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On the Distribution of the Product of Inverse Pareto and Exponential Random Variables

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Abstract

This article considers Inverse Pareto and Exponential distributions to create the distribution of the product. The researchers derived its properties, such as; survival functions and hazard functions, and used the model criterion such as Sum Square Error (SSE), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) in estimating the parameters for deriving the best joint distribution between monthly precipitation and temperature in the Philippines from 1974 to 2013. The results showed that considering the monthly precipitation and temperature data, the distribution of the product of Inverse Pareto and Exponential outperformed the other existing distribution of the product.

Keywords: product distribution, inverse Pareto, exponential

1.0 Introduction

Determining the distributions and function of a random variable is a critical issue, and it has piqued the interest of multiple academics due to its numerous applications (Ly et al., 2019; Nadarajah & Espejo, 2006). Also, it has importance in biological and physical sciences, engineering, physics, various theories, economics, and other scientific fields.

Many practical research issues include product distribution in two continuous random variables. The study of Grubel (1968) considers the typical portfolio model and the scenario of many overseas investments where the domestic currency is related to the product of random variables. In creating a forecasted model, Feldstein (1971) studied that the dependent variable was proportional to the product random variable. Another great importance of the product of random variables is that finds application in a broad range of wireless communication systems (Tse & Viswanath, 2005). Also, Thomas (2013) finds applications in the financial sector as they modeled the density of products as the total amount of money received by the company for goods sold during a certain period of time. Pizon and Paluga (2022) derived the best joint distribution between the meteorological data.

Several writers have looked at product distribution, mainly when independent random variables are from the same or distinct families. These include the beta family (Tang & Gupta, 1984), student's t family (Wallgren, 1980), uniform family (Sakamoto, 1943), gamma family (Stuart, 1962), exponential family (Malik & Trudel, 1986), Inverse Burr distribution (Pizon & Arcede, 2018), and Lomax distribution (Arcede & Macalos, 2016).

This article investigated the distribution of the product of independent random variable XY following Inverse Pareto (X) and Exponential

(Y) distribution. The study also investigated the properties such as; survival functions and hazard functions. Moreover, the model criterion, such as SSE, AIC, and BIC, was applied to determine the best distribution of the product.

The researchers used several custom functions in the computations in this work. The upper incomplete gamma function is included in this formula:

$$\Gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt. \qquad s \tag{1}$$

The following results of Gradshteyn and Ryzhik (2014) are required in the following discussions.

Lemma 1. *For $p \in R$,*

$$\frac{d}{dp}\Gamma\big(t, v(p)\big) = -v(p)^{t-1} e^{-v(p)} \frac{d}{dp} v(p) \tag{2}$$

Lemma 2. *For $a > 0$, $b > 0$, and $p > 0$,*

$$\int_0^\infty \frac{e^{-px}}{(ax \pm b)^n} dx = \frac{p^{n-1} e^{\pm\frac{pb}{a}}}{a^n} \Gamma(-n+1, \pm\frac{pb}{a}) \tag{3}$$

2.0 Probability Density and Cumulative Functions

If X follows Inverse Pareto distribution, the PDF and CDF have the form

$$f_X(x;\theta,\tau) = \begin{cases} \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \\ 0 \end{cases} \tag{4}$$

and

$$F_X(x;\theta,\tau) = \begin{cases} \left(\frac{x}{x+\theta}\right)^{\tau} \\ 0 \end{cases} \tag{5}$$

respectively, positive values of x, τ, and θ.

If Y follows Exponential distribution, the PDF and CDF have the form

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta}\, e^{-\frac{y}{\theta}} \\ 0 \end{cases} \tag{6}$$

and

$$F_Y(y;\theta) = \begin{cases} 1 - e^{-\frac{y}{\theta}} \\ 0 \end{cases} \tag{7}$$

respectively, positive values of y and θ.

3.0 Distribution of the Product XY

Theorem 1. Suppose X and Y are independent and distributed as shown in Equations (4) and (6). Then the *CDF* of $Z = XY$ may thus be represented as follows for $z > 0$.

$$F_Z(z) = \begin{cases} 0, \ if\ z \le 0 \\ \frac{z^\tau e^{\frac{z}{\theta^2}}}{\theta^{2\tau}} \Gamma\left(-\tau+1, \frac{z}{\theta^2}\right), \ if\ z \ge 0 \end{cases} \tag{8}$$

Proof: The corresponding *CDF* to equation (4) is $F_X(x) = \left(\frac{x}{x+\theta}\right)^\tau$. As a result, the *CDF* of XY can be written as:

$$F_Z(z) = \int_0^\infty F_X\left(\frac{Z}{y}\right) f(y) dy = \frac{1}{\theta}\int_0^\infty \frac{e^{-\frac{y}{\theta}}}{(z+\theta y)^\tau} dy$$

Applying Lemma 2 in the above integral gives us,

$$F_Z(z) = \frac{z^\tau e^{\frac{z}{\theta^2}}}{\theta^{2\tau}} \Gamma\left(-\tau+1, \frac{z}{\theta^2}\right).$$

■

Theorem 2. Assume that X and Y are independent and, as shown in Equations (4) and (6). Then the PDF of $Z = XY$ may thus be represented as follows for $z > 0$.

$$f_Z(z) = \begin{cases} \Gamma\left(-\tau+1, \frac{z}{\theta^2}\right) \tau z^{\tau-1} e^{\frac{z}{\theta^2}} \theta^{-2\tau} \\ \qquad 0, \ if\ z \le 0 \\ +\Gamma\left(-\tau+1, \frac{z}{\theta^2}\right) z^\tau e^{\frac{z}{\theta^2}} \theta^{-2\tau-2} - \theta^{-2}, \ if\ z \ge 0 \end{cases} \tag{9}$$

Proof: The probability density function $f_Z(z)$ in Equation (9) follows by differentiation using Lemma 1. ■

Figures 1 and 2 depict the graphs of Equations (8) and (9), respectively, for various values of the parameters and. The influence of parameters is clear.

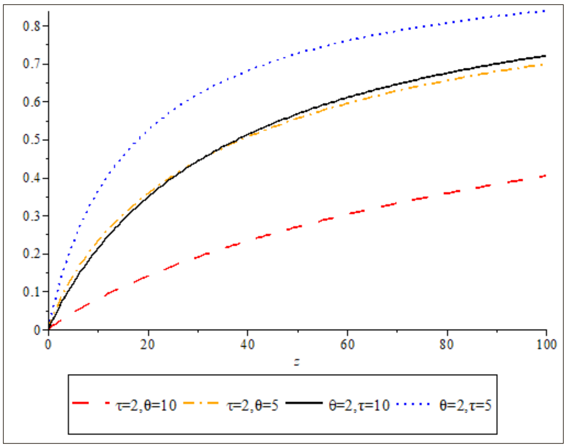

Figure 1. *Plots of Cumulative Density Function*

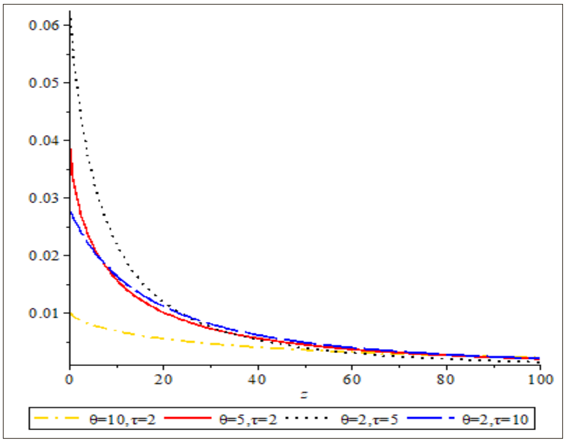

Figure 2. *Plots of the Probability Density Function*

Corollary 3. Assume that X and Y are independent and distributed according to Equations (4) and (6). The survival function of $Z = XY$ may thus be defined as follows for $z > 0$.

$$S_Z(z) = \begin{cases} 0, \; if \; z \leq 0 \\ 1 - \frac{z^{\tau} e^{\frac{z}{\theta^2}}}{\theta^{2\tau}} \Gamma\left(-\tau + 1, \frac{z}{\theta^2}\right), \; if \; z \geq 0 \end{cases} \quad (10)$$

Proof: The results follow directly by applying the formula of survival function $S_Z(z) = 1 - F_Z(z)$.

Corollary 4. Assume that X and Y are independent and distributed according to Equations (4) and (6). The hazard function of $Z = XY$ may thus be defined as follows for $z > 0$.

$$H_Z(z) = \begin{cases} 0, \; if \; z \leq 0 \\ \frac{\left(\Gamma\left(-\tau+1, \frac{z}{\theta^2}\right)\tau z^{\tau-1} e^{\frac{z}{\theta^2}} \theta^{-2\tau} + \Gamma\left(-\tau+1, \frac{z}{\theta^2}\right) z^{\tau} e^{\frac{z}{\theta^2}} \theta^{-2\tau-2} - \theta^{-2}\right)}{1 - \frac{z^{\tau} e^{\frac{z}{\theta^2}}}{\theta^{2\tau}} \Gamma\left(-\tau+1, \frac{z}{\theta^2}\right)}, \; if \; z \geq 0 \end{cases} \quad (11)$$

Proof: By definition of hazard function $H_Z(z) = \frac{f_Z(z)}{F_Z(z)}$ ■

4.0 Data Fitting and CDF Model Selection

This study used secondary data obtained from the available online website NASA Langley Research Center (n.d.), which provides the data sets containing average monthly precipitation (mm) and temperature (^{o}C) in the province of Iloilo, Philippines specifically in the Municipality of Conception. The data consists of 468 (months) starting from January 1974 to December 2012 and it was used to fit the CDF model.

Table 1 displays the descriptive statistics of monthly precipitation and temperature. The table revealed that the Philippines has minimum monthly precipitation of 0.36 and maximum monthly precipitation of 629.82, and it happened in March 2012 and December 1986, respectively. This indicate that a potential low rainfall during March 2012 and heavy rainfall during December 1986. Similarly, the results also revealed that the Philippines has a minimum monthly temperature of 24.77 and a maximum monthly temperature of 30.09, and it happened in January 1974 and May 2012, respectively. This suggests a notably cool period for January 1974 and warm period during the month of May 2012.

Table 1. *Descriptive Statistics of Monthly Precipitation and Temperature*

	N	Minimum	Maximum	Mean	Standard Deviation
Precipitation	468	0.36	629.82	169.63	121.18
Temperature	468	24.77	30.09	27.54	1.09

This part presents the procedures for analysis and results for the model fitting with the data. The researcher used maple software in estimating the parameters to obtain the SSE, AIC, and BIC values. The steps are the following:

Step 1: Obtain the empirical frequency of each data point $x_1, x_2, x_3, \ldots, x_n$ using the empirical formula of *CDF* defined a

$$F_E(x_j) = \frac{1}{p}\sum_{k=1}^{p} 1(x_k \leq x_j).$$

for $j = 1, 2, 3, \ldots, p$.

Step 2: Obtain the sum of the square error

$$SSE = \sum_{j=1}^{p}\left(F(x_j) - F_E(x_j)\right)^2$$

in which p is the data points, F is the Fitted CDF, and F_E is the empirical CDF.

Step 3: Minimize the SSE by applying the optimization method from the calculus. The researchers obtained the ∂SSE concerning each parameter, set the $\partial SSE = 0$, and solved for the parameters. The calculated values of the parameters will minimize the SSE.

Step 4: Substitute the value of the parameters to the CDF to get the maximum value of log-likelihood $l(\theta)$.

Step 5: The CDF with the smallest value of SSE, AIC, and BIC is the best-fit model.

Table 2 presents the CDF fitting among the Inverse Pareto and Exponential, Pareto and Exponential, Pareto and Rayleigh. It can be seen from the table that the generated CDF which is the Inverse Pareto and Exponential has the smallest value of SSE = 5.8308, AIC = 9085.8487 and BIC = 9087.1892. The results imply that Inverse Pareto and Exponential have the suited fit.

Table 2. *Cumulative Density Function Fitting of Precipitation and Temperature*

CDF	Parameters	SSE	AIC	BIC
Inverse Pareto and Exponential	$\tau = 31.23$ $\theta = 9.64$	5.83	9085.85	9087.19
Pareto and Exponential	$a = 7.43$ $c = 0.56$ $\gamma = 0.24$	105.74	10608.34	10609.68
Pareto and Rayleigh	$a = 17.32$ $b = 0.68$ $c = 0.56$	117.85	10914.63	10915.97

5.0 Conclusion

The probability density function, cumulative density function, survival function, and hazard function for the distributions of the product XY where X and Y are Inverse Pareto and Exponential random variables distributed independently of each other have been determined analytically in this study. The researchers presented the findings in the form of graphs depicting product distributions. Finally, they illustrated an application from the derived CDF with given specific data, the Inverse Pareto and Exponential distribution outperformed the other existing product distribution.

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