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Investigation of Adjoint of Linear Transformation and Some Its Important Properties

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ABSTRACT

Linear operator on inner product space is including adjoint operator, self adjoint operator, unitary operator, normal operator ,… (self adjoint operator and unitary operator is normal operator but convers is not true at all) in this paper I discussed about adjoint operator and self adjoint operator of linear transformation and some important properties.

In this paper first I defined the linear transformation, inner product space, adjoint linear transformation, self adjoint operator and very important relevant properties and theorem.

Keywords- vector space, linear map, dimension, field, inner product space, operator, orthonormal basis, orthogonal, diagonal, positive definite.

I. INTRODUCTION

Definition 1: let V and W be vector spaces over field K , A mapping from V to W denoted by $T: V \to W$ (1) is called linear transformation if T hold the following properties
1: $T(u+v) = T(u) + T(v)$ for all $u, v \in V$
2: $T(\alpha u) = \alpha T(u)$ for $\alpha \in K$

Definition 2: Let V be vector space over field K , an inner product over vector space V denoted by $\langle , \rangle$ is a map from $V \times V \to F$ satisfy the given properties
1:$\langle u, v\rangle = \overline{\langle u, v\rangle}$, the complex conjugate of $\langle u, v\rangle$ $\forall\, u, v \in V$
2: $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \ \forall\, u, v, w \in V$
3: $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle \ \forall u, v \in V$ and $\alpha \in F$
4: $\langle u, v\rangle \geq 0\ \forall u \in V$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$

Definition 3: let V be vector space with inner product $\langle , \rangle$ then $(V, \langle\ \rangle)$ is called an inner product space, Robert Messer, [1993]

Theorem: let V , W be finite dimensional inner product space and let $T \in L(V, W)$, then there exist a unique linear map $T^*: W \to V$ (3) such that for all $v \in V$ and $w \in W$ we have $\langle Tv, w\rangle = \langle v, T^* w\rangle$.

Definition 4: in (3) T^* is called ad joint of T , Axler sheldon (2015).

Proof: for fixed $w \in W$, we have the function $f_w: V \to W$ defined by $F_w(v) = \langle Tv, w\rangle$ (1) which is a linear functional on V by reisz representation theorem there exist a unique $u \in V$ such that $F_w = \langle v, u\rangle$ (2) if we set $u = T^*(w)$ in (2) we have $F_w(v) = \langle v, T^* w\rangle$ (3) from (1) , (3) I can write the result bellow
$\langle Tv, w\rangle = \langle v, T^* w\rangle\ \forall v \in V$, since $u \in V$ is unique, T^* is unique.
Now I want to show T^* is linear.
Let $(x, y) \in W$ and $(\alpha, \beta) \in K$,
$\langle v, T^*(\alpha x + \beta y)\rangle = \langle Tv, \alpha x + \beta y\rangle = \langle Tv, \alpha x\rangle +$
$\langle Tv, \beta y\rangle = \bar{\alpha}\langle Tv, x\rangle + \bar{\beta}\langle Tv, y\rangle = \bar{\alpha}\langle v, T^* x\rangle +$

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$\bar{\beta}\langle v, T^*y\rangle = \langle v, \alpha T^*x\rangle + \langle v, \beta T^*y\rangle \Longrightarrow \langle v, T^*(\alpha x + \beta y)\rangle = \langle v, \alpha T^*x\rangle + \langle v, \beta T^*y\rangle$

Preposition: let V, W be finite dimensional inner product space over field K then

1: if $(S,T) \in L(V,W)$ then $(S+T)^* = S^* + T^*$ and for $\alpha \in K$, $(\alpha S)^* = \bar{\alpha} S^*$

2: if $S \in L(V,W)$ then $S^{**} = S$ where $S^{**} = S$

3: if $S,T \in L(V)$ then $(ST)^{**} = T^*S^*$Vikas Bist , Vivek sahai (2017)

Proof 1: for $v \in V$, $w \in W$ and by definition of ad joint operator we have

$$\begin{aligned}\langle v, (S+T)^*w\rangle &= \langle (S+T)v, w\rangle = \langle Sv + Tv, w\rangle \\ &= \langle Sv, w\rangle + \langle Tv, w\rangle \\ &= \langle v, S^*w\rangle + \langle v, T^*w\rangle \\ &= \langle v, (S^* + T^*)w\rangle = \langle v, (S+T)^*w\rangle\end{aligned}$$

By uniqueness of adjoint mapping we have $(S+T)^* = S^* + T^*$

2: if $S \in L(V,W)$ then by definition of adjoint mapping, for $v \in V$, $w \in W$ $\langle S^*w, v\rangle = \langle w, S^{**}v\rangle$ (1) and also $\langle S^*w, v\rangle = \overline{\langle v, S^*w\rangle} = \overline{\langle Sv, w\rangle} = \langle w, Sv\rangle$ (2) , by (1) , (2) I can write $\langle w, S^{**}v\rangle = \langle w, Sv\rangle$ for all $v \in V, w \in W$, hence $S^{**} = S$.

3: for $(u,v) \in V$ $\langle u, (ST)^*v\rangle = \langle (ST)u, v\rangle = \langle S(Tu), v\rangle = \langle Tu, S^*v\rangle = \langle u, T^*S^*v\rangle$ hence $(ST)^* = T^*S^*$.

Preposition: let V, W be finite dimensional inner product space over field K and let $T \in L(V,W)$ if B_1, B_2 be ordered orthonormal basis of V, W respectively, then the matrix representation of T^* with respect to these basis is the conjugate transpose of the matrix representation of T with respect to the given basis.

Proof: let $B_1 = \{v_1, v_2, v_3, \dots v_m\}$ and $B_2 = \{w_1, w_2, w_3, \dots, w_n\}$ be ordered orthonormal basis of V, W respectively

Let $[T; B_2, B_1] = A_{m\times n} = (a_{ij})$ and let Let $[T^*; B_2, B_1] = B_{m\times n} = (b_{ij})$

Now $\langle Tv_j, w_i\rangle = \langle \sum_{k=1}^{m} a_{kj} w_k, w_i\rangle = (a_{ij})$ entry of $\in K^{m\times n}$

$(b_{ij}) = \langle T^*w_j, v_i\rangle = \langle w_j, Tv_i\rangle = \overline{\langle Tv_\iota, w_i\rangle} = (a_{ij})^* \Rightarrow$ $B = A^*$ ie ($[T^*; B_2, B_1] = [T; B_1, B_2]^*$)

Preposition: let V, W be finite dimensional inner product space over K and let $T \in L(V,W)$ then Vikas Bist , vivek Sahai (2017)

1: $\ker T^* = (img\ T)^\perp$ and $\ker T^\perp = (img\ T)^*$

2: $V = \ker T \oplus img\ T^*$ and $W = img\ T \oplus \ker T^*$

3: $\ker T^*T = \ker T$ and $img\ T^*T = img\ T^*$

Proof 1: let $w \in (img\ T)^\perp \Leftrightarrow \langle Tv, w\rangle = 0 = \langle v, T^*w\rangle = 0$ $\forall v \in V$ $T^*w = 0$ (ie $w \in \ker T^*$). Hence $(img\ T)^\perp = \ker T^*$ (1) ,now replace T by T^* in (1) we can get $\ker T^{**} = (img\ T^*)^\perp \Rightarrow \ker T = (img\ T^*)^\perp \Rightarrow$ $\ker T^\perp = \{(img\ T^*)^\perp\}^\perp \Rightarrow \ker T^\perp = img\ T^*$ (2)

Proof 2: note, let V be vector space over field K and let W be subspace of V and $W^\perp$ be orthogonal complement set of W then we can write $V = W \oplus W^\perp$ (3)

Now $\ker T = \{v \in V : T(v) = 0\}$ is subspace of V and $\ker T^\perp$ is its orthogonal complement set then by (3) I can write $V = \ker T \oplus \ker T^\perp$ (4), by (2) and (4) I can get the result $V = \ker T \oplus img\ T^{**}$, by same way I can write $W = imgT \oplus (img\ T)^\perp$,(ie $img\ T$ is subspace of W) so by (3) the result follows.

Then by (1) and (4) I can write $W = img\ T \oplus \ker T^*$

Proof 3: let $u \in \ker T \Rightarrow T(u) = 0 \Rightarrow T^*(Tu) = 0 \Rightarrow T^*T(u) = 0 \Rightarrow u \in \ker T^*T$, hence $\ker T \subseteq \ker T^*T$ (5). If $v \in \ker T^*T \Rightarrow \langle v, T^*Tv\rangle = 0 \Rightarrow \langle Tv, Tv\rangle = \|Tv\|^2 = 0 \Rightarrow v \in \ker T$, hence $\ker T^*T \subseteq \ker T$ by (5) and (6) I can write the result $\ker T^*T = \ker T$

Let $w \in img\ T^*T \Rightarrow w = T^*T(u)$ for some $u \in V$, $w = T^*(Tu) \Rightarrow w \in img\ T^*$, hence $img\ T^*T \subseteq img\ T^*$ (6). $\dim(img\ T^*) = \dim(\ker T)^\perp$ (7). by (2) and also we have

$\dim(\ker T)^\perp = \dim V - \dim T$ (8) , by (7) and (8) I can write

$$\begin{aligned}&\dim(img\ T^*) = \dim V - \dim(\ker T) \Rightarrow \\ &\dim(img\ T^*) = \dim V - \dim(\ker T^*T) \\ &\quad \Rightarrow \dim(img\ T^*) = \dim(img\ T^*T) \quad (9)\end{aligned}$$

Hence by (6) and (9) I can write $img\ T^*T = img\ T^*$.

II. SELF ADJOINT OPERATOR

Let V be inner product space over field K and $T \in L(V)$, we say T is self adjoint if and only if $T = T^*$(T is Hermitian), Marc Lars Lipson, Seymour Lipschutz,(2018)

Preposition: let S, T be self adjoint operator on an inner product space V then we have

a: $S + T$ is self adjoint,

b: ST is self adjoint if and only if $ST = TS$,

c: T^{-1} is self adjoint if T is invertible.

Proof:

a: it is given that S and T are self adjoint (ie $S^* = S$ and $T = T^*$)

$(S+T)^* = S^* + T^* = S + T$ hence the sum of two self adjoint operator is self adjoint operator.

b: let (ST) be self adjoint operator, I want to show that $ST = TS$

since (ST) is self adjoint, I have $(ST)^* = (ST)$ (1) and also $(ST)^* = T^*S^* = TS$ (2), by (1) and (2) I can write $ST = TS$ the invers is also true.

c: the invers an invertible self adjoint operator is also self adjoint $(T^{-1})^* = T^{-1}$

Preposition: Let T be self adjoint operator on the finite dimensional inner product space $V(K)$, then the root of characteristic polynomial of T are real.

Proof: suppose that $K = \mathbb{C}$ = (complex number), let λ be an eigenvalue of T and v corresponding eigenvector (ie $Tv = \lambda v$ $v \neq 0$) let λ be a root of characteristic polynomial of T , then we have $\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle$ hence $\lambda = \bar{\lambda}$ = (ie $\lambda\langle v, v\rangle = \bar{\lambda}\langle v, v\rangle \Rightarrow (\lambda - \bar{\lambda})\langle v, v\rangle = 0, \langle v, v\rangle = \|v\|^2 \neq 0 \therefore v \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$)and $\lambda \in \mathbb{R}$ so λ is real.

If V is inner product space over $\mathbb{R}$ ($K = \mathbb{R}$) then $C_T(x) \in \mathbb{R}[x]$, so it is possible that the root of $C_T(x)$ can be

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complex number but now I want to show that all the roots of $C_T(x)$ are real let B be an orthonormal basis of V and $T \in L(V)$ and let $[T]_B = A$ since T is self adjoint operator, consider $A \in F^{m\times n}$ as linear operator on the standard inner product space C^n, A is self adjoint operator and $C_T(x) = C_A(x)$ and all root of $C_A(x)$ are real, hence the result follows.

Theorem: A self adjoint operator T on a finite dimensional inner product space V is orthogonally diagonalizable, Marc Lars Lipson, Seymour Lipschutz (2018)

Proof: let $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k) \in \mathbb{R}$ be eigenvalues of T with multiplicities $m_1, m_2, m_3, \dots, m_k$ $m_T(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}(x-\lambda_3)^{m_3} \dots (x-\lambda_k)^{m_k}$, by primary decomposition theorem $V = \ker(T-\lambda_1 I)^{m_1} \oplus \ker(T-\lambda_2 I)^{m_2} \oplus \ker(T-\lambda_3 I)^{m_3} \oplus \dots \oplus \ker(T-\lambda_k I)^{m_k}$ since $\lambda_i \in \mathbb{R}$ and $(T-\lambda_2 I)$ is also self adjoint ($(T-\lambda_i I)^* = T^* = \lambda_i I^* = T - \lambda_i I$).

Thus for $i = 1,2,3,\dots,k$, $v_i \in \ker(T-\lambda_i I)^{m_i} \Rightarrow (T-\lambda_i I)v_i = 0$ ($T^k v = 0 \Rightarrow Tv = 0$), $\Rightarrow v_i \in \ker(T = \lambda_i I)$

Note that $\ker(T-\lambda_i I)^{m_i} = \ker(T-\lambda_i I)$ for $i = 1,2,3,\dots,k$ then we have $V = \ker(T-\lambda_1 I) \oplus \ker(T-\lambda_2 I) \oplus \ker(T-\lambda_3 I) \oplus \dots \oplus \ker(T-\lambda_k I)$ and $\dim\{\ker(T-\lambda_i I)\} = 1 \therefore [T]_B = diag(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k)$ so T is diagonalizable.

Note that: let V be finite dimensional inner product space over K and let $T \in L(v)$ be self adjoint operator, then

Preposition

1: T is positive definite if and only if eigenvalue of T are positive

2: if and only if there is a positive definite operator $S \in L(v)$ such that $T = SS$ (ie $T = S^2$)

3: if and only if there is an invertible operator $S \in L(V)$ such that $T = S^*S$ Axler Sheldon (2015)

Proof 1 : If λ is an eigenvalue of T then $\lambda \in \mathbb{R}$

Now $Tv = \lambda v \Rightarrow \lambda\langle v, v\rangle > 0$

Conversely: let $B = \{v_1, v_2, v_3, \dots v_n\}$ be orthonormal basis of V consisting of eigenvector of T , it is given that $Tv_i = \lambda_i v_i$, for $v \in V$ we have $v = \sum_{i=1}^n \lambda_i v_i$ (unique)

$$\langle Tv, v\rangle = \langle \sum_{i=1}^n \alpha_i T v_i, \sum_{j=1}^n \alpha_j v_j\rangle = \langle \sum_{i=1}^n \alpha_i \lambda_i v_i, \sum_{j=1}^n \alpha_j v_j\rangle$$

$$= \sum_{i=1}^n \sum_{i=1}^n \lambda_i \alpha_i \overline{\alpha_j}\langle v_i, v_j\rangle \Rightarrow \langle Tv, v\rangle > 0$$

2: consider $T = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \dots + \lambda_k p_k$, $\lambda_i > 0$ $i = 1,2,3,\dots,k$

Note that $T = S^2$,so we can have $S = \sqrt{\lambda_1}p_1 + \sqrt{\lambda_2}p_2 + \sqrt{\lambda_3}p_3 + \dots + \sqrt{\lambda_k}p_k$ and $\sqrt{\lambda_i} > 0$ for $i = 1,2,3,\dots,k$ so the result follows.

Convers is obvious from above proof

3: note that $\langle Tv, v\rangle = \langle S^*Sv, v\rangle = \langle Sv, S^*v\rangle = \|Sv\|^2 \geq 0$, since S is invertible so $S \neq 0$ hence $\langle Tv, v\rangle = \|Sv\|^2 > 0 \Rightarrow \langle Tv, v\rangle$, and T is positive definite.

III. CONCLOSION

Undoubtedly, many of the discussed spaces in the fields of engineering are vector spaces, where inner product spaces and their operators are very important. The properties discussed in this article can help us in solving the problems in these spaces.

Adjoint of linear transformation is widely used in solving systems of linear equations, systems of differential equations in the fields of engineering and computer science. In this article, some of its properties have been researched and investigated, which can help us in these areas.

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