{2}]$. Then (X, p) , where $p(x, y) = \max\{x, y\}$, is a complete partial metric space. Define mappings f and T on X by

$$f(x) = x \quad \text{and} \quad T(x) = \frac{x}{2}.$$

Then f and T satisfy all the conditions of the Theorem 2, and zero is the only coincidence point of f and T .

Remark 4. Theorem 2 represents Theorem 1 for $f = I$ (identity mapping). Moreover, Theorem 5.3 of [22], Theorem 3 of [21], Theorem 1 of [18] and Theorem 2.1 of [12] are special cases of Theorem 2.

Now we recall the notions of orbitally continuous mappings and orbitally complete partial metric spaces defined by Karapinar and Erhan [19].

Definition 3 ([19]). Let (X, p) be a partial metric space. A mapping $T : X \rightarrow X$ is called orbitally continuous if

$$\lim_{i \rightarrow \infty} p(T^{n_i}x, z) = p(z, z) \quad \Rightarrow \quad \lim_{i \rightarrow \infty} p(TT^{n_i}x, Tz) = p(Tz, Tz), \quad \text{for each } x \in X.$$

Definition 4 ([19]). A partial metric space (X, p) is called orbitally complete if every Cauchy sequence $\{T^{n_i}x\}$ converges in (X, p) , i.e., if

$$\lim_{i, j \rightarrow \infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i \rightarrow \infty} p(T^{n_i}x, z) = p(z, z), \quad \text{for each } x \in X.$$

We further prove the following fixed point result under the assumption of orbitally continuous mapping on orbitally complete partial metric spaces.

Theorem 3. *Let (X, p) be an orbitally complete partial metric space and let $T : X \rightarrow X$ be an orbitally continuous mapping satisfying*

$$(15) \quad \begin{aligned} p(Tx, Ty) &\leq \alpha p(x, y) + \beta[p(x, Tx) + p(y, Ty)] \\ &\quad + \gamma[p(x, Ty) + p(y, Tx)] \\ &\quad + \delta[M_p(x, y) + h m_p(x, y)], \end{aligned}$$

for all $x, y \in X$, where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$, while $M_p(x, y)$ and $m_p(x, y)$ are defined as in Theorem 1. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary. We define a sequence $\{x_n \in X : n \in \mathbb{N} \cup \{0\}\}$ such that $x_{n+1} = Tx_n = T^n x_0$. If there exist $n \in \mathbb{N} \cup \{0\}$ such that $p(x_n, x_{n-1}) = 0$, then by (p_2) , we have $p(x_{n-1}, x_{n-1}) = p(x_n, x_n)$. Thus, by (p_1) , we get $x_{n-1} = x_n = Tx_{n-1}$. Suppose that $p(x_n, x_{n+1}) > 0$ for all $n \geq 0$. Now, using (15), we get

$$(16) \quad \begin{aligned} p(x_{n+1}, x_{n+2}) &\leq \alpha p(x_n, x_{n+1}) + \beta[p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})] \\ &\quad + \gamma[p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})] \\ &\quad + \delta[M_p(x_n, x_{n+1}) + h m_p(x_n, x_{n+1})]. \end{aligned}$$

We consider the following two cases:

Case 1: If $\max\{p(x_n, x_{n+2}), p(x_{n+1}, x_{n+1})\} = p(x_n, x_{n+2})$, then (16) implies

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &\leq (\alpha + \beta + \gamma + \delta) p(x_n, x_{n+1}) + (\beta + \gamma + \delta) p(x_{n+1}, x_{n+2}) \\ &\quad - (1 - h)\delta p(x_{n+1}, x_{n+1}) \\ &\leq \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta}\right) p(x_n, x_{n+1}) \quad (\text{since } h < 1) \\ &= k_1 p(x_n, x_{n+1}), \text{ where } k_1 = \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta}\right) < 1. \end{aligned}$$

Case 2: If $\max\{p(x_n, x_{n+2}), p(x_{n+1}, x_{n+1})\} = p(x_{n+1}, x_{n+1})$, then (16) implies

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &\leq (\alpha + \beta + \gamma + h\delta) p(x_n, x_{n+1}) + (\beta + \gamma + \delta) p(x_{n+1}, x_{n+2}) \\ &\leq \left(\frac{\alpha + \beta + \gamma + h\delta}{1 - \beta - \gamma - \delta}\right) p(x_n, x_{n+1}) \\ &= k_2 p(x_n, x_{n+1}), \text{ where } k_2 = \left(\frac{\alpha + \beta + \gamma + h\delta}{1 - \beta - \gamma - \delta}\right) < 1. \end{aligned}$$

Thus, in both cases, we conclude that $p(x_{n+1}, x_{n+2}) \leq k p(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, where $k = \min\{k_1, k_2\}$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X . Without loss of generality, we assume that $n > m$. Then, using above inequality and (p_4) ,

we have

$$\begin{aligned}
 p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+m}) \\
 &\quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) \\
 &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots + p(x_{n+m-1}, x_{n+m}) \\
 &\leq k^n p(x_0, x_1) + k^{n+1} p(x_0, x_1) + \dots + k^{n+m-1} p(x_0, x_1) \\
 &\leq k^n \left(\frac{1 - k^{m-1}}{1 - k} \right) p(x_0, x_1).
 \end{aligned}$$

Making $n, m \rightarrow \infty$, we get $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. Thus, by (p_2) , we have $\lim_{n \rightarrow \infty} p(x_n, x_n) = 0$ and $\lim_{m \rightarrow \infty} p(x_m, x_m) = 0$. Hence $p^s(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \rightarrow 0$ as $n \rightarrow \infty$. So, we conclude that $\{x_n\} = \{T^n x_0\}$ is a Cauchy sequence in (X, p^s) . Since (X, p) is orbitally complete, the sequence $\{T^n x_0\}$ converges in the metric space (X, p^s) to $z \in X$, i.e., $\lim_{n \rightarrow \infty} p^s(T^n x_0, z) = 0$. Again from Lemma 1, we have

$$(17) \quad p(z, z) = \lim_{n \rightarrow \infty} p(T^n x_0, z) = \lim_{n, m \rightarrow \infty} p(T^n x_0, T^m x_0) = 0.$$

Suppose that $p(z, Tz) > 0$. Since T is orbitally continuous, by (17) we get

$$(18) \quad \lim_{n \rightarrow \infty} p(T^n x_0, z) = p(z, z) \Rightarrow \lim_{n \rightarrow \infty} p(TT^n x_0, Tz) = p(Tz, Tz).$$

Now, we have

$$\begin{aligned}
 p(z, Tz) &\leq p(z, T^{n+1} x_0) + p(T^{n+1} x_0, Tz) - p(T^{n+1} x_0, T^{n+1} x_0) \\
 &\leq p(z, T^{n+1} x_0) + p(T^{n+1} x_0, Tz).
 \end{aligned}$$

Taking $n \rightarrow \infty$ and using Lemma 2 with (18), we get

$$p(z, Tz) \leq \lim_{n \rightarrow \infty} p(z, x_{n+2}) + \lim_{n \rightarrow \infty} p(T^{n+1} x_0, Tz) = p(Tz, Tz).$$

From (p_2) this is possible only if

$$(19) \quad p(z, Tz) = p(Tz, Tz).$$

Hence, by (15), we get

$$\begin{aligned}
 p(z, Tz) &\leq p(z, x_{n+1}) + p(x_{n+1}, Tz) - p(x_{n+1}, x_{n+1}) \\
 &\leq p(z, x_{n+1}) + \alpha p(x_n, z) + \beta [p(x_n, x_{n+1}) + p(z, Tz)] \\
 &\quad + \gamma [p(x_n, Tz) + p(z, Tx_n)] + \delta [M_p(x, y) + h m_p(x, y)].
 \end{aligned}$$

We also consider the following two cases:

Case 1: If $M_p(x, y) = p(x_n, Tz)$, then

$$p(z, Tz) \leq \left(\frac{1 + \beta + \gamma + h\delta}{1 - \beta - \gamma - \delta} \right) p(z, x_{n+1}) + \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta} \right) p(x_n, z).$$

Case 2: If $M_p(x, y) = p(z, Tx_n)$, then

$$p(z, Tz) \leq \left(\frac{1 + \beta + \gamma + \delta}{1 - \beta - \gamma - h\delta} \right) p(z, x_{n+1}) + \left(\frac{\alpha + \beta + \gamma + h\delta}{1 - \beta - \gamma - h\delta} \right) p(x_n, z).$$

Thus, taking $n \rightarrow \infty$ in both cases, we get

$$(20) \quad p(z, Tz) \leq 0 \quad \Rightarrow \quad p(z, Tz) = 0.$$

Using (17), (19) and (20), we have

$$(21) \quad p(Tz, Tz) = p(z, Tz) = p(z, z) = 0.$$

From (p_1) and (21), we get $z = Tz$, i.e., z is a fixed point of T .

Furthermore, let t be another fixed point of T , then from (15), we have

$$\begin{aligned} p(z, t) &= p(Tz, Tt) \leq \alpha p(z, t) + \beta [p(z, Tz) + p(t, Tt)] \\ &\quad + \gamma [p(z, Tt) + p(t, Tz)] + \delta [\max\{p(z, Tt), p(t, Tz)\}] \\ &\quad + h \min\{p(z, Tt), p(t, Tz)\} \\ &\leq (\alpha + 2\gamma + (1 + h)\delta) p(z, t) \\ p(z, t) &\leq 0 \quad \Rightarrow \quad p(z, t) = 0, \quad \text{that is } z = t, \end{aligned}$$

which completes the proof. \square

In 2012, Aydi et al. [7] introduced the concept of a partial Hausdorff metric with $CB^p(X)$, the collection of all non-empty closed and bounded subsets of X with respect to partial metric p . Thereafter, in 2016, Abdesslem Benterki [11] used the concept of collection of closed subsets of X with respect to partial metric p (say, $CL^p(X)$) instead of closed and bounded subsets of X . For any $A, B, C \in CL^p(X)$ (resp. $CB^p(X)$):

- (i) $p(a, C) = \inf\{p(a, x) : x \in C\}$.
- (ii) $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$.
- (iii) $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$.
- (iv) $\rho_p(A, B) = \sup\{p(a, b) : a \in A, b \in B\}$.

Moreover, for all $A, B \in CL^p(X)$ (resp. $CB^p(X)$), we have

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Then H_p is called the generalized partial Hausdorff metric (resp. partial Hausdorff metric) induced by p .

We also cite the following definitions which are very useful to our next result (see also, N. Chandra et al. [13] and references therein).

Definition 5. Let T be a self mapping of a partial metric space X . Then, an orbit of T at a point $x_0 \in X$ is a sequence $\{x_n : x_n \in Tx_{n-1}\}$.

Definition 6. Let T be a multi-valued mapping on a partial metric space X . Then X is said to be T -orbitally complete if every Cauchy sequence of the form $\{x_n : x_n \in Tx_{n-1}\}$ is convergent in X in the sense of the partial metric.

Definition 7. Let f and T be a self mapping and a multi-valued mapping on a partial metric space (X, p) , respectively. If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ such that $fx_n \in Tx_{n-1}$, $n \in \mathbb{N}$, then $O_f(x_0) = \{fx_n : n \in \mathbb{N}\}$ is called the orbit of (T, f) at x_0 . Furthermore, $O_f(x_0)$ is called a regular orbit of (T, f) at x_0 , if $p(fx_n, fx_{n+1}) \leq H_p(Tx_{n-1}, Tx_n)$, for every $n \in \mathbb{N}$.

Definition 8. Let f and T be a self mapping and a multi-valued mapping on a partial metric space (X, p) , respectively. Then X is said to be (T, f) -orbitally complete if every Cauchy sequence of the form $\{fx_n : fx_n \in Tx_{n-1}\}$ is convergent in the sense of the partial metric.

Lemma 3 ([7, 8, 13]). *Let (X, p) be a partial metric space. Then,*

- (i) (X, p) is T -orbitally complete if and only if (X, p^s) is T -orbitally complete;
- (ii) (X, p) is (T, f) -orbitally complete if and only if (X, p^s) is (T, f) -orbitally complete.

Lemma 4 ([7, 8, 13]). *Let $A, B \in CLP(X)$ and $a \in A$. Then, for any $\epsilon > 0$ there exists a point $b \in B$ such that $p(a, b) \leq H_p(A, B) + \epsilon$.*

Furthermost, we prove the following result for a pair of mappings under the assumption of (T, f) -orbitally complete as follows.

Theorem 4. *Let (X, p) be a partial metric spaces. Suppose that $T : X \rightarrow CLP(X)$ and $f : X \rightarrow X$ are such that $T(X) \subseteq f(X)$. If $f(X)$ is (T, f) -orbitally complete and, for all $x, y \in X$, the following condition is satisfying:*

$$(22) \quad \begin{aligned} H_p(Tx, Ty) &\leq \alpha p(fx, fy) + \beta [p(fx, Tx) + p(fy, Ty)] \\ &\quad + \gamma [p(fx, Ty) + p(fy, Tx)] \\ &\quad + \delta [M_p^f(x, y) + h m_p^f(x, y)], \end{aligned}$$

where $0 < h < 1$ and $\alpha \geq 0, \beta, \gamma, \delta > 0$ with $\alpha + 2\beta + 2\gamma + 2\delta \leq k < 1$, while $M_p^f(x, y)$ and $m_p^f(x, y)$ are defined as in Theorem 2. Then T and f have a coincidence point, i.e., there exists a point $z \in X$ such that $fz \in Tz$.

Proof. Let $\epsilon > 0$ such that $\mu = k + \epsilon < 1$, and x_0 be an arbitrary point of X . Since $T(X) \subset f(X)$, we choose a point $x_1 \in X$ such that $fx_1 \in Tx_0$. Clearly $p(fx_1, Tx_1) \geq 0$. If $p(fx_1, Tx_1) = 0$, then nothing to prove because in this case $fx_1 \in Tx_1$. Assume $p(fx_1, Tx_1) > 0$, then by Lemma 4, there exists $fx_2 \in Tx_1$ such that $p(fx_1, fx_2) \leq H_p(Tx_0, Tx_1) + \epsilon p(fx_1, Tx_1)$. Similarly, if $p(fx_2, Tx_2) > 0$, then there exists $fx_3 \in Tx_2$ such that $p(fx_2, fx_3) \leq H_p(Tx_1, Tx_2) + \epsilon p(fx_2, Tx_2)$. Continuing this process, we construct a sequence $\{fx_n\} \subset f(X)$ such that $p(fx_{n+1}, fx_{n+2}) \leq H_p(Tx_n, Tx_{n+1}) + \epsilon p(fx_n, Tx_n)$ and $fx_{n+1} \in Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Using (22), we get

$$\begin{aligned}
p(fx_{n+1}, fx_{n+2}) &\leq H_p(Tx_n, Tx_{n+1}) + \epsilon p(fx_n, Tx_n) \\
&\leq \alpha p(fx_n, fx_{n+1}) + \beta [p(fx_n, Tx_n) + p(fx_{n+1}, Tx_{n+1})] \\
&\quad + \gamma [p(fx_n, Tx_{n+1}) + p(fx_{n+1}, Tx_n)] \\
&\quad + \delta [\max\{p(fx_n, Tx_{n+1}), p(fx_{n+1}, Tx_n)\}] \\
&\quad + h \min\{p(fx_n, Tx_{n+1}), p(fx_{n+1}, Tx_n)\} \\
&\quad + \epsilon p(fx_n, Tx_n) \\
&\leq \alpha p(fx_n, fx_{n+1}) + \beta [p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2})] \\
&\quad + \gamma [p(fx_n, fx_{n+2}) + p(fx_{n+1}, Tx_n)] + \delta [p(fx_n, fx_{n+2}), \\
&\quad + h p(fx_{n+1}, Tx_n)] + \epsilon p(fx_n, fx_{n+1}) \\
&\leq \alpha p(fx_n, fx_{n+1}) + \beta [p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2})] \\
&\quad + \gamma p(fx_n, fx_{n+2}) + \delta p(fx_n, fx_{n+2}) + \epsilon p(fx_n, fx_{n+1}).
\end{aligned}$$

Now, if $p(fx_n, fx_{n+1}) < p(fx_{n+1}, fx_{n+2})$ for some $n \in \mathbb{N}$, then $p(fx_{n+1}, fx_{n+2}) \leq \alpha p(fx_{n+1}, fx_{n+2}) < p(fx_{n+1}, fx_{n+2})$, which is a contradiction.

Hence $p(fx_{n+1}, fx_{n+2}) \leq p(fx_n, fx_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, and so

$$p(fx_{n+1}, fx_{n+2}) \leq \mu p(fx_n, fx_{n+1}).$$

Continuing in this way, we get

$$p(fx_{n+1}, fx_{n+2}) \leq \mu^n p(fx_0, fx_1).$$

Since $\mu < 1$, taking $n \rightarrow \infty$, we get $p(fx_{n+1}, fx_{n+2}) \rightarrow 0$ implies $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Since $f(X)$ is (T, f) -orbitally complete, the sequence $\{fx_n\}$ is convergent to a point $a \in f(X)$, that is,

$$(23) \quad p(fx_{n+1}, fx_{n+2}) = p(a, a) = 0.$$

Also, $a \in f(X)$ implies that there exists a point $z \in X$ such that $fz = a$. Now, we get

$$\begin{aligned}
p(fz, Tz) &\leq p(fz, fx_{n+1}) + p(fx_{n+1}, Tz) \\
&\leq p(fz, fx_{n+1}) + H_p(Tx_n, Tz) \\
&\leq p(fz, fx_{n+1}) + \alpha p(fx_n, fz) + \beta [p(fx_n, Tx_n) + p(fz, Tz)] \\
&\quad + \gamma [p(fx_n, Tz) + p(fz, Tx_n)] + \delta [M_p(x, y) + h m_p(x, y)].
\end{aligned}$$

We now consider the following cases:

Case 1: If $M_p(x, y) = p(fx_n, Tz)$, then

$$\begin{aligned}
p(fz, Tz) &\leq p(fz, fx_{n+1}) + \alpha p(fx_n, fz) + \beta [p(fx_n, fx_{n+1}) + p(fz, Tz)] \\
&\quad + \gamma [p(fx_n, Tz) + p(fz, fx_n) + p(fx_n, fx_{n+1})] + \delta [p(fx_n, Tz) \\
&\quad + h \{p(fz, fx_n) + p(fx_n, fx_{n+1})\}].
\end{aligned}$$

Making $n \rightarrow \infty$, we get $p(fz, Tz) \leq (\beta + \gamma + \delta)p(fz, Tz)$ which implies $p(fz, Tz) = 0$, since $\beta + \gamma + \delta < 1$.

Case 2: If $M_p(x, y) = p(fz, Tx_n)$, then

$$\begin{aligned} p(fz, Tz) &\leq p(fz, fx_{n+1}) + \alpha p(fx_n, fz) + \beta[p(fx_n, fx_{n+1}) + p(fz, Tz)] \\ &\quad + \gamma[p(fx_n, Tz) + p(fz, fx_n) + p(fx_n, fx_{n+1})] + \delta[p(fz, fx_n) \\ &\quad + p(fx_n, fx_{n+1}) + hp(fx_n, Tz)]. \end{aligned}$$

Making $n \rightarrow \infty$, we get $p(fz, Tz) \leq (\beta + \gamma + h\delta)p(fz, Tz)$ which implies $p(fz, Tz) = 0$, since $\beta + \gamma + h\delta < 1$.

Thus, in both cases, we conclude that $fz \in Tz$, i.e., z is the coincidence point of f and T . \square

Remark 5. Theorem 4 is a proper generalization of Theorem 3 whenever $f = I$ (identity mapping) and X is T -orbitally complete. Moreover, Theorem 5.3 of [22], Theorems 3 and 4 of [21], Theorems 1 and 2 of [18] and Theorem 2.2 of [12] are also special cases of Theorem 4.

3. AN APPLICATION TO INTEGRAL EQUATIONS

In this section, we study the existence of the solution to a nonlinear integral equation by using a result obtained in the previous section. First, we consider the following integral equation:

$$(24) \quad u(t) = \int_0^T G(t, s)f(s, u(s))ds, \quad \text{for all } t \in [0, T],$$

where $T > 0$, $f : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $G : [0, T] \times [0, T] \rightarrow [0, \infty)$ are continuous functions. Let $X = C([0, T])$ be the set of positive real continuous functions on $[0, T]$. We endow X with partial metric D_p defined by

$$D_p(u, v) = \max_{t \in [0, T]} \{u(t), v(t)\}, \quad \text{for all } u, v \in X.$$

Obviously, (X, D_p) is a complete partial metric space. Let $(\alpha, \beta) \in X \times X$, $(\alpha_0, \beta_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ be such that

$$(25) \quad \alpha_0 \leq \alpha(t) \leq \beta(t) \leq \beta_0, \quad \text{for all } t \in [0, T].$$

Assume that for all $t \in [0, T]$, we have

$$(26) \quad \alpha(t) \leq \int_0^T G(t, s)f(s, \beta(s))ds$$

and

$$(27) \quad \beta(t) \geq \int_0^T G(t, s)f(s, \alpha(s))ds.$$

Let $f(s, \cdot)$ be a decreasing function for all $s \in [0, T]$, that is,

$$(28) \quad x, y \in \mathbb{R}^+ \text{ with } y \leq x \Rightarrow f(s, x) \leq f(s, y).$$

Assume that

$$(29) \quad \max_{t \in [0, T]} \int_0^T G(t, s) ds \leq 1.$$

Moreover, for any mapping $T : X \rightarrow X$, we suppose that for all $s \in [0, T]$, and for all $x, y \in \mathbb{R}^+$ with $(x \leq \beta_0$ and $y \geq \alpha_0)$ or $(x \geq \alpha_0$ and $y \leq \beta_0)$,

$$(30) \quad \begin{aligned} \max \{ f(s, x), f(s, y) \} &\leq \alpha \max \{ x, y \} \\ &+ \beta \left(\max \{ x, Tx \} + \max \{ y, Ty \} \right) \\ &+ \gamma \left(\max \{ x, Ty \} + \max \{ y, Tx \} \right) \\ &+ \delta \left(M(x, y) + hm(x, y) \right), \end{aligned}$$

where $0 < h < 1$ and $\alpha \geq 0$, $\beta, \gamma, \delta > 0$ such that $\alpha + 2\beta + 2\gamma + 2\delta < 1$, and

$$\begin{aligned} M(x, y) &= \max \left\{ \max \{ x, Ty \}, \max \{ y, Tx \} \right\}, \\ m(x, y) &= \min \left\{ \max \{ x, Ty \}, \max \{ y, Tx \} \right\}. \end{aligned}$$

Theorem 5. *Under assumptions (25)-(30), the integral equation (24) has a solution in $\{u \in C([0, T]) : \alpha \leq u(t) \leq \beta, \text{ for all } t \in [0, T]\}$.*

Proof. We consider the closed subsets A_1 and A_2 of X , defined by

$$A_1 = \{u \in X : u \leq \beta\}$$

and

$$A_2 = \{u \in X : u \geq \alpha\}.$$

Also, we define a mapping $T : X \rightarrow X$ by

$$Tu(t) = \int_0^T G(t, s) f(s, u(s)) ds, \quad \text{for all } t \in [0, T].$$

Let us prove that

$$(31) \quad T(A_1) \subset A_2 \quad \text{and} \quad T(A_2) \subset A_1.$$

Suppose that $u \in A_1$, that is, $u(s) \leq \beta(s)$ for all $s \in [0, T]$. Since $G(t, s) \geq 0$ for all $t, s \in [0, T]$, by (28) we obtain

$$G(t, s) f(s, u(s)) \geq G(t, s) f(s, \beta(s)), \quad \text{for all } t, s \in [0, T].$$

This inequality and condition (26) imply that

$$\int_0^T G(t, s) f(s, u(s)) ds \geq \int_0^T G(t, s) f(s, \beta(s)) ds \geq \alpha(t), \quad \text{for all } t \in [0, T].$$

Hence, $Tu \in A_2$. Similarly, let $u \in A_2$, that is, $u(s) \geq \alpha(s)$, for all $s \in [0, T]$. Since $G(t, s) \geq 0$ for all $t, s \in [0, T]$, by (28) we obtain

$$G(t, s) f(s, u(s)) \geq G(t, s) f(s, \alpha(s)), \quad \text{for all } t, s \in [0, T].$$

This inequality and condition (27) imply that

$$\int_0^T G(t, s) f(s, u(s)) ds \leq \int_0^T G(t, s) f(s, \alpha(s)) ds \leq \beta(t), \quad \text{for all } t \in [0, T].$$

Hence $Tu \in A_1$, and therefore condition (31) holds.

Now, let $(u, v) \in A_1 \times A_2$, that is, $u(t) \leq \beta(t)$, $v(t) \geq \alpha(t)$, for all $t \in [0, T]$. Thus condition (25) implies that

$$u(t) \leq \beta_0, \quad v(t) \geq \alpha_0, \quad \text{for all } t \in [0, T].$$

By conditions (29) and (30), for all $t \in [0, T]$, we have

$$\begin{aligned} \max\{Tx, Ty\} &= \max \left\{ \int_0^T G(t, s) f(s, x(s)) ds, \int_0^T G(t, s) f(s, y(s)) ds \right\} \\ &\leq \int_0^T G(t, s) \max\{f(s, x(s)), f(s, y(s))\} ds \\ &\leq \int_0^T G(t, s) \left[\alpha \max\{x, y\} \right. \\ &\quad \left. + \beta \left(\max\{x, Tx\} + \max\{y, Ty\} \right) \right. \\ &\quad \left. + \gamma \left(\max\{x, Ty\} + \max\{y, Tx\} \right) \right. \\ &\quad \left. + \delta \left(M(x, y) + hm(x, y) \right) \right] ds \\ &= \int_0^T G(t, s) \left[\alpha \left(\max\{x, y\} \right) \right] ds \\ &\quad + \int_0^T G(t, s) \left[\beta \left(\max\{x, Tx\} + \max\{y, Ty\} \right) \right] ds \\ &\quad + \int_0^T G(t, s) \left[\gamma \left(\max\{x, Ty\} + \max\{y, Tx\} \right) \right] ds \\ &\quad + \int_0^T G(t, s) \left[\delta \left(M(x, y) + hm(x, y) \right) \right] ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} D_P(Tx, Ty) &\leq \left[\alpha D_p(x, y) \right. \\ &\quad \left. + \beta \left(D_p(x, Tx) + D_p(y, Ty) \right) + \gamma \left(D_p(x, Ty) + D_p(y, Tx) \right) \right. \\ &\quad \left. + \delta \left(M(x, y) + hm(x, y) \right) \right] \times \int_0^T G(t, s) ds \\ &\leq \alpha D_p(x, y) + \beta \left[D_p(x, Tx) + D_p(y, Ty) \right] \\ &\quad + \gamma \left[D_p(x, Ty) + D_p(y, Tx) \right] + \delta \left[M(x, y) + hm(x, y) \right]. \end{aligned}$$

Now, if $M(x, y) = \max\{x, Ty\}$, we have

$$\begin{aligned} D_P(Tx, Ty) &\leq \alpha D_p(x, y) + \beta \left[D_p(x, Tx) + D_p(y, Ty) \right] \\ &\quad + \gamma \left[D_p(x, Ty) + D_p(y, Tx) \right] \\ &\quad + \delta \left[D_p(x, Ty) + hD_p(y, Tx) \right], \end{aligned}$$

and if $M(x, y) = \max\{y, Tx\}$, we have

$$\begin{aligned} D_P(Tx, Ty) &\leq \alpha D_p(x, y) + \beta \left[D_p(x, Tx) + D_p(y, Ty) \right] \\ &\quad + \gamma \left[D_p(x, Ty) + D_p(y, Tx) \right] \\ &\quad + \delta \left[D_p(y, Tx) + hD_p(x, Ty) \right]. \end{aligned}$$

By a similar method, we can show that the above inequality holds if $(u, v) \in A_2 \times A_1$. Thus, all the conditions of Theorem 1 hold, and therefore, T has a unique fixed point z in the set

$$A_1 \cap A_2 = \{u \in C([0, T]) : \alpha \leq u(t) \leq \beta, \quad \text{for all } t \in [0, T]\}.$$

That is, $z \in A_1 \cap A_2$ is the solution to (24). □

4. CONCLUSION

We have established some fixed and coincidence point results for generalized contractive type conditions in complete partial metric spaces. Also, we prove the results are valid for analogous contractive conditions on orbitally complete and (T, f) -orbitally complete partial metric spaces. Established theorems extend and generalize several theorems due to various researchers in the literature to partial metric spaces, for example, see G. S. Saluja [26,27], M. Dinarvand [15], R. Kumar et al. [20,21], and others. Moreover, we have given some examples for justification as well as provided an application for our obtained results. Henceforth, our theorems open a direction to new fixed point results and applications in partial metric spaces.

5. COMPLIANCE WITH ETHICAL STANDARDS

5.1. Conflict of interest. The authors declare that they have no conflict of interest.

5.2. Ethical approval. This article does not contain any studies with human participants or animals performed by any of the authors.

5.3. Informed consent. Informed consent was obtained from all individual participants included in the study.

6. DATA AVAILABILITY

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

7. ACKNOWLEDGMENTS

The authors are grateful to the anonymous referees for their precise remarks and suggestions which led to the improvement of the paper.

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