Some Results on Brownian Motion Perturbed by Alternating Jumps in Biological Modeling

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Abstract. We consider the model of random evolution on the real line consisting in a Brownian motion perturbed by alternating jumps. We give the probability density of the process and pinpoint a connection with the limit density of a telegraph process subject to alternating jumps. We study the first-crossing-time probability in two special cases, in the presence of a constant upper boundary.

Keywords: Brownian motion, alternating jumps, first-crossing time.

1 Introduction

In certain biological contexts some phenomena can be viewed as subject to streams of perturbations of various nature and at different scales. We consider systems which evolve according to the Brownian motion and are subject to perturbations driven by suitable stochastic processes. Usually such perturbations produce abrupt changes on the state of the process and can be described by jumps. These phenomena can be often modeled as the superposition of Brownian motion and a pure jump process.

Numerous examples of random motions perturbed by jumps arise in the biological literature. For instance Berg and Brown [1] described the motion of microorganisms performed as gradual or abrupt changes in direction. Moreover, a general framework for the dispersal of cell or organisms is provided in Othmer *et al.* [10], where a position jump process is proposed to describe a motion consisting of sequence of alternative pauses and jumps. See also the general mechanistic movement-model framework employed for biological populations by Lutscher *et al.* [8]. Furthermore we recall the paper by Garcia *et al.* [7], where Brownian-type hopping motions of various *Daphnia* species are studied in detail.

In neuronal modeling framework the Brownian motion process describes the dynamics of the membrane potential in an integrate-and-fire model. This is characterized by the superposition of downward and upward jumps that correspond respectively to the effect of excitatory and inhibitory pulses in a neuronal network (see, for instance, Sacerdote and Sirovich [12] and [13], and references therein).

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A further model is provided by the description of mechanisms of acto-myosin interaction, that is responsible for the force generation during muscle contraction. In this context the rising phase dynamics can be viewed as the superposition of a Brownian motion and a jump process (see, e.g. Buonocore *et al.* [3] and [4]).

Stimulated by the need to give formal and analytical tools to describe biological phenomena perturbed by dichotomous streams, in this paper we study some features of a Brownian motion perturbed by jumps driven by an alternating renewal process. In Section 2 we provide the probability law, mean and variance of such stochastic process, and notice a connection with the jump-telegraph process. Section 3 is devoted to the first-crossing-time problem through a constant boundary. We study the probability that the first-crossing time occurs before or at the occurrence of the first jump.

2 Brownian Motion Perturbed by Alternating Jumps

Let $\{W(t), t > 0\}$ be a Wiener process with drift $\mu \in \mathbb{R}$ and infinitesimal variance σ^2, with $\sigma > 0$. We consider a particle moving on the real line according to $W(t)$, and perturbed by alternating jumps driven by a Poisson process $\{N(t), t > 0\}$ with parameter $\lambda > 0$. Assume that $N(t)$ is independent of $W(t)$. The jumps have constant size $\alpha > 0$, and are directed forward and backward alternately. Moreover, the sequence of jumps is regulated by a Bernoulli random variable B, such as at the k-th event of $N(t)$ the particle performs a displacement $(-1)^{k+B}\alpha$, for $k = 1, 2, 3, \ldots$, with B independent of processes $W(t)$ and $N(t)$. Hence, if $B = 1$ (with probability p) then the first jump is forward and thus the sequence of jumps is $\alpha, -\alpha, \alpha, -\alpha, \ldots$, whereas if $B = 0$ (with probability $1 - p$) the sequence of jumps is $-\alpha, \alpha, -\alpha, \alpha, \ldots$.

Let us consider the stochastic process $\{X(t), t > 0\}$, where

$$X(t) = W(t) + \alpha \sum_{k=1}^{N(t)} (-1)^{k+B}, \qquad t > 0. \tag{1}$$

According to the above assumptions, $X(t)$ gives the position of the particle at time t. A sample-path of $X(t)$ is shown in Figure 1, where the first jump is upward. For $x \in \mathbb{R}$ and $t > 0$ let the probability density of $X(t)$ be denoted as

$$f_X(x,t) = \frac{\partial}{\partial x} P\{X(t) \le x\}. \tag{2}$$

Hereafter we express the density (2) as a time-varying mixture of three Gaussian densities. It involves the following probabilities, for $t > 0$:

$$\begin{aligned}
\pi_{\mathrm{o}}(t) &= P\{N(t)\,\text{odd}\} = \sum_{n=0}^{+\infty} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^{2n+1}}{(2n+1)!} = \mathrm{e}^{-\lambda t}\sinh(\lambda t) = \frac{1-\mathrm{e}^{-2\lambda t}}{2}, \\
\pi_{\mathrm{e}}(t) &= P\{N(t)\,\text{even}\} = \sum_{n=0}^{+\infty} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^{2n}}{(2n)!} = \mathrm{e}^{-\lambda t}\cosh(\lambda t) = \frac{1+\mathrm{e}^{-2\lambda t}}{2},
\end{aligned} \tag{3}$$

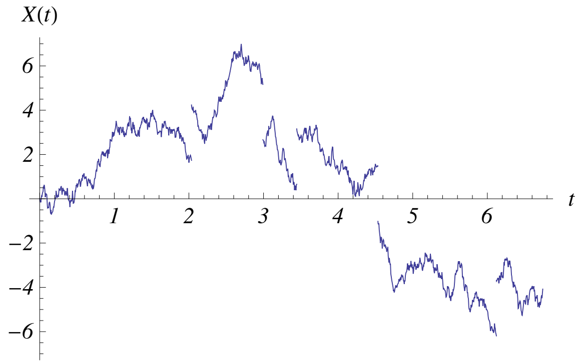

Fig. 1. A simulated sample-path of $X(t)$, for $\mu = 0$, $\sigma = 2$, $\alpha = 2.5$ and $\lambda = 1$

and the Gaussian probability density of process $W(t)$, given by

$$f_W(x,t) := \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{(x-\mu t)^2}{2\sigma^2 t}\right\}, \quad x \in \mathbb{R},\ t > 0. \tag{4}$$

Proposition 1. *For $x \in \mathbb{R}$, $t > 0$, the probability density of $X(t)$ is:*

$$f_X(x,t) = \pi_{\rm e}(t)\, f_W(x,t) + \pi_{\rm o}(t)\left[p\, f_W(x-\alpha,t) + (1-p) f_W(x+\alpha,t)\right]. \tag{5}$$

Moreover, mean and variance of process (1) are, for $t > 0$,

$$E[X(t)] = \mu t + \pi_{\rm o}(t)(2p-1)\alpha, \qquad Var[X(t)] = \sigma^2 t. \tag{6}$$

Remark 1. It is worthwhile noting that when $p = 1/2$, $\mu = 0$ and in the limit as $\lambda \to +\infty$, the density (5) tends to the function $g_1(x,t)$ given in Proposition 4.3 of Di Crescenzo and Martinucci [5], which is the limit density of a telegraph process subject to deterministic jumps occurring at velocity reversals. This confirms that under suitable scaling assumptions the density of the Wiener process with alternating jumps is the limit of the density of a telegraph process subject to the same kind of jumps, this being in agreement to analogous results holding in the absence of jumps (see Orsingher [9]).

3 First-Crossing-Time Problem

In this section we consider the first-crossing-time problem of process $X(t)$ through a constant boundary. We aim to determine the first-crossing-time probability in two different cases, i.e. when the first passage of $X(t)$ through the boundary occurs (i) before the first jump, or (ii) at the occurrence of first jump.

Let J_k be the random time in which the moving particle performs the k-th jump, with $k = 1, 2, \ldots$. We denote by

$$T_\beta^X = \inf\{t > 0 : X(t) > \beta\}, \qquad X(0) = 0 \text{ a.s.}, \tag{7}$$

the first-crossing time of process $X(t)$ through the upper boundary $\beta > 0$, so that the first crossing occurs from below. In the following, we aim to investigate the probability that the first-crossing time of $X(t)$ through β occurs

(i) before the first jump takes place, i.e. $P[T_\beta^X < J_1]$,

(ii) at the occurrence of the first jump, i.e. $P[T_\beta^X = J_1]$.

The case $T_\beta^X > J_1$ will be the object of a subsequent investigation.

Proposition 2. *For $\beta > 0$, $\lambda > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$ we have*

$$P[T_\beta^X < J_1] = \exp\left\{-\frac{\beta}{\sigma^2}\left[\sqrt{\mu^2 + 2\lambda\sigma^2} - \mu\right]\right\}. \tag{8}$$

Proof. Under the given assumptions, since J_1 has exponential distribution with parameter λ, we get

$$P[T_\beta^X < J_1] = E[P[T_\beta^X < J_1 | T_\beta^X]] = E[\mathrm{e}^{-\lambda T_\beta^X}].$$

Hence, recalling that the probability density of (7) is given by (cf. Eq. 2.0.2, p. 223, of Borodin and Salminen [2])

$$f_{T_\beta^X}(t) = \frac{\beta}{\sqrt{2\pi\sigma^2 t^3}} \exp\left\{-\frac{(\beta - \mu t)^2}{2\sigma^2 t}\right\}, \qquad t > 0,$$

rearranging the terms we have

$$\begin{aligned} P[T_\beta^X < J_1] &= \int_0^{+\infty} \mathrm{e}^{-\lambda t} f_{T_\beta^X}(t)\,\mathrm{d}t \\ &= \mathrm{e}^{\beta\mu/\sigma^2} \int_0^{+\infty} \mathrm{e}^{-\left(\lambda + \frac{\mu^2}{2\sigma^2}\right)t} \frac{\beta}{\sqrt{2\pi\sigma^2 t^3}}\, \mathrm{e}^{-\beta^2/(2\sigma^2 t)}\,\mathrm{d}t. \end{aligned}$$

Recalling Eq. (28) of § 4.5 of Erdélyi *et al.* [6] we thus obtain

$$P[T_\beta^X < J_1] = \mathrm{e}^{\beta\mu/\sigma^2}\, \mathrm{e}^{-\sqrt{\frac{2\beta^2}{\sigma^2}\left(\lambda + \frac{\mu^2}{2\sigma^2}\right)}},$$

so that Eq. (8) immediately follows.

From Eq. (8) we note that $P[T_\beta^X < J_1]$ is increasing in μ and σ, whereas it is decreasing in β and λ. Figure 2 shows some plots of such probability.

Proposition 3. *For $\beta > 0$, $0 < p < 1$, $\lambda > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$ we have*

$$P[T_\beta^X = J_1] = \begin{cases} p\,\mathrm{e}^{\frac{\beta(\mu-\gamma)}{\sigma^2}} \left\{\mathrm{e}^{-\frac{\alpha\mu}{\sigma^2}}\left[\cosh\left(\frac{\alpha\gamma}{\sigma^2}\right) + \frac{\mu}{\gamma}\sinh\left(\frac{\alpha\gamma}{\sigma^2}\right)\right] - 1\right\}, & 0 < \alpha \le \beta, \\ p\left\{1 - \mathrm{e}^{\frac{\beta(\mu-\gamma)}{\sigma^2}} - \left(1 - \frac{\mu}{\gamma}\right)\mathrm{e}^{-\frac{\alpha(\gamma+\mu)}{\sigma^2}}\,\mathrm{e}^{\frac{\beta\mu}{\sigma^2}}\sinh\left(\frac{\beta\gamma}{\sigma^2}\right)\right\}, & \alpha \ge \beta, \end{cases} \tag{9}$$

where we have set $\gamma = \sqrt{\mu^2 + 2\lambda\sigma^2}$.

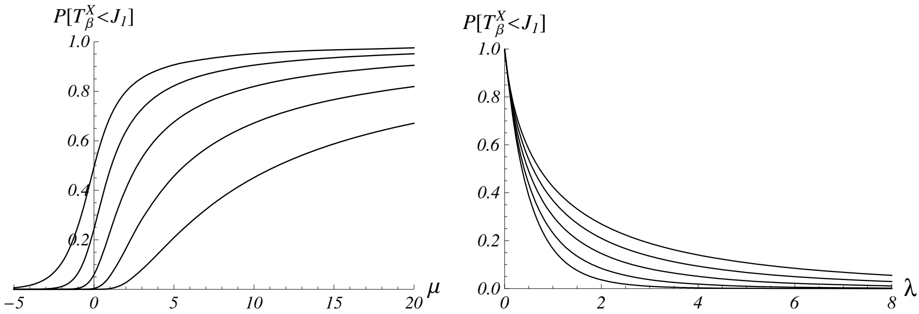

Fig. 2. Probability (8) for $-5 \le \mu \le 20$, when $\lambda = 1$, $\sigma = 1$ and $\beta = 0.5, 1, 2, 4, 8$ (from top to bottom, left plot) and for $0 \le \lambda \le 8$, when $\beta = 2$, $\mu = 1$ and $\sigma = 0.5, 1, 1.5, 2, 2.5$ (from bottom to top, right plot)

Proof. By conditioning on the instant of the first jump and exploiting the properties of the first-crossing times we have

$$\begin{aligned} P[T_\beta^X = J_1] &= E[P[T_\beta^X = J_1 | J_1]] \\ &= E[P[\beta - \alpha \le W(J_1^-) < \beta, T_\beta^W > J_1^-, B = 1 | J_1]]. \end{aligned} \tag{10}$$

Let us consider the β-avoiding density of process $W(t)$:

$$\begin{aligned} f_W^{\langle\beta\rangle}(x,t) &:= \frac{\partial}{\partial x} P[W(t) \le x, T_\beta^W > t] \\ &= f_W(x,t) - \exp\left\{-\frac{2\mu(\beta - x)}{\sigma^2}\right\} f_W(2\beta - x, t), \end{aligned} \tag{11}$$

where f_W is the density of $W(t)$, given in (4), and where the last equality follows from a suitable symmetry property of $W(t)$ (see, for instance, Example 5.4 of Ricciardi *et al.* [11]). Due to (10) we thus have

$$P[T_\beta^X = J_1] = p \int_0^{+\infty} \lambda \mathrm{e}^{-\lambda t} \int_{\beta-\alpha}^{\beta} f_W^{\langle\beta\rangle}(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Making use of Eq. (11) after some calculations we get, for $\gamma = \sqrt{\mu^2 + 2\lambda\sigma^2}$,

$$\begin{aligned} P[T_\beta^X = J_1] = \frac{\lambda p}{\sqrt{\mu^2 + 2\lambda\sigma^2}} \int_{\beta-\alpha}^{\beta} \Bigg[&\exp\left\{\frac{\mu x - |x|\gamma}{\sigma^2}\right\} \\ &- \exp\left\{-\frac{2\mu}{\sigma^2}(\beta - x)\right\} \exp\left\{\frac{\mu x - |x - 2\beta|\gamma}{\sigma^2}\right\}\Bigg] \mathrm{d}x. \end{aligned} \tag{12}$$

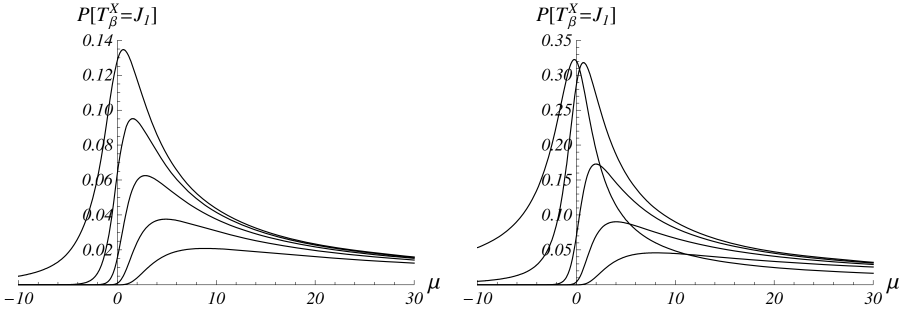

Fig. 3. Probability (9) for $-10 \le \mu \le 30$, when $p = 1$, $\lambda = 1$, $\sigma = 1$ and $\beta = 0.5, 1, 2, 4, 8$ (from top to bottom near the origin), for $\alpha = 0.5$ (left plot) and $\alpha = 1$ (right plot)

Noting that

$$\int_{\beta-\alpha}^{\beta} \exp\left\{\frac{\mu x - |x|\gamma}{\sigma^2}\right\} \mathrm{d}x$$
$$= \begin{cases} \frac{(\gamma+\mu)}{2\lambda}\left[\exp\left\{\frac{(\beta-\alpha)(\mu-\gamma)}{\sigma^2}\right\} - \exp\left\{\frac{\beta(\mu-\gamma)}{\sigma^2}\right\}\right], & 0 < \alpha \le \beta, \\ \frac{\gamma}{\lambda} - \frac{1}{2\lambda}\left[(\gamma-\mu)\exp\left\{\frac{(\beta-\alpha)(\gamma+\mu)}{\sigma^2}\right\} + (\gamma+\mu)\exp\left\{\frac{\beta(\mu-\gamma)}{\sigma^2}\right\}\right], & \alpha \ge \beta, \end{cases}$$

and

$$\int_{\beta-\alpha}^{\beta} \exp\left\{-\frac{2\mu}{\sigma^2}(\beta - x)\right\} \exp\left\{\frac{\mu x - |x-2\beta|\gamma}{\sigma^2}\right\} \mathrm{d}x$$
$$= \frac{\gamma-\mu}{2\lambda} \exp\left\{\frac{\beta(\mu-\gamma)}{\sigma^2}\right\}\left[1 - \exp\left\{-\frac{\alpha(\mu+\gamma)}{\sigma^2}\right\}\right],$$

Eq. (9) thus finally follows from Eq. (12).

Remark 2. From Eq. (9) we have:

$$\lim_{\lambda\to+\infty} P[T_\beta^X = J_1] = \begin{cases} 0, & \alpha < \beta, \\ p/2, & \alpha = \beta, \\ p, & \alpha > \beta, \end{cases} \qquad \lim_{\alpha\to+\infty} P[T_\beta^X = J_1] = p\left\{1 - \mathrm{e}^{\frac{\beta(\mu-\gamma)}{\sigma^2}}\right\},$$

$$\lim_{\beta\to+\infty} P[T_\beta^X = J_1] = \lim_{\mu\to+\infty} P[T_\beta^X = J_1] = \lim_{\sigma\to+\infty} P[T_\beta^X = J_1] = 0.$$

Moreover,

$$\lim_{\sigma\to 0} P[T_\beta^X = J_1] = P(\beta-\alpha < \mu J_1 < \beta) = \begin{cases} p\,\mathrm{e}^{-\beta\lambda/\mu}\left(\mathrm{e}^{\alpha\lambda/\mu} - 1\right), & \mu \ge 0,\ \alpha \le \beta, \\ 0, & \mu \le 0,\ \alpha \le \beta, \\ p\left(1 - \mathrm{e}^{-\beta\lambda/\mu}\right), & \mu \ge 0,\ \alpha \ge \beta, \\ p\left(1 - \mathrm{e}^{-(\beta-\alpha)\lambda/\mu}\right), & \mu \le 0,\ \alpha \ge \beta. \end{cases} \tag{13}$$

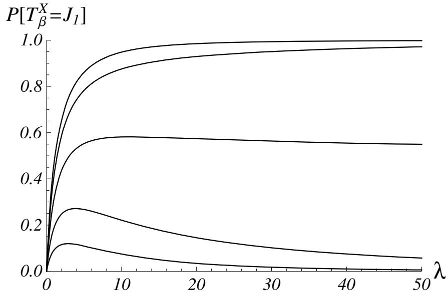

Fig. 4. Probability (9) for $0 \leq \lambda \leq 50$, when $p = 1$, $\mu = 1$, $\beta = 1$ and $\alpha = 0.5, 0.75, 1, 1.25, 1.5$ (from bottom to top), for $\sigma = 1$ (left plot) and $\sigma = 2$ (right plot)

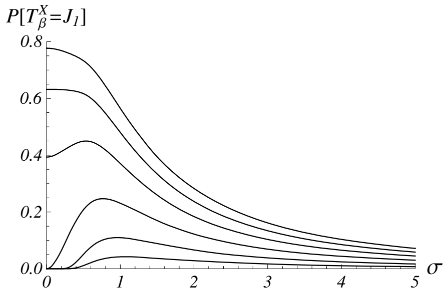

Fig. 5. Probability (9) for $0 \leq \sigma \leq 5$, with $p = 1$, $\beta = 1$, $\lambda = 1$ and $\alpha = 0.5, 0.75, 1, 1.25, 1.5, 1.75$ (from bottom to top), for $\mu = -0.5$ (left plot) and $\mu = 0.1$ (right plot)

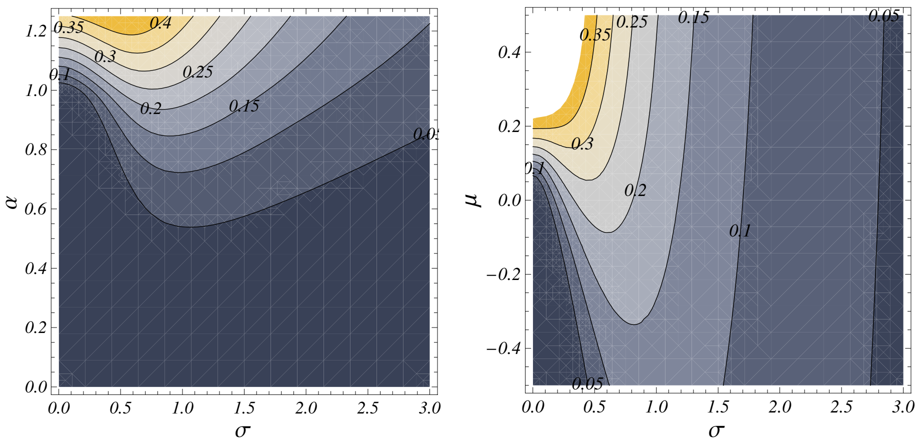

Fig. 6. Contour plot of (9) for $0 \leq \sigma \leq 3$, $p = 1$, $\beta = 1$, $\lambda = 1$, with $0 \leq \alpha \leq 1.25$, $\mu = -0.5$ (left plot), and $-0.5 \leq \mu \leq 0.5$, $\alpha = 0.8$ (right plot)

Various plots of probability (9) are given in Figures 4 and 5, for different choices of the parameters. In particular, Figure 5 shows that, for some choices of the involved constants, $P[T_\beta^X = J_1]$ attains a maximum for a positive value of σ. This is of special interest in problems where optimal values of the "noise parameter" σ are relevant for biological systems in which the maximization of certain utility functions is significant. The non-monotonic behaviour of probability (9) with respect to σ is also confirmed by the contour plot of $P[T_\beta^X = J_1]$ given in Figure 6.

References

1. Berg, H.C., Brown, D.A.: Chemotaxis in Escherichia coli analysed by three-dimensional tracking. Nature 239, 500–504 (1972)
2. Borodin, A.N., Salminen, P.: Handbook of Brownian motion – facts and formulae. Probability and its Applications. Birkhäuser Verlag, Basel (1996)
3. Buonocore, A., Di Crescenzo, A., Martinucci, B.: A stochastic model for the rising phase of myosin head displacements along actin filaments. In: Capasso, V. (ed.) 5th ESMTB Conference on Mathematical Modelling & Computing in Biology and Medicine, pp. 121–126. MIRIAM Proj., 1, Esculapio, Bologna (2003)
4. Buonocore, A., Di Crescenzo, A., Martinucci, B., Ricciardi, L.M.: A stochastic model for the stepwise motion in actomyosin dynamics. Sci. Math. Japon. 58, 245–254 (2003)
5. Di Crescenzo, A., Martinucci, B.: On the generalized telegraph process with deterministic jumps. Methodol. Comput. Appl. Probab. 15, 2012–2235 (2013)
6. Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Tables of Integral Transforms, vol. I. McGraw-Hill Publisher, New York (1954)
7. Garcia, R., Moss, F., Nihongi, A., Strickler, J.R., Göller, S., Erdmann, U., Schimansky-Geier, L., Sokolov, I.M.: Optimal foraging by zooplankton within patches: the case of Daphnia. Math. Biosci. 207, 165–188 (2007)
8. Lutscher, F., Pachepsky, E., Lewis, M.A.: The effect of dispersal patterns on stream populations. SIAM J. Appl. Math. 65, 1305–1327 (2005)
9. Orsingher, E.: Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws. Stoch. Proc. Appl. 34, 49–66 (1990)
10. Othmer, H.G., Dunbar, S.R., Alt, W.: Models of dispersal in biological systems. J. Math. Biol. 26, 263–298 (1988)
11. Ricciardi, L.M., Di Crescenzo, A., Giorno, V., Nobile, A.G.: An outline of theoretical and algorithmic approaches to first passage time problems with applications to biological modeling. Math. Japon. 50(2), 247–322 (1999)
12. Sacerdote, L., Sirovich, R.: Multimodality of the interspike interval distribution in a simple jump-diffusion model. Sci. Math. Japon. 58, 307–322 (2003)
13. Sacerdote, L., Sirovich, R.: A Wiener process with inverse Gaussian time distributed jumps as a model for neuronal activity. In: Capasso, V. (ed.) 5th ESMTB Conference on Mathematical Modelling & Computing in Biology and Medicine, pp. 134–140. MIRIAM Proj., 1, Esculapio, Bologna (2003)