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Expressing Solutions of Bessel Differential Equation in terms of Hypergeometric Functions

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ABSTRACT

The BDE (Bessel differential equation) is a second-order linear ordinary differential equation (ODE), and it is considered one of the most significant differential equations because of its extensive applications. The solutions of this differential equation are called Bessel functions. These solutions can be expressed in terms of hypergeometric functions, and in this study, the Bessel functions of the first kind are worked. Hypergeometric functions are the sum of a hypergeometric series. A series $\sum u_n$ is known as hypergeometric when the ratio u_{n+1}/u_n is a rational function of n. BDE has coefficients with variables; therefore, it is solved by the Frobenius approach. We implement some mathematical steps based on the definition of hypergeometric functions to express the solutions in terms of hypergeometric function. The result shows that the solution of the BDE in terms of hypergeometric function is $f(x) = a_0 x^p {}_1F_2(1; 1 + \frac{p+\rho}{2}, 1 + \frac{p-\rho}{2}; x^2/4)$ and the two linearly independent solutions for the BDE of order ρ in terms of hypergeometric function are $f_1(x) = a_0 x^\rho {}_0F_1(1+\rho; x^2/4)$ and $f_2(x) = a_0 x^\rho {}_0F_1(1-\rho; x^2/4)$.

Keywords- Bessel Differential Equation, Frobenius Approach, Hypergeometric Functions, Hypergeometric Series.

I. INTRODUCTION

Differential equations comprise a number of derivatives of a single unknown function; for one variable, they are defined ODEs, and for more than one variable, they are defined PDEs (partial differential equations) [1]. These equations and their solutions have been considered problems in pure mathematics since the time when Newton and Leibniz invented calculus in the 17th century [2]. In addition, differential equations are used in many other disciplines of applied mathematics, for instance, engineering science, physics science, economics, and biology [3].

The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \rho^2)y = 0\,, \tag{1}$$

is called BDE, where $\rho \geq 0$ is the order and it is a second-order linear ODE [4]. This differential equation is considered one of the most significant differential equations because of its extensive applications [5]. This well-known differential equation was first studied in connection with heavy chain oscillations and circular membrane vibrations [3]. In addition, this equation is seen in heat transfer, stress analysis, fluid mechanics, and vibrations [6]. Linear ODEs with constant coefficients could simply be solved with functions recognized from

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calculus; when a linear ODE has coefficients with variables like a BDE, it should be solved by applying suitable approaches [3]. There are several approaches to solve BDE. Laplace's transform approach, known as mathematician Pierre-Simon Laplace, is a suitable integral transform approach to find solutions set for the BDE for certain initial conditions [1]. Another suitable approach is the power series approach, which is a very common approach for solving linear ODEs [6]. The BDE is frequently solved on the Frobenius approach based on the power series [7].

The first and second kinds Bessel functions are the solutions set for this BDE. For $n = \rho \notin \mathbb{Z}^+ \cup \{0\}$, $J_\rho(x)$ and $J_{-\rho}(x)$ are two independent solutions (first kind Bessel functions) for the equation, and

$$y(x) = AJ_\rho(x) + BJ_{-\rho}(x) ,$$

where A and B are constants, is the general solution for the equation. For $n = \rho \ with \ n = 0 \ or \ n = \mathbb{Z}^+$, the equation has just one Bessel function $J_n(x)$ of the first kind, another independent solution is $Y_n(x)$,which is second kind Bessel function, and

$$y(x) = AJ_n(x) + BY_n(x),$$

is the general solution for equation [8]. Firstly, these functions were defined by Daniel Bernoulli and then generalized by Friedrich Bessel mathematicians [1]. It is necessary to mention here that in this paper, we consider expressing solutions (first-kind Bessel functions) of BDE in terms of hypergeometric functions, which are important special functions. Special functions have an important role in mathematical science [1]. Actually, in solving numerous problems, most analytical solutions are identified based on some of the special functions [3]. In applied science and engineering, scientists face several applications of differential equations, identifying the role and importance of the special functions as a mathematical tool [8].

Hypergeometric functions and their various generalizations appear in several branches of applied mathematics and their applications [9]. The generalized hypergeometric functions are extremely useful, and most of the special functions of mathematical science can be represented based on these functions [11]. A generalized hypergeometric function is defined based on the sum of a hypergeometric series as

$${}_rF_s(a_1, a_2, \dots a_r; b_1, b_2, \dots b_s; x) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i, n)x^n}{\prod_{j=1}^{s}(b_j, n)n!}, \qquad (2)$$

which have r numerator parameters and s denominator parameters [11]. Here $|x| < 1$ and $(a, n) = \frac{\Gamma(a+n)}{\Gamma(a)}$ where (a, n) is Appell's symbol for $a(a + 1)(a + 2) \dots (a + n - 1)$. Base on this notation, for $r = 2$ and $= 1$, in 1836, Gauss defined ${}_2F_1(a, b; c; x)$ hypergeometric function which have two numerator parameters and one denominator parameters. For $r = 1 = s$, the ${}_1F_1(a; b; x)$ confluent hypergeometric function is defined by Kummar in 1836, which have one numerator parameters and one denominator parameters [11].

The contents of this paper are organized in four sections. The first section comprises the introduction and objective of the study. The second section is assigned for preliminaries and methods, which include shifted factorial, Appell's Symbol, hypergeometric functions and series, the Frobenius approach for solving linear ODEs. The third section comprises results which express solutions of BDE in terms of hypergeometric functions. Lastly, the study is concluded in section four.

II. PRELIMINARIES AND METHODS

The factorial function $n! = 1 \cdot 2 \cdot 3 \cdots n$ is firstly defined for a positive integer, and then it is defined for a positive real number by Euler in 1729 , in terms of gamma function, but a simpler generalized form of $n!$ is called shifted factorial function.

2.1 Shifted Factorial and Appell's Symbol

The following function with its derivatives

$$f(x) = (1 - x)^{-a},$$
$$f'(x) = a(1 - x)^{-a-1},$$
$$f''(x) = a(a + 1)(1 - x)^{-a-2},$$
$$\vdots \qquad \vdots \qquad \vdots \quad ,$$
$$f^n(x) = a(a + 1)(a + 2) \cdots (a + n - 1)(1 - x)^{-a-n},$$

are considered, here the product of factors a , $(a + 1)$, $(a + 2)$, $\cdots$, $(a + n - 1)$ are called shifted factorial and simply denoted by Appell's symbol (a, n) . Hence for $n - 1 \in \mathbb{N}$ and $\in \mathbb{R}$,

$$(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1). \qquad (3)$$

For special case when $a = 1$,

$$(1, n) = n! ,$$

and some other special cases are

$$(0,0) = 1 , (a, 1) = a \ \ and \ (-n, m) = 0 , \ \ m > n \geq 0 \cdot$$

Generally, for any real number a and the integers m and n the shifted factorial has the following important properties which called addition, reflection and duplication formulae respectively.

$$(a, m + n) = (a, m)(a + m, n), \qquad (4)$$
$$(a, -n) = \frac{(-1)^n}{(1-a,n)}, \qquad (5)$$
$$(2a, 2n) = 2^{2n}(a, n)\left(a + \tfrac{1}{2}, n\right). \qquad (6)$$

The shifted factorial can be expressed based on gamma function,

$$(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}. \qquad (7)$$

Hypergeometric functions are simply represented based on Appell's symbol (a, n) .

2.2 Hypergeometric functions

Hypergeometric functions are the sum of a hypergeometric series. A series $\sum u_n$ is recognized as hypergeometric when the ratio $\frac{u_{n+1}}{u_n}$ is a rational function of n. For $n = 0,1,2, \dots$, one can conclude that

$$\frac{u_{n+1}}{u_n} = \frac{(n + a_1)(n + a_2) \cdots (n + a_r)x}{(n + b_1)(n + b_2) \cdots (n + b_s)(n + 1)} . \qquad (8)$$

Let $a_1 = a$, $a_2 = b$ and $b_1 = c$, then u_n can be written as,

$$1 + \frac{ab}{1 \cdot c}x + \frac{a(a + 1)b(b + 1)}{1 \cdot 2 \cdot c(c + 1)}x^2$$

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$$+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c\cdot (c+1)\cdot (c+2)}x^3+\cdots . (9)$$

Based on (8), it is clear that $\frac{u_{n+1}}{u_n}=\frac{(n+a)(n+b)x}{(n+b)(n+1)}$ is a rational function of n. Sum of this hypergeometric series in terms of shifted factorial can be expressed as

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!},\qquad |x|<1 \qquad (10)$$

where a, b and c are real or complex parameters, $c\neq 0,-1,-2,\ldots$ and x can be real or complex variable. The function (10) is called Gauss hypergeometric function.

A generalized hypergeometric function is defined based on the sum of a hypergeometric series as,

$${}_rF_s(a_1,a_2,\ldots a_r;b_1,b_2,\ldots b_s;x)={}_rF_s\begin{bmatrix}a_1 & \ldots & a_r\\ b_1 & \ldots & b_s\end{bmatrix};z\Big]=$$

$$\sum_{n=0}^{\infty}\frac{(a_1,n)\ldots(a_r,n)}{(b_1,n)\ldots(b_s,n)}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}(a_i,n)x^n}{\prod_{j=1}^{s}(b_j,n)n!},\qquad (11)$$

which has r numerator parameters and s denominator parameters. Here $a_1,a_2,\ldots a_r$ and $b_1,b_2,\ldots b_s$ are real or complex parameters, $b_j\neq 0,-1,-2,\ldots$ and x can be real or complex variable.

The series

$$S=x+\frac{1}{3\cdot 5}x^3+\frac{1\cdot 3}{5\cdot 7\cdot 9}x^5+\frac{1\cdot 3\cdot 5}{7\cdot 9\cdot 11\cdot 13}x^7+\cdots,$$

can be expressed as a ${}_rF_s$. For u_n it can be written

$$u_n=\frac{1\cdot 3\cdots(2n-1)x^{2n+1}}{(2n+1)(2n+3)\cdots(4n+1)}.\qquad n\in\mathbb{N}$$

Based on (6) , it is clear that

$$\frac{u_{n+1}}{u_n}=\frac{(2n+1)^2x^2}{(4n+3)(4n+5)}=\frac{(\frac{1}{2}+n)(\frac{1}{2}+n)}{(\frac{3}{4}+n)(\frac{5}{4}+n)}\frac{(n+1)x^2}{(n+1)4},$$

is a rational function of n. Since $u_0=x$, then

$$x\,{}_3F_2\left(\tfrac{1}{2},\tfrac{1}{2},1;\tfrac{3}{4},\tfrac{5}{5};\tfrac{x^2}{4}\right)=x\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)(\frac{1}{2},n)(1,n)}{(\frac{3}{4},n)(\frac{5}{4},n)}\frac{x^{2n}}{4\cdot n!}.$$

Many functions can be expressed in terms of hypergeometric function. For instance, for $y=e^x$ and $y=\sin x$,

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$={}_0F_0(x),$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$=x\,{}_0F_1\left(\tfrac{3}{2};-\tfrac{x^2}{4}\right).$$

The error function is another example which can be expressed in terms of hypergeometric function.

$$Erf(x)=\int_0^x e^{-t^2}\,dt=\int_0^x\sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!}\,dt$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x e^{2n}\,dt$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{t^{2n+1}}{2n+1}\Big|_{t=0}^{x}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}.$$

Since $u_n=\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}$, and based on (8) , it is clear that

$$\frac{u_{n+1}}{u_n}=\frac{(-1)^{n+1}x^{2(n+1)+1}}{(n+1)!\,[2(n+1)+1]}\cdot\frac{n!\,(2n+1)}{(-1)^n x^{2n+1}}$$
$$=-\frac{x^2(2n+1)}{(n+1)(2n+3)}=-\frac{(n+\frac{1}{2})}{(n+\frac{3}{2})}\frac{-x^2}{(n+1)}$$

is a rational function of n.

Since $u_0=x$, then

$$Erf(x)=x\,{}_1F_1\left(\tfrac{1}{2};\tfrac{3}{2};-x^2\right)=x\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)}{(\frac{3}{2},n)}\frac{(-x^{2m})}{n!}.$$

The $y=\sinh^{-1}x$ function is as well simply expressed in terms of hypergeometric function.

$y=\sinh^{-1}x$

$$=\int_0^x\frac{du}{\sqrt{1+u^2}}=\int_0^x(1+u^2)^{-\frac{1}{2}}\,du$$
$$=\int_0^x\sum_{n=0}^{\infty}(\tfrac{1}{2},n)\frac{-u^{2n}}{n!}\,du=\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)}{n!}\int_0^x\sum_{n=0}^{\infty}-u^{2n}\,du$$
$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)}{n!}\left(\frac{-u^{2n+1}}{2n+1}\Big|_{u=0}^{x}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)}{n!}\left(\frac{(-x)^{2n+1}}{2n+1}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)(-x)^{2n+1}}{n!\,(2n+1)}=x\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)}{(2n+1)n!}(-x^2)^n.$$

Here $u_n=\frac{(\frac{1}{2},n)}{(2n+1)n!}(-x^2)^n$, and based on (8) , it is clear that,

$$\frac{u_{n+1}}{u_n}=\frac{\frac{(\frac{1}{2},n+1)}{[2(n+1)+1](n+1)!}(-x^2)^{n+1}}{\frac{(\frac{1}{2},n)}{(2n+1)n!}(-x^2)^n}$$
$$=\frac{(\frac{1}{2},n+1)}{[2(n+1)+1](n+1)!}(-x^2)^{n+1}\cdot\frac{(2n+1)n!}{(\frac{1}{2},n)(-x^2)^n}$$
$$=\frac{(\frac{1}{2},n+1)(2n+1)(-x^2)}{(2n+3)(\frac{1}{2},n)(n+1)}$$
$$=\frac{(\frac{1}{2},n+1)(2n+1)}{(2n+3)(\frac{1}{2},n)}\frac{-x^2}{n+1},$$

is a rational function of n . Applying equation (4) , we can write

$$\frac{u_{n+1}}{u_n}=\frac{(n+\frac{1}{2})(2n+1)}{(2n+3)}\cdot\frac{-x^2}{n+1}$$
$$=\frac{(n+\frac{1}{2})(n+\frac{1}{2})}{(n+\frac{3}{2})}\cdot\frac{-x^2}{n+1}.$$

Hence, based on equation (10) it can be expressed as a Gauss hypergeometric function,

$$y=\sinh^{-1}x=x\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)(\frac{1}{2},n)}{(\frac{3}{2},n)}\frac{(-x^2)^n}{n!}$$

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$$= {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-x^2\right).$$

The Bessel functions, which are solutions for BDE, can be expressed in terms of hypergeometric functions. The differential equation (1) is called BDE, where $\rho \geq 0$ is the order, and it is a second-order linear ODE. Linear ODEs with constant coefficients could simply be solved with functions recognized from calculus; when a linear ODE has coefficients with variables like the Bessel ODE, it should be solved by applying suitable approaches. There are several approaches to solving Bessel ODE; the power series approach is a suitable and common approach to solving linear ODEs. The Bessel differential equation is frequently solved using the Frobenius approach based on power series. A form of

$$\sum_{n=0}^{\infty} a_n(x-h)^n = a_0 + a_1(x-h) + a_2(x-h)^2 + \cdots, \qquad (12)$$

is called power series, where a_n denotes coefficients, which are usually constants, and h is center of the series.

2.3 Frobenius approach

The Frobenius approach is applied to answer ODEs with variable coefficients. If an ODE has the form

$$x^2\frac{d^2y}{dx^2} + xb(x)\frac{dy}{dx} + c(x)y = 0, \qquad (13)$$

where for $x = 0$, $b(x)$ and $c(x)$ are analytic functions. If the equation is represented in the standard form

$$\frac{d^2y}{dx^2} + \frac{b(x)}{x}\frac{dy}{dx} + \frac{c(x)}{x^2}y = 0. \qquad (14)$$

Then, if $\frac{b(x)}{x}$ and $\frac{c(x)}{x^2}$ are analytic at $=0$, solution for the differential equation is also analytic for $x = 0$, which could be denoted in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \qquad (15)$$

Hence, if both $\frac{b(x)}{x}$ and $\frac{c(x)}{x^2}$ are not analytic at $x = 0$, then for $x = 0$ it has a singular point, in this case solution could not be denoted in terms of conventional power series, therefore we need to applied power series update approach, and it is known as Frobenius approach. In this approach the solution is represented as

$$y(x) = x^p\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+p}. \qquad (16)$$

Here, a_n denotes unknown coefficients and p is as well unknown. The unknown p is found through the solution process so that $a_0 \neq 0$. The first and second differentiations of the series for $y(x)$ in equation (16), we obtain

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+p)a_n x^{n+p-1}, \qquad (17)$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+p)(n+p-1)a_n x^{n+p-2}. \qquad (18)$$

We suppose that $b(x)$ and $c(x)$ are simple polynomials, constants or series terms with the following forms.

$$b(x) = \sum_{n=0}^{\infty} b_n x^n \quad , \qquad c(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (19)$$

Now we substitute the equations (17),(18), and (19) into equation (13) , and get the following equation,

$$x^2\left(\sum_{n=0}^{\infty}(n+p)(n+p-1)a_n x^{n+p-2}\right) + x\left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty}(n+p)a_n x^{n+p-1}\right) + \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{n+p}\right) = 0. \qquad (20)$$

The first few terms of the (20) , can be written for each series as

$$p(p-1)a_0x^p + (1+p)pa_1x^{p+1} + (2+p)(1+p)a_2x^{p+2} + \cdots + b_0pa_0x^p + [b_1pa_0 + b_0(1+p)a_1]x^{p+1} + [b_2pa_0 + b_1(1+p)a_1 + b_0(2+p)a_2]x^{p+2} + \cdots + c_0a_0x^p + [c_1a_0 + c_0a_1]x^{p+1} + [c_2a_0 + c_1a_1 + c_0a_2]x^{p+2} + \cdots = 0. \qquad (21)$$

Then, we need that the coefficient of x^p should be zero.

$$[p(p-1) + b_0p + c_0]a_0 = 0$$

For this case $a_0 \neq 0$ is accepted, therefore we have that

$$p(p-1) + b_0p + c_0 = 0$$

If we solve the quadratic equation, we will find two possible values for p,

$$p = \frac{1 - b_0 \pm \sqrt{(1-b_0)^2 - 4c_0}}{2}.$$

It follows that there are two solutions and a basis for all solutions to ODE. They are

$$y_1(x) = x^{p_1}\sum_{n=0}^{\infty} a_n x^n \quad , \qquad y_2(x) = x^{p_2}\sum_{n=0}^{\infty} A_n x^n. \qquad (22)$$

The coefficients, a_n and A_n for $y_1(x)$ and $y_2(x)$ in (22) , are different each other. Applying (8) and (11), the functions in (22), are expressible as a hypergeometric functions. For double root $p_1 = p_2 = p$, there are the following solutions

$$y_1(x) = x^p\sum_{n=0}^{\infty} a_n x^n \quad ,$$

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} A_n x^n. \qquad (23)$$

If the roots, p_1 and p_2 are differ by an integer, the possible two solutions can be written as

$$y_1(x) = x^{p_1}\sum_{n=0}^{\infty} a_n x^n \quad ,$$

$$y_2(x) = ky_1(x)\ln(x) + x^{p_2}\sum_{n=0}^{\infty} A_n x^n. \qquad (24)$$

III. FINAL RESULTS

We consider the equation (1) and normalizing, then we have

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$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{\rho^2}{x^2}\right)y=0. \qquad (25)$$

Let $b(x)=\frac{1}{x}$ and $(x)=1-\frac{\rho^2}{x^2}$. It is clear that both $b(x)$ and $c(x)$ are not analytic at $x=0$, so $x=0$ is not ordinary point of the equation. Let $xb(x)=x\cdot\frac{1}{x}$ and $x^2c(x)=x^2\left(1-\frac{\rho^2}{x^2}\right)$. It is clear that both $xb(x)$ and $x^2c(x)$ are not analytic at $x=0$. Hence $x=0$ is a regular singular point of the equation. Then we can say that the BDE has solution in the form of Frobenius approach about $x=0$.If we substitute (16), (17) and (18) in equation (25), then

$$\sum_{n=0}^{\infty}(n+p)(n+p-1)a_nx^{n+p-2}$$
$$+\frac{1}{x}\sum_{n=0}^{\infty}(n+p)\,a_nx^{n+p-1}+\left(1-\frac{\rho^2}{x^2}\right)\sum_{n=0}^{\infty}a_n\,x^{n+p}=0,$$
$$x^2\sum_{n=0}^{\infty}(n+p)(n+p-1)a_nx^{n+p-2}$$
$$+x\sum_{n=0}^{\infty}(n+p)a_n\,x^{n+p-1}-(x^2+\rho^2)\sum_{n=0}^{\infty}a_n\,x^{n+p}=0,$$
$$\sum_{n=0}^{\infty}(n+p)(n+p-1)a_nx^{n+p}+\sum_{n=0}^{\infty}(n+p)a_nx^{n+p}-$$
$$\sum_{n=0}^{\infty}a_nx^{n+p+2}-\sum_{n=0}^{\infty}\rho^2a_nx^{n+p}=0,$$
$$\sum_{n=0}^{\infty}[(n+p)+(n+p)(n+p-1)-\rho^2]a_nx^{n+p}-$$
$$\sum_{n=0}^{\infty}a_nx^{n+p+2}=0.$$

The first few terms of the latest equation can be written as

$$[p+p(p-1)-\rho^2]a_0x^p+$$
$$[(1+p)+(1+p)(p)-\rho^2]a_1x^{1+p}$$
$$+\sum_{n=2}^{\infty}[(n+p)+(n+p)(n+p-1)-\rho^2]a_nx^{n+p}-$$
$$\sum_{n=0}^{\infty}a_nx^{n+p+2}=0.$$

By replacing $n=m-2=0$, then we have

$$[p+p(p-1)-\rho^2]a_0x^p+$$
$$[(1+p)+(1+p)(p)-\rho^2]a_1x^{1+p}$$
$$+\sum_{n=2}^{\infty}[(n+p)+(n+p)(n+p-1)-\rho^2]a_nx^{n+p}-$$
$$\sum_{m=2}^{\infty}a_{m-2}x^{n+p}=0,$$
$$[p^2-\rho^2]a_0x^p+[p^2+2p+1-\rho^2]a_1x^{1+p}+$$
$$\sum_{m=2}^{\infty}([(m+p)+(m+p)(m+p-1)-\rho^2]a_m$$
$$-a_{m-2})x^{m+p}=0.$$

Now we compare the coefficients, then

$p^2-\rho^2=0,$
$(p+\rho)(p-\rho)=0,$
$\rho=\pm p.$

It is clear that

$a_1=a_3=a_5=\cdots=0$, and

$$[(m+p)^2-\rho^2]a_m=a_{m-2}$$
$$a_m=\frac{1}{[(m+p)^2-\rho^2]}a_{m-2}$$
$$=\frac{1}{(m+p+\rho)(m+p-\rho)}a_{m-2}.$$

As $a_1=a_3=a_5=\cdots=a_{2m+1}=0$,then
$u_m=a_{2m}x^{2m+p}$, $u_0=a_0x^p$, $m=0$.

Based on (8) , it is clear that

$$\frac{u_{m+1}}{u_m}=\frac{a_{2m+2}x^{2m+2+p}}{a_{2m}x^{2m+p}}$$
$$=\frac{a_{2m+2}}{a_{2m}}x^2$$
$$=\frac{x^2}{(2m+2+p+\rho)(2m+2+p-\rho)}$$
$$=\frac{x^2}{4\left(m+1+\frac{p+\rho}{2}\right)\left(m+1+\frac{p-\rho}{2}\right)}$$
$$=\frac{m+1}{\left(1+\frac{p+\rho}{2}+m\right)\left(1+\frac{p-\rho}{2}+m\right)}\cdot\frac{\frac{x^2}{4}}{(m+1)},$$

is a rational function of m . Hence, based on equation (10), it can be expressed as a hypergeometric function

$$y(x)=u_0\,{}_1F_2\left(1;1+\frac{p+\rho}{2},1+\frac{p-\rho}{2};\frac{x^2}{4}\right)$$
$$=a_0x^p\,{}_1F_2\left(1;1+\frac{p+\rho}{2},1+\frac{p-\rho}{2};\frac{x^2}{4}\right).$$

The two linearly independent solutions for the Bessel differential equation of order ρ are

$$y_1(x)=a_0x^\rho\,{}_1F_2\left(1;1+\frac{\rho+\rho}{2},1+\frac{\rho-\rho}{2};\frac{x^2}{4}\right)$$
$$=a_0x^\rho\,{}_1F_2\left(1;1+\rho,1;\frac{x^2}{4}\right)$$
$$=a_0x^\rho\,{}_0F_1\left(1+\rho;\frac{x^2}{4}\right)$$
$$=J_\rho(x),$$

and

$$y_2(x)=a_0x^{-\rho}\,{}_1F_2\left(1;1+\frac{-\rho+\rho}{2},1+\frac{-\rho-\rho}{2};\frac{x^2}{4}\right)$$
$$=a_0x^{-\rho}\,{}_1F_2\left(1;1,1-\rho;\frac{x^2}{4}\right)$$
$$=a_0x^{-\rho}\,{}_0F_1\left(1-\rho;\frac{x^2}{4}\right)$$
$$=J_{-\rho}(x).$$

For $\rho\notin\mathbb{Z}^+\cup\{0\}$, $J_\rho(x)$ and $J_{-\rho}(x)$ are two independent solutions for the equation (25) , and

$y(x)=AJ_\rho(x)+BJ_{-\rho}(x)$

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$y(x) = Aa_0 x^{\rho}\, {}_0F_1\left(1+\rho; \frac{x^2}{4}\right) + Ba_0 x^{-\rho}\, {}_0F_1\left(1-\rho; \frac{x^2}{4}\right)$,
where A and B are constants is the general solution for the equation.

IV. CONCLUSION

In this study, we implement hypergeometric functions. Many functions can be expressed in terms of hypergeometric functions. The Frobenius approach is used to solve the Bessel differential equation because it has coefficients with variables. Then we applied some properties of hypergeometric functions and series to express the solution in terms of ${}_rF_s$. We find the solution in terms of hypergeometric functions in the form ${}_1F_2$ and the two linearly independent solutions in the form of ${}_0F_1$. In this paper, it can see that this representation of Bessel functions of the first kind in terms of hypergeometric functions and hypergeometric series are the easier and simpler notations.

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