HOMOGENEITY OF MAGNETIC TRAJECTORIES IN THE REAL SPECIAL LINEAR GROUP

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Dedicated to professor Iskander Asanovich Taimanov on the occasion of his 60th birthday

Abstract. We prove the homogeneity of contact magnetic curves in the real special linear group of degree 2. Every contact magnetic trajectory is a product of a homogeneous geodesic and a charged Reeb flow.

Introduction

According to Thurston's *Geometrization Conjecture* (proved by Perelman), any compact orientable 3-fold can be cut in a special way into pieces admitting one of the eight geometric structures: the Euclidean, spherical, and hyperbolic structures ($\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$), two product manifolds ($\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$), and the 3-dimensional Lie groups (Nil_3, Sol_3 and $\widetilde{\mathrm{SL}}_2\mathbb{R}$). Each model spaces admits compatible Riemannian metric. The resulting Riemannian 3-manifolds are naturally reductive homogeneous spaces except Sol_3. Notice that compact Riemannian symmetric spaces are normal homogeneous spaces. It should be remarked that the class of naturally reductive homogeneous spaces is strictly wider than that of normal homogeneous space. Indeed, among the eight model spaces of Thurston geometry, the only normal homogeneous spaces are spheres. The class of simply connected 3-dimensional naturally reductive homogeneous space is exhausted by the above seven model spaces together with Berger 3-spheres.

Dynamical systems on 3-manifolds, especially, the model spaces (and their compact quotients) of Thurston geometry have been paid much attention of mathematicians. An important study concerning the relation between the integrability and the vanishing of the topological entropy of the geodesic flow is done by Bolsinov and Taimanov. In [7], they gave a $\mathbb{T}^2$-bundle over a circle $\mathbb{S}^1$ which provides an example of Liouville integrable geodesic flow with positive topological entropy and vanishing Liouville entropy.

We concentrate our attention to $\widetilde{\mathrm{SL}}_2\mathbb{R}$ and its quotients. The special linear group $\mathrm{SL}_2\mathbb{R}$ is a *non-normal* naturally reductive homogeneous space. Thus all of geodesics are *homogeneous*. This fact implies that geodesic flows of $\mathrm{SL}_2\mathbb{R}$ are *integrable* in non-commutative sense (see [5, 15]). According to [8], the fact that the modular 3-fold $\mathrm{PSL}_2\mathbb{R}/\mathrm{PSL}_2\mathbb{Z}$ is topologically equivalent to the complement of the trefoil $\mathcal{K}$ in the 3-sphere was first observed by Quillen (see Milnor [17]). Bolsinov, Veselov and Ye [8] proved that the periodic geodesics on the

2000 *Mathematics Subject Classification.* 53C15, 53C25, 53C30, 37J45, 53C80.

Key words and phrases. magnetic field; hyperbolic Sasakian space form; homogeneous geodesic

Proc. Amer. Math. Soc. **152** (2024), 1287-1300.

modular 3-fold $\mathrm{SL}_2\mathbb{R}/\mathrm{SL}_2\mathbb{Z}$ with sufficiently large values of $\mathcal{C} = \kappa^2/16$ represent trefoil cable knots in $\mathbb{S}^3 \setminus \mathcal{K}$, where κ is the geodesic curvature of the projected curve in the hyperbolic surface. Any trefoil cable knot can be described in this way. These facts really prove that geodesics flows in $\mathrm{SL}_2\mathbb{R}$ and its compact quotients are very attractive dynamical system from both differential geometry and low-dimensional topology perspectives.

It would be interesting to perturbe geodesic flows of $\mathrm{SL}_2\mathbb{R}$. Concerning this viewpoint, Arnol'd [1] suggested to perturbe the symplectic form of the Hamiltonian system of geodesic flow of arbitrary Riemannian manifolds by a closed 2-form on the configuration manifold. The 2-form is regarded as a *static magnetic field* of the configuration manifold. Since then magnetic trajectories on Riemannian manifolds have been paid much attention of mathematicians. In particular, analysis and classification of periodic trajectories have been studied extensively, see *e.g.*, Bahri and Taimanov [4]. On the special linear group $\mathrm{SL}_2\mathbb{R}$, there exists a unit Killing vector field ξ (Reeb vector field) which is identified with the magnetic field (called the *contact magnetic field*) whose potential is the canonical contact structure. The differential equation of the magnetic trajectory strongly reflects geometric properties of $\mathrm{SL}_2\mathbb{R}$. In this sense the system of contact magnetic trajectory is a nice perturbation of geodesic flow. In our previous work [13] we investigated periodicity of contact magnetic trajectories in $\mathrm{SL}_2\mathbb{R}$.

In [6], Bolsinov and Jovanović studied magnetic trajectories in normal homogeneous Riemannian spaces. They showed that every magnetic trajectory starting at the origin is homogeneous (see [6, Remark 1]). We expect the homogeneity of magnetic trajectories on more wider class of homogeneous Riemannian spaces. As our first attempt, in this article we prove that magnetic trajectories in the special linear group $\mathrm{SL}_2\mathbb{R}$ (derived from the canonical contact structure) are homogeneous.

More precisely, we show that every contact magnetic trajectory γ of charge q starting at the origin Id of $\mathrm{SL}_2\mathbb{R}$ with initial velocity X and with charge q is the product of the homogeneous geodesic $\gamma_X(s)$ with initial velocity X and the *charged Reeb flow* $\exp_{\mathrm{SO}(2)}\{s(2q\xi)\}$.

1. Preliminaries

1.1. **Homogeneous geometry.** Let $N = \mathcal{G}/\mathcal{H}$ be a homogeneous space. Denote by $\mathfrak{G}$ and $\mathfrak{H}$ the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$, respectively. Then N is said to be *reductive* if there exits a linear subspace $\mathfrak{p}$ of $\mathfrak{G}$ complementary to $\mathfrak{H}$ and satisfies $[\mathfrak{H}, \mathfrak{p}] \subset \mathfrak{p}$. It is known that every homogeneous Riemannian space is reductive.

Now let $N = \mathcal{G}/\mathcal{H}$ be a homogeneous Riemannian space with reductive decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{p}$ and a $\mathcal{G}$-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$. Then N is said to be *naturally reductive* (with respect to $\mathfrak{p}$) if it satisfies

$$\langle [X, Y]_{\mathfrak{p}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{p}} \rangle = 0 \tag{1.1}$$

for any X, Y, $Z \in \mathfrak{p}$. Here we denote the $\mathfrak{p}$ part of a vector $X \in \mathfrak{G}$ by $X_{\mathfrak{p}}$.

A homogeneous Riemannian space $N = \mathcal{G}/\mathcal{H}$ is said to be *normal* if G is compact semi-simple and the metric is derived from the restriction of the bi-invariant Riemannian metric of G. Normal homogeneous spaces are naturally reductive.

As a generalization of naturally reductive homogeneous space, the notion of Riemannian g. o. space was introduced by Kowalski and Vanhecke [16].

According to [16], a homogeneous Riemannian manifold $N = \mathcal{G}/\mathcal{H}$ is called a *space with homogeneous geodesics* or a *Riemannian g.o. space* if every geodesic $\gamma(s)$ of M is an orbit of a one-parameter subgroup of the *largest* connected group of isometries. Naturally reductive homogenous spaces are typical examples of Riemannian g.o. spaces. (For more informations, we refer to [3]).

1.2. **The Euler-Arnold equation.** Let G be a Lie group equipped with a left invariant Riemannian metric $\langle\cdot,\cdot\rangle$. Denote by $\mathfrak{g}$ the Lie algebra of G. The bi-invariance obstruction U is a symmetric bilinear map $\mathsf{U} : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ defined by

$$2\langle \mathsf{U}(X,Y),Z\rangle = -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle,\ X,Y\in\mathfrak{g} \tag{1.2}$$

The Levi-Civita connection ∇ is described as

$$\nabla_X Y = \frac{1}{2}[X,Y] + \mathsf{U}(X,Y), \quad X,Y\in\mathfrak{g}.$$

On the Lie algebra $\mathfrak{g}$, we define the linear operator $\mathrm{ad}^* : \mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ by

$$\langle \mathrm{ad}(X)Y,Z\rangle = \langle Y,\mathrm{ad}^*(X)Z\rangle, \quad X,Y,Z\in\mathfrak{g}.$$

One can see that

$$-2\mathsf{U}(X,Y) = \mathrm{ad}^*(X)Y + \mathrm{ad}^*(Y)X.$$

Take a curve $\gamma(s)$ starting at the origin (identity Id) of G. Set $\Omega(s) = \gamma(s)^{-1}\dot\gamma(s)$, then one can check that

$$\nabla_{\dot\gamma}\dot\gamma = \gamma\left(\dot\Omega - \mathrm{ad}^*(\Omega)\Omega\right). \tag{1.3}$$

This implies the following fundamental fact [2, Appendix 2]:

Proposition 1.1. *The curve $\gamma(s)$ is a geodesic if and only if*

$$\dot\Omega = \mathrm{ad}^*(\Omega)\Omega.$$

Now let us assume that G is semi-simple. Then G admits a bi-invariant semi-Riemannian metric B. It is a constant multiple of the Killing form. On this reason we call the metric by the name (normalized) *Killing metric* and denote it by B. Introduce an endomorphism field $\mathcal{I}$ (called the *moment of inertia tensor field*) on $\mathfrak{g}$ by

$$\langle X,Y\rangle = \mathsf{B}(\mathcal{I}X,Y), \quad \mathcal{I}X,Y\in\mathfrak{g}$$

and the (*anglar*) *momentum* (see [2, Appendix 2.C]) by

$$\mu = \mathcal{I}\Omega.$$

Then $\dot\Omega = \mathcal{I}^{-1}\dot\mu$ and hence

$$\langle\nabla_{\dot\gamma}\dot\gamma,\gamma Z\rangle = \langle\mathcal{I}^{-1}(\dot\mu - [\mu,\Omega]),Z\rangle.$$

Thus we arrive at the so-called *Euler-Arnold equation* [18] (also called the *Euler-Poincaré equation* [12])

$$\dot\mu - [\mu,\Omega] = 0. \tag{1.4}$$

When $G = \mathrm{SO}(3)$ equipped with Killing metric, $\mathit{\Omega}_{\mathsf{c}} := \gamma(t)^{-1}\dot{\gamma}(t)$ and $\mathit{\Omega}_{\mathsf{s}} := \dot{\gamma}(t)\gamma(t)^{-1}$ are *anglar velocity in the body* (*corpus*) and *angular velocity in the space*, respectively. The terminologies are due to Arnold and are justified by the fact that the motion of a rigid body in Euclidean 3-space under inertia is a geodesic in SO(3) equipped with Killing metric.

1.3. **Static magnetism.** Let (M, g) be a Riemannian manifold. A (static) *magnetic field* is a closed 2-form F on M. The *Lorentz force* φ derived from F is an endomorphim field defined by

$$F(X, Y) = g(\varphi X, Y), \quad X, Y \in \mathit{\Gamma}(TM).$$

A curve $\gamma(s)$ is said to be a *magnetic trajectory* under the influence of F if it obeys the *Lorentz equation*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\varphi\dot{\gamma},$$

where ∇ is the Levi-Civita connection of (M, g), $\dot{\gamma}$ is the velocity of $\gamma(s)$ and q is a constant called the *charge*.

The Lorentz equation implies that magnetic trajectories are of constant speed. Note that when $F = 0$ or $q = 0$, magnetic trajectories reduce to geodesics. For more information on magnetic trajectories, we refer to [14].

2. The special linear group

2.1. **Iwasawa decomposition.** Let $\mathrm{SL}_2\mathbb{R}$ be the real special linear group of degree 2:

$$\mathrm{SL}_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

The Iwasawa decomposition $\mathrm{SL}_2\mathbb{R} = NAK$ of $\mathrm{SL}_2\mathbb{R}$;

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \ x \in \mathbb{R} \right\}, \qquad \text{(Nilpotent part)}$$

$$A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \middle| \ y > 0 \right\}, \qquad \text{(Abelian part)}$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \ 0 \leq \theta < 2\pi \right\} = \mathrm{SO}(2), \quad \text{(Maximal torus)}$$

allows to introduce the following global coordinate system (x, y, θ) of $\mathrm{SL}_2\mathbb{R}$:

$$(x, y, \theta) \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.1}$$

The mapping

$$\psi : \mathbb{H}^2(-4) \times \mathbb{S}^1 \to \mathrm{SL}_2\mathbb{R}; \quad \psi(x, y, \theta) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is a diffeomorphism onto $\mathrm{SL}_2\mathbb{R}$. Hereafter, we shall refer (x, y, θ) as a global coordinate system of $\mathrm{SL}_2\mathbb{R}$. Hence $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^1$ and hence diffeomorphic to $\mathbb{R}^3 \setminus \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}^+$ is diffeomorphic to open unit disk $\mathbb{D}$, then $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to open solid torus $\mathbb{D} \times \mathbb{S}^1$.

The Iwasawa decomposition of $\mathrm{SL}_2\mathbb{R}$ can be carried out explicitly:

Proposition 2.1. *The Iwasawa decomposition of an element* $p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in \mathrm{SL}_2\mathbb{R}$ *is given explicitly by* $p = n(p)a(p)k(p)$*, where*

$$n(p) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(p) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}, \quad k(p) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with

$$x = \frac{p_{11}p_{21} + p_{12}p_{22}}{(p_{21})^2 + (p_{22})^2}, \quad y = \frac{1}{(p_{21})^2 + (p_{22})^2}, \quad e^{i\theta} = \frac{p_{22} - ip_{21}}{\sqrt{(p_{21})^2 + (p_{22})^2}}.$$

2.2. **The standard Riemannian metric.** The Lie algebra $\mathfrak{sl}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$ is given explicitly by

$$\mathfrak{sl}_2\mathbb{R} = \left\{ X \in \mathrm{M}_2\mathbb{R} \;\middle|\; \mathrm{tr}\, X = 0 \right\}.$$

We take the following basis of $\mathfrak{sl}_2\mathbb{R}$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This basis satisfies the commutation relations:

$$[E, F] = H, \quad [F, H] = 2F, \quad [H, E] = 2E.$$

The Lie algebra $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ of the closed subgroups N, A and K are given by

$$\mathfrak{n} = \mathbb{R}E, \quad \mathfrak{a} = \mathbb{R}H, \quad \mathfrak{k} = \mathbb{R}(E - F).$$

The Lie algebra $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}_2\mathbb{R}$. Moreover $\mathfrak{n}$ and $\mathbb{R}F$ are root spaces with respect to $\mathfrak{h}$. The decomposition $\mathfrak{sl}_2\mathbb{R} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathbb{R}F$ is the root space decomposition (or Gauss decompostion) of $\mathfrak{sl}_2\mathbb{R}$.

Hereafter we use the left invariant frame field $\{e_1 = E - F, e_2 = E + F, e_3 = H\}$. These left invariant vector fields are given explicitly by

$$\begin{aligned} e_1 &= \frac{\partial}{\partial\theta}, \\ e_2 &= \cos(2\theta)\left(2y\frac{\partial}{\partial x} - \frac{\partial}{\partial\theta}\right) + \sin(2\theta)\left(2y\frac{\partial}{\partial y}\right), \\ e_3 &= -\sin(2\theta)\left(2y\frac{\partial}{\partial x} - \frac{\partial}{\partial\theta}\right) + \cos(2\theta)\left(2y\frac{\partial}{\partial y}\right). \end{aligned}$$

Define an inner product $\langle\cdot,\cdot\rangle$ so that $\{e_1, e_2, e_3\}$ is orthonormal with respect to $\langle\cdot,\cdot\rangle$. By left-translating this inner product, we equip a left invariant Riemannian metric

$$g = \frac{dx^2 + dy^2}{4y^2} + \left(d\theta + \frac{dx}{2y}\right)^2.$$

The one form

$$\eta = d\theta + \frac{dx}{2y}$$

is a globally defined contact form on $\mathrm{SL}_2\mathbb{R}$. The Reeb vector field ξ of the contact form η is $\xi = e_1$.

The universal covering space of $(\mathrm{SL}_2\mathbb{R}, g)$ is one of the model space of Thurston geometry [19].

2.3. **The hyperbolic Hopf fibering.** The special linear group $\mathrm{SL}_2\mathbb{R}$ acts transitively and isometrically on the upper half plane:

$$\mathbb{H}^2(-4) = \left(\{(x, y) \in \mathbb{R}^2 \mid y > 0\}, \frac{dx^2 + dy^2}{4y^2}\right)$$

of constant curvature -4 by the linear fractional transformation as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Here we regard a point $(x, y) \in \mathbb{H}^2(-4)$ as a complex number $z = x + yi$.

The isotropy subgroup of $\mathrm{SL}_2\mathbb{R}$ at $i = (0, 1)$ is the rotation group $\mathrm{SO}(2)$. The natural projection $\pi : (\mathrm{SL}_2\mathbb{R}, g) \to \mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2) = \mathbb{H}^2(-4)$ is given explicitly by

$$\pi(x, y, \theta) = (x, y) \in \mathbb{H}^2(-4)$$

in terms of the global coordinate system (2.1).

The tangent space $T_i\mathbb{H}^2(-4)$ at the origin $i = (0, 1)$ is identified with the vector subspace $\mathfrak{m}$ defined by

$$\mathfrak{m} = \{X \in \mathfrak{sl}_2\mathbb{R} \mid {}^tX = X\}.$$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$ has the orthogonal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This splitting can be carried out explicitly as

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}}, \quad X_{\mathfrak{k}} = \frac{1}{2}(X - {}^tX), \quad X_{\mathfrak{m}} = \frac{1}{2}(X + {}^tX).$$

It is easy to see that the projection $\pi : (\mathrm{SL}_2\mathbb{R}, g) \to \mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2) = \mathbb{H}^2(-4)$ is a Riemannian submersion with totally geodesic fibres. This submersion $\pi : (\mathrm{SL}_2\mathbb{R}, g) \to \mathbb{H}^2(-4)$ is called the *hyperbolic Hopf fibering* of $\mathbb{H}^2(-4)$.

Under the identification $\mathfrak{k} \cong \mathbb{R}$, the contact form η is regarded as a connection form of the principal circle bundle $\mathrm{SL}_2\mathbb{R} \to \mathbb{H}^2(-4)$.

2.4. **The naturally reductive structure.** On the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$, the inner product $\langle \cdot, \cdot \rangle$ at the identity induced from g is written as

$$\langle X, Y \rangle = \frac{1}{2}\,\mathrm{tr}\,({}^tXY), \quad X, Y \in \mathfrak{sl}_2\mathbb{R}.$$

One can see that the metric g is not only invariant by $\mathrm{SL}_2\mathbb{R}$-left translation but also right translations by $\mathrm{SO}(2)$. Hence the Lie group $\mathcal{G} = \mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)$ with multiplication:

$$(a, b)(a', b') = (aa', bb')$$

acts isometrically on $\mathrm{SL}_2\mathbb{R}$ via the action:

$$(\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)) \times \mathrm{SL}_2\mathbb{R} \to \mathrm{SL}_2\mathbb{R}; \quad (a, b) \cdot X = aXb^{-1}.$$

Furthermore, this action of $\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$ on $\mathrm{SL}_2\mathbb{R}$ is transitive, hence $\mathrm{SL}_2\mathbb{R}$ is a homogeneous Riemannian space of $\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$. The isotropy subgroup $\mathcal{H}$ of $\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$ at the identity matrix Id is the diagonal subgroup

$$\Delta K = \{(k,k) \mid k \in K\} \cong K$$

of $K\times K$. The coset space $(\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2))/\mathrm{SO}(2)$ is a reductive homogeneous space. The Lie algebra $\mathfrak{G}$ of the product group $\mathcal{G} = G\times K$ is $\mathfrak{G} = \mathfrak{g}\oplus\mathfrak{k}$. On the other hand the Lie algebra $\mathfrak{H}$ of $\mathcal{H} = \Delta K$ is

$$\Delta\mathfrak{k} = \{(W,W) \mid W \in \mathfrak{k}\} \cong \mathfrak{k}.$$

The tangent space $T_{\mathrm{Id}}\mathrm{SL}_2\mathbb{R}$ of $\mathcal{G}/\mathcal{H} = (\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2))/\Delta K$ is the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$. This tangent space is identified with the vector subspace $\mathfrak{p}$ of $\mathfrak{G} = \mathfrak{g}\times\mathfrak{k}$ defined by

$$\mathfrak{p} = \{(V+W, 2W) \mid V \in \mathfrak{m},\ W \in \mathfrak{k}\}.$$

The Lie algebra $\mathfrak{g}\oplus\mathfrak{k}$ is decomposed as $\mathfrak{g}\oplus\mathfrak{k} = \Delta\mathfrak{k}\oplus\mathfrak{p}$. One can see that this decomposition is reductive. Every $(X,Y) \in \mathfrak{g}\oplus\mathfrak{k}$ is decomposed as

$$(X,Y) = (2X_{\mathfrak{k}} - Y, 2X_{\mathfrak{k}} - Y) + (X_{\mathfrak{m}} + (Y - X_{\mathfrak{k}}), 2(Y - X_{\mathfrak{k}})).$$

One can see that the linear subspace $\mathfrak{p}$ satisfies (1.1). Thus $(\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2))/\mathrm{SO}(2)$ is naturally reductive with respect to the decomposition $\mathfrak{g}\oplus\mathfrak{k} = \Delta\mathfrak{k}\oplus\mathfrak{p}$. It is known as the only naturally reductive space representation of $\mathrm{SL}_2\mathbb{R}$ up to isomorphism (see [11]).

2.5. **Curvatures.** The commutation relations of $\{e_1, e_2, e_3\}$ are

$$[e_1,e_2] = 2e_3, \quad [e_2,e_3] = -2e_1, \quad [e_3,e_1] = 2e_2.$$

The Levi-Civita connection ∇ of is given by

$$\begin{array}{ccc} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = 3e_3, & \nabla_{e_1}e_3 = -3e_2 \\ \nabla_{e_2}e_1 = e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = -e_1 \\ \nabla_{e_3}e_1 = -e_2, & \nabla_{e_3}e_2 = e_1 & \nabla_{e_3}e_3 = 0. \end{array}$$

The Riemannian curvature R defined by

$$R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

is given by

$$R(e_1,e_2)e_1 = -e_2, \quad R(e_1,e_2)e_2 = e_1,$$
$$R(e_2,e_3)e_2 = 7e_3, \quad R(e_2,e_3)e_3 = -7e_2,$$
$$R(e_1,e_3)e_1 = -e_3, \quad R(e_1,e_3)e_3 = e_1.$$

The basis $\{e_1, e_2, e_3\}$ diagonalizes the Ricci tensor. The principal Ricci curvatures are given by

$$\rho_1 = 2, \quad \rho_2 = \rho_3 = -6.$$

The bi-invariance obstruction U defined by

$$2\langle \mathsf{U}(X,Y), Z\rangle = -\langle X, [Y,Z]\rangle + \langle Y, [Z,X]\rangle, \quad X,Y,Z \in \mathfrak{sl}_2\mathbb{R}$$

is given by

$$\mathsf{U}(e_1,e_2) = 2e_3, \quad \mathsf{U}(e_1,e_3) = -2e_2. \tag{2.2}$$

All the other components are zero.

From these we obtain

$$\mathsf{U}(X,Y) = [X_{\mathfrak{k}}, Y_{\mathfrak{m}}] + [Y_{\mathfrak{k}}, X_{\mathfrak{m}}], \quad X, Y \in \mathfrak{g}.$$

The Levi-Civita connection is rewritten as

$$\nabla_X Y = \frac{1}{2}[X,Y] + \mathsf{U}(X,Y) = \frac{1}{2}[X,Y] + [X_{\mathfrak{k}}, Y_{\mathfrak{m}}] + [Y_{\mathfrak{k}}, X_{\mathfrak{m}}], \quad X, Y \in \mathfrak{g}. \tag{2.3}$$

2.6. **The contact magnetic field.** The contact form η gives a left invariant magnetic field $F = d\eta$ called the *contact magnetic field* (see [13, 14]). The Lorentz force φ of F is given by

$$\varphi e_1 = 0, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2.$$

Then the quartet (φ, ξ, η, g) satisfies the following relations:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \nabla_X \xi = \varphi X,$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X,$$

for all X, $Y \in \Gamma(T\mathrm{SL}_2\mathbb{R})$. These formulas imply that the structure (φ, ξ, η) is a left invariant Sasakian structure of $\mathrm{SL}_2\mathbb{R}$. The structure (φ, ξ, η) is called the *canonical Sasakian structure* of $\mathrm{SL}_2\mathbb{R}$. The Riemannian curvature tensor R of the metric g is described by the following formulas:

$$\begin{aligned} R(X,Y)Z = &- g(Y,Z)X + g(Z,X)Y \\ &- 2\,\{\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X \\ &+ g(Z,X)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &- g(Y,\varphi Z)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\,\} \end{aligned}$$

in terms of the canonical Sasakian structure.

3. The Euler-Arnold equation of $\mathrm{SL}_2\mathbb{R}$

3.1. In this section we deduce the Euler-Arnold equation for $\mathrm{SL}_2\mathbb{R}$. First of all we recall the notion of contact angle. Let $\gamma(s)$ be an arc length parametrized curve, then its *contact angle* $\sigma(s)$ is the angle function between Reeb vector field and $\dot\gamma(s)$, *i.e.*, $\cos\sigma(s) = g(\dot\gamma(s), \xi)$. An arc length parametrized cure is said to be a *slant curve* if its contact angle is constant. In particular, $\gamma(s)$ is said to be a *Reeb flow* if $\sin\sigma(s) = 0$ along $\gamma(s)$. An arc length parametrized curve is called a *Legendre curve* if $\cos\sigma(s) = 0$. One can see that every geodesic is a slant curve.

3.2. The Killing form of $\mathrm{SL}_2\mathbb{R}$ is given by

$$4\mathrm{tr}(XY), \quad X, Y \in \mathfrak{sl}_2\mathbb{R}.$$

In this article we use the normalized Killing metric defined by

$$\mathsf{B}(X,Y) = \frac{1}{2}\mathrm{tr}(XY), \quad X, Y \in \mathfrak{sl}_2\mathbb{R}.$$

As is well known the normalized Killing metric is a bi-invariant Lorentzian metric of constant curvature -1. Thus $(\mathrm{SL}_2\mathbb{R}, \mathsf{B})$ is identified with the anti de Sitter 3-space $\mathbb{H}^3_1$. Here we explain the reason why we choose the normalization as above. The naturally reductive Riemannian metric g is directly connected to the anti de Sitter metric. Indeed, the anti de Sitter metric is given by

$$g - 2\eta \otimes \eta = \frac{dx^2 + dy^2}{4y^2} - \left(d\theta + \frac{dx}{2y}\right)^2.$$

Moreover the endomorphism field $\mathcal{I}$ is given by a simple form

$$\mathcal{I}(X) = {}^t X, \quad X \in \mathfrak{g}.$$

Hence the Euler-Arnold equation is rewritten as

$$\dot{\mu} = [\mu, {}^t\mu]. \tag{3.1}$$

3.3. Let us take an arc length parametrized curve $\gamma(s) = (x(s), y(s), \theta(s))$ in $G = \mathrm{SL}_2\mathbb{R}$. Then the anglar velocity Ω is computed as

$$\Omega = Ae_1 + Be_2 + Ce_3,$$

where

$$A = \dot{\theta} + \frac{\dot{x}}{2y} = \eta(\Omega) = \cos\sigma, \quad B = \frac{\dot{x}}{2y}\cos(2\theta) + \frac{\dot{y}}{2y}\sin(2\theta), \quad C = -\frac{\dot{x}}{2y}\sin(2\theta) + \frac{\dot{y}}{2y}\cos(2\theta).$$

Since $\mathcal{I}$ is the transpose operation, we have

$$\mathcal{I}(e_1) = -e_1, \quad \mathcal{I}(e_2) = e_2, \quad \mathcal{I}(e_3) = e_3.$$

Hence the momentum is given by

$$\mu = -Ae_1 + Be_2 + Ce_3.$$

Hence $[\mu, {}^t\mu] = [\mu, \Omega]$ is computed as

$$[\mu, {}^t\mu] = 4A(Ce_2 - Be_3).$$

Hence the Euler-Arnold equation is

$$\dot{A} = 0, \quad \dot{B} = 4AC, \quad \dot{C} = -4AB.$$

The first equation means the constancy of the contact angle.

Hence the Euler-Arnold equation reduces to the matrix-valued ODE:

$$\frac{d}{ds}\begin{pmatrix} B \\ C \end{pmatrix} = -4\cos\sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} B \\ C \end{pmatrix}.$$

Thus the coefficient functions $B(s)$ and $C(s)$ are given explicitly by

$$\begin{aligned} \begin{pmatrix} B(s) \\ C(s) \end{pmatrix} &= \exp_K(4s\cos\sigma\, e_1)\begin{pmatrix} B(0) \\ C(0) \end{pmatrix} \\ &= \begin{pmatrix} \cos(4s\cos\sigma) & \sin(4s\cos\sigma) \\ -\sin(4s\cos\sigma) & \cos(4s\cos\sigma) \end{pmatrix}\begin{pmatrix} B(0) \\ C(0) \end{pmatrix}. \end{aligned} \tag{3.2}$$

Split Ω as $\Omega = \Omega_{\mathfrak{k}} + \Omega_{\mathfrak{m}}$ along $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, we notice that $\Omega_{\mathfrak{k}} = Ae_1 = A\xi$,

$$\exp_K(4s\cos\sigma\, e_1) = \exp_K(4s\Omega_{\mathfrak{k}}).$$

Let us demand the initial condition

$$\gamma(0) = (x(0), y(0), \theta(0)) = (0, 1, 0)$$

and

$$\Omega(0) = A(0)e_1 + B(0)e_2 + C(0)e_3 = \cos\sigma\, e_1 + B_0 e_2 + C_0 e_3,$$

where

$$B_0 = \frac{\dot{x}(0)}{2}, \quad C_0 = \frac{\dot{y}(0)}{2}, \quad \cos\sigma = \dot{\theta}(0) + B_0.$$

Proposition 3.1. *Let* $\gamma(s)$ *be a geodesic of* $\mathrm{SL}_2\mathbb{R}$ *starting at the identity* Id *whose initial angular velocity is* $\Omega(0) = \cos\sigma e_1 + B_0 e_2 + C_0 e_3$. *Then the angular velocity* $\Omega(s)$ *of* $\gamma(s)$ *is given by*

$$\Omega(s) = \cos\sigma\, e_1 + B(s)e_2 + C(s)e_3,$$

where σ *is a constant and*

$$\begin{cases} B(s) = B_0\,\cos(4s\cos\sigma) + C_0\sin(4s\cos\sigma), \\ C(s) = C_0\,\cos(4s\cos\sigma) - B_0\sin(4s\cos\sigma). \end{cases} \tag{3.3}$$

3.4. We wish to solve the ODE $\dot{\gamma}(s) = \gamma(s)\Omega(s)$ where $\Omega(s)$ is the one given in Proposition 3.1. For this purpose, let us consider curves of the form:

$$\gamma(s) = \exp_G(sW)\exp_K(-sV), \quad W \in \mathfrak{sl}_2\mathbb{R},\ V \in \mathfrak{so}(2).$$

Here $\exp_G$ and $\exp_K$ are the exponential maps

$$\exp_G : \mathfrak{sl}_2\mathbb{R} \to \mathrm{SL}_2\mathbb{R}, \quad \exp_K : \mathfrak{so}(2) \to \mathrm{SO}(2),$$

respectively. Express W and V as

$$W = ae_1 + be_2 + ce_3 = \begin{pmatrix} c & a+b \\ -a+b & -c \end{pmatrix}, \quad V = ue_1 = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}.$$

By using the relations

$$\begin{aligned} \mathrm{Ad}(\exp_K(sV))e_1 =& e_1, \\ \mathrm{Ad}(\exp_K(sV))e_2 =& \cos(2us)\, e_2 + \sin(2us)\, e_3, \\ \mathrm{Ad}(\exp_K(sV))e_2 =& -\sin(2us)\, e_2 + \cos(2us)\, e_3, \end{aligned}$$

the angular velocity is computed as

$$\Omega(s) = \gamma(s)^{-1}\dot{\gamma}(s) = \mathrm{Ad}(\exp_K(sV))W - V = w_1(s)e_1 + w_2(s)e_2 + w_3(s)e_3,$$

where

$$\begin{cases} w_1(s) = a - u, \\ w_2(s) = b\cos(2us) - c\,\sin(2us), \\ w_3(s) = b\sin(2us) + c\,\cos(2us). \end{cases} \tag{3.4}$$

Obviously w_1 is constant along γ. Thus $\gamma(s)$ is a geodesic if and only if

$$\dot{w}_2 - 4w_1w_3 = 2(u-2a)\{b\,\cos(2us) + c\sin(2us)\} = 0,$$
$$\dot{w}_3 + 4w_1w_2 = -2(u-2a)\{b\,\cos(2us) - c\sin(2us)\} = 0.$$

Thus we conclude that $\gamma(s) = \exp_G(sW)\exp_K(-sV)$ is a geodesic if and only if $u = 2a$. It follows that

$$\gamma(s) = \exp_G\{s(ae_1 + be_2 + ce_3)\}\ \exp_K\{s(-2ae_1)\}.$$

Set

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}}, \quad X_{\mathfrak{k}} = -ae_1, \quad X_{\mathfrak{m}} = be_2 + ce_3.$$

Then $\gamma(s)$ is rewritten as

$$\gamma(s) = \exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{m}})\} \exp_K\{s(2X_{\mathfrak{k}})\}.$$

Comparing the systems (3.3) and (3.4), the uniqueness of geodesics implies $a = -\cos\sigma$ and $b = B_0$ and $c = C_0$ and hence the initial velocity is

$$X = \cos\sigma e_1 + B_0 e_2 + C_0 e_3 \in \mathfrak{sl}_2\mathbb{R}.$$

Thus we retrive the following fundamental result.

Theorem 3.1 ([9, 10, 11])**.** *The geodesic $\gamma_X(s)$ starting at the origin* Id *of* $\mathrm{SL}_2\mathbb{R}$ *with initial velocity $X = X_{\mathfrak{k}} + X_{\mathfrak{m}} \in T_{\mathrm{Id}}\mathrm{SL}_2\mathbb{R} = \mathfrak{g}$ is given explicitly by*

$$\gamma_X(s) = \exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{m}})\} \exp_K\{s(2X_{\mathfrak{k}})\}.$$

Here we give homogeneous geometric interpretation for this fundamental theorem.

The exponential map $\exp_{G\times K} : \mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(2) \to \mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)$ is given by

$$\exp_{G\times K}(X, Y) = (\exp_G X, \exp_K Y), \quad (X, Y) \in \mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(2).$$

Take a vector $(X, Y) \in \mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(2)$, then the orbit of Id under the one-parameter subgroup $\{\exp_{G\times K}\{s(X, Y)\}\}_{s\in\mathbb{R}} \subset G \times K$ is given by

$$\exp_{G\times K}\{s(X, Y)\} \cdot \mathrm{Id} = \exp_G(sX) \cdot \mathrm{Id} \cdot (\exp_K(sY))^{-1} = \exp_G(sX) \exp_K(-sY).$$

Since $\mathrm{SL}_2\mathbb{R} = (\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2))/\mathrm{SO}(2)$ is naturally reductive, every geodesic starting at the origin Id is an orbit of Id under the one-parameter subgroup of $G\times K$ with initial velocity $X \in T_{\mathrm{Id}}\mathrm{SL}_2\mathbb{R} = \mathfrak{g}$. Under the identification of $T_{\mathrm{Id}}\mathrm{SL}_2\mathbb{R}$ with $\mathfrak{p}$, the initial velocity $X = X_{\mathfrak{k}} + X_{\mathfrak{m}}$ is identified with $(X_{\mathfrak{m}} - X_{\mathfrak{k}}, -2X_{\mathfrak{k}})$. Thus the geodesic

$$\gamma_X(s) = \exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{m}})\} \exp_K\{s(2X_{\mathfrak{k}})\}.$$

is nothing but the orbit $\exp_{G\times K}(sX)\,\mathrm{Id}$.

According to [8], the fact that the modular 3-fold $\mathrm{PSL}_2\mathbb{R}/\mathrm{PSL}_2\mathbb{Z}$ is topologically equivalent to the complement of the trefoil $\mathcal{K}$ in the 3-sphere was first observed by Quillen (see Milnor [17]). Bolsinov, Veselov and Ye [8] proved that the periodic geodesics on the modular 3-fold $\mathrm{SL}_2\mathbb{R}/\mathrm{SL}_2\mathbb{Z}$ with sufficiently large values of $\mathcal{C} = \kappa^2/16$ represent trefoil cable knots in $\mathbb{S}^3 \setminus \mathcal{K}$, where κ is the geodesic curvature of the projected curve in the hyperbolic surface. Any trefoil cable knot can be described in this way.

4. Homogeneous magnetic trajectories in $\mathrm{SL}_2\mathbb{R}$

4.1. The magnetized Euler-Arnold equation. Now we magnetize the Euler-Arnold equation. Since φ is left invariant,

$$\varphi\dot{\gamma} = \gamma\varphi\Omega.$$

Hence

$$\varphi\Omega = \mathcal{I}^{-1}\mathcal{I}\varphi\Omega = \mathcal{I}^{-1}(\mathcal{I}\varphi\mathcal{I}\mu).$$

One can confirm that $\mathcal{I}\varphi\mathcal{I} = \varphi$. Hence the Lorentz equation is rewritten as the following *magnetized Euler-Arnold equation*:

$$\dot{\mu} - [\mu, {}^t\mu] = q\varphi\mu. \tag{4.1}$$

Express $\mu = -Ae_1 + Be_2 + Ce_3$ as before, then the left hand side of the magnetized Euler-Arnold equation is

$$-\dot{A}e_1 + (\dot{B} - 4AC)e_2 + (\dot{C} + 4AB)e_3.$$

On the other hand,

$$\varphi\mu = -Ce_2 + Be_3.$$

Hence the magnetized Euler-Arnold equation is the system

$$\dot{A} = 0,$$
$$\dot{B} = (4A - q)C,$$
$$\dot{C} = -(4A - q)B.$$

One can check that this system coincides the system obtained in [13, p. 2181]. The first equation implies the constancy of the contact angle $A = \cos\sigma$. Under the initial condition $B(0) = B_0$ and $C(0) = C_0$, we obtain

$$\begin{aligned} \begin{pmatrix} B(s) \\ C(s) \end{pmatrix} &= \exp_K(s(4\cos\sigma - q)\, e_1) \begin{pmatrix} B_0 \\ C_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos((4\cos\sigma - q)s) & \sin((4\cos\sigma - q)s) \\ -\sin((4\cos\sigma - q)s) & \cos((4\cos\sigma - q)s) \end{pmatrix} \begin{pmatrix} B_0 \\ C_0 \end{pmatrix}. \end{aligned} \tag{4.2}$$

4.2. To solve the ODE system (4.2), here we determine contact magnetic curves of the form

$$\gamma(s) = \exp_G(sW)\exp_K(-sV), \quad (W, V) \in \mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(2).$$

Note that when $(W, V) \in \mathfrak{p}$, then $\gamma(s)$ is a geodesic. Express W and V as

$$W = ae_1 + be_2 + ce_3 = \begin{pmatrix} c & a+b \\ -a+b & -c \end{pmatrix}, \quad V = ue_1 = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}.$$

Use the same notations as before, that is $\Omega(s) = \gamma(s)^{-1}\dot{\gamma}(s) := w_1(s)e_1 + w_2(s)e_2 + w_3(s)e_3$. Thus $\gamma(s)$ is a contact magnetic trajectory if and only if

$$\begin{aligned} \dot{w}_2 - 4w_1w_3 + qw_3 &= \ (2u - 4a + q)(b\sin(2us) + c\,\cos(2us)) = 0, \\ \dot{w}_3 + 4w_1w_2 - qw_2 &= -(2u - 4a + q)(b\cos(2us) - c\,\sin(2us)) = 0. \end{aligned}$$

Thus we conclude that $\gamma(s) = \exp_G(sW)\exp_K(-sV)$ is a contact magnetic trajectory when and only when $u = 2a - q/2$. Namely

$$\gamma(s) = \exp_G\{s(ae_1 + be_2 + ce_3)\}\ \exp_K\left\{-s(2a - \tfrac{q}{2})e_1\right\}.$$

Set

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}}, \quad X_{\mathfrak{k}} = -ae_1, \quad X_{\mathfrak{m}} = be_2 + ce_3.$$

Then $\gamma(s)$ is rewritten as

$$\begin{aligned} \gamma(s) =& \exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{m}})\}\exp_K\{s(2X_{\mathfrak{k}} + \tfrac{q}{2}\xi)\} \\ =& \exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{m}})\}\exp_K\{s(2X_{\mathfrak{k}})\}\exp_K\{s(\tfrac{q}{2}\xi)\} \\ =& \gamma_X(s)\ \exp_K\{s(\tfrac{q}{2}\xi)\}. \end{aligned}$$

We may state the following result.

Theorem 4.1. *Every contact magnetic curve γ of $\mathrm{SL}_2\mathbb{R}$ is homogeneous.*

More precisely, we may explicitly describe contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$.

Theorem 4.2. *Every contact magnetic curve γ starting at the origin* Id *of* $\mathrm{SL}_2\mathbb{R}$ *with initial velocity $X \in T_{\mathrm{Id}}\mathrm{SL}_2\mathbb{R} = \mathfrak{g}$ and with charge q is the product of the homogeneous geodesic $\gamma_X(s)$ and the* charged Reeb flow $\exp_K\{s(\frac{q}{2}\xi)\}$*, namely*

$$\gamma(s) = \gamma_X(s)\, \exp_K\{s(\tfrac{q}{2}\xi)\}.$$

Remark 4.1. Recall that contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$ are slant (*e.g.*, [13]). Theorem 4.2 is another justification of this fact.

Acknowledgement. The first named author was partially supported by JSPS KAKENHI JP19K03461 and JP23K03081. The second named author was partially supported by Romanian Ministry of Research, Innovation and Digitization, within Program 1 – Development of the national RD system, Subprogram 1.2 – Institutional Performance – RDI excellence funding projects, Contract no.11PFE/30.12.2021.

References

[1] V. I. Arnol'd, *Some remarks on flows of line elements and frames*, Dokl. Akad. Nauk. SSSR **138** (1961), 255–257. English translation: Sov. Math. Dokl. **2** (1961), 562–564.

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Second Edition, Grad. Texts in Math. **60**, Springer-Verlag, 1989.

[3] A. Arvanitoyeorgos, *Homogeneous manifolds whose geodesics are orbits. Recent results and some open problems*, Irish Math. Soc. Bull. **79** (2017) 5–29 (see also `arXiv:1706.09618`).

[4] A. Bahri, I. A. Taimanov, *Periodic orbits in magnetic fields and Ricci curvature of Lagrangian systems*, Trans. Amer. Math. Soc. **350** (1998), no. 7, 2697–2717.

[5] A. V. Bolsinov, B. Jovanović, *Noncommutative integrability, moment map and geodesic flows*, Annals of Global Analysis and Geometry **23** (2003), 305–322.

[6] A. V. Bolsinov, B. Jovanović, *Magnetic flows on homogeneous spaces*, Comment. Math. Helv. **83** (2008), no. 3, 679–700.

[7] A. V. Bolsinov, I. A. Taimanov, *Integrable geodesic flows with positive topological entropy*, Invent. Math. **140** (2000), no. 3, 639–650.

[8] A. V. Bolsinov, A. P. Veselov, Y. Ye, *Chaos and integrability in* $\mathrm{SL}(2,\mathbb{R})$*-geometry*, Russian Math. Surv. **76** (2021), no. 4, 557–586.

[9] J. E. D'Atri, W. Ziller, *Naturally Reductive Metrics and Einstein Metrics on Compact Lie Groups*, Mem. Amer. Math. Soc. **215** (1979).

[10] C. S. Gordon, *Naturally reductive homogeneous Riemannian manifolds*, Canadian J. Math. **37** (1985), no. 3, 467–487.

[11] S. Halverscheid, A. Iannuzzi, *On naturally reductive left-invariant metrics of* $\mathrm{SL}(2,\mathbb{R})$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **5** (2006), no. 2, 171–187.

[12] D. D. Holm, J. E. Marsden, T. S. Ratiu, *The Euler–Poincaré equations and semidirect products with applications to continuum theories*, Advances in Math. **137** (1998), no. 1, 1–81.

[13] J. Inoguchi and M. I. Munteanu, *Magnetic curves in the real special linear group*, Adv. Theor. Math. Phys. **23** (2019), no. 8, 2161–2205.

[14] J. Inoguchi and M. I. Munteanu, *Slant curves and magnetic curves*, in: Contact Geometry of Slant Submanifolds, Springer Nature Singapore, Singapore, 2022, pp. 199–259.

[15] B. Jovanović, Geodesic flows on Riemannian g. o. spaces, *Regul. Chaot. Dyn.* **16** (2011), 504–513.

[16] O. Kowalski, L. Vanhecke, *Riemannian manifolds with homogeneous geodesics*, Boll. Un. Math. Ital. B (7) **5** (1991), no. 1, 189–246.

[17] J. Milnor, *Introduction to Algebraic K-Theory*, Ann. of Math. Stud., vol. 72, Princeton Univ. Press, Princeton, NJ; Univ. of Tokyo Press, Tokyo 1971.

[18] K. Modin, M. Perlmutter, S. Marsland, R. McLachlan, *On Euler–Arnold equations and totally geodesic subgroups*, J. Geom. Phys. **61** (2011), no. 8, 1446–1461.

[19] W. M. Thurston, *Three-dimensional Geometry and Topology* I, Princeton Math. Series., vol. 35 (S. Levy ed.), 1997.

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