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DUAL STABILIZATION OF THE MULTIDIMENSIONAL REGRESSION OBJECT AT THE GIVEN LEVEL

VLADIMIR S. MUKHA, NANCY F. KAKO

Belarusian State University of Informatics and Radioelectronics (Minsk, Republic of Belarus)

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Abstract. The statement of the problem of the dual control of the regression object with multidimensional-matrix input and output variables and dynamic programming functional equations for its solution are given. The problem of the dual stabilization of the regression object at the given level is considered. The purpose of control is reaching the given value of the output variable by sequential control actions in production operation mode. In order to solve the problem, the regression function of the object is supposed to be affine in input variables, and the inner noise is supposed to be Gaussian. The sequential solution of the functional dynamic programming equations is performed. As a result, the optimal control action at the last control step is obtained. It is shown also that the obtaining of the optimal control actions at the other control steps is connected with big difficulties and impossible both analytically and numerically. The control action obtained at the last control step is proposed to be used at the arbitrary control step. This control action is called the control action with passive information accumulation. The dual control algorithm with passive information accumulation was programmed for numerical calculations and tested for a number of objects. It showed acceptable results for the practice. The advantages of the developed algorithm are theoretical and algorithmical generality.

Keywords: dual control, multidimensional-matrix regression object, dynamic programming, passive information accumulation.

Conflict of interests. The authors declare no conflict of interests.

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ДУАЛЬНАЯ СТАБИЛИЗАЦИЯ МНОГОМЕРНОГО РЕГРЕССИОННОГО ОБЪЕКТА НА ЗАДАННОМ УРОВНЕ

В. С. МУХА, Н. Ф. КАКО

Белорусский государственный университет информатики и радиоэлектроники (г. Минск, Республика Беларусь)

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Аннотация. Приводятся постановка задачи дуального управления регрессионным объектом с многомерно-матричными входной и выходной переменными и функциональные уравнения динамического программирования для ее решения. Рассматривается задача дуальной стабилизации объекта на заданном уровне. Целью управления является вывод выходной переменной объекта на требуемый уровень и поддержание ее на этом уровне с помощью последовательных управляющих воздействий в режиме нормальной эксплуатации. Для решения задачи функция регрессии объекта аппроксимируется аффинной по входному воздействию функцией, а внутренний шум объекта предполагается аддитивным Гауссовским. Выполнено последовательное решение функциональных уравнений динамического программирования, в результате чего получено управляющее воздействие на последнем шаге управления. Показано, что отыскание управляющего воздействия на других шагах управления связано с большими трудностями и невыполнимо как аналитически, так и численно. Управляющее воздействие, полученное на последнем шаге, предлагается использовать на любом шаге управления. Такой алгоритм назван алгоритмом дуального управления с пассивным накоплением информации. Этот алгоритм запрограммирован для численных расчетов, апробирован на ряде объектов и показал приемлемые для практики результаты. Важным достоинством алгоритма является его теоретическая и алгоритмическая общность.

Ключевые слова: дуальное управление, многомерно-матричный регрессионный объект, динамическое программирование, пассивное накопление информации.

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Introduction

The problem of the dual control of the multidimensional regression object is formulated as follows [1–5]. The control system with controlled object O, controller C, feedback path and driving action g_s is considered (Fig. 1).

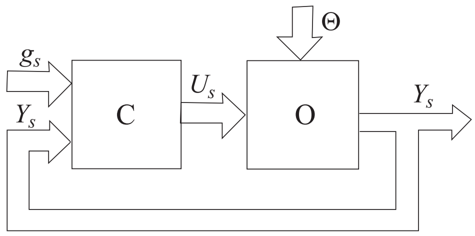

Fig. 1. To the statement of the dual control problem

The controlled object O is described at the instant of time *s* by the probability density function

$$f_{Y_s}\left(y_s, \Theta, U_s\right),\ s = 0, 1, 2, ..., n,$$

where $Y_s = (Y_{i_1,i_2,...,i_p,s})$ is the *p*-dimensional matrix of the output of the object at the instant of time *s*; $U_s = (U_{i_1,i_2,...,i_q,s})$ is the *q*-dimensional matrix of the input of the object at the instant of time *s* (control action); $\Theta = \{\Theta_1,...,\Theta_m\}$ is a set of the parameters of the controlled object consisting of the random multidimensional matrices $\Theta_1,...,\Theta_m$ with known priory joint probability density function $f_{\Theta,0}(\theta)$.

We will call the set $\Theta = \{\Theta_1,...,\Theta_m\}$ a generalized parameter of the object O. It is supposed, that the generalized parameter Θ takes constant value for all of the instants of time $s = 0, 1, ..., n$. The driving action g_s is supposed to be known deterministic multidimensional-matrix sequence.

The quality of the functioning of the system at each instant of time *s* is estimated by a specific loss function $W_s(Y_s, g_s)$, depending of output Y_s and, might, driving action g_s. A system, for which the total for $n+1$ instants of time total average risk

$$R = E\left\{\sum_{s=0}^{n} W_s\left(Y_s, g_s\right)\right\} = \sum_{s=0}^{n} R_s,\ R_s = E\left(W_s\left(Y_s, g_s\right)\right), \qquad (1)$$

is minimal, is called optimal system.

There $E(\cdot)$ means the mathematical expectation, $R_s = E(W_s(Y_s, g_s))$ is a specific risk. The control action U_s belongs to some permissible area. The controller C uses all of the past information in the form of observations $\vec{u}_{s-1} = (u_0, u_1, ..., u_{s-1})$, $\vec{y}_{s-1} = (y_0, y_1, ..., y_{s-1})$ of the input and output values of the object to determine the control action u_s at the instant of time *s*.

The task consists of determining the strategies of the controller C, i. e. sequence of the conditional probability density functions $f_{U_s}(u_s / \vec{u}_{s-1}, \vec{y}_{s-1})$, $i = 0, 1, ..., n$, for which the total average risk R (1) is minimal.

As it is known [2–5], the optimal strategies of the controller C are not randomized, i. e. the control actions U_s are not random and will be denoted u_s. In this conditions the controller C will be described by conditional probability density function $f_{Y_s}(y_s / \theta, u_s)$, where u_s is the fixed value of the variable U_s. We will use the following simplified notation: $f_{\Theta,0}(\theta) = f_0(\theta)$, $f_{Y_s}(y_s / \theta, u_s) = f(y_s / \theta, u_s)$.

The optimal control algorithm, i. e. the sequence of the control actions $u_n, u_{n-1}, ..., u_0$ is determined in pointed inverse order from the following functional equations:

$$f_n^*\left(\vec{u}_{n-1}, u_n^*, \vec{y}_{n-1}\right) = \min_{u_n \in \bigcup} \varphi_n\left(\vec{u}_n, \vec{y}_{n-1}\right); \tag{2}$$

$$f_{n-m}^*\left(\vec{u}_{n-m-1}, u_{n-m}^*, \vec{y}_{n-m-1}\right) =$$

$$= \min_{u_{n-m} \in \bigcup}\left[\varphi_{n-m}\left(\vec{u}_{n-m}, \vec{y}_{n-m-1}\right) + \int_{\Omega(y_{n-m})} f_{n-m+1}^*\left(\vec{u}_{n-m}, u_{n-m+1}^*, \vec{y}_{n-m}\right) f\left(y_{n-m} / \vec{u}_{n-m}, \vec{y}_{n-m-1}\right) d\Omega\right],$$

$$m = 1, 2, ..., n, \tag{3}$$

where φ_s is determined by expression

$$\varphi_s\left(\vec{u}_s, \vec{y}_{s-1}\right) = \int_{\Omega(\vec{y}_s)} W_s\left(y_s, g_s\right) f\left(y_s / \vec{u}_s, \vec{y}_{s-1}\right) d\Omega, \ s = 0, ..., n, \tag{4}$$

in which

$$f\left(y_s / \vec{u}_s, \vec{y}_{s-1}\right) = \int_{\Omega(\theta)} f\left(y_s / \theta, u_s\right) f_s(\theta) d\Omega; \tag{5}$$

$$f_s(\theta) = \frac{f_0(\theta)\prod_{v=0}^{s-1} f\left(y_v / \theta, u_v\right)}{\int_{\Omega(\theta)} f_0(\theta)\prod_{v=0}^{s-1} f\left(y_v / \theta, u_v\right) d\Omega}, \tag{6}$$

and u_{n-m+1}^* is optimal control action for the instant of time $(n - m + 1)$.

Note. The notation $\min_{u_n \in \bigcup} \varphi_n(\vec{u}_n, \vec{y}_{n-1})$ means the following:

$$\min_{u_n \in \bigcup} \varphi_n\left(\vec{u}_n, \vec{y}_{n-1}\right) = \varphi_n\left(u_n^*, \vec{u}_{n-1}, \vec{y}_{n-1}\right).$$

Stabilization of the object at the given level

We will consider the task of reaching the required value of the regression function by sequential control actions in production operation mode and stabilization of the regression function at this level. The task is formulated in this case as follows.

The controlled object is described at the *s*-th instant of time by Gaussian probability density function

$$f_{Y_s}\left(y_s / C, u_s\right) = N\left(\psi\left(C, u_s\right), d_Y\right), \tag{7}$$

where $\psi(C, u_s)$ is a regression function; d_Y is a variance-covariance matrix of the inner noise; u_s is a *q*-dimensional matrix; y_s is a *p*-dimensional matrix; *C* is a generalized parameter of the object.

Note, that we denote now the geleralized prameter C instead of θ in expresions (5), (6). Let us approximate the regression function by affine function:

$$y = \psi(C) = C_0 + {}^{0,q}(C_1 u) = C_{t,0} + {}^{0,q}(u C_{t,1}) = \psi(C_t), \tag{8}$$

or

$$y = \psi(C) = \sum_{i=0}^{m} {}^{0,iq}\left(C_i u^i\right) = \sum_{i=0}^{m} {}^{0,iq}\left(u^i C_{t,i}\right) = \psi\left(C_t\right), \ m = 1,$$

where C_k, $k = \overline{0, m}$, are *kq*-dimensional random matrices $C_{t,k} = (C_k)^{B_{p+kq,kq}}$; $C_k = (C_{t,k})^{H_{p+kq,kq}}$, and $H_{p+kq,kq}$, $B_{p+kq,kq}$ are the transpose substitutions of the type “back” and “onward” respectively (in the article, the multidimensional-matrix notation is used [6]).

Let us combine the matrices C_k into a one-dimensional cell $C=\{C_k\}$, $k=0{,}1$. Provided the regression function (8), the probability density function of the object (7) take the following form:

$$f\left(y_n / c_t, u_n\right)=\frac{1}{\sqrt{(2\pi)^{k_Y}\,|d_Y|}}\exp\left(-\frac{1}{2}\,{}^{0,p}\left(d_Y^{-1}\left(y_n-\sum_{i=0}^{m}{}^{0,iq}\left(u_n^i c_{t,i}\right)\right)^2\right)\right), \tag{9}$$

where k_Y is the number of the elements of the matrix y_n.

For the task of the object stabilization at the level g we choose the loss function in the form of $W(Y_s)=\|Y_s-g\|^2$, where $\|\cdot\|$ is the Euclidean norm of a multidimensional matrix.

Let the random cell $C_t=\{C_{t,k}\}$, $k=\overline{0,m}$, has the Gaussian priory probability denity function described by the following expresion [7]:

$$f(c_t)=M_\Xi \exp\left(-\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m}{}^{0,q_j}\left({}^{0,q_i}\left((c_{t,i}-\nu_{c_t,i})d_{c_t}^{i,j}\right)(\xi_j-\nu_{c_t,j})\right)\right)=$$

$$=M_{c_t}\exp\left\{-\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m}{}^{0,q_j}\left({}^{0,q_i}(c_{t,i}d_{c_t}^{i,j})c_{t,j}\right)+\sum_{i=0}^{m}\sum_{j=0}^{m}{}^{0,q_j}\left({}^{0,q_i}(c_{t,i}d_{c_t}^{i,j})\nu_{c_t,j}\right)-\right.$$

$$\left.-\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m}{}^{0,q_j}\left({}^{0,q_i}(\nu_{c_t,i}d_{c_t}^{i,j})\nu_{c_t,j}\right)\right\},\ M_{c_t}=\frac{1}{\sqrt{(2\pi)^{n_c}\,|d_{c_t}|}},\ q_i=p+iq,\ i=\overline{0,m},$$

where the two-dimensional cell $d_{c_t}=\{d_{c_t,i,j}\}$ $\left(i,j=\overline{0,m}\right)$ is the variance-covriance cell of the random cell C_t [7]; $d_{c_t,i,j}=E\left({}^{0,0}\left((C_{t,i}-\nu_{c_t,i})(C_{t,j}-\nu_{c_t,j})\right)\right)$ is the $((iq+p)+(jq+p))$-dimensional matrix; $d_{c_t}^{-1}=\{d_{c_t}^{i,j}\}$ $\left(i,j=\overline{0,m}\right)$ is the cell invere to the cell d_{c_t}; $\nu_{c_t}=\{\nu_{c_t,0},\nu_{c_t,1},\ldots,\nu_{c_t,m}\}=\{\nu_{c_t,i}\}$ $\left(i,j=\overline{0,m}\right)$ is the one-dimensional cell of the mathematical expectation of the random cell C_t; ($\nu_{c_t,i}=E(C_{t,i})$ is the $(iq+p)$-dimensional matrix); n_c is the number of the scalar elements of the cell c_t.

The calculation of the control actions $u_n,u_{n-1},\ldots,u_0$ is connected with the formulae (2)–(6).

1. The posterior probability density function $f_n(c)$ (6) is defined by the expression [7]:

$$f\left(c_t/\vec{y}_{n-1},\vec{u}_{n-1}\right)=\frac{1}{\sqrt{(2\pi)^{n_y}\left|D_{c_t}\right|}}\exp\left(-\frac{1}{2}\,{}^{0,2}\left\{D_{c_t}^{-1}\,{}^{0,0}\left\{c_t-N_{c_t}\right\}^2\right\}\right)=f_n\left(c_t\right), \tag{10}$$

in which $D_{c_t}=\left\{D_{c_t,i,j}\right\}$,

$$D_{c_t}^{-1}=\left\{D_{c_t}^{i,j}\right\}=\left\{d_{c_t}^{i,j}+S_{i,j}\right\}=\left\{d_{c_t}^{i,j}+\left({}^{0,0}(d_Y^{-1}S_{u^iu^j})\right)^{T_{i,j}}\right\},\ i,j=\overline{0,m}; \tag{11}$$

$$B=\left\{B_i\right\}=\left\{\sum_{j=0}^{m}{}^{0,jq+p}\left(d_{c_t}^{i,j}\nu_{c_t,j}\right)+{}^{0,p}\left(d_\varepsilon^{-1}S_{yu^i}\right)^{T_i}\right\},\ i=\overline{0,m}; \tag{12}$$

$$N_{c_t}=\left\{N_{c_t,i}\right\}={}^{0,1}\left\{D_{c_t}B\right\}=\left\{\sum_{j=0}^{m}{}^{0,p+jq}\left(D_{c_t,i,j}B_j\right)\right\},\ i=\overline{0,m}; \tag{13}$$

$$S_{u^ku^\lambda}=\sum_{\mu=1}^{n-1}{}^{0,0}\left(u_\mu^k u_\mu^\lambda\right);\ S_{yu^\lambda}=\sum_{\mu=1}^{n-1}{}^{0,0}\left(y_\mu u_\mu^\lambda\right); \tag{14}$$

$$\vec{y}_{n-1}=\left(y_1,y_2,\ldots,y_{n-1}\right);\ \vec{u}_{n-1}=\left(u_1,u_2,\ldots,u_{n-1}\right);\ \vec{u}_{n-1}=(u_1,u_2,\ldots,u_{n-1}).$$

The substitutions of transpose $T_{i,j}$ in (11) and T_i in (12) have the following forms:

$$T_{i,j}=\begin{pmatrix}\bar{i}_1,\bar{i}_2,\ldots,\bar{i}_i,\bar{\lambda},\bar{j}_1,\bar{j}_2,\ldots,\bar{j}_j,\bar{\mu},\\ \bar{\lambda},\bar{\mu},\bar{i}_1,\bar{i}_2,\ldots,\bar{i}_i,\bar{j}_1,\bar{j}_2,\ldots,\bar{j}_j\end{pmatrix},\ i,j=\overline{0,m};\ T_i=\begin{pmatrix}\bar{i}_1,\bar{i}_2,\ldots,\bar{i}_i,\bar{\mu}\\ \bar{\mu},\bar{i}_1,\bar{i}_2,\ldots,\bar{i}_i\end{pmatrix},\ i=\overline{0,m},$$

where the multi-indexes $\bar{j}_1,\bar{j}_2,\ldots,\bar{j}_j,\bar{i}_1,\bar{i}_2,\ldots,\bar{i}_i$ contain by q indexes and the multi-indexes $\bar{\lambda},\bar{\mu}$ contain by p indexes.

There are no multi-indexes $\bar{\lambda}, \bar{\mu}$ in these substitutions in the case of $p=0$, and substitutions $T_{i,j}$, T_i in this case are identical [6].

The two-dimensional cell $D_{c_t}^{-1} = \{D_{c_t}^{i,j}\},\ i,j=\overline{0,m}$, (11) has the same dimension as the two-dimensional cell $D_{c_t} = \{D_{c_t,i,j}\}$, i. e. $D_{c_t}^{i,j}$ is the $((iq+p)+(jq+p))$-dimensional matrix. The element B_i of the one-dimensional cell $B=\{B_i\},\ i=\overline{0,m}$, (12) is the $(iq+p)$-dimensional matrix.

It is of interest in dual control to use the single measurements for updating the estimations (10)–(14). We will have for this the expressions $S_{u_s^k u_s^\lambda} = {}^{0,0}(u_s^{k+\lambda})$, $S_{y_s u_s^\lambda} = {}^{0,0}(y_s u_s^\lambda)$, determined by single measurement (u_s, y_s), instead of the expressions (14).

2. Let us find the probability density function $f(y_n / \vec{u}_n, \vec{y}_{n-1})$ by the formula (5)

$$f\left(y_n / \vec{u}_n, \vec{y}_{n-1}\right) = \int\limits_{\Omega(C)} f\left(y_n / c_t, u_n\right) f_n\left(c_t\right) d\Omega, \tag{15}$$

where $f_n(c_t)$ is determined by the formula (10).

We will use for this the following theorem from [7].

Theorem (total probability formula for the joint Gaussian distribution of the multidimensional random matrices). Let $\Xi = \{\Xi_i\}$, $i=1,2,...,m'$, be an one-dimensional random cell, composed of the q_i-dimensional matrices Ξ_i, k_i the number of the scalar components of the matrix Ξ_i, $f(\xi)$ the probability density function of the cell Ξ, $k_\Xi = k_1 + k_2 + ... + k_{m'}$ the number of the scalar components of the cell Ξ, $f(y/\xi)$ the condition probability density function of a *p*-dimensional matrix *Y*, k_Y the number of the scalar components of the matrix *Y*, E^{k_Ξ} the k_Ξ-dimensional Euclidean space. If in the total probability formula

$$f(y) = \int\limits_{E^{k_\Xi}} f\left(y/\xi\right) f(\xi) d\xi \tag{16}$$

the conditional probability density function $f(y/\xi)$ has the following form

$$f\left(y/\xi\right) = \frac{1}{\sqrt{(2\pi)^{k_Y} \, |d_Y|}} \exp\left(-\frac{1}{2}\, {}^{0,p}\!\left(d_Y^{-1}\left(y - \sum_{i=1}^{m'} {}^{0,q_i}\!\left(h_i \xi_i\right)\right)^2\right)\right), \tag{17}$$

where h_i is a $(p+q_i)$-dimensional matrix, allowing the multiplication ${}^{0,q_i}(h_i\xi_i)$, and the probability density function $f(\xi)$ has the following form

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^{k_\Xi} \, |d_\Xi|}} \exp\left\{-\frac{1}{2} \sum_{i=1}^{m'} \sum_{j=1}^{m'} {}^{0,q_j}\!\left({}^{0,q_i}\!\left(\left(\xi_i - \nu_{\Xi_i}\right) d_\Xi^{i,j}\right)\left(\xi_j - \nu_{\Xi_j}\right)\right)\right\}$$

then the integral (16) (the total probability formula) is defined by the following expression:

$$f(y) = \int\limits_{E^{k_\Xi}} f(y/\xi) f(\xi) d\xi = \frac{1}{\sqrt{(2\pi)^{k_Y} \, |D_Y|}} \exp\left(-\frac{1}{2}\, {}^{0,p}\!\left(D_Y^{-1}\left(y - \sum_{i=1}^{m'} {}^{0,q_i}\!\left(h_i \nu_{\Xi,i}\right)\right)^2\right)\right), \tag{18}$$

where $D_Y = d_Y + \sum_{i=1}^{m'} \sum_{j=1}^{m'} {}^{0,q_j}\!\left({}^{0,q_i}\!\left(h_i d_{\Xi,i,j}\right) h_j\right)$.

Let us replace ξ by c_t and $f(\xi)$ by $f_n(c_t)$ (10) in this theorem and compare the expression (9) with the expression (17) from theorem. We realize that $p_i = iq,\ h_i = u_n^i$. In accordance with formula (18) of the theorem we obtain the following expression for the integral (15):

$$f\left(y_n / \vec{u}_n, \vec{y}_{n-1}\right) = \int\limits_{E^{k_C}} f\left(y_n / c_t, u_n\right) f_n\left(c_t\right) dc_t = \frac{1}{\sqrt{(2\pi)^{k_Y} \, |D_Y|}} \exp\left(-\frac{1}{2}\, {}^{0,p}\!\left(D_Y^{-1}\left(y_n - N_Y\right)^2\right)\right), \tag{19}$$

where

$$D_Y = d_Y + \sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,jq}\left({}^{0,iq}\left(u_n^i D_{c_t,i,j}\right) u_n^j \right); \tag{20}$$

$$N_Y = \sum_{i=0}^{m} {}^{0,iq}\left(u_n^i N_{c_t,i}\right). \tag{21}$$

The matrices $D_{c_t,i,j}$ and $N_{c_t,i}$ in (20), (21) are defined by the expressions (11), (13).

The further calculations are connected with formula (4) of the functional equations. When the loss function is $W(Y_s) = \| Y_s - g \|^2$, then we need to calculate the integral

$$\varphi_n\left(\vec{u}_n, \vec{y}_{n-1}\right) = \int_{E^{n_y}} \| y_n - g \|^2 \, f\left(y_n / \vec{u}_n, \vec{y}_{n-1}\right) dy_n,$$

with weight function $f(y_n / \vec{u}_n, \vec{y}_{n-1})$ (19). In accordance with the theorem from Appendix 1 we get:

$$\varphi_n(\vec{u}_n, \vec{y}_{n-1}) = E\left(\| Y_n - g \|^2\right) = \int_{E^{n_y}} \| y_n - g \|^2 \, f\left(y_n / \vec{u}_n, \vec{y}_{n-1}\right) dy_n = tr\left(D_Y + \left(N_Y - g\right)^2\right),$$

where D_Y and N_Y are determined by the formulae (20), (21). The variables D_Y and N_Y in this function $\varphi_n(\vec{u}_n, \vec{y}_{n-1})$ depend on u_n. These dependencies for our affine regression function have the following forms:

$$D_Y = d_Y + \sum_{i=0}^{1}\sum_{j=0}^{1} {}^{0,jq}\left({}^{0,iq}\left(u_n^i D_{c_t,i,j}\right) u_n^j \right) = d_Y + \sum_{j=0}^{1}\left({}^{0,jq}\left({}^{0,0}\left(u_n^0 D_{c_t,0,j}\right) u_n^j \right) + {}^{0,jq}\left({}^{0,q}\left(u_n^1 D_{c_t,1,j}\right) u_n^j \right)\right) =$$

$$= d_Y + \sum_{j=0}^{1}\left({}^{0,jq}\left(D_{c_t,0,j}\, u_n^j\right) + {}^{0,jq}\left({}^{0,q}\left(u_n^1 D_{c_t,1,j}\right) u_n^j \right)\right) =$$

$$= d_Y + D_{c_t,0,0} + {}^{0,q}\left(D_{c_t,0,1}\, u_n\right) + {}^{0,q}\left(u_n D_{c_t,1,0}\right) + {}^{0,q}\left({}^{0,q}\left(D_{c_t,1,1}^{H_q}\, u_n\right) u_n \right) =$$

$$= d_Y + D_{c_t,0,0} + {}^{0,q}\left(D_{c_t,0,1}\, u_n\right) + {}^{0,q}\left(D_{c_t,1,0}^{H_q} u_n\right) + {}^{0,2q}\left(D_{c_t,1,1}^{H_q}\, u_n^2\right);$$

$$N_Y - g = \sum_{i=0}^{1} {}^{0,iq}\left(u_n^i N_{c_t,i}\right) - g = \left(N_{c_t,0} - g\right) + {}^{0,q}\left(u_n N_{c_t,1}\right);$$

$$\left(N_Y - g\right)^2 = \left(N_{c_t,0} - g\right)^2 + {}^{0,0}\left(\left(N_{c_t,0} - g\right) {}^{0,q}\left(u_n N_{c_t,1}\right)\right) +$$

$$+ {}^{0,0}\left({}^{0,q}\left(u_n N_{c_t,1}\right)\left(N_{c_t,0} - g\right)\right) + {}^{0,0}\left({}^{0,q}\left(u_n N_{c_t,1}\right) {}^{0,q}\left(u_n N_{c_t,1}\right)\right).$$

Let us combine the similar terms in the last expression. We transform for this the summands in the expression for $\left(N_Y - g\right)^2$. We get for the second summand:

$${}^{0,0}\left(\left(N_{c_t,0} - g\right) {}^{0,q}\left(u_n N_{c_t,1}\right)\right) = {}^{0,0}\left(\left(N_{c_t,0} - g\right) {}^{0,q}\left(N_{c,1} u_n\right)\right) = {}^{0,q}\left({}^{0,0}\left(\left(N_{c_t,0} - g\right) N_{c,1}\right) u_n \right).$$

Let us transform the third summand as follows:

$${}^{0,0}\left({}^{0,q}\left(u_n N_{c_t,1}\right)\left(N_{c_t,0} - g\right)\right) = {}^{0,q}\left(u_n \, {}^{0,0}\left(N_{c_t,1}\left(N_{c_t,0} - g\right)\right)\right).$$

Since the p-dimensional matrix u_n is fully convoluted here, than we can use the known formula for transpose the product [7] and continue:

$${}^{0,0}\left({}^{0,q}\left(u_n N_{c_t,1}\right)\left(N_{c_t,0} - g\right)\right) = {}^{0,q}\left(u_n \, {}^{0,0}\left(N_{c_t,1}\left(N_{c_t,0} - g\right)\right)\right) = {}^{0,q}\left({}^{0,0}\left(N_{c_t,1}\left(N_{c_t,0} - g\right)\right)^{H_q} u_n \right).$$

We transform now the fourth summand:

$${}^{0,0}\left({}^{0,q}\left(u_n N_{c_t,1}\right) {}^{0,q}\left(u_n N_{c_t,1}\right)\right) = {}^{0,0}\left({}^{0,q}\left(u_n N_{c_t,1}\right) {}^{0,q}\left(N_{c,1} u_n\right)\right) = {}^{0,q}\left({}^{0,q}\left(u_n \, {}^{0,0}\left(N_{c_t,1} N_{c,1}\right)\right) u_n \right) =$$

$$= {}^{0,2q}\left({}^{0,0}\left(N_{c_t,1} N_{c,1}\right)^{H_q} u_n^2 \right).$$

Finally, we get for $\left(N_Y - g\right)^2$ the following expression:

$$\left(N_Y - g\right)^2 = \left(N_{c_t,0} - g\right)^2 +$$
$$+ {}^{0,q}\left({}^{0,0}\left(\left(N_{c_t,0}-g\right)N_{c,1}\right)u_n\right) + {}^{0,q}\left({}^{0,0}\left(N_{c_t,1}\left(N_{c_t,0}-g\right)\right)^{H_q}u_n\right) + {}^{0,2q}\left({}^{0,0}\left(N_{c_t,1}N_{c,1}\right)^{H_q}u_n^2\right).$$

If we denote $F(u_n) = D_Y + (N_Y - g)^2$, then we get the function $\varphi_n(\vec{u}_n, \vec{y}_{n-1}) = tr\left(F(u_n)\right)$, where $F(u_n) = K_0 + {}^{0,q}(K_1 u_n) + {}^{0,2q}(K_2 u_n^2)$;

$$K_0 = d_Y + D_{c_t,0,0} + \left(N_{c_t,0} - g\right)^2; \tag{22}$$

$$K_1 = D_{c_t,0,1} + D_{c_t,1,0}^{H_q} + {}^{0,0}\left(\left(N_{c_t,0}-g\right)N_{c,1}\right) + {}^{0,0}\left(N_{c_t,1}\left(N_{c_t,0}-g\right)\right)^{H_q}; \tag{23}$$

$$K_2 = D_{c_t,1,1}^{H_q} + {}^{0,0}\left(N_{c_t,1}N_{c,1}\right)^{H_q}. \tag{24}$$

The necessary condition of the extremum of the function $\varphi_n(\vec{u}_n, \vec{y}_{n-1})$ is the following equation:

$$\frac{d\varphi_n(\vec{u}_n, \vec{y}_{n-1})}{du_n} = {}^{0,2p}\left(\frac{dtr\left(F\right)}{dF}\frac{dF}{du_n}\right) = 0. \tag{25}$$

Since $dtr\left(F\right)/dF = E(0,p)$, $dF/du_n = K_1 + 2\,{}^{0,q}(K_2 u)$, then the condition (25) take the form:

$$\frac{d\varphi_n(\vec{u}_n, \vec{y}_{n-1})}{du_n} = {}^{0,2p}\left(E(0,p)K_1\right) + 2\,{}^{0,q}\left({}^{0,2p}(E(0,p)K_2)u\right) = L_1 + 2\,{}^{0,q}\left(L_2 u\right) = 0, \tag{26}$$

where $L_1 = {}^{0,2p}\left(E(0,p)K_1\right)$, $L_2 = {}^{0,2p}(E(0,p)K_2)$.

From the equation (26) we get the optimal value u_n^* of the control action at the last n-th instant of time:

$$u_n^* = \arg\min_{u_n} \varphi_n\left(\vec{u}_n, \vec{y}_{n-1}\right) = -\frac{1}{2}\,{}^{0,q}\left(L_2^{-1}L_1\right). \tag{27}$$

The minimal value of the function $F(u_n)$ is defined by the expression (Appendix 2) $F(u_n^*) = K_0 - {}^{0,q}\left(K_1\,{}^{0,q}\left({}^{0,q}K_2^{-1}K_1\right)\right)/4$, and the minimal value of the function $f_n^*(\vec{u}_{n-1}, u_n^*, \vec{y}_{n-1})$ – by the expression:

$$f_n^*(\vec{u}_{n-1}, u_n^*, \vec{y}_{n-1}) = \min_{u_n \in \bar{\cup}} \varphi_n(\vec{u}_n, \vec{y}_{n-1}) = tr\left(F(u_n^*)\right) = tr\left(K_0 - \frac{1}{4}\,{}^{0,q}\left(K_1\,{}^{0,q}\left({}^{0,q}K_2^{-1}K_1\right)\right)\right). \tag{28}$$

The search of the optimal control action u_n^* at the last n-th instant of time finished there and the search of the optimal control action u_{n-1}^* at the penultimate $(n-1)$-th instant of time starts. The control action u_{n-1}^* is defined by the following expression (the formula (3)):

$$u_{n-1}^* = \arg\min_{u_{n-1} \in \bar{\cup}}\left[\varphi_{n-1}\left(\vec{u}_{n-1}, \vec{y}_{n-21}\right) + \int\limits_{\Omega(y_{n-1})} f_n^*(\vec{u}_{n-1}, u_n^*, \vec{y}_{n-1}) f(y_{n-1}/\vec{u}_{n-1}, \vec{y}_{n-2})\,d\Omega\right]. \tag{29}$$

The function $f_n^*(\vec{u}_{n-1}, u_n^*, \vec{y}_{n-1})$ (28) in (29) is integrated by y_{n-1} with weight function $f(y_{n-1}/\vec{u}_{n-1}, \vec{y}_{n-2})$ and then minimized by u_{n-1} in sum with $\varphi_{n-1}\left(\vec{u}_{n-1}, \vec{y}_{n-21}\right)$. One can understand, that the matrices K_0, K_1, K_2 in (28) depend by u_{n-1} by means the $N_{c_t,0}$, $N_{c_t,1}$, $N_{c_t,2}$, $D_{c_t,0,0}$, $D_{c_t,0,1}$, $D_{c_t,1,0}$, $D_{c_t,1,1}$ in (22)–(24), which are determined by formulae (11)–(13) provided $S_{u^k u^\lambda} = {}^{0,0}(u_{n-1}^k u_{n-1}^\lambda)$, $S_{yu^\lambda} = {}^{0,0}(y_{n-1}u_{n-1}^\lambda)$:

$$D_{c_t}^{-1} = \left\{D_{c_t}^{i,j}\right\} = \left\{d_{c_t}^{i,j} + S_{i,j}\right\} = \left\{d_{c_t}^{i,j} + {}^{0,0}\left(d_Y^{-1}S_{u^i u^j}\right)\right\},\ i,j = \overline{0,1};$$

$$B = \{B_i\} = \left\{ \sum_{j=0}^{m} {}^{0,jq+p}\left(d_{c_t}^{i,j} \nu_{c_t,j}\right) + {}^{0,p}\left(d_Y^{-1} S_{yu^i}\right) \right\},\ i = \overline{0,1};$$

$$N_{c_t} = \left\{N_{c_t,i}\right\} = {}^{0,1}\left\{D_{c_t} B\right\} = \left\{ \sum_{j=0}^{m} {}^{0,p+jq}\left(D_{c_t,i,j} B_j\right) \right\},\ i = \overline{0,1};$$

$$S_{u^k u^\lambda} = {}^{0,0}\left(u_{n-1}^k u_{n-1}^\lambda\right);\ S_{yu^\lambda} = {}^{0,0}\left(y_{n-1} u_{n-1}^\lambda\right).$$

Let us write down these expressions in details:

$$D_{c_t}^{-1} = \begin{Bmatrix} d_{c_t}^{0,0} + d_Y^{-1} & d_{c_t}^{0,1} + {}^{0,0}(d_Y^{-1} u_{n-1}) \\ d_{c_t}^{1,0} + {}^{0,0}(d_Y^{-1} u_{n-1}) & d_{c_t}^{1,1} + {}^{0,0}(d_Y^{-1} u_{n-1}^2) \end{Bmatrix};$$

$$B = \{B_i\} = \begin{Bmatrix} {}^{0,0}(d_{c_t}^{0,0} \nu_{c_t,0}) + {}^{0,q}(d_{c_t}^{0,1} \nu_{c_t,1}) + {}^{0,0}(d_Y^{-1} y_{n-1}) \\ {}^{0,0}(d_{c_t}^{1,0} \nu_{c_t,0}) + {}^{0,q}(d_{c_t}^{1,1} \nu_{c_t,1}) + {}^{0,0}(d_Y^{-1} y_{n-1} u_{n-1}) \end{Bmatrix} = \begin{Bmatrix} B_0 \\ B_1 \end{Bmatrix};$$

$$N_{c_t,i} = \sum_{j=0}^{1} {}^{0,jq}(D_{c_t,i,j} B_j) = {}^{0,0}(D_{c_t,i,0} B_0) + {}^{0,q}(D_{c_t,i,1} B_1);$$

$$N_{c_t,0} = {}^{0,0}(D_{c_t,0,0} B_0) + {}^{0,q}(D_{c_t,0,1} B_1);\ N_{c_t,1} = {}^{0,0}(D_{c_t,1,0} B_0) + {}^{0,q}(D_{c_t,1,1} B_1);\ N_{c_t,2} = {}^{0,0}(D_{c_t,2,0} B_0) + {}^{0,q}(D_{c_t,2,1} B_1).$$

It is impossible to write down the explicit expressions for the $D_{c_t,i,j}$ as the functions by u_{n-1}, since they are the elements of the cell inverse to the cell $D_{c_t}^{-1}$. As a result, it is impossible to perform the analytical minimization in the expression (29). The numerical minimization in the expression (29) is impossible too.

However, the control action (27), obtained at the last instant of time, can be used at any instant of time. We will call the expression (27) the algorithm of the optimal dual control with passive information storage. The developed algorithm (27) has the theoretical and algorithmical generality.

Computer simulation

The algorithm of the optimal dual control with passive information storage (27) was realized programmatically, utilized at a number of objects and showed results acceptable for practice. For instance, the regression object with vector input and output variables and affine regression function (8) was simulated with following coefficients and variance-covariance matrix of the inner noise:

$$c_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix};\ c_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix};\ d_Y = \begin{pmatrix} 0,001 & 0 \\ 0 & 0,001 \end{pmatrix}. \tag{30}$$

The prior characteristics of the coefficients of the approximating polynomial (8) and initial control action u_0 are simulated as random.

The sequence of the control actions is showed in a Fig. 2 for some variant of the simulation.

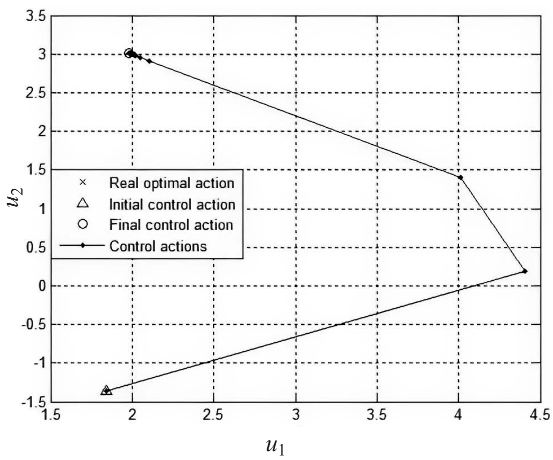

Fig. 2. The sequence of the control actions for the simulated instance

The Fig. 2 corresponds to the following priory characteristics of the coefficients of the approximating polynomial (8): priory mathematical expectations

$$\nu_{c_0} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; \ \nu_{c_1} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

and priory variance-covariance matrices

$$d_{c,1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \ d_{c,2,2} = \begin{pmatrix} \overbrace{1}^{1,1,1,1} & \overbrace{0}^{1,1,2,1} & \overbrace{0}^{1,1,1,2} & \overbrace{0}^{1,1,2,2} \\ \overbrace{0}^{2,1,1,1} & \overbrace{1}^{2,1,2,1} & \overbrace{0}^{2,1,1,2} & \overbrace{0}^{2,1,2,2} \\ \overbrace{0}^{1,2,1,1} & \overbrace{0}^{1,2,2,1} & \overbrace{1}^{1,2,1,2} & \overbrace{0}^{1,2,2,2} \\ \overbrace{0}^{2,2,1,1} & \overbrace{0}^{2,2,2,1} & \overbrace{0}^{2,2,1,2} & \overbrace{1}^{2,2,2,2} \end{pmatrix}.$$

The four-dimensional matrix $d_{c,2,2}$ is presented by an associated with it two-dimensional matrix. The covariance matrices $d_{c,1,2}$ and $d_{c,2,1}$ are taken as zero matrices of appropriate sizes. The Fig. 2 illustrates the stabilization of the regression function at the level $g = (9 \quad 20)$. As it follows from the object description (30), the regression function has the value $y = g = (9 \quad 20)$ provided control action $u = (2 \quad 3)$. One can see in the Fig. 2 that this value of the control action is reached.

Appendix 1

Theorem. If $z = (z_{i_1, i_2, ..., i_p})$ be a p-dimensional random matrix with mathematical expectation $E(z) = N_z$ and variance-covariance matrix D_z, then the mathematical expectation of the square of the Euclidean norm of the matrix z is defined by the following expression:

$$E\left(\| z \|^2\right) = E\left({}^{0,p}(zz)\right) = tr\left(D_z + N_z^2\right),$$

where $tr(\cdot)$ means the trace of the matrix; ${}^{0,p}(zz)$ is the $(0, p)$-convoluted square of the matrix z; N_z^2 is the $(0,0)$-convoluted square of the matrix N_z.

Proof. The square of the Euclidean norm of the matrix z is defined by the formula

$$\| z \|^2 = \sum_{i_1, i_2, ..., i_p} z^2_{i_1, i_2, ..., i_p} = \sum_{i_1, i_2, ..., i_p} z_{i_1, i_2, ..., i_p} z_{i_1, i_2, ..., i_p} = {}^{0,p}(zz).$$

Then

$$E\left(\| z \|^2\right) = E\left(\sum_{i_1, i_2, ..., i_p} z^2_{i_1, i_2, ..., i_p}\right) = \sum_{i_1, i_2, ..., i_p} E\left(z^2_{i_1, i_2, ..., i_p}\right) = \sum_{i_1, i_2, ..., i_p} \left(E^2(z_{i_1, i_2, ..., i_p}) + D(z_{i_1, i_2, ..., i_p})\right),$$

where $D(z_{i_1, i_2, ..., i_p})$ is the variation of the random variable $z_{i_1, i_2, ..., i_p}$.

Thus $E\left(\| z \|^2\right) = \sum_{i_1, i_2, ..., i_p} E^2(z_{i_1, i_2, ..., i_p}) + \sum_{i_1, i_2, ..., i_p} D(z_{i_1, i_2, ..., i_p}) = tr\left(E^2(z) + D(z)\right)$. The theorem is proved.

If $y = z - g$, where g is a constant matrix, then $D_y = D_z$, $E(y) = N_z - g$, and $E\left(\| z - g \|^2\right) = tr\left(D_z + (N_z - g)^2\right)$.

Appendix 2

Let $x = (x_{j_{(q)}})$, $j_{(q)} = (j_1, j_2, ..., j_q)$, be a q-dimensional matrix, that is the argument of a p-dimensional-matrix function $y = (y_{i_{(p)}})$, $i_{(p)} = (i_1, i_2, ..., i_p)$, and this function has the form $y = \varphi(x) = c_0 + {}^{0,q}(c_1 x) + {}^{0,2q}(c_2 x^2) = c_{t,0} + {}^{0,q}(x c_{t,1}) + {}^{0,2q}(x^2 c_{t,2})$, where c_k, $k = 0,1,2$, are the $(p + kq)$-dimensional-matrix coefficients of the function $\varphi(x)$, and c_2 is symmetric relative its last q-multi-indexes. Let it be required to find the extremum of this function.

Optimal value of x can be found from the equation $\partial\varphi(x) / \partial x = 0$. Differentiating of $\varphi(x)$ gives the equation $\widehat{c}_1 + 2\,{}^{0,q}(c_2 x) = 0$. Hence $x_n^* = -\,{}^{0,q}({}^{0,q}c_2^{-1} c_1) / 2$, where ${}^{0,q}c_2^{-1}$ is the matrix $(0,q)$-inverse to the matrix c_2.

Let us to find the minimum value $y^* = \varphi(x^*)$ of the function $\varphi(x)$. Since ${}^{0,2q}(c_2 x^2) = {}^{0,q}({}^{0,q}(c_2 x)x)$ and the equation ${}^{0,q}(c_2 x_n^*) = -c_1 / 2$ for $x = x^*$ is fullfilled, we have ${}^{0,2q}(c_2 (x_n^*)^2) = -{}^{0,q}(c_1 x_n^*) / 2$ and $y^* = \varphi(x^*) = c_0 + {}^{0,q}(c_1 x^*) + {}^{0,2q}(c_2 (x^*)^2) = c_0 + {}^{0,q}(c_1 x^*) - \frac{1}{2}\, {}^{0,q}(c_1 x^*) = c_0 + {}^{0,q}(c_1 x^*)$.

Substituting x_n^* into this expression dives

$$y^* = c_0 - \frac{1}{4}\, {}^{0,q}\left(c_1\, {}^{0,q}\left({}^{0,q}c_2^{-1} c_1\right)\right).$$

Conclusions

To sum up, the general solution to the problem of the dual stabilization of the multidimensional regression object at the given level with passive information storage in the Gaussian case was obtained for the first time. The important advantages of the developed algorithm are theoretical and algorithmical generality. This solution can be applied to control of the various technological processes with many input and output variables, but each of them requires separate consideration.

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Authors' contribution

Mukha V. S. developed and wrote the article.
Kako N. F. was directly involved in developing and writing the article.

Information about the authors

Mukha V. S., D. Sci., Professor, Professor at the Department of Information Technologies of Automated Systems of the Belarusian State University of Informatics and Radioelectronics.

Kako N. F., Postgraduate at the Belarusian State University of Informatics and Radioelectronics.

Address for correspondence

220013, Republic of Belarus,
Minsk, P. Brovki St., 6
Belarusian State University
of Informatics and Radioelectronics
Tel.: +375 44 781-16-51
E-mail: mukha@bsuir.by
Mukha Vladimir Stepanovich