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Research Article

Proof of a conjecture on edge coloring of the Kneser graph $K(t, 2)$

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Abstract

In this paper, it is proved that the Kneser graph $K(t, 2)$ is Class 1 for $t \equiv 1 \pmod 4) \geq 9$. This result proves the conjecture posed in [C. M. H. de Figueiredo, C. S. R. Patrão, D. Sasaki, M. Valencia-Pabon, *J. Combin. Optim.* **44** (2022) 119–135].

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1. Introduction

All graphs considered here are simple and finite. We denote the *complete graph* and the *complete bipartite graph* by K_n and $K_{n,n}$ respectively. For a graph G and $S \subseteq V(G)$, the subgraph of G *induced by* S is denoted by $\langle S \rangle$. Similarly, for $E' \subseteq E(G)$, the subgraph of G *induced by* E' is denoted by $\langle E' \rangle$. The *complement of a graph* G is denoted by $\overline{G}$ with $V(\overline{G}) = V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G.

An r-*factor* of a graph G is an r-regular spanning subgraph of G. A graph G is said to be 1-*factorable* if $E(G)$ can be partitioned into perfect matchings. A 2-*factorization* of G is a partition of $E(G)$ into edge-disjoint 2-factors. The *union* of two graphs G_1 and G_2 is a new graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. In other words, the resulting graph G contains all the vertices and edges of both G_1 and G_2 without duplicating any common elements. Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{r-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{r-1}\}$. If G contains the set of edges $F_i(X, Y) = \{x_j\, y_{j+i} \,|\, 0 \leq j \leq r-1\}, 0 \leq i \leq r-1$, where addition in the subscript is taken modulo r, then we say that G has the *1-factor of jump i from X to Y*. Note that $F_i(X, Y) = F_{r-i}(Y, X)$. Clearly, if $G = K_{r,r}$, then $E(G) = \bigcup_{i=0}^{r-1} F_i(X, Y)$. Let $E(X, Y)$ denote the set of edges of G having one end in X and the other end in Y, where $X, Y \subset V(G)$ and $X \cap Y = \emptyset$. A *circulant* graph $\Gamma = C(n; L)$ is a graph with $V(\Gamma) = \{1, 2, \ldots, n\}$ and $E(\Gamma) = \{i\; i+\ell \,|\, 1 \leq i \leq n \text{ and } \ell \in L\}$, where $L \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and the addition is taken modulo n with residues $1, 2, \ldots, n$. The elements of L are called the distances of the circulant graph Γ and L is called the *set of distances*. Let $GP(n, j)$ denote the *generalized Petersen graph* with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_iu_{i+1}, v_iv_{i+j}, u_iv_i \,|\, 1 \leq i \leq n\}$, where the addition in the subscripts are taken modulo n with residues $1, 2, \ldots, n$. A proper k-edge coloring of a graph G is a function $f : E(G) \to \{1, 2, \ldots, k\}$ such that adjacent edges receive distinct colors. The smallest integer k for which graph G has a proper k-edge coloring is called the *edge-chromatic number* or *chromatic index* of G. It is denoted by $\chi'(G)$. The Vizing theorem proved by Vizing [17] and independently by Gupta [10] states that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G. If $\chi'(G) = \Delta(G)$ (respectively, $\Delta(G) + 1$), then the graph G is said to be *Class 1* (respectively, *Class 2*). The *line graph* of a graph G, denoted by $L(G)$, is the graph with $V(L(G)) = E(G)$ and the edge $e_1e_2 \in E(L(G))$ if and only if both the edges e_1 and e_2 of G incident at a vertex. Let $\mathcal{P}_k(t)$ be the set of all k-element subsets of a t-element set. The *Kneser graph* $K(t, k)$ is defined as follows: $V(K(t, k)) = \mathcal{P}_k(t)$ and $E(K(t, k)) = \{A\,B \,|\, A, B \in \mathcal{P}_k(t) \text{ and } A \cap B = \emptyset\}$. Note that, when $k = 1$, $K(t, k) \cong K_t$ and the graph $K(t, 2)$ is isomorphic to $\overline{L(K_t)}$. Definitions which are not given here can be found in [3]. The Kneser graph was introduced by Kneser, see [12]. Initially, Kneser conjectured that if $t \geq 2k$, then the chromatic number $\chi(K(t, k)) = t - 2k + 2$. This conjuncture was settled using different proof techniques, see [4, 9, 12, 15, 16]. Other coloring parameters of Kneser graphs have been studied by many authors, see [1, 2, 8, 11].

In 1983, Leven and Galil [14] showed that the proper edge coloring problem is NP-complete even for regular graphs of degree at least 3. Cao et al. [5] presented a detailed survey on edge coloring of graphs. Recently, de Figueiredo et al. [7] showed that $K(t, 2)$ is Class 1 for $t \equiv 0 \pmod 4$ and posed a conjecture for $t \equiv 1 \pmod 4$.

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Conjecture 1.1. [7] *For $t \geq 9$, the Kneser graph $K(t, 2)$ with $t \equiv 1 \pmod 4$ is Class 1.*

We state the following theorem for our reference.

Theorem 1.1. [13] *Every bipartite graph is Class 1.*

2. Proof of Conjecture 1.1

In this section, we present the proof of Conjecture 1.1.

Theorem 2.1. *For $t \geq 9$, the Kneser graph $K(t, 2)$ with $t \equiv 1 \pmod 4$ is Class 1.*

Proof. Let $t = 4k + 1, k \geq 2$. Consider the complete graph K_t with vertex set $\{1, 2, \ldots, t\}$. Clearly, $K_t = \bigcup_{i=1}^{2k} C(t; \{i\})$, where $C(t; \{i\})$ is the circulant graph with distance set $\{i\}$ and vertex set $\{1, 2, \ldots, t\}$. For $1 \leq i \leq 2k$, let V_i be the 2-element vertex subset of $K(t, 2)$ that corresponds to the ends of the t edges of $C(t; \{i\})$, that is,

$$V_i = \{\{1,\, 1+i\},\, \{2,\, 2+i\},\, \{3,\, 3+i\}, \ldots, \{t,\, t+i\}\},$$

where the addition is taken modulo t with residues $1, 2, \ldots, t$. Clearly, $\mathcal{P}_2(t) = \bigcup_{i=1}^{2k} V_i = V(K(t, 2))$ and

$$E(K(t,2)) = \left\{ \bigcup_{i=1}^{2k} E(\langle V_i \rangle) \right\} \cup \left\{ \bigcup_{1 \leq i < j \leq 2k} E(V_i,\, V_j) \right\}.$$

Since $V(K(t, 2)) = \mathcal{P}_2(t)$ and $E(K(t, 2)) = \{X\,Y \mid X, Y \in \mathcal{P}_2(t) \text{ and } X \cap Y = \emptyset\}$, that is, they share no points, on the other hand, two vertices in the $L(K_t)$ are adjacent by an edge if they share a common endpoint in K_t, but such edge is not in $\overline{L(K_t)}$. Therefore, $K(t, 2) \cong \overline{L(K_t)} = \overline{L(\cup_{i=1}^{2k} C(t; i))}$, each of the subgraphs $\langle V_i \rangle$, $1 \leq i \leq 2k$, of $K(t, 2)$ is isomorphic to $\langle E(K_t) - E(C(t; \{i\})) \rangle$. It is easy to check that the edge induced subgraph $\langle E(V_i,\, V_j) \rangle$, $1 \leq i < j \leq 2k$, of $K(t, 2)$ is isomorphic to $K_{t,\,t} - \{F_0(V_i,\, V_j) \cup F_i(V_i,\, V_j) \cup F_{t-j}(V_i,\, V_j) \cup F_{t+i-j}(V_i,\, V_j)\}$, where the addition in the subscript of F is calculated modulo t, see Figure 1.

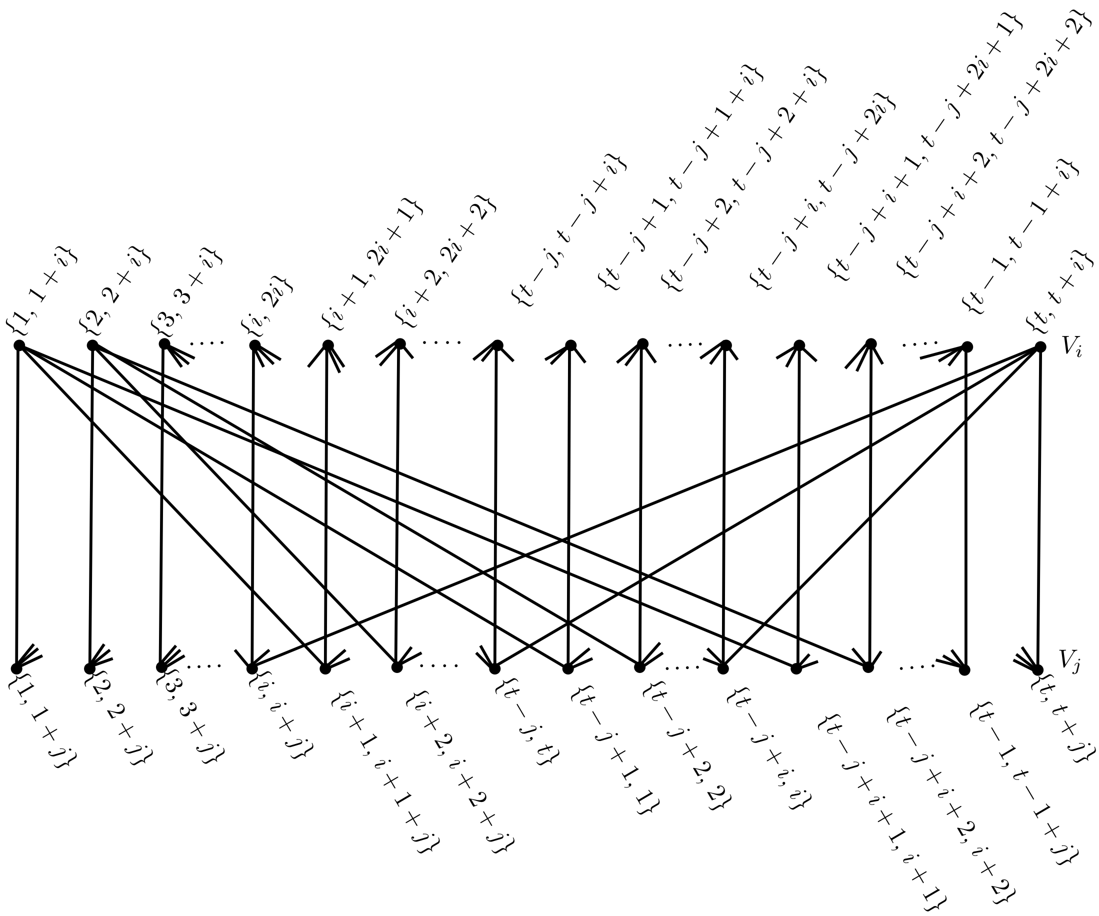

Figure 1: Graph $\langle \overline{E(V_i,\, V_j)} \rangle$, $1 \leq i < j \leq 2k$. Observe that the vertex $\{i,\, i+j\}$ is same as $\{t+i,\, i+j\}$.

Consider the complete graph K_{2k} with $V(K_{2k}) = \{v_1, v_2, \ldots, v_{2k}\}$. Let $\{F_1, F_2, \ldots, F_{2k-1}\}$ be a 1-factorization of K_{2k}. Corresponding to each of these 1-factors of K_{2k}, we associate a regular spanning subgraph of $K(t, 2)$ as follows: let

$$H_1 = \left\{ \bigcup_{v_i v_j \,\in\, E(F_1)} \langle E(V_i,\, V_j)\rangle \right\} \cup \left\{ \bigcup_{i=1}^{2k} \langle V_i \rangle \right\}$$

and let $H_i = \{\bigcup_{v_r v_s \,\in\, E(F_i)} \langle E(V_r,\, V_s)\rangle\}$, $2 \le i \le 2k-1$. Clearly, $K(t, 2) = H_1 \cup H_2 \cup \cdots \cup H_{2k-1}$. Since the subgraph $\langle E(V_i,\, V_j)\rangle$ is a $t-4 = (4k-3)$-regular bipartite graph, it is Class 1 by Theorem 1.1. As each H_i, $2 \le i \le 2k-1$, is the union of k vertex disjoint $(4k-3)$-regular bipartite graphs $\langle E(V_i,\, V_j)\rangle$, it is Class 1 by Theorem 1.1.

To complete the proof, it is enough to obtain a 1-factorization of H_1, and also graph H_1 is going to be analyzed in several Class 1 subgraphs. It is clear that for each edge $v_i v_j \in E(F_1)$, $i < j$, of K_{2k}, there corresponds a component $H_{ij} = \langle V_i \rangle \cup \langle V_j \rangle \cup \langle E(V_i,\, V_j)\rangle$ in H_1. Now our aim is to obtain a 1-factorization of H_{ij}. Let X_a^i (respectively, Y_b^j) denote the circulant graph isomorphic to $C(t;\, \{a\})$ (respectively, $C(t;\, \{b\})$) contained in $\langle V_i \rangle$ (respectively, $\langle V_j \rangle$). As pointed out earlier,

$$\langle V_i \rangle \cong K_t - E(C(t;\, \{i\})) = \bigcup_{j\,=\,1,\, j\,\ne\, i}^{2k} E(C(t;\, \{j\})),$$

that is, $\langle V_i \rangle = \{\bigcup_{a=1}^{2k} X_a^i\} - E(X_i^i)$ and $\langle V_j \rangle = \{\bigcup_{b=1}^{2k} Y_b^j\} - E(Y_j^j)$. Recall that

$$\langle E(V_i,\, V_j)\rangle = K_{t,\,t} - \{F_0(V_i,\, V_j) \cup F_i(V_i,\, V_j) \cup F_{t-j}(V_i,\, V_j) \cup F_{t+i-j}(V_i,\, V_j)\} = \bigcup_{r=1}^{4k-3} F_{\alpha_r}(V_i,\, V_j),$$

where $\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_{4k-3}$ is the arrangement of the integers in $\{0,\, 1,\, 2,\, \ldots,\, 4k\} \setminus \{0,\, i,\, t-j,\, t+i-j\}$ in the increasing order.

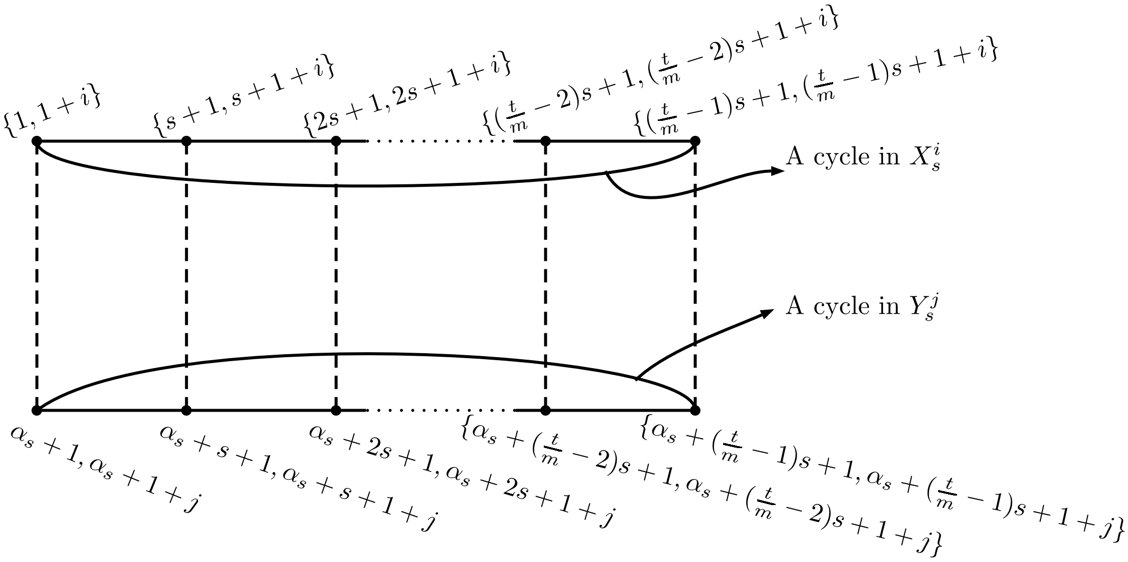

Figure 2: One of the copies of the prisms in H_s' is shown here.

First, we obtain a 1-factorization of H_{ij} for $i \ne 1$ ($i = 1$ will be considered later). Let H_{ij} be the graph defined above. To each $s \in \{1,\, 2,\, 3,\, \ldots,\, 2k\} \setminus \{1,\, i,\, j\}$, consider the subgraph $H_s' = X_s^i \cup Y_s^j \cup F_{\alpha_s}(V_i,\, V_j)$ of H_{ij} (the values of $s = 1$, i and j will be considered later). Observe that X_s^i (respectively, Y_s^j) is a 2-factor of $\langle V_i \rangle$ (respectively, $\langle V_j \rangle$) having m cycles each of length $\frac{t}{m}$, where $m = gcd(t,\, s)$. Consequently, H_s' is isomorphic to the union of m disjoint copies of the prism over the cycle of length $\frac{t}{m}$, where the *prism over a cycle* length $\frac{t}{m}$ is the cubic graph obtained by taking two disjoint copies of the cycle $C_{\frac{t}{m}}$ and joining their corresponding vertices, see Figure 2. As the prism over a cycle is hamiltonian, H_s' is 1-factorable. The edges of H_{ij} which are not on the H_s', $s \in \{1,\, 2,\, 3,\, \ldots,\, 2k\} \setminus \{1,\, i,\, j\}$, are the edges of $E(X_1^i) \cup E(X_j^i) \cup E(Y_1^j) \cup E(Y_i^j)$ and the edges of the 1-factors of jumps $F_{\alpha_1}(V_i,\, V_j)$, $F_{\alpha_i}(V_i,\, V_j)$, $F_{\alpha_j}(V_i,\, V_j)$, $F_{\alpha_{2k+1}}(V_i,\, V_j)$, $F_{\alpha_{2k+2}}(V_i,\, V_j)$, $\ldots$, $F_{\alpha_{4k-3}}(V_i, V_j)$, as X_i^i (respectively, Y_j^j) is not in $\langle V_i \rangle$ (respectively, $\langle V_j \rangle$), X_1^i, X_j^i, Y_1^j and Y_i^j can not be in the above H_s' and the 1-factors (from V_i to V_j) in H_{ij} which are not listed here are in H_s'. Now consider two subgraphs $X_1^i \cup Y_i^j \cup F_{\alpha_1}(V_i,\, V_j)$ and $X_j^i \cup Y_1^j \cup F_{\alpha_i}(V_i,\, V_j)$ of H_{ij}. By the definition of X_1^i, it is a cycle of length t; Y_i^j is a 2-factor of $\langle V_j \rangle$ having ℓ cycles of length $\frac{t}{\ell}$, where $\ell = gcd(t,\, i)$. Further, $F_{\alpha_1}(V_i,\, V_j)$ is the 1-factor of jump α_1 from V_i to V_j. Clearly, $X_1^i \cup Y_i^j \cup F_{\alpha_1}(V_i,\, V_j) \cong GP(t,\, i)$ and $X_j^i \cup Y_1^j \cup F_{\alpha_i}(V_i,\, V_j) \cong GP(t,\, j)$, see Figure 3.

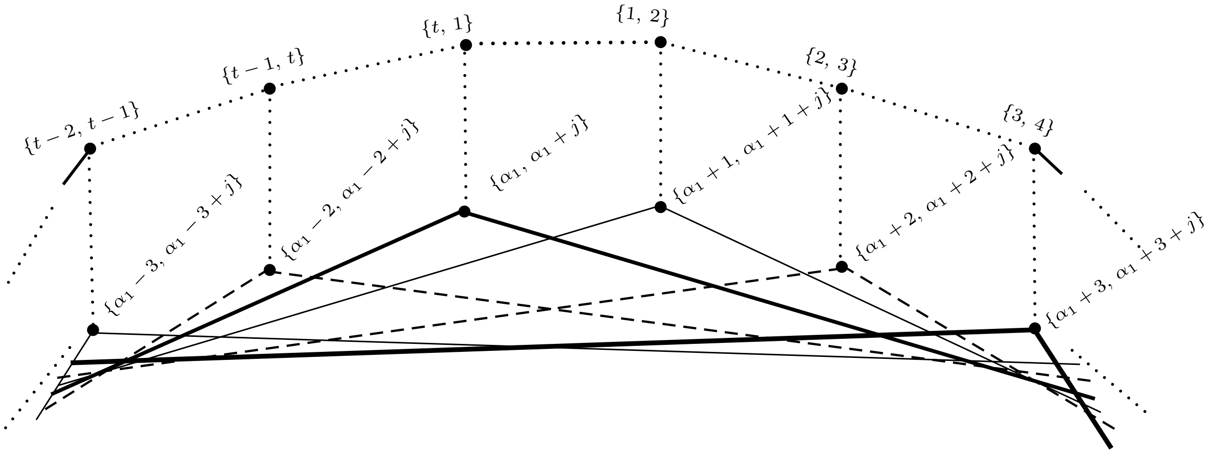

Figure 3: A part of the subgraph of $X_1^i \cup Y_i^j \cup F_{\alpha_1}(V_i, V_j) = GP(t, i)$, where Y_i^j is the union of three disjoint cycles shown in normal edges, bold edges, and broken edges.

The generalized Petersen graphs $GP(t,\, i)$ and $GP(t,\, j)$ admit Tait colorings, and hence they are 1-factorable, see [6]. The remaining edges of H_{ij} are the edges of the 1-factors of jumps $F_{\alpha_j}(V_i,\, V_j)$, $F_{\alpha_{2k+1}}(V_i,\, V_j)$, $F_{\alpha_{2k+2}}(V_i,\, V_j)$, $\ldots$, $F_{\alpha_{4k-3}}(V_i,\, V_j)$ and they are 1-factors of H_{ij}.

Finally, we obtain a 1-factorization of H_{ij} for $i = 1$. As above, consider the subgraph $H'_s = X_s^1 \cup Y_s^j \cup F_{\alpha_s}(V_1,\, V_j)$ of H_{1j} for each $s \in \{1,\, 2,\, 3,\, \ldots,\, 2k\} \setminus \{1,\, j\}$. As discussed above, each H'_s is 1-factorable. Thus

$$E(H_{1j}) - E(\cup_{s=2,\, s\neq j}^{2k} H'_s) = E(X_j^1) \cup E(Y_1^j) \cup F_{\alpha_1}(V_i,\, V_j) \cup F_{\alpha_j}(V_i,\, V_j) \cup \{\cup_{r=2k+1}^{4k-3} F_{\alpha_r}(V_i,\, V_j)\}.$$

Now we consider the subgraph $X_j^1 \cup Y_1^j \cup F_{\alpha_1}(V_i,\, V_j)$ of H_{1j}; (here Y_1^j is a cycle of length t). This subgraph is isomorphic to the generalized Petersen graph $GP(t,\, j)$ and hence it is 1-factorable, see [6]. The remaining edges of H_{1j} are $\{\bigcup_{r=2k+1}^{4k-3} F_{\alpha_r}(V_i,\, V_j)\} \cup F_{\alpha_j}(V_i,\, V_j)$; this is the edge disjoint union of $2k-2$ 1-factors of H_{1j}. This completes the proof. □

3. Conclusion

Basic necessary condition for the Kneser graph $K(t, 2)$ to be Class 1 is $t \equiv 0$ or $1 \pmod 4$. In [7], de Figueiredo et al. proved that Kneser graph $K(t, 2)$ is Class 1 when $t \equiv 0 \pmod 4$ and posed the case $t \equiv 1 \pmod 4$ as a conjecture. In this paper, we proved that $K(t, 2)$ is Class 1 when $t \equiv 1 \pmod 4$, which completely settled the conjecture of de Figueiredo et al. [7]. Thus, our result (Theorem 2.1) together with the result of de Figueiredo et al. [7] proves that $K(t, 2)$ is Class 1 for all possible t except $t = 5$, because when $t = 5$, $K(t, 2)$ is isomorphic to Petersen graph, which is Class 2. These results significantly advance our understanding of Kneser graphs and their properties.

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References

[1] G. Araujo-Pardo, J. C. Díaz-Patiño, C. Rubio-Montiel, Achromatic numbers of Kneser graphs, *Ars Math. Contemp.* **21** (2021) #P1.05.
[2] R. Balakrishnan, T. Kavaskar, b-coloring of Kneser graphs, *Discrete Appl. Math.* **160** (2012) 9–14.
[3] R. Balakrishnan, K. Ranganathan, *A Textbook of Graph Theory*, 2nd Edition, Springer, New York, 2012.
[4] I. Bárány, A short proof of Kneser's conjecture, *J. Combin. Theory Ser. A* **25** (1978) 325–326.
[5] Y. Cao, G. Chen, G. Jing, M. Stiebitz, B. Toft, Graph edge coloring: a survey, *Graphs Combin.* **35** (2019) 33–66.
[6] F. Castagna, G. Prins, Every generalized Petersen graph has a Tait coloring, *Pacific J. Math.* **40** (1972) 53–58.
[7] C. M. H. de Figueiredo, C. S. R. Patrão, D. Sasaki, M. Valencia-Pabon, On total and edge coloring some Kneser graphs, *J. Combin. Optim.* **44** (2022) 119–135.
[8] R. Fidytek, H. Furmańczyk, P. Żyliński, Equitable coloring of Kneser graphs, *Discuss. Math. Graph Theory* **29** (2009) 119–142.
[9] J. E. Greene, A new short proof of Kneser's conjecture, *Amer. Math. Monthly* **109** (2002) 918–920.
[10] R. P. Gupta, The chromatic index and the degree of a graph, *Notices Amer. Math. Soc.* **13** (1966) 449.

[11] H. Hajiabolhassan, X. Zhu, Circular chromatic number of Kneser graphs, *J. Combin. Theory Ser. B* **88** (2003) 299–303.
[12] M. Kneser, Aufgabe 360. Jahresbericht der deutschen mathematiker-vereinigung 2, *Abteilung* **58** (1955) 27.
[13] D. König, Über Graphen und ihre Anwendungen auf Determinantentheorie und Mengenlehre, *Math. Ann.* **77** (1916) 453-465.
[14] D. Leven, Z. Galil, NP-completeness of finding the chromatic index of regular graphs, *J. Algorithms* **4** (1983) 35–44.
[15] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *J. Combin. Theory Ser. A* **25** (1978) 319–324.
[16] J. Matoušek, A combinatorial proof of Kneser's conjecture, *Combinatorica* **24** (2004) 163–170.
[17] V. G. Vizing, On an estimate of the chromatic class of a p-graph (in Russian), *Diskret. Analiz.* **3** (1964) 25–30.