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ON IMPLICATIVE AND POSITIVE IMPLICATIVE GE ALGEBRAS

Abstract

GE algebras (generalized exchange algebras), transitive GE algebras (tGE algebras, for short) and aGE algebras (that is, GE algebras verifying the antisymmetry) are a generalization of Hilbert algebras. Here some properties and characterizations of these algebras are investigated. Connections between GE algebras and other classes of algebras of logic are studied. The implicative and positive implicative properties are discussed. It is shown that the class of positive implicative GE algebras (resp. the class of implicative aGE algebras) coincides with the class of generalized Tarski algebras (resp. the class of Tarski algebras). It is proved that for any aGE algebra the property of implicativity is equivalent to the commutative property. Moreover, several examples to illustrate the results are given. Finally, the interrelationships between some classes of implicative and positive implicative algebras are presented.

Keywords: GE algebra, tGE algebra, BCK algebra, Hilbert algebra, (positive) implicativity.

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1. Introduction

L. Henkin [6] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, A. Monteiro [16] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, K. Iséki [9] introduced a new notion called a BCK

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algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [15], and generalize the concept of implicative algebras (see [1]). In 2021, P. Cintula and C. Noguera [4] presented of the most important logics that one can find in the literature. In particular, they considered the $\mathcal{BCK}$ logic and its many extensions. To solve some problems on BCK algebras, Y. Komori [14] introduced BCC algebras. These algebras (also called BIK$^+$-algebras) are an algebraic model of $\mathcal{BIK}^+$ logic. In [12], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim defined BE algebras. In 2008, A. Walendziak [18] defined commutative BE algebras and proved that they are BCK algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [3] introduced the notion of pre-BCK algebra. A BCK algebra is just a pre-BCK algebra satisfying also the antisymmetry. In 2016, A. Iorgulescu [7] introduced new generalizations of BCK and Hilbert algebras (RML, aBE, pi-BE, pimpl-RML algebras and many others). Recently, R. Bandaru et al. [2] introduced the concepts of GE algebra (generalized exchange algebra) and transitive GE algebra (tGE algebra for short). These algebras are a generalization of Hilbert algebras.

In 1978, K. Iséki and S. Tanaka [10] introduced the concepts of implicativity and positive implicativity in the theory of BCK algebras. The present paper is a continuation of the author's paper [19], where the property of implicativity for various generalizations of BCK algebras was studied. Implicative BE algebras were presented in [21] (see also [23]).

Here we consider RML, BE, GE, tGE, pre-BCC and pre-BCK algebras and investigate the implicative and positive implicative properties for these algebras. We obtain some characterizations of GE and transitive GE algebras. We study connections between GE algebras and other classes of algebras of logic. We show that the class of positive implicative GE algebras (resp. the class of implicative GE algebras satisfying the property of antisymmetry) coincides with the class of generalized Tarski algebras (resp. the class of Tarski algebras). We prove that for any GE algebra with the antisymmetry the property of implicativity is equivalent to the commutative property. Moreover, we give several examples to illustrate the results. Finally, we present the interrelationships between the classes of implicative and positive implicative algebras considered here.

2. Preliminaries

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We consider the following list of properties ([7]) that can be satisfied by $\mathcal{A}$ (the properties in the list are the most important properties satisfied by a BCK algebra):

(An) (Antisymmetry) $x \rightarrow y = 1 = y \rightarrow x \Longrightarrow x = y$,

(B) $(y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,

(BB) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,

(C) $[x \rightarrow (y \rightarrow z)] \rightarrow [y \rightarrow (x \rightarrow z)] = 1$,

(D) $y \rightarrow [(y \rightarrow x) \rightarrow x] = 1$,

(Ex) (Exchange) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(K) $x \rightarrow (y \rightarrow x) = 1$,

(L) (Last element) $x \rightarrow 1 = 1$,

(M) $1 \rightarrow x = x$,

(Re) (Reflexivity) $x \rightarrow x = 1$,

(Tr) (Transitivity) $x \rightarrow y = 1 = y \rightarrow z \Longrightarrow x \rightarrow z = 1$,

(*) $y \rightarrow z = 1 \Longrightarrow (x \rightarrow y) \rightarrow (x \rightarrow z) = 1$,

(**) $y \rightarrow z = 1 \Longrightarrow (z \rightarrow x) \rightarrow (y \rightarrow x) = 1$.

The following lemma will be used many times throughout the rest of this paper.

Lemma 2.1 ([7], Proposition 2.1 and Theorem 2.7). *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) *(M) + (B) imply (Re), (*) and (**),*

(ii) *(M) + (*) imply (Tr),*

(iii) *(M) + (**) imply (Tr),*

(iv) *(An) + (C) imply (Ex),*

(v) *(M) + (BB) imply (B).*

Definition 2.2 ([7]).

1. A *RML algebra* is an algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ verifying (Re), (M), (L).
2. A *BE algebra* is a RML algebra verifying (Ex).
3. An *aBE algebra* is a BE algebra verifying (An).
4. A *pre-BCC algebra* is a RML algebra verifying (B).
5. A *pre-BCK algebra* is a pre-BCC algebra verifying (Ex).
6. A *BCC algebra* is a pre-BCC algebra verifying (An).
7. A *BCK algebra* is a pre-BCK algebra verifying (An).

Denote by **RML**, **BE**, **aBE**, **pre-BCC**, **pre-BCK**, **BCC** and **BCK** the classes of RML, BE, aBE, pre-BCC, pre-BCK, BCC and BCK algebras, respectively.

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \to y = 1.$$

It is known that $\leq$ is an order relation in BCC and BCK algebras. By definition, in RML and BE algebras, $\leq$ is a reflexive relation; in aBE algebras, $\leq$ is reflexive and antisymmetric. By Lemma 2.1 (i) and (ii), in pre-BCC and pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

Definition 2.3 ([2]). A *GE algebra* (*generalized exchange algebra*) is an algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ verifying (Re), (M) and

(GE) $x \to (y \to z) = x \to [y \to (x \to z)]$.

Following [2],

- a *transitive GE algebra* (*tGE algebra*, for short) is a GE algebra verifying (B),
- an *aGE algebra* is a GE algebra verifying (An).

Denote by **GE**, **tGE** and **aGE** the classes of all GE algebras, transitive GE algebras and aGE algebras, respectively.

Proposition 2.4. Any GE algebra satisfies the folowing property

(pi) $x \to y = x \to (x \to y)$.

Proof: Let $\mathcal{A}$ be a GE algebra and $x, y \in A$. We have $x \to y \overset{(M)}{=} x \to (1 \to y) \overset{(GE)}{=} x \to [1 \to (x \to y)] \overset{(M)}{=} x \to (x \to y)$, that is, (pi) holds in $\mathcal{A}$. □

Example 2.5. Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	d	d	1	1
c	a	1	1	1	1	1
d	a	1	1	1	1	1
e	a	1	1	1	1	1
1	a	b	c	d	e	1

We can observe that the properties (Re), (M), (L), (GE) (hence (pi)) are satisfied. Therefore, $(A, \to, 1)$ is a GE algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, c)$; (Tr) and (B) for $(x, y, z) = (a, e, c)$. Then, $\mathcal{A}$ is not transitive.

Example 2.6. Let $A = \{a, b, c, d, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	1
a	1	1	c	c	1
b	1	1	d	d	1
c	a	a	1	1	1
d	b	b	1	1	1
1	a	b	c	d	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies (Re), (M), (L), (GE), (B). It does not verify (An) for $x = a, y = b$; (Ex) for $x = a, y = b, z = c$. Thus $\mathcal{A}$ is a tGE algebra which is not a pre-BCK algebra.

Following [7], a *pi-RML algebra* (respectively: *pi-BE, pi-aBE, pi-pre-BCC, pi-pre-BCK, pi-BCC, pi-BCK algebra*) is a RML algebra (respectively: BE, aBE, pre-BCC, pre-BCK, BCC, BCK algebra) verifying (pi).

Denote by **pi-RML**, **pi-BE**, **pi-aBE**, **pi-pre-BCC**, **pi-pre-BCK**, **pi-BCC**, **pi-BCK** the classes of pi-RML, pi-BE, pi-aBE, pi-pre-BCC, pi-pre-BCK, pi-BCC, pi-BCK algebras, respectively.

PROPOSITION 2.7. *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) (Re) + (pi) *imply* (L),

(ii) (Ex) + (pi) *imply* (GE).

PROOF: (i) It follows immediately from Proposition 6.4 (ii) of [7].

(ii) Let $x, y, z \in A$. We obtain

$$x \rightarrow (y \rightarrow z) \overset{\text{(pi)}}{=} x \rightarrow [x \rightarrow (y \rightarrow z)] \overset{\text{(Ex)}}{=} x \rightarrow [y \rightarrow (x \rightarrow z)].$$

Thus (GE) holds. □

By Propositions 2.4 and 2.7 (i), we have

COROLLARY 2.8. *Any GE algebra is a pi-RML algebra.*

By Proposition 2.7 (ii), we get

COROLLARY 2.9. *Any pi-BE algebra is a GE algebra.*

Remark 2.10. By Corollaries 2.8 and 2.9, **pi-BE** $\subset$ **GE** $\subset$ **pi-RML**. Observe that these inclusions are proper. Indeed, the algebra given in Example 2.5 is a GE algebra not satisfying (Ex). The algebra from Example 10.1 of [8] is a pi-RML algebra that is not a GE algebra.

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if **X** and **Y** are classes of algebras, then $\mathbf{X} \longrightarrow \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

In [17], S. Tanaka introduced the notion of commutativity in the theory of BCK algebras. A BCK algebra $\mathcal{A} = (A, \rightarrow, 1)$ is called *commutative* if, for all $x, y \in A$,

(Com) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.

H. Yutani [22] proved that the class of commutative BCK algebras is equationally definable. A. Walendziak [18] showed that any commutative

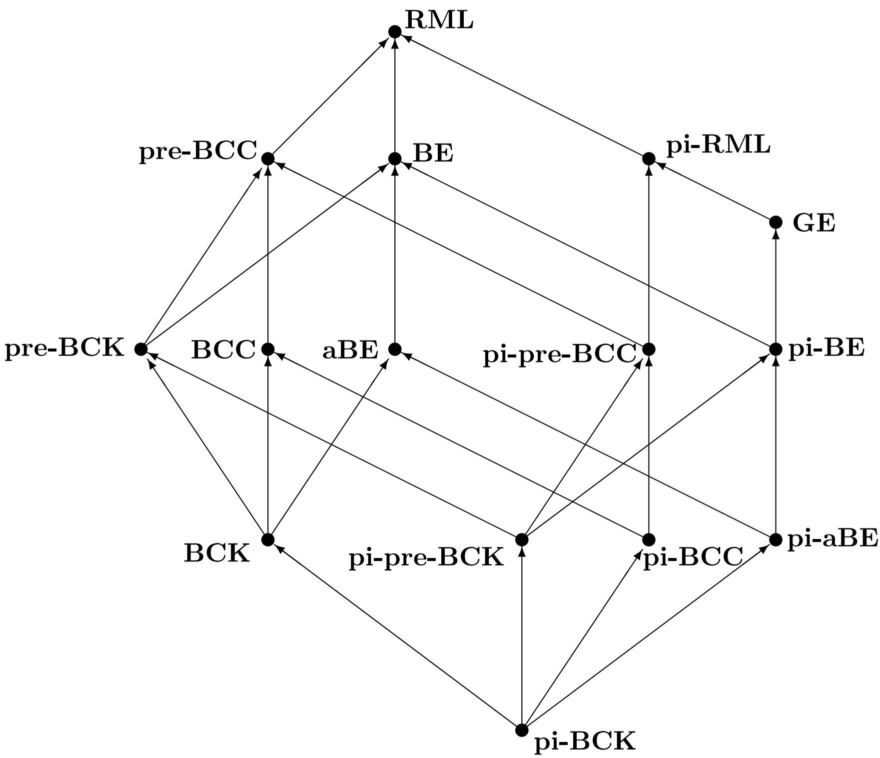

Figure 1.

BE algebra is a BCK algebra. The property of commutativity for other generalizations of BCK algebras was investigated in [20].

As in the case of BCK algebras, we define:

DEFINITION 2.11. A RML algebra $\mathcal{A} = (A, \to, 1)$ is called *commutative* if it satisfies (Com).

Denote by **com-RML** the class of commutative RML algebras. Similarly, if **X** is a subclass of the class **RML**, then **com**–**X** denotes the class of all commutative algebras belonging to **X**.

Remark 2.12. Since every commutative BE algebra is a BCK algebra, we have **com-BE** = **com-BCK**. Moreover, following [20], we obtain

com-BE = **com-pre-BCC** = **com-pre-BCK** = **com-BCC** = **com-BCK**.

As a preparation for the next results we need the following

LEMMA 2.13. ([20], Proposition 3.3) *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2,0)$ *verifying* (M) *and* (Com). *Then* $\mathcal{A}$ *verifies* (An).

Remark 2.14. Note that commutative GE algebras were introduced and studied in [2].

3. On GE and transitive GE algebras

First we present the following

PROPOSITION 3.1. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2,0)$. Then the following hold:

(i) (K) + (GE) *imply* (C),

(ii) (Re) + (GE) + (L) *imply* (D) *and* (K),

(iii) (GE) + (K) + (An) *imply* (Ex),

(iv) (C) + (D) + (M) + (Tr) *imply* (**).

PROOF: (i) Let $x, y, z \in A$. We have $[x \to (y \to z)] \to [y \to (x \to z)] \overset{\text{(GE)}}{=} [x \to (y \to z)] \to [y \to (x \to (y \to z))] \overset{\text{(K)}}{=} 1$, that is, (C) holds in $\mathcal{A}$.

(ii) Let $x, y \in A$. We obtain

$$y \to [(y \to x) \to x] \overset{\text{(GE)}}{=} y \to [(y \to x) \to (y \to x)] \overset{\text{(Re)}}{=} y \to 1 \overset{\text{(L)}}{=} 1,$$

that is, (D) holds in $\mathcal{A}$.

Now, applying (GE), (Re) and (L), we get $x \to (y \to x) = x \to [y \to (x \to x)] = 1$, that is, (K) holds in $\mathcal{A}$.

(iii) It follows from above (i) and Lemma 2.1 (iv).

(iv) Let $x, y, z \in A$ and $y \leq z$. By (D), $z \leq (z \to x) \to x$. Applying (Tr), we get $y \leq (z \to x) \to x$. From (C) it follows that

$$1 = y \to [(z \to x) \to x] \leq (z \to x) \to (y \to x).$$

Hence, by (M), $(z \to x) \to (y \to x) = 1$. Therefore $z \to x \leq y \to x$, thus (**) holds in $\mathcal{A}$. □

From Propositions 2.4, 2.7 (i) and 3.1 (i), (ii) we have

COROLLARY 3.2. Any GE algebra satisfies *(pi), (L), (C), (D)* and *(K).*

COROLLARY 3.3. In GE algebras, we have

$$\textit{(Tr)} \Longleftrightarrow \textit{(**)}.$$

PROOF: Let $\mathcal{A}$ be a GE algebra verifying (Tr). By Proposition 3.1 (iv), $\mathcal{A}$ verifies (**). The converse follows from Lemma 2.1 (iii). □

Remark 3.4. Applying Proposition 3.1 (iii), we have **aGE** $\subseteq$ **pi-aBE**. Since **pi-BE** $\subset$ **GE**, see Remark 2.9, we get **pi-aBE** $\subseteq$ **aGE**. Consequently, **pi-aBE** $=$ **aGE**.

Since (M) + (B) imply (Re), see Lemma 2.1 (i), we obtain

PROPOSITION 3.5. An algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ is a transitive GE algebra if and only if it satisfies *(M), (GE), (B).*

Now we consider the following properties; they are the most important properties satisfied by a Hilbert algebra:

(p-1) $x \to (y \to z) \leq (x \to y) \to (x \to z)$,

(p-2) $(x \to y) \to (x \to z) \leq x \to (y \to z)$,

(pimpl) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Remark 3.6. It is easy to see that (p-1) + (p-2) + (An) imply (pimpl).

PROPOSITION 3.7. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following hold:

(i) (pi) + (pimpl) *imply* (GE),

(ii) (Re) + (M) + (pimpl) *imply* (pi),

(iii) (Re) + (M) + (pimpl) *imply* (GE),

(iv) (M) + (C) + (B) +(pi) *imply* (p-1),

(v) (M) + (K) + (C) + (**) *imply* (p-2),

(vi) (K) + (Tr) + (p-1) *imply* (B),

(vii) (M) + (L) + (p-1) *imply* (*).

Proof: (i) Let $x, y, z \in A$. We obtain

$$\begin{aligned} x \to (y \to z) &\overset{\text{(pimpl)}}{=} (x \to y) \to (x \to z) \\ &\overset{\text{(pi)}}{=} (x \to y) \to [x \to (x \to z)] \\ &\overset{\text{(pimpl)}}{=} x \to [y \to (x \to z)]. \end{aligned}$$

Thus (GE) holds.

(ii) Follows from Proposition 6.4 (iii) of [7].

(iii) Follows from above (i) and (ii).

(iv) By Lemma 2.1 (i) and (ii), $\mathcal{A}$ satisfies (Tr). Let $x, y, z \in A$. From (C) it follows $x \to (y \to z) \leq y \to (x \to z)$. Applying (B) and (pi), we get $y \to (x \to z) \leq (x \to y) \to [x \to (x \to z)] = (x \to y) \to (x \to z)$. By (Tr), $x \to (y \to z) \leq (x \to y) \to (x \to z)$, that is, (p-1) holds.

(v) Let $x, y, z \in A$. By (K), $y \leq x \to y$, and hence, using (**), we obtain

$$(x \to y) \to (x \to z) \leq y \to (x \to z). \tag{3.1}$$

By (C),

$$y \to (x \to z) \leq x \to (y \to z). \tag{3.2}$$

Since $\mathcal{A}$ satisfies (M) and (**), from Lemma 2.1 (iii) we see that (Tr) holds in $\mathcal{A}$. Therefore, applying (3.1) and (3.2), we get $(x \to y) \to (x \to z) \leq x \to (y \to z)$, that is, (p-2) holds.

(vi) Let $x, y, z \in A$. By (K) and (p-1), $y \to z \leq x \to (y \to z)$ and $x \to (y \to z) \leq (x \to y) \to (x \to z)$. Then, from (Tr) we have $y \to z \leq (x \to y) \to (x \to z)$. Thus (B) holds.

(vii) Let $x, y, z \in A$ and $y \to z = 1$. Using (L) and (p-1), we get $1 = x \to (y \to z) \leq (x \to y) \to (x \to z)$. By (M), $(x \to y) \to (x \to z) = 1$. Therefore (*) holds. □

Since (M) + (B) imply (**), see Lemma 2.1 (i), from Prposition 3.7 (iv), (v) we obtain

Corollary 3.8. *Any transitive GE algebra verifies properties (p-1) and (p-2).*

Proposition 3.9. In GE algebras, *(p-1)* $\Longrightarrow$ *(p-2)*.

Proof: Let $\mathcal{A}$ be a GE algebra verifying (p-1). By Proposition 3.7 (vii), $\mathcal{A}$ verifies (*). Therefore, (Tr) also holds, and hence $\mathcal{A}$ verifies (**), by Proposition 3.1 (iv). Applying Proposition 3.7 (v), we get (p-2). $\square$

Theorem 3.10. *In GE algebras, we have*

$$\text{(BB)} \Longleftrightarrow \text{(B)} \Longleftrightarrow \text{(p-1)} \Longleftrightarrow (*).$$

Proof: By Lemma 2.1 (v), (BB) $\Longrightarrow$ (B), and, by Proposition 3.7 (iv), (vii), we conclude that (B) $\Longrightarrow$ (p-1) and (p-1) $\Longrightarrow$ (*). Let $\mathcal{A}$ be a GE algebra with (*). Let $x, y, z \in A$. From (C) we see that $x \to [(y \to z) \to z] \leq (y \to z) \to (x \to z)$, and hence

$$(x \to y) \to [x \to ((y \to z) \to z)] \leq (x \to y) \to [(y \to z) \to (x \to z)]$$

by (*). Observe that

$$(x \to y) \to [x \to ((y \to z) \to z)] = 1. \tag{3.3}$$

Indeed, from (D) we conclude that $y \leq (y \to z) \to z$. Applying (*), we obtain (3.3). Therefore, $x \to y \leq (y \to z) \to (x \to z)$, that is, (BB) holds. $\square$

Corollary 3.11. An algebra $\mathcal{A} = (A, \to, 1)$ *of type* $(2, 0)$ is a transitive GE algebra if and only if $\mathcal{A}$ verifies *(Re), (M), (GE)* and *(p-1)*.

Corollary 3.12. Any transitive GE algebra verifies *(B), (BB), (*), (**), (Tr), (p-1), (p-2)*.

4. Implicative and positive implicative GE algebras

The well-known implicative and positive implicative BCK algebras were introduced by K. Iséki and S. Tanaka [10].

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We first consider the following property:

(im) $(x \to y) \to x = x$.

Proposition 4.1. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then:

(i) (Re) + (im) *imply* (M),

(ii) (M) + (im) *imply* (L),

(iii) (im) *implies* (pi),

(iv) (Re) + (pimpl) *imply* (L) *and* (B),

(v) (Re) + (M) + (pimpl) + (An) *imply* (Ex).

Proof: (i)–(iii) follow from Proposition 3.5 of [19].

(iv) and (v) follow from Propositions 6.4, 6.9 and Theorem 6.16 of [7]. □

Similarly as in the case of BCK algebras, we say that a RML algebra (in particular, a GE algebra) $\mathcal{A} = (A, \to, 1)$ is *implicative* if it satisfies (im).

A *positive implicative RML algebra* ([7]), or a *pimpl-RML algebra* for short, is a RML algebra verifying (pimpl).

Remark 4.2. Note that from Theorem 8 of [10] it follows that for BCK algebras, (pimpl) and (pi) are equivalent. By Theorem 9 of [10], a commutative BCK algebra is implicative if and only if it is positive implicative.

Denote by **im-RML** and **pimpl-RML** the classes of implicative and positive implicative RML algebras, respectively; similarly for subclasses of the class of all RML algebras.

It is easy to check that the algebra from Example 2.6 is an implicative tGE algebra. However, the algebra given in Example 2.5 is not implicative, since $(b \to a) \to b = 1 \neq b$.

Example 4.3. Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\to$:

$\to$	a	b	c	d	1
a	1	b	b	1	1
b	a	1	1	a	1
c	a	1	1	a	1
d	1	c	c	1	1
1	a	b	c	d	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies (Re), (M), (L), (GE) (hence (C), (D), (K), (pi)), (B) (hence (*), (**), (Tr)) and (pimpl). It does not verify (An) for b, c; (Ex) for a, d, b; (im) for c, a. Thus $\mathcal{A}$ is a pimpl-tGE algebra which is not implicative.

Remark 4.4. Any implicative RML and pimpl-RML algebra is a pi-RML algebra by Propositions 4.1 (iii) and 3.7 (ii).

We recall the following definitions:

DEFINITION 4.5. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$.

8. $\mathcal{A}$ is a *Hilbert algebra* ([5]) if it verifies (An), (K) and (p-1).
9. $\mathcal{A}$ is a generalized *Hilbert algebra* (*GH-algebra* for short) if it verifies (Re), (M), (Ex) and (pimpl).
10. $\mathcal{A}$ is a *Tarski algebra* ([11])if it verifies (Re), (M), (pimpl) and (Com).
11. $\mathcal{A}$ is a *generalized Tarski algebra* (*GT-algebra* for short) if it verifies (Re), (M) and (pimpl).

Denote by **H**, **GH**, **T** and **GT** the classes of Hilbert algebras, GH-algebras, Tarski algebras and GT-algebras, respectively.

Remark 4.6. Hilbert algebras were introduced in 1950, in a dual form, by L. Henkin [6], under the name "implicative model". A. Monteiro has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models (see [6, 2]). In [5], A. Diego proved that the class of all Hilbert algebras is a variety. From Remarks 6.18 and 6.19 of [7] and Remark 3.4 we conclude that

$$\begin{aligned}\textbf{pimpl-BCC} &= \textbf{pimpl-BCK}\\ &= \textbf{pi-BCK}\\ &= \textbf{pimpl-aBE}\\ &= \textbf{pimpl-aGE}\\ &= \textbf{H}.\end{aligned}$$

PROPOSITION 4.7 ([7], Corollary 6.17). Any algebra $(A, \rightarrow, 1)$ verifying (Re), (M), (An) and (pimpl) is a Hilbert algebra.

Remark 4.8. By definition, generalized Hilbert algebras coincide with positive implicative BE algebras, that is, **pimpl-BE** = **GH**. Note that a self-distributive BE algebra (see [12]) is in fact our pimpl-BE algebra. By Remark 6.19 of [7], **pimpl-BE** = **pimpl-pre-BCK** (= **GH**).

Remark 4.9. Note that GT-algebras were introduced and studied in [13]. Since (Re) + (pimpl) imply (L) and (B), see Proposition 4.1 (iv), we have $\mathbf{GT} = \mathbf{pimpl\text{-}RML} = \mathbf{pimpl\text{-}pre\text{-}BCC}$. By Proposition 3.7 (iii), $\mathbf{GT} = \mathbf{pimpl\text{-}GE} = \mathbf{pimpl\text{-}tGE}$.

Remark 4.10. A Tarski algebra is in fact a commutative GT-algebra. By Lemma 2.13, a Tarski algebra verifies (An), hence, by Proposition 4.7, it is a Hilbert algebra. Therefore, Tarski algebras coincide with commutative Hilbert algebras, and with commutative GE algebras by Theorem 3.9 of [2]. Thus $\mathbf{T} = \mathbf{com\text{-}GT} = \mathbf{com\text{-}GE} = \mathbf{com\text{-}H}$, where **com-GT**, **com-GE** and **com-H** denote commutative GT, commutative GE and commutative Hilbert algebras, respectively.

By above remarks, we obtain that

$$\mathbf{T} = \mathbf{com\text{-}H} \overset{a)}{\subset} \mathbf{H} = \mathbf{pimpl\text{-}aBE} \overset{b)}{\subset} \mathbf{GH} = \\ \mathbf{pimpl\text{-}BE} \overset{c)}{\subset} \mathbf{GT} = \mathbf{pimpl\text{-}RML} = \mathbf{pimpl\text{-}tGE} \overset{d)}{\subset} \mathbf{GE}.$$

These inclusions are proper; see Examples 3.10 [2], for a), 10.8 [8], for b); 4.3, for c); and finally, Example 2.5, for d).

By definition, we have

$$\mathbf{im\text{-}BCK} \subset \mathbf{im\text{-}pre\text{-}BCK} \subset \mathbf{im\text{-}tGE} \subset \\ \mathbf{im\text{-}pre\text{-}BCC} \subset \mathbf{im\text{-}RML} \subset \mathbf{pi\text{-}RML}.$$

These inclusions are proper; see Examples 4.11, 2.6, 4.12, 4.13 and Example 10.1 of [8].

Example 4.11. Let $A = \{a, b, c, d, e, 1\}$ and $\rightarrow$ be defined as follows:

$\rightarrow$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	d	d	d	1
c	1	1	1	1	1	1
d	a	b	b	1	1	1
e	a	a	a	1	1	1
1	a	b	c	d	e	1

It is easy to see that the properties (Re), (M), (L), (Ex), (B), (im) (hence (pi)) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$, (pimpl) is not satisfied for $(x, y, z) = (a, b, c)$, Therefore, $(A, \rightarrow, 1)$ is an implicative pre-BCK algebra that is not positive implicative.

Example 4.12. Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\rightarrow$ given by the following table:

$\rightarrow$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	c	d	d	1
c	b	b	1	1	1	1
d	a	b	1	1	1	1
e	a	a	1	1	1	1
1	a	b	c	d	e	1

We can observe that the properties (Re), (M), (L), (B) and (im) are satisfied. Hence, $(A, \rightarrow, 1)$ is an implicative pre-BCC algebra. It does not satisfy (An) for $(x, y) = (a, b)$; (Ex) and (GE) for $(x, y, z) = (a, b, c)$; (pimpl) for $(x, y, z) = (a, b, e)$.

Example 4.13. ([19], Example 3.24) Let $A = \{a, b, c, d, 1\}$ and $\rightarrow$ be defined as follows:

$\rightarrow$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	a	a	1
c	1	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

It is easy to see that the properties (Re), (M), (L) and (im) (hence (pi)) are satisfied; (An) is not satisfied for $(x, y) = (c, d)$, (GE), (Ex) and (pimpl) are not satisfied for $(x, y, z) = (b, a, d)$, (Tr) is not satisfied for $(x, y, z) = (d, c, a)$. Therefore, $(A, \rightarrow, 1)$ is an implicative RML algebra (hence also a pi-RML algebra) that is not a pre-BCC algebra.

PROPOSITION 4.14 ([19], Proposition 3.14). Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra verifying (Re), (D), (**) and (im). Then

$$y \leq x \Longrightarrow (x \rightarrow y) \rightarrow y \leqslant x \tag{4.1}$$

for all $x, y \in A$.

Theorem 4.15. *If $\mathcal{A} = (A, \to, 1)$ is an implicative GE algebra with* (Tr), *then $\mathcal{A}$ satisfies the following condition:*

(wCom) $(x \to y) \to y \leq (y \to x) \to x$.

Proof: Let $\mathcal{A}$ be an implicative GE algebra verifying (Tr). By Proposition 4.14, $\mathcal{A}$ satisfies (4.1). Let $x, y \in A$. From (K) we have $x \leq (y \to x) \to x$. Applying (**) twice, we obtain

$$(x \to y) \to y \leq (((y \to x) \to x) \to y) \to y. \tag{4.2}$$

By (D), $y \leq (y \to x) \to x$, and hence, using (4.1), we get

$$(((y \to x) \to x) \to y) \to y \leq (y \to x) \to x. \tag{4.3}$$

Since $\mathcal{A}$ satisfies (Tr), from inequalities (4.2) and (4.3) we have (wCom). □

Proposition 4.16 ([23]). Implicative aBE algebras satisfy (Tr).

Proposition 4.17. Implicative aGE algebras concide with implicative aBE algebras.

Proof: From Remark 3.4 it follows that **pi-aBE** = **aGE**. Since (im) implies (pi), we have **im-aBE** = **im-aGE**. □

Proposition 4.18. In GE algebras, we have

$$\text{(im)} + \text{(An)} \iff \text{(Com)}.$$

Proof: Let $\mathcal{A} = (A, \to, 1)$ be a GE algebra. Assume that (im) and (An) hold in $\mathcal{A}$. By Propositions 4.16 and 4.17, $\mathcal{A}$ satisfies (Tr). From Theorem 4.15 we conclude that $\mathcal{A}$ is commutative.

Conversely, suppose that $\mathcal{A}$ satisfies (Com). By Lemma 2.13, (An) is satisfied. To prove (im), let $x, y \in A$. We have $((x \to y) \to x) \to x \overset{\text{(Com)}}{=} (x \to (x \to y)) \to (x \to y) \overset{\text{(pi)}}{=} (x \to y) \to (x \to y) \overset{\text{(Re)}}{=} 1$. Then $(x \to y) \to x \leq x$. Applying Proposition 3.1 (ii), we see that $\mathcal{A}$ satisfies (K). Therefore, $x \leq (x \to y) \to x$. Then, using (An), we obtain $x = (x \to y) \to x$, that is, (im) holds in $\mathcal{A}$. □

Corollary 4.19. Let $\mathcal{A}$ be a GE algebra satisfying (An). Then the property of implicativity is equivalent to the commutative property.

From Corollary 4.19 it follows that **com-GE** = **im-aGE**. Since $\mathbf{T}$ = **com-GE** (see Remark 4.10), we have $\mathbf{T}$ = **im-aGE**. Hence we obtain

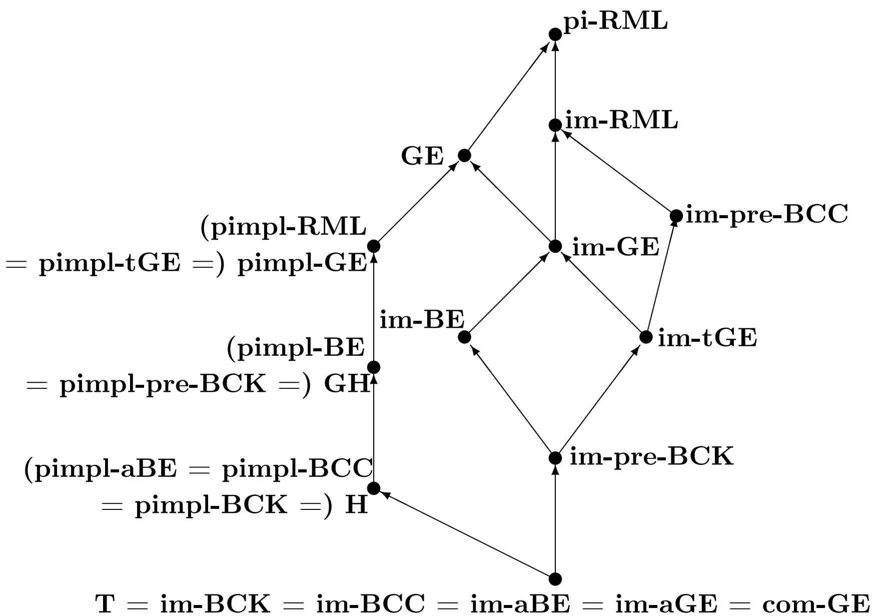

Figure 2.

Corollary 4.20. Any implicative aGE algebra is a Tarski algebra.

We draw now the interrelationships between some classes of implicative and positive implicative algebras mentioned before (see Figure 2).

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