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Suspension bridge model with laminated beam

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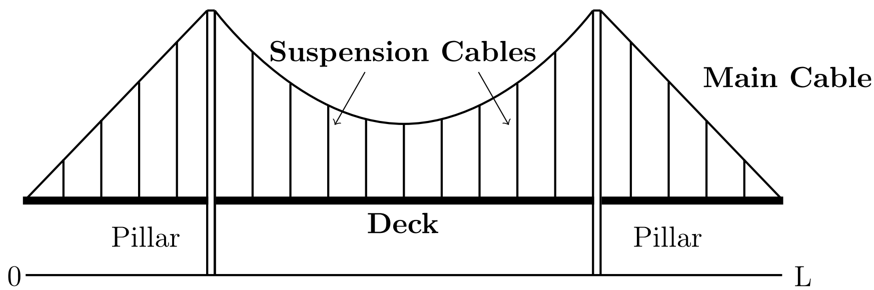
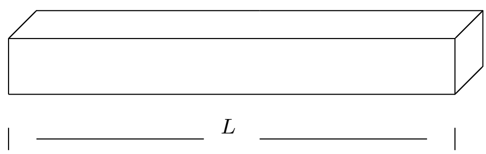
Abstract. This manuscript introduces a suspension bridge system where laminated beams model the deck. The action of frictional damping is considered. Well-posedness is proved using the Lumer-Phillips theorem, and the exponential stability is obtained by applying the Gearhart-Huang-Prüss theorem.

1. Introduction

A suspension bridge is a mechanical structure that carries vertical loads through the main cables modeled by an elastic string $u = u(x,t)$, which is coupled to the deck employing suspension cables, where x denotes the distance along the center line of the beam in its equilibrium configuration and t the time variable.



Considering that the deck has negligible transversal section dimensions compared to the length (span of the bridge), it is modeled in Timoshenko's theory as a one-dimensional extensible beam of length L.



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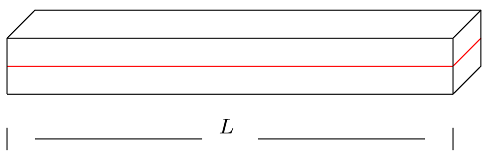
Denoting by $\varphi = \varphi(x,t)$ the displacement of the cross-section on the point $x \in (0, L)$, by $\psi = \psi(x,t)$ the rotation angle of the cross-section, we have the following coupled system

$$\begin{aligned} u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 u_t &= 0, \\ \rho_1 \varphi_{tt} - k(\varphi_x - \psi)_x + \lambda(\varphi - u) + \gamma_2 \varphi_t &= 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma_3 \psi_t &= 0. \end{aligned}$$

The suspender cables are assumed to be linear elastic springs with standard stiffness $\lambda > 0$. The constant $\alpha > 0$ is the elastic modulus of the string (holding the main cable to the deck). The positive coefficients ρ_1 and ρ_2 are the mass density and the moment of mass inertia of the beam, respectively. Moreover, b represents the cross section's rigidity coefficient, and k represents the elasticity's shear modulus.

In this manuscript, we introduce a model of a suspension bridge where the deck is modeled by laminated beams,



The laminated beam system of length L considered here, is a model proposed by Hansen and Spies [9, 10] for two-layered beams in which a slip $s = s(x,t)$ can occur at the interface of contact, in red at figure above, given by

$$\rho \varphi_{tt} + G(\psi - \varphi_x)_x + \lambda(\varphi - u) = 0, \tag{1}$$

$$I_\rho(3S_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - \varphi_x) = 0, \tag{2}$$

$$I_\rho 3S_{tt} - D3S_{xx} + 3G(\psi - \varphi_x) + 4\gamma_0 S + 4\delta_0 S_t = 0. \tag{3}$$

The positive parameters ρ, I_ρ, G, D, and γ_0, are the density, mass moment of inertia, shear stiffness, flexural rigidity, and adhesive stiffness, respectively.

We consider the action of frictional dampings on each component in (1)-(3), and, as in [16], to simplify the computations, we make the following replacements: $s(x,t) = 3S(x,t)$, $\xi = (3S - \psi)(x,t)$ $\rho_1 = \rho$, $\rho_2 = I_\rho$, $k = G$, $b = D$, $3\gamma_{=}4\gamma_0$, $3\mu_4 = 4\delta_0$. Then, we introduce a suspension bridge model in laminated beams as follows,

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \mu_1 u_t = 0, \tag{4}$$

$$\rho_1 \varphi_{tt} + k(s - \xi - \varphi_x)_x + \lambda(\varphi - u) + \mu_2 \varphi_t = 0, \tag{5}$$

$$\rho_2 \xi_{tt} - b\xi_{xx} - k(s - \xi - \varphi_x) + \mu_3 \xi_t = 0, \tag{6}$$

$$\rho_2 s_{tt} - b s_{xx} + 3k(s - \xi - \varphi_x) + \gamma s + \mu_4 s_t = 0. \tag{7}$$

The non-negative parameters $\mu_i > 0$, $i = 1, 2, 3, 4$, are the coefficients of the damping force, and μ_4 is called the adhesive damping. System (4)–(7) is subject to the initial data

$$(u(x,0), \varphi(x,0), \xi(x,0), s(x,0)) = (u_0(x), \varphi_0(x), \xi_0(x), s_0(x)),\ x \in (0,L),$$
$$(u_t(x,0), \varphi_t(x,0), \xi_t(x,0), S_t(x,0)) = (u_1(x), \varphi_1(x), \xi_1(x), s_1(x)),\ x \in (0,L),$$

and Dirichlet boundary conditions

$$u(0,t) = \varphi(0,t) = \xi(0,t) = s(0,t) = 0, t \geq 0,$$
$$u(L,t) = \varphi(L,t) = \xi(L,t) = s(L,t) = 0,\ t \geq 0.$$

In [18], it was proved that the structural damping created by the interfacial slip alone is insufficient to stabilize the laminated beam system exponentially to its equilibrium state. The question arises of studying the action of additional stabilizing mechanisms located in other system equations. A natural damping is the internal (frictional damping). A laminated beam with friction damping was considered in [16]. The internal damping combined with thermoelastic damping given by Fourier lay was considered in [3] for the suspension bridge, modeled by the Timoshenko system, where the cable is supposed to be thermally insulated, and it was proved that the solution decays exponentially to zero.

The critical feature in the present paper is introducing dampings in the system to avoid resonance and produce exponential stabilization for the structure. Mechanical resonance occurs when the energy transfers from one object to another with the same natural or resonant frequency. Strong vibrations can cause lots of damage to structures and can cause materials to collapse apart.

The collapse in the mechanical structures can be considered a resonance effect, as it strongly occurred in the Tacoma Narrows bridge, that collapsed in 1940, the same year it opened, after being hit by strong winds. In 1831, the bridge at Broughton, Manchester, collapsed when a company of British Army Fusile Corps marched across the bridge in synchronized steps. In 2000, the Millennium Bridge, a steel suspension bridge for pedestrian use linking Bankside with the City of London, suffered an unexpected and excessive lateral vibration due to a structural resonant response, causing it to close two days after opening. Problems like these are solved by introducing stabilizing mechanisms in the structure. That is what happened to the Rio-Niterói Bridge in Brazil, built in 1968. The bridge vibrated frequently, causing discomfort to anyone traveling over the bridge. In 2004 synchronized dynamic attenuators were installed, and this application prevented discomfort, damage, or outright structural failure.

This paper introduces a suspension bridge system where laminated beams model the deck under the action of frictional damping. The manuscript has three sections: Introduction, well-posedness, and exponential stability.

2. Well-posedness

Now, we introduce the vector function

$$U = (u, w, \varphi, \phi, \xi, \eta, s, z)^T,$$

where $\xi = s - \psi$, $w = u_t$, $\phi = \varphi_t$, $\eta = \xi_t$ and $z = s_t$.

The system (4)-(7) can be written as

$$\begin{cases} U_t - \mathcal{A}U = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{8}$$

where the linear operator

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

is defined by

$$\mathcal{A}\begin{pmatrix} u \\ w \\ \varphi \\ \phi \\ \xi \\ \eta \\ s \\ z \end{pmatrix} = \begin{pmatrix} w \\ -\left[-\alpha u_{xx} - \lambda(\varphi - u) + \mu_1 w\right] \\ \phi \\ -\dfrac{1}{\rho_1}\left[k(s - \xi - \varphi_x)_x + \lambda(\varphi - u) + \mu_2\phi\right] \\ \eta \\ -\dfrac{1}{\rho_2}\left[-b\xi_{xx} - k(s - \xi - \varphi_x) + \mu_3\eta\right] \\ z \\ -\dfrac{1}{\rho_2}\left[-bs_{xx} + k(s - \xi - \varphi_x) + \gamma s + \mu_4 z\right] \end{pmatrix},$$

on energy space

$$\mathcal{H} = [H_0^1(0,L) \times L^2(0,L)]^4$$

and

$$D(\mathcal{A}) = [H_0^1(0,L) \cap H^2(0,L) \times H_0^1(0,L)]^4.$$

We denote the $L^2(0,L)$ inner product by

$$(f, \bar{g}) = \int_0^L f(x)\bar{g}(x)dx, \ \forall\ f, g \in L^2(0,L) \text{ and } (f, \bar{f}) = \|f\|^2.$$

In $\mathcal{H}$ we consider the following inner product

$$\begin{aligned} \langle U, \tilde{U}\rangle_{\mathcal{H}} = {} & (w, \bar{\tilde{w}}) + \alpha(u_x, \bar{\tilde{u}}_x) \\ & + \lambda(\phi - u, \bar{\tilde{\phi}} - \bar{\tilde{u}}) \\ & + \rho_1(\phi, \bar{\tilde{\phi}}) + \rho_2(\eta, \bar{\tilde{\eta}}) + \rho_2(z, \bar{\tilde{z}}) \\ & + k(s - \xi - \varphi_x, \bar{s} - \bar{\tilde{\xi}} - \bar{\tilde{\varphi}}_x) \\ & + b(\xi_x, \bar{\tilde{\xi}}_x) + b(s_x, \bar{\tilde{s}}_x) + \gamma(z, \bar{\tilde{s}}). \end{aligned}$$

Clear $D(\mathcal{A})$ is dense in $\mathcal{H}$, and $\mathcal{H}$ is a Hilbert space with norm

$$\|U\|_{\mathcal{H}}^2 = \langle U, U\rangle_{\mathcal{H}}.$$

Proposition 1. *The operator $\mathcal{A}$ is dissipativo on $\mathcal{H}$, that is,*

$$\mathfrak{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\mu_1\|w\|^2 - \mu_2\|\phi\|^2 - \mu_3\|\eta\|^2 - \mu_4\|z\|^2. \tag{9}$$

Proof.

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= (-\left[-\alpha u_{xx} - \lambda(\varphi - u) + \mu_1 w\right], \bar{w}) + \alpha(w_x, \bar{u}_x)\\ &+ \lambda(\phi - w, \bar{\varphi} - \bar{u})\\ &+ \rho_1(-\frac{1}{\rho_1}\left[k(s - \xi - \varphi_x)_x + \lambda(\varphi - u) + \mu_2\phi\right], \bar{\phi})\\ &+ \rho_2(-\frac{1}{\rho_2}\left[-b\xi_{xx} - k(s - \xi - \varphi_x) + \mu_3\eta\right], \bar{\eta})\\ &+ \rho_2(-\frac{1}{\rho_2}\left[-bs_{xx} + k(s - \xi - \varphi_x) + \gamma s + \mu_4 z\right], \bar{z})\\ &+ k(z - \eta - \phi_x, \bar{s} - \bar{\xi} - \bar{\varphi}_x)\\ &+ b(\eta_x, \bar{\xi}_x) + b(z_x, \bar{s}_x) + \gamma(z, \bar{s}).\end{aligned}$$

Performing integration by parts, we get

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\mu_1(w, \bar{w}) - \mu_2(\phi, \bar{\phi}) - \mu_3(\eta, \bar{\eta}) - \mu_4(z, \bar{z})\\ &+ \lambda\, 2i\mathfrak{Im}(\phi - w, \bar{\varphi} - \bar{u})\\ &+ k\, 2i\mathfrak{Im}(z - \eta - \phi_x, \bar{s} - \bar{\xi} - \bar{\varphi}_x)\\ &+ b\, 2i\mathfrak{Im}(\eta_x, \bar{\xi}_x) + b\, 2i\mathfrak{Im}(z_x, \bar{s}_x)\\ &+ \alpha\, 2i\mathfrak{Im}(w_x, \bar{u}_x) + \gamma\, 2i\mathfrak{Im}(z, \bar{s}).\end{aligned}$$

Taking the real part we obtain (9). □

Lemma 1. $0 \in \rho(\mathcal{A})$, *the resolvent set of* $\mathcal{A}$.

Proof. Given $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8)^T \in \mathcal{H}$, the resolvent equation $-AU = F$ in $\mathcal{H}$, in terms of the component coordinates of U and F, leads to

$$-w = f^1 \text{ in } H_0^1(0, L), \tag{10}$$

$$-\alpha u_{xx} - \lambda(\varphi - u) + \mu_1 w = f^2 \text{ in } L^2(0, L), \tag{11}$$

$$-\phi = f^3 \text{ in } H_0^1(0, L), \tag{12}$$

$$k(s - \xi - \varphi_x)_x + \lambda(\varphi - u) + \mu_2\phi = \rho_1 f^4 \text{ in } L^2(0, L), \tag{13}$$

$$-\eta = f^5 \text{ in } H_0^1(0, L), \tag{14}$$

$$-b\xi_{xx} - k(s - \xi - \varphi_x) + \mu_3\eta = \rho_2 f^6 \text{ in } L^2(0, L). \tag{15}$$

$$-z = f^7 \text{ in } H_0^1(0, L), \tag{16}$$

$$-bs_{xx} + k(s - \xi - \varphi_x) + \gamma s + \mu_4 z = \rho_2 f^8 \text{ in } L^2(0, L). \tag{17}$$

Replacing (10) in (11), (12) in (13), (14) in (15) and (16) in (17), we get

$$-\alpha u_{xx} - \lambda(\varphi - u) = \mu_1 f^1 + f^2 := g^1 \in L^2(0, L), \tag{18}$$

$$k(s-\xi-\varphi_x)_x+\lambda(\varphi-u)=\mu_2 f^3+\rho_1 f^4:=g_2\in L^2(0,L), \tag{19}$$

$$-b\xi_{xx}-k(s-\xi-\varphi_x)=\mu_3 f^5+\rho_2 f^6:=g^3\in L^2(0,L). \tag{20}$$

$$-bs_{xx}+k(s-\xi-\varphi_x)+\gamma s=\mu_4 f^7+\rho_2 f^8:=g^4\in L^2(0,L). \tag{21}$$

We denote $V=H^{-1}(0,L)\times H_0^1(0,L)$. From Sobolev spaces, the following embeddings hold,

$$H_0^1(0,L)\hookrightarrow L^2(0,L)\equiv[L^2(0,L)]'\hookrightarrow H^{-1}(0,L).$$

Multiplying (18) by $\tilde{u}\in H_0^1(0,L)$, (19) by $\tilde{\varphi}\in H_0^1(0,L)$, (20) by $\tilde{\xi}\in H_0^1(0,L)$, (21) by $\tilde{s}\in H_0^1(0,L)$, respectively, and integrating by parts, we obtain

$$\alpha\langle u_x,\tilde{u}_x\rangle_V+\lambda\langle\varphi-u,-\tilde{u}\rangle_V=\langle g^1,\tilde{u}\rangle_V, \tag{22}$$

$$k\langle s-\xi-\varphi_x,-\tilde{\varphi}_x\rangle_V+\lambda\langle\varphi-u,\tilde{\varphi}\rangle_V:=\langle g^2,\tilde{\varphi}\rangle_V, \tag{23}$$

$$b\langle\xi_x,\tilde{\xi}_x\rangle_V+k\langle s-\xi-\varphi_x,-\tilde{\xi}\rangle_V:=\langle g^3,\tilde{\xi}\rangle_V, \tag{24}$$

$$b\langle s_x,\tilde{s}_x\rangle_V+k\langle s-\xi-\varphi_x,\tilde{s})+\gamma(s,\tilde{s}\rangle_V:=\langle g^4,\tilde{s}\rangle_V. \tag{25}$$

Denoting $\mathcal{V}=[H_0^1(0,L)]^4$ and adding (22), (23), (24) and (25), we build a variational problem

$$\mathbb{B}((u,\varphi,\xi,s),(\tilde{u},\tilde{\varphi},\tilde{\xi},\tilde{s}))=\mathbb{L}(\tilde{u},\tilde{\varphi},\tilde{\xi},\tilde{s}), \tag{26}$$

where $\mathbb{B}:\mathcal{V}\times\mathcal{V}\to\mathbb{C}$ is given by

$$\begin{aligned}\mathbb{B}((u,\varphi,\xi,s),(\tilde{u},\tilde{\varphi},\tilde{\xi},\tilde{s}))=\ &\alpha\langle u_x,\tilde{u}_x\rangle_V+\lambda\langle\varphi-u,\tilde{\varphi}-\tilde{u}\rangle_V\\ &+k\langle s-\xi-\varphi_x,\tilde{s}-\tilde{\xi}-\tilde{\varphi}_x\rangle_V\\ &+b\langle\xi_x,\tilde{\xi}_x\rangle_V+b\langle s_x,\tilde{s}_x\rangle_V\end{aligned}$$

and, $\mathbb{L}:\mathcal{V}\to\mathbb{C}$ is continuous and linear operator

$$\mathbb{L}(\tilde{u},\tilde{\varphi},\tilde{\xi},\tilde{s})=\langle g^1,\tilde{u}\rangle_V+\langle g^2,\tilde{\varphi}\rangle_V+\langle g^3,\tilde{\xi}\rangle_V+\langle g^4,\tilde{s}\rangle_V.$$

We define in $\mathcal{V}$ the norm $||(u,\varphi,\xi,s)||_{\mathcal{V}}^2=B((u,\varphi,\xi,s),(u,\varphi,\xi,s))$. It is easy to see that with this norm, $\mathbb{B}$ is a continuous coercive sesquilinear form on $\mathcal{V}$. Therefore, by Lax-Milgram theorem, there exists a unique $(u,\varphi,\xi,s)\in\mathcal{V}$ solution of (26), for all $(\tilde{u},\tilde{\varphi},\tilde{\xi},\tilde{s})\in\mathcal{V}$. By the standad theory in the eliptic equations, see [13], chapter 1, (18), (19), (20) and (21) lead to $u,\varphi,\xi,s\in H^2(0,L)$, and then, $u,\varphi,\xi,s\in H_0^1(0,L)\cap H^2(0,L)$. From (10), (12), (14) and (16) we have $w,\phi,\eta,z\in H_0^1(0,L)$. So, we have $U\in D(\mathcal{A})$ and the unique solution of $-AU=F$ follows. Note that $\|U\|_{\mathcal{H}}\leq K\|F\|_{\mathcal{H}}$ with K a positive constant independent of U, that is $\|A^{-1}F\|_{\mathcal{H}}\leq K\|F\|_{\mathcal{H}}$. Thus, we conclude that $0\in\rho(\mathcal{A})$. □

The well-posedness of (4)-(7) is ensured by the following theorem.

Theorem 1. *For $U_0 \in \mathcal{H}$, there exists a unique weak solution U of* (8) *satisfying*

$$U \in C^0((0,\infty);\mathcal{H}). \tag{27}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C^0((0,\infty);D(\mathcal{A})) \cap C^1((0,\infty);\mathcal{H}). \tag{28}$$

Proof. As $D(\mathcal{A})$ is dense in $\mathcal{H}$, $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$. As consequence of the Lumer-Philips theorem, see [13], Theorem 1.2.4, p. 3., we have that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. From semigroup theory, see [14], p. 100. $U(t) = e^{t\mathcal{A}}U_0$ is the unique solution of (8) satisfying (27) and (28). □

3. Stability

Consider the following results.

Theorem 2 (Gagliardo-Niremberg)**.** *Let j and m be integers satisfying $0 \leq j < m$, and let $1 \leq q, r \leq \infty$, and $p \in \mathbb{R}$, $\frac{j}{m} \leq a \leq 1$ such that*

$$\frac{1}{p} - \frac{j}{n} = a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

Then, for any $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, there are two positive constants C_1, C_2 such that

$$|D^j u|_{L^p(\Omega)} \leq C_1 |D^m u|^a_{L^r(\Omega)} |u|^{1-a}_{L^q(\Omega)} + C_2 |u|_{L^q(\Omega)}.$$

In particular, for any $u \in W_0^{m,r}(\Omega) \cap L^q(\Omega)$, the constant C_2 can be taken as zero.

Theorem 3 (Gearhart-Huang-Prüss)**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then, $S(t)$ is exponentially stable if, only if,*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \text{ the resolvent set of } \mathcal{A}.$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty.$$

Proof. See [7, 12, 15]. □

Lemma 2. *Let H be a Hilbert space and $B, L : H \to H$ bounded linear operators, where L has a limited inverse. If $||B||_H < \dfrac{1}{||L^{-1}||_H}$, then $B + L$ is a bounded and invertible linear operator.*

Proof. First we prove that $B+L$ is invertible, i. e., $B+L$ is bijective. Let $y \in H$. For $x \in H$, $P(x) = L^{-1}y - L^{-1}Bx$ is a bounded linear operator. On the other hand,

$$\begin{aligned} ||P(z) - P(x)||_H &= ||L^{-1}Bz - L^{-1}Bx||_H \\ &\leq ||L^{-1}||_H ||Bz - Bx||_H \\ &\leq ||L^{-1}||_H ||B||_H ||z - x||_H \\ &\leq C||z - x||_H. \end{aligned}$$

Since $C = ||L^{-1}||_H ||B||_H$ we have $0 < C < 1$. By contraction mapping theorem, there exists a unique point $x \in X$ such that $P(x) = x$. Since $L^{-1}y - L^{-1}Bx = x$ we get $Lx = y - Bx$ and then x is the unique solution of the problem $(B+L)x = y$. We have that $B+L$ is surjective. To see that $B+L$ is also injective, note that $(B+L)x = 0$ leads to $x = L^{-1}Bx$ and then $||x||_H \leq C||x||_H$. As $C < 1$ we get $x = 0$. Finally, as $B+L$ is bounded, by the closed graph theorem, it follows that $(B+L)^{-1}$ is also bounded. □

Lemma 3. *$i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. Defining $L = -I : H \to H$ we obtain $||L|| = ||L^{-1}|| = 1$. As $0 \in \rho(\mathcal{A})$ we have that $\mathcal{A}$ is invertible, and we can define $B = i\lambda\mathcal{A}^{-1}$. For

$$|\lambda| < \frac{1}{||\mathcal{A}^{-1}||}, \tag{29}$$

follows that

$$||B|| = ||i\lambda\mathcal{A}^{-1}|| < 1 = \frac{1}{||L^{-1}||}.$$

From Lemma 2 the operator $B + L = i\lambda\mathcal{A}^{-1} - I$ is invertible. Writing

$$i\lambda I - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - I)$$

we deduce that $i\lambda I - \mathcal{A}$ is invertible because it is a composition of invertible operators.

From (29) we get that the real function $||(i\lambda I - A)^{-1}||$ is continuous in the interval

$$\left(-\frac{1}{||\mathcal{A}^{-1}||}, \frac{1}{||\mathcal{A}^{-1}||}\right).$$

If $i\mathbb{R} \subset \rho(\mathcal{A})$ is not true, then there exists $\theta \in \mathbb{R}$ with $\dfrac{1}{\|\mathcal{A}^{-1}\|} \leq |\theta| < \infty$, such that

$$\{i\beta \, : \, |\beta| < |\theta|\} \tag{30}$$

satisfies

$$\sup\Big\{||(i\beta - A)^{-1}|| \, : \, |\beta| < |\theta|\Big\} = \infty. \tag{31}$$

By (30) we can extract a sequence $\beta^n \to \theta$, $|\beta^n| < |\theta|$ and a sequence

$$U^n = (i\beta^n I - \mathcal{A})^{-1} F_n \subset D(\mathcal{A}), \; F^n \in \mathcal{H}, \; \|U^n\|_{\mathcal{H}} = 1.$$

Since $|\beta^n| < |\theta|$ from (31) we deduce that

$$|| \, (i\beta^n I - \mathcal{A})^{-1} \, F^n || \to \infty, \text{ as } n \to \infty,$$

or equivalently,

$$|| \, (i\beta^n I - \mathcal{A}) \, U^n || \to 0, \text{ as } n \to \infty,$$

that is,

$$i\beta^n u^n - w^n \to 0 \text{ in } H_0^1(0,L), \tag{32}$$

$$i\beta^n w^n - \alpha u_{xx}^n - \lambda(\varphi^n - u^n) + \mu_1 w^n \to 0 \text{ in } L^2(0,L), \tag{33}$$

$$i\beta^n \varphi^n - \phi^n \to 0 \text{ in } H_0^1(0,L), \tag{34}$$

$$i\beta^n \phi^n + \frac{1}{\rho_1}\left[k(s^n - \xi^n - \varphi_x^n)_x + \lambda(\varphi^n - u^n) + \mu_2 \phi^n\right] \to 0 \text{ in } L^2(0,L), \tag{35}$$

$$i\beta^n \xi^n - \eta^n \to 0 \text{ in } H_0^1(0,L), \tag{36}$$

$$i\beta^n \eta^n + \frac{1}{\rho_2}\left[-b\xi_{xx}^n - k(s^n - \xi^n - \varphi_x^n) + \mu_3 \eta^n\right] \to 0 \text{ in } L^2(0,L). \tag{37}$$

$$i\beta^n s^n - z^n \to 0 \text{ in } H_0^1(0,L), \tag{38}$$

$$i\beta^n z^n + \frac{1}{\rho_2}\left[-b s_{xx}^n + k(s^n - \xi^n - \varphi_x^n) + \gamma s^n + \mu_4 z^n\right] \to 0 \text{ in } L^2(0,L). \tag{39}$$

Multiplying $(i\beta^n I - A)U^n$ by U^n in $\mathcal{H}$, and taking the real part, and using (9), we get

$$\mathfrak{Re}\langle (i\beta^n I - A)U^n, U^n\rangle_{\mathcal{H}} = -\mu_1\|w^n\|^2 - \mu_2\|\phi^n\|^2 - \mu_3\|\eta^n\|^2 - \mu_4\|z^n\|^2.$$

As U^n bounded and $(i\beta^n I - A)U^n \to 0$ we obtain

$$w^n \to 0 \text{ in } L^2(0,L), \tag{40}$$

$$\phi^n \to 0 \text{ in } L^2(0,L), \tag{41}$$

$$\eta^n \to 0 \text{ in } L^2(0,L), \tag{42}$$

$$z^n \to 0 \text{ in } L^2(0,L). \tag{43}$$

From (40) and (32) we have $i\beta^n u^n \to 0$ in $L^2(0,L)$, and using $\beta^n \to \theta$, we get

$$u^n \to 0 \text{ in } L^2(0,L). \tag{44}$$

We need to prove that $u^n \to 0$ in $H_0^1(0,L)$. Using (41) in (34), (42) in (36) and (43) in (38), we found

$$\varphi^n \to 0 \text{ in } L^2(0,L), \tag{45}$$

$$\xi^n \to 0 \text{ in } L^2(0,L). \tag{46}$$

$$s^n \to 0 \text{ in } L^2(0,L). \tag{47}$$

Using (40), (44) and (45) in (33), we obtain

$$u_{xx}^n \to 0 \text{ in } L^2(0,L), \tag{48}$$

and integrating from 0 to x we have

$$u_x^n - u_x^n(0) \to 0 \text{ in } L^2(0, L). \tag{49}$$

Applying Gagliardo-Niremberg inequality with $\Omega = (0, L)$, $m = 2$, $a = \frac{1}{2}$, $p = 2$, $r = 2$, $j = 1$, $n = 1$, $q = 2$, we deduce that

$$||u_x^n(0)||_{L^2(0,L)} \leq C_1 ||u_{xx}^n(0)||^{\frac{1}{2}}_{L^2(0,L)} ||u^n(0)||^{\frac{1}{2}}_{L^2(0,L)} + C_2 ||u^n(0)||_{L^2(0,L)}.$$

From (44), $u^n(0) \to 0$ in $L^2(0, L)$ and by (48), u_{xx}^n is bounded in $L^2(0, L)$. So, we have $u_x^n(0) \to 0$ in $L^2(0, L)$ and by (49) $u_x^n \to 0$ in $L^2(0, L)$. As $u^n \to 0$ in $L^2(0, L)$ and $u_x^n \to 0$ in $L^2(0, L)$, then

$$u^n \to 0 \text{ in } H_0^1(0, L). \tag{50}$$

Applying same idea, we can deduce that

$$\varphi^n \to 0 \text{ in } H_0^1(0, L). \tag{51}$$

$$\xi^n \to 0 \text{ in } H_0^1(0, L). \tag{52}$$

$$s^n \to 0 \text{ in } H_0^1(0, L). \tag{53}$$

From (40), (41), (56), (50), (51), (52) and (53), we obtain $||U^n||_{\mathcal{H}} \to 0$, contradicting $||U^n||_{\mathcal{H}} = 1$. □

Now, we present and prove our main result.

Theorem 4. *The semigroup $S(t) = e^{\mathcal{A}t}$, $t \geq 0$, generated by $\mathcal{A}$ is exponentially stable.*

Proof. As $i\mathbb{R} \subset \rho(\mathcal{A})$, it remains to prove that

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty. \tag{54}$$

If (54) is not true, there exists a sequence $\beta^n \to \infty$, without loss of generality $\beta^n > 0$, a sequence of complex vectors $F^n \in \mathcal{H}$ and a corresponding sequence $U^n = (u^n, w^n, \varphi^n, \phi^n, \xi^n, \eta^n, s^n, z^n)^T \in D(\mathcal{A})$, with $\|U^n\|_{\mathcal{H}} = 1$,

$$U^n = (i\beta I - \mathcal{A})^{-1} F^n, \tag{55}$$

such that,

$$\frac{||\,(i\beta^n I - \mathcal{A})^{-1}\, F^n||_{\mathcal{H}}}{||F^n||_{\mathcal{H}}} > n, \;\; \forall\, n > n_0,$$

or, equivalently

$$||F_n||_{\mathcal{H}} < \frac{||\,(i\beta^n I - \mathcal{A})^{-1}\, F^n||_{\mathcal{H}}}{n} = \frac{||U^n||_{\mathcal{H}}}{n} = \frac{1}{n}, \;\; \forall\, n > n_0,$$

from where follows that $F_n \to 0$ in $\mathcal{H}$ and from (55),

$$\|(i\beta^n I - \mathcal{A})\, U^n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty. \tag{56}$$

Taking the inner product of $(i\beta^n I - \mathcal{A})\, U^n$ with U^n in $\mathcal{H}$ we have

$$\langle (i\beta^n I - \mathcal{A})\, U^n, U^n\rangle_{\mathcal{H}} = \langle F^n, U^n\rangle_{\mathcal{H}}$$

that is,

$$i\beta^n ||U^n||_{\mathcal{H}} - \langle \mathcal{A}U^n, U^n\rangle_{\mathcal{H}} = \langle F^n, U^n\rangle_{\mathcal{H}}$$

Taking the real part and using (9) we get

$$\mu_1\|w^n\|^2 + \mu_2\|\phi^n\|^2 + \mu_3\|\eta^n\|^2 + \mu_4\|z^n\|^2 = \mathfrak{Re}\,\langle F^n, U^n\rangle_{\mathcal{H}}$$

As U^n is bounded and $F^n \to 0$ in $\mathcal{H}$ we have that

$$w^n \to 0 \text{ in } L^2(0,L), \tag{57}$$

$$\phi^n \to 0 \text{ in } L^2(0,L), \tag{58}$$

$$\eta^n \to 0 \text{ in } L^2(0,L), \tag{59}$$

$$z^n \to 0 \text{ in } L^2(0,L), \tag{60}$$

From (56) we obtain,

$$i\beta^n u^n - w^n \to 0 \text{ in } H_0^1(0,L), \tag{61}$$

$$i\beta^n w^n - \alpha u^n_{xx} - \lambda(\varphi^n - u^n) + \mu_1 w^n \to 0 \text{ in } L^2(0,L), \tag{62}$$

$$i\beta^n \varphi^n - \phi^n \to 0 \text{ in } H_0^1(0,L), \tag{63}$$

$$i\beta^n\phi^n + \frac{1}{\rho_1}\left[k(s^n - \xi^n - \varphi^n_x)_x + \lambda(\varphi^n - u^n) + \mu_2\phi^n\right] \to 0 \text{ in } L^2(0,L), \tag{64}$$

$$i\beta^n \xi^n - \eta^n \to 0 \text{ in } H_0^1(0,L), \tag{65}$$

$$i\beta^n\eta^n + \frac{1}{\rho_2}\left[-b\xi^n_{xx} - k(s^n - \xi^n - \varphi^n_x) + \mu_3\eta^n\right] \to 0 \text{ in } L^2(0,L). \tag{66}$$

$$i\beta^n s^n - z^n \to 0 \text{ in } H_0^1(0,L), \tag{67}$$

$$i\beta^n z^n + \frac{1}{\rho_2}\left[-bs^n_{xx} + k(s^n - \xi^n - \varphi^n_x) + \gamma s^n + \mu_4 z^n\right] \to 0 \text{ in } L^2(0,L). \tag{68}$$

Using (57) in (61), (58) in (63), (59) in (65) and (60) in (67), we obtain

$$\beta^n u^n \to 0 \text{ in } L^2(0,L),$$

$$\beta^n \varphi^n \to 0 \text{ in } L^2(0,L),$$

$$\beta^n \xi^n \to 0 \text{ in } L^2(0,L),$$

$$\beta^n s^n \to 0 \text{ in } L^2(0,L).$$

Taking into account that $\beta^n \to \infty$, the last convergence leads to

$$u^n \to 0 \text{ in } L^2(0,L), \tag{69}$$

$$\varphi^n \to 0 \text{ in } L^2(0,L), \tag{70}$$

$$\xi^n \to 0 \text{ in } L^2(0,L), \tag{71}$$

$$s^n \to 0 \text{ in } L^2(0,L), \tag{72}$$

however, we need to prove that

$$u^n \to 0 \text{ in } H_0^1(0,L), \tag{73}$$

$$\varphi^n \to 0 \text{ in } H_0^1(0,L), \tag{74}$$

$$\xi^n \to 0 \text{ in } H_0^1(0,L), \tag{75}$$

$$s^n \to 0 \text{ in } H_0^1(0,L). \tag{76}$$

From (57), (69) and (70), we deduce from (62) that

$$u_{xx}^n \to 0 \text{ in } L^2(0,L).$$

Now, integrating from 0 to x and applying Gagliardo-Niremberg inequality as in the proof of lemma 3, we obtain $u_x^n \to 0$ in $L^2(0,L)$ and then, we obtain the following convergence

$$u^n \to 0 \text{ in } H_0^1(0,L). \tag{77}$$

Applying the same idea, we can prove that

$$\varphi^n \to 0 \text{ in } H_0^1(0,L), \tag{78}$$

$$\phi^n \to 0 \text{ in } L^2(0,L), \tag{79}$$

$$\xi^n \to 0 \text{ in } H_0^1(0,L), \tag{80}$$

$$\eta^n \to 0 \text{ in } L^2(0,L), \tag{81}$$

$$s^n \to 0 \text{ in } H_0^1(0,L), \tag{82}$$

$$z^n \to 0 \text{ in } L^2(0,L). \tag{83}$$

Thus, it follows from (57), (77), (78), (79), (80), (81), (82) and(83) that $||U^n||_{\mathcal{H}} \to 0$, which is a contradiction with $||U^n||_{\mathcal{H}} = 1$. □

FINAL REMARKS AND OPEN PROBLEMS

The instability of suspension bridges is a worrisome question for engineering, mathematics and physics. Several models were introduced in the literature to solve each instability issue, for instance, [1, 2, 4–6, 8, 11]. The most used model for beams in the literature is given by Timoshenko theory, which deals with both the effect of rotary inertia and shear deformation. We introduce the system in which the deck is modeled by laminated beams. It consists of two Timoshenko beams connected by an adhesive layer and proves the exponential stability by using just the internal damping given by friction. Other stabilizing mechanisms can be introduced for the model proposed in this manuscript, and consequently, the stability analysis in each situation is an important and open question to be considered.

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