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Enriched lower separation axioms and the principle of enriched continuous extension ☆

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Abstract

This paper presents a version of the lower separation axioms and the principle of enriched continuous extension for quantale-enriched topological spaces. As a remarkable result, among other things, we point out that in the case of commutative Girard quantales the principle of continuous extension holds for projective modules in Sup.

Keywords: Unital quantale; Modules in Sup; Quantale-enriched topological space; Closed presheaves; Lower separation axioms; Convergence of quantale-enriched filters; Extension by quantale-enriched continuity

1. Introduction

Let $\mathfrak{Q}$ be a unital quantale — i.e. a monoid in the symmetric and monoidal closed category Sup of complete lattices and join-preserving maps. The concept of $\mathfrak{Q}$-enriched topological spaces arises from the topologization of closed left (right) ideal lattices of not necessarily commutative C^*-algebras (cf. [11,12]). In this paper we introduce closed $\mathfrak{Q}$-enriched presheaves, lower separation axioms and regularity with the goal to formulate the principle of continuous extension for $\mathfrak{Q}$-enriched topological spaces. It is well known that the set-theoretical version of this principle goes back to Bourbaki and Dieudonné (cf. [4]). A frame-theoretical extension by continuity was provided by Banaschewski and Hong (cf. [1]), which is also a generalization of Bourbaki's and Dieudonné's result from 1939. Since in the quantale-enriched setting there are in general more «open covers» than in the frame-theoretical one, the respective convergence theories involved in these principles are different, and consequently the convergence of a quantale-enriched filter

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cannot in general be characterized by the property that it meets every «open cover» (cf. Remark 6.3). In this sense the respective principles of continuous extension are unrelated to each other.

After these general comments on convergence theory we continue our presentation with some more detailed information on separation axioms. First we note that some lower separation axioms of quantale-enriched topology have their origin in many-valued topology — e.g. the Kolmogorov and Fréchet separation axiom (cf. [25,34,30,29]), while the Hausdorff separation axiom is an extension of the $\mathbb{1}$-Hausdorff axiom of probabilistic topological spaces to the general setting of $\mathfrak{Q}$-enriched topological spaces (cf. [16]).

The regularity axiom uses the cocompleteness of $\mathfrak{Q}$-enriched topologies viewed as $\mathfrak{Q}^{op}$-enriched categories and requires the concept of closed $\mathfrak{Q}$-enriched presheaves. In contrast to various approaches in many-valued topology the concept of closedness is here based on the concept of $\mathfrak{Q}$-enriched adherence operator on a given set X, which is a coclosure operator on the right $\mathfrak{Q}$-module on the dual lattice of the free right $\mathfrak{Q}$-module on X in the sense of Sup (cf. Section 4). If the underlying quantale $\mathfrak{Q}$ has a dualizing element, then $\mathfrak{Q}$-adherence operators and $\mathfrak{Q}$-interior operators are equivalent concepts. This equivalence extends to open and closed $\mathfrak{Q}$-presheaves (cf. Proposition 4.6).

With regard to the historical background, we point out that also the regularity axiom has its origin in many-valued topology. If $\mathfrak{Q}$ is a commutative Girard quantale, then in the case of cotensored $\mathfrak{Q}$-enriched topologies the regularity axiom is equivalent to H-R-regularity — an axiom formulated by Hutton and Reilly in 1980 (cf. [19]). Moreover, in the general case of unital and commutative quantales the regularity of cotensored $\mathfrak{Q}$-enriched topologies is equivalent to probabilistic regularity. In this setting the regularity axiom has its origin in probabilistic topology (cf. [16, Def. 3.3]).

All the lower separation axioms for $\mathfrak{Q}$-enriched topological spaces (including regularity) are preserved and reflected by the change of base conveyed by the embedding $\mathbf{2} \hookrightarrow \mathfrak{Q}$.

Forced by the non-idempotency of the quantale multiplication (cf. Example 6.10) we introduce a completely new and weaker form of regularity. It is interesting to see that under the assumption of this weak regularity axiom the Kolmogoroff, Fréchet and Hausdorff separation axioms are all equivalent (cf. 6.19). Further, and more importantly, weak regularity is sufficient for the formulation of the principle of continuous extension. As a remarkable result in this context we point out that, in the case of commutative Girard quantales, every projective $\mathfrak{Q}$-module in Sup provided with the interval $\mathfrak{Q}$-topology is a Hausdorff separated and weakly regular $\mathfrak{Q}$-enriched topological space (cf. Example 6.21).

The paper is organized as follows. Starting from some preliminaries with a certain weight on quantales with a dualizing element we explain the interrelationship between $\mathfrak{Q}$-preorders (i.e. hom-object assignments with values in $\mathfrak{Q}$) and right $\mathfrak{Q}$-modules in Sup. This approach can be seen as a preparation of the module-theoretical properties of open and closed $\mathfrak{Q}$-presheaves and leads to a very natural concept of adherent point and density. Subsequently we develop the announced lower separation axioms, regularity and weak regularity for $\mathfrak{Q}$-enriched topological spaces and show that in the case of quantales with a dualizing element the principle of continuous extension holds for any weakly T_3 space.

2. Preliminaries on quantales

First we recall that the usual tensor product $\otimes$ of complete lattices (cf. [7, Sect. 2.1.2]) gives rise to the monoidal structure on Sup (cf. [20]). A semigroup in Sup is called a *quantale* $\mathfrak{Q} = (\mathfrak{Q}, *)$, and a monoid in Sup is called a *unital* quantale $\mathfrak{Q} = (\mathfrak{Q}, *, e)$, where the unit is always denoted by e. Since Sup is a symmetric monoidal category, the *opposite quantale* of a unital quantale $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ is given by $\mathfrak{Q}^{op} = (\mathfrak{Q}, *^{op}, e)$ with $\alpha *^{op} \beta = \beta * \alpha$ for all $\alpha, \beta \in \mathfrak{Q}$. A unital quantale is *integral* if $e = \top$.

If $\alpha, \beta \in \mathfrak{Q}$ then the right and left implications are determined by:

$$\alpha \searrow \beta = \bigvee \{ \gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta \} \quad \text{and} \quad \beta \swarrow \alpha = \bigvee \{ \gamma \in \mathfrak{Q} \mid \gamma * \alpha \leq \beta \}.$$

It is well known that the associativity of the quantale multiplication $*$ is equivalent to each of the following properties (cf. [7, (E) in the proof of Thm. 2.3.6]) for all $\alpha, \beta, \gamma \in \mathfrak{Q}$:

$$\alpha \searrow (\beta \searrow \gamma) = (\beta * \alpha) \searrow \gamma, \tag{2.1}$$

$$(\gamma \swarrow \beta) \swarrow \alpha = \gamma \swarrow (\alpha * \beta), \tag{2.2}$$

$$(\alpha \searrow \beta) \swarrow \gamma = \alpha \searrow (\beta \swarrow \gamma). \tag{2.3}$$

An element δ of a quantale $\mathfrak{Q}$ is

– *dualizing* if it satisfies the following condition for all $\alpha \in \mathfrak{Q}$:

$$(\delta \swarrow \alpha) \searrow \delta = \alpha = \delta \swarrow (\alpha \searrow \delta). \tag{2.4}$$

– *cyclic* if the equivalence $\alpha * \beta \le \delta \iff \beta * \alpha \le \delta$ holds for all $\alpha, \beta \in \mathfrak{Q}$.

Every quantale with a dualizing element δ is unital, and the unit e is given by $\delta \searrow \delta = e = \delta \swarrow \delta$. A quantale with a dualizing element δ is integral if and only if $\delta = \bot$, hence in this case the dualizing element is unique.

A *Girard* quantale is a quantale which has a cyclic and dualizing element δ. Hence in any Girard quantale $\mathfrak{Q}$ the relation $\delta \swarrow \alpha = \alpha \searrow \delta$ holds for all $\alpha \in \mathfrak{Q}$ and consequently every cylic and dualizing element induces an order-reversing involution on $\mathfrak{Q}$ sending α to $\alpha \searrow \delta$.

Example 2.1. On the 3-chain $C_3 = \{\bot, a, \top\}$ with $\bot < a < \top$ there exist exactly two Girard quantales $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Both are commutative. The quantale $\mathfrak{Q}_1$ is *integral* and is the well known 3-valued MV-algebra, in which $a * a = \bot$ holds and $\bot$ is necessarily the dualizing element. In contrast to $\mathfrak{Q}_1$, the quantale $\mathfrak{Q}_2$ is *non-integral* and *idempotent*, in which the unit and the dualizing element coincide and are given by a.

From a historical point of view, $\mathfrak{Q}_1$ has its origin in the Łukasiewicz three-valued logic in 1920 (cf. [27, The principles of implication]), while the quantale multiplication of $\mathfrak{Q}_2$ appears already as Peirce's Ψ-operator in context of triadic logic in 1909 (cf. [8, Plate 2 of Peirce's Logic Notebook]).

Let now $\mathfrak{Q}$ be a quantale with a dualizing element δ. With regard to (2.1), (2.4), (2.3) and (2.2) we observe:

$$\begin{aligned}((\delta \swarrow \alpha) * (\delta \swarrow \beta)) \searrow \delta &= (\delta \swarrow \beta) \searrow ((\delta \swarrow \alpha) \searrow \delta) = (\delta \swarrow \beta) \searrow \alpha \\ &= ((\delta \swarrow \beta) \searrow \delta) \swarrow (\alpha \searrow \delta) = \beta \swarrow (\alpha \searrow \delta) \\ &= (\delta \swarrow (\beta \searrow \delta)) \swarrow (\alpha \searrow \delta) = \delta \swarrow ((\alpha \searrow \delta) * (\beta \searrow \delta)).\end{aligned}$$

Therefore we shall use the notation

$$\alpha *_\delta \beta := ((\delta \swarrow \alpha) * (\delta \swarrow \beta)) \searrow \delta = \delta \swarrow \big((\alpha \searrow \delta) * (\beta \searrow \delta)\big). \tag{2.5}$$

Remarks 2.2. (1) Let $\alpha, \beta \in \mathfrak{Q}$ and $\delta \in \mathfrak{Q}$ be a dualizing element. If in

$$\alpha *_\delta \beta = (\delta \swarrow \beta) \searrow \alpha = \beta \swarrow (\alpha \searrow \delta)$$

we replace β by $\beta \searrow \delta$ and α by $\delta \swarrow \alpha$, then we also have:

$$\beta \searrow \alpha = \alpha *_\delta (\beta \searrow \delta) \quad \text{and} \quad \beta \swarrow \alpha = (\delta \swarrow \alpha) *_\delta \beta.$$

(2) For each dualizing element $\delta \in \mathfrak{Q}$ the maps $\mathfrak{Q} \xrightarrow{(\cdot)\searrow\delta} \mathfrak{Q}$ and $\mathfrak{Q} \xrightarrow{\delta\swarrow(\cdot)} \mathfrak{Q}$ are *order-reversing*, *bijective* and *inverse to each other*. Then the following hold (cf. [7, Prop. 2.6.2 (iii) and (iv)]):

$$\Big(\bigwedge_{i\in I} \alpha_i\Big) \searrow \delta = \bigvee_{i\in I} (\alpha_i \searrow \delta) \quad \text{and} \quad \delta \swarrow \Big(\bigwedge_{i\in I} \alpha_i\Big) = \bigvee_{i\in I} (\delta \swarrow \alpha_i), \qquad \{\alpha_i\}_{i\in I} \subseteq \mathfrak{Q}.$$

(3) It follows immediately from (1) or (2) that

$$\Big(\bigwedge_{i\in I} \alpha_i\Big) *_\delta \beta = \bigwedge_{i\in I} (\alpha_i *_\delta \beta) \quad \text{and} \quad \alpha *_\delta \Big(\bigwedge_{i\in I} \beta_i\Big) = \bigwedge_{i\in I} (\alpha *_\delta \beta_i).$$

Moreover, $(\alpha *_\delta \beta) \searrow \delta = (\alpha \searrow \delta) * (\beta \searrow \delta)$ and $\delta \swarrow (\alpha * \beta) = (\delta \swarrow \alpha) *_\delta (\delta \swarrow \beta)$, and $\delta \swarrow (\alpha *_\delta \beta) = (\delta \swarrow \alpha) * (\delta \swarrow \beta)$ and $(\alpha * \beta) \searrow \delta = (\alpha \searrow \delta) *_\delta (\beta \searrow \delta)$.

Hence, if $\mathfrak{Q}^\dagger = (\mathfrak{Q}, \le^{op})$ is the dual lattice of $\mathfrak{Q}$, then $\big(\mathfrak{Q}^\dagger, *_\delta\big) \xrightarrow{(\cdot)\searrow\delta} (\mathfrak{Q}, *)$, $\big(\mathfrak{Q}^\dagger, *_\delta\big) \xrightarrow{\delta\swarrow(\cdot)} (\mathfrak{Q}, *)$, $(\mathfrak{Q}, *) \xrightarrow{\delta\swarrow(\cdot)} \big(\mathfrak{Q}^\dagger, *_\delta\big)$ and $(\mathfrak{Q}, *) \xrightarrow{(\cdot)\searrow\delta} \big(\mathfrak{Q}^\dagger, *_\delta\big)$ are unital quantale isomorphisms. Note that δ is the unit in $(\mathfrak{Q}^\dagger, *_\delta)$, while e is a dualizing element and $\top$ the zero element. We shall refer to this unital quantale as $\mathfrak{Q}_\delta = (\mathfrak{Q}^\dagger, *_\delta, \delta)$.

There exists a large class of quantales with dualizing elements. Typical commutative and integral quantales with a dualizing element are given by the MacNeille completion of MV-algebras (cf. [7, Thm. 2.6.8 and p. 183])), while non-commutative and non-integral quantales with a dualizing element arise frequently from the MacNeille completion of partially ordered groups. In particular, we have the following:

Example 2.3. Every group G can be considered as a partially ordered group with respect to the discrete order on G (see [2, Exercise 1 on p. 291]) and its MacNeille completion $\mathfrak{Q}_G = G \cup \{\bot, \top\}$ can be endowed with the extension of the group multiplication to a quantale multiplication $*$:

$$\top * \alpha = \alpha * \top = \top * \top = \top, \qquad \alpha \in G.$$

The unit e of G is also the unit of $\mathfrak{Q}_G$, hence $\mathfrak{Q}_G$ is a *non-integral*, unital quantale and it is commutative if and only if G is commutative. Moreover, every $\delta \in G$ is a dualizing element in $\mathfrak{Q}_G$ (cf. [14, Example 5.2]). Consequently dualizing elements are in general not unique.

An element p of a complete lattice L is *completely prime* (cf. [6, Def. 10.26]) if $p \neq \top$ and for all $A \subseteq L$ such that $\bigwedge A \le p$ there exists $\alpha \in A$ such that $\alpha \le p$. We will need the following:

Lemma 2.4. *If a dualizing element δ of a quantale $\mathfrak{Q}$ is completely prime, then every dualizing element of $\mathfrak{Q}$ is completely prime.*

Proof. Let δ be a completely prime and dualizing element of $\mathfrak{Q}$. Further, let $\tilde{\delta}$ be an arbitrary dualizing element and $A \subseteq \mathfrak{Q}$ such that $\bigwedge A \le \tilde{\delta}$. Then

$$e = \delta \searrow \delta \le \bigvee_{\alpha \in A} \big((\delta \swarrow \tilde{\delta}) * \alpha\big) \searrow \delta = \bigvee_{\alpha \in A} (\alpha \searrow \tilde{\delta}),$$

which is equivalent to $\bigwedge_{\alpha \in A} (\delta \swarrow \tilde{\delta}) * \alpha \le \delta$. Since δ is completely prime, there exists $\alpha \in A$ such that $(\delta \swarrow \tilde{\delta}) * \alpha \le \delta$. Hence $\alpha \le \tilde{\delta}$. $\square$

The following notion of a quasi-magma on a unital quantale has been introduced in [13, Def. 1] and plays a crucial role in the «intersection axiom» of $\mathfrak{Q}$-enriched topologies.

Definition 2.5. Let $\mathfrak{Q}$ be a unital quantale and $\diamond$ be an isotone binary operation on $\mathfrak{Q}$. Then $(\mathfrak{Q}, \diamond)$ is called a *quasi-magma* on $\mathfrak{Q}$ if it satisfies the following conditions for all $\alpha, \beta, \gamma \in \mathfrak{Q}$:

$$\alpha * (\beta \diamond \gamma) \le (\alpha * \beta) \diamond \gamma \quad \text{and} \quad (\alpha \diamond \beta) * \gamma \le \alpha \diamond (\beta * \gamma).$$

A quasi-magma $(\mathfrak{Q}, \diamond)$ is *strict* if there exist elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathfrak{Q}$ such that the condition

$$(\alpha_1 \diamond \beta_1) \vee (\alpha_2 \diamond \beta_2) \neq (\alpha_1 \vee \alpha_2) \diamond (\beta_1 \vee \beta_2)$$

holds.

Remark 2.6. If $(\mathfrak{Q}, \diamond)$ is a quasi-magma, δ is a dualizing element of $\mathfrak{Q}$ and $\mathfrak{Q}_\delta = (\mathfrak{Q}^\dagger, *_\delta, \delta)$ is the unital quantale constructed in Remarks 2.2, then we define a binary operation on $\mathfrak{Q}_\delta$ by

$$\alpha \diamond_\delta \beta := \delta \swarrow \big((\alpha \searrow \delta) \diamond (\beta \searrow \delta)\big), \qquad \alpha, \beta \in \mathfrak{Q}. \tag{2.6}$$

Since the order on $\mathfrak{Q}^\dagger$ is the dual order $\le^{op}$ w.r.t. the given order on $\mathfrak{Q}$, we observe for $\alpha, \beta, \gamma \in \mathfrak{Q}$:

$$\begin{aligned} \alpha *_\delta (\beta \diamond_\delta \gamma) &= \delta \swarrow \Big((\alpha \searrow \delta) * \big((\beta \searrow \delta) \diamond (\gamma \searrow \delta)\big)\Big) \\ &\le^{op} \delta \swarrow \Big(\big((\alpha \searrow \delta) * (\beta \searrow \delta)\big) \diamond (\gamma \searrow \delta)\Big) = (\alpha *_\delta \beta) \diamond_\delta \gamma. \end{aligned}$$

Analogously we prove $(\beta \diamond_\delta \gamma) *_\delta \alpha \le^{op} \beta \diamond_\delta (\gamma *_\delta \alpha)$. Hence $(\mathfrak{Q}^\dagger, \diamond_\delta)$ is a quasi-magma.

Moreover, if $(\mathfrak{Q}, \diamond)$ is strict and $\alpha_1, \alpha_2, \beta_1$ and β_2 are elements of $\mathfrak{Q}$ witnessing its strictness, then so do $(\delta \swarrow \alpha_1)$, $(\delta \swarrow \alpha_2)$, $(\delta \swarrow \beta_1)$ and $(\delta \swarrow \beta_2)$ with respect to $\big(\mathfrak{Q}^\dagger, \diamond_\delta\big)$.

Examples 2.7. (1) If $\mathfrak{Q}$ coincides with the unique unital quantale $\mathbf{2} = \{0, 1\}$, then 0 is the unique dualizing element, and every strict quasi-magma coincides with $(\mathbf{2}, \wedge)$. Hence $\diamond$ coincides with the binary meet $\wedge$ and $\diamond_\perp$ with the binary join $\vee$ on $\mathbf{2}$. In this sense the notion of strict quasi-magma covers also the algebraic and order-theoretic basis for traditional topology.
(2) More generally, let $\mathfrak{Q}$ be a unital quantale with a dualizing element δ. If we choose $\diamond = *$, then $(\mathfrak{Q}, *)$ is a strict quasi-magma (cf. [13, Example 1 (b)] and [15, Rem. 2.3 (3)]). In this case $\diamond_\delta = *_\delta$.
We anticipate here that this type of strict quasi-magma will play an important role for example in 4.2, 4.3, 5.9, 5.10 (2), 5.12 (2) and (3a), 6.8, 6.9, 6.10, 6.18 — 6.23.
(3) Let $\mathfrak{Q}$ be a unital quantale with a dualizing element δ. Then we refer to [15, Subsec. 2.2] and conclude that $(\mathfrak{Q}, \diamond)$ is a strict quasi-magma with

$$\alpha \diamond \beta := (\alpha * \top) \wedge (\top * \beta), \qquad \alpha, \beta \in \mathfrak{Q}.$$

The strictness follows from $(\perp \diamond \top) \vee (\top \diamond \perp) = \perp \neq \top = (\perp \vee \top) \diamond (\top \vee \perp)$. Moreover, the following special cases of $\mathfrak{Q}$ are interesting:

(i) if $\mathfrak{Q}$ is an integral quantale, then $\perp$ is the unique dualizing element, $\diamond$ coincides with the binary meet $\wedge$ and $\diamond_\perp$ with the binary join $\vee$ on $\mathfrak{Q}$ (cf. [15, 2.2]);
(ii) if $\mathfrak{Q} = \mathfrak{Q}_G$ is the MacNeille completion of a group G (cf. Example 2.3), then $\diamond$ and $\diamond_\delta$ have the following form:

$$\alpha \diamond \beta = \begin{cases} \perp, & \text{if } \alpha = \perp \text{ or } \beta = \perp, \\ \top, & \text{otherwise,} \end{cases} \qquad \alpha \diamond_\delta \beta = \begin{cases} \top, & \text{if } \alpha = \top \text{ or } \beta = \top, \\ \perp, & \text{otherwise.} \end{cases}$$

(4) Let $\mathfrak{Q} = (\mathfrak{Q}, *)$ be a complete MV-algebra with square roots — i.e. an integral, divisible and commutative Girard quantale such that every element of $\mathfrak{Q}$ is a square w.r.t. $*$ (cf. [7, Sec. 2.7]). The formation of square roots

$$\alpha^{1/2} = \bigvee \{\beta \in \mathfrak{Q} \mid \beta * \beta \leq \alpha\}, \qquad \alpha \in \mathfrak{Q},$$

induces a further isotone and binary operation $\circledast$ on $\mathfrak{Q}$ by

$$\alpha \circledast \beta := \alpha^{1/2} * \beta^{1/2}, \qquad \alpha, \beta \in \mathfrak{Q}.$$

Then $(\mathfrak{Q}, \circledast)$ is a strict quasi-magma (cf. [13, Example 1 (c)]). Now we refer to [7, Thm. 2.7.16 (i)] and observe

$$\perp^{1/2} \to \perp = ((\top \to \perp)^{1/2} * (\perp \to \perp)^{1/2}) \to \perp = \top \circledast_\perp \perp.$$

Hence $\circledast_\perp = \circledast$ if and only if $\perp^{1/2} = \perp^{1/2} \to \perp$ holds (cf. [17, Prop. 2.22 (2)]).

3. Modules in Sup

Let $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ be a unital quantale. A *$\mathfrak{Q}$-preordered set* (X, p) is a set X provided with a $\mathfrak{Q}$-preorder — i.e. a map $X \times X \xrightarrow{p} \mathfrak{Q}$ satisfying the following axioms:

$$e \leq p(x, x) \quad \text{and} \quad p(x, y) * p(y, z) \leq p(x, z), \qquad x, y, z \in X.$$

Since $\mathfrak{Q}$-preorders coincide with $\mathfrak{Q}^{op}$-valued hom-object assignments, $\mathfrak{Q}$-preordered sets are equivalent to $\mathfrak{Q}^{op}$-enriched categories (cf. [21]), where $\mathfrak{Q}^{op}$ is viewed as a biclosed monoidal category. In particular, every nonempty set is provided with the discrete $\mathfrak{Q}$-preorder and is consequently understood as a discrete $\mathfrak{Q}^{op}$-enriched category.

Every $\mathfrak{Q}$-preorder p has an *underlying preorder* $\leq_p$ corresponding to the associated ordinary category of a $\mathfrak{Q}^{op}$-enriched category (cf. [21]). In particular $\leq_p$ is given by:

$$\leq_p = \{(x, y) \in X \times X \mid e \leq p(x, y)\}.$$

Let (X, p) and (Y, q) be $\mathfrak{Q}$-preordered sets. A $\mathfrak{Q}^{op}$-functor is a map $X \xrightarrow{\varphi} Y$ satisfying the condition $p(x_1, x_2) \leq q(\varphi(x_1), \varphi(x_2))$ for all $x_1, x_2 \in X$. Sometimes $\mathfrak{Q}^{op}$-functors are also called $\mathfrak{Q}$-homomorphisms (cf. [7]).

A *right* $\mathfrak{Q}$*-module* in Sup is a complete lattice M provided with a *right action* $M \otimes \mathfrak{Q} \xrightarrow{\boxdot} M$ (cf. [28, p. 174] and [7, p. 204]). Due to the universal property of the tensor product $\otimes$ in Sup every right action on M can be identified with a map $M \times \mathfrak{Q} \xrightarrow{\boxdot} M$ which is join-preserving in each variable separately and satisfies the following axioms:

$$m \boxdot e = m \quad \text{and} \quad (m \boxdot \alpha) \boxdot \beta = m \boxdot (\alpha * \beta), \qquad m \in M,\ \alpha, \beta \in \mathfrak{Q}.$$

Analogously, we can introduce *left* $\mathfrak{Q}$*-modules* in Sup as complete lattices M provided with a *left* action $\mathfrak{Q} \otimes M \xrightarrow{\odot} M$ (cf. [7, p. 204]). Similarly to the previous arguments, every left action on M can be identified with a map $\mathfrak{Q} \times M \xrightarrow{\odot} M$, which is join-preserving in each variable separately and satisfies the following axioms:

$$e \odot m = m \quad \text{and} \quad \beta \odot (\alpha \odot m) = (\beta * \alpha) \odot m, \qquad m \in M,\ \alpha, \beta \in \mathfrak{Q}.$$

Every left $\mathfrak{Q}$-module is a right $\mathfrak{Q}^{op}$-module and vice verse.

Right (resp. *left*) $\mathfrak{Q}$*-module morphisms* (cf. [28, p. 174]) are join-preserving maps, which also preserve the respective right (resp. left) actions. Obviously, right (resp. left) $\mathfrak{Q}$-modules and right (resp. left) $\mathfrak{Q}$-module morphisms form a category denoted by $\mathsf{Mod}_r(\mathfrak{Q})$ (resp. $\mathsf{Mod}_\ell(\mathfrak{Q})$).

Since the monoidal category Sup is symmetric and has a self-duality given by the construction of right adjoint maps, we can compute the right adjoints of left and right actions leading to the following situation:

(1) The right adjoint $\boxdot^{\vdash} : M \to M \otimes \mathfrak{Q}$ of a right action $\boxdot$ has the form:

$$\boxdot^{\vdash}(m) = \bigvee \{ n \otimes \alpha \in M \otimes \mathfrak{Q} \mid n \boxdot \alpha \le m \}, \qquad m \in M,$$

and the evaluation at $n \in M$ determines a $\mathfrak{Q}$-preorder p on M as follows (cf. [7]):

$$p(n,m) = \bigvee \{ \alpha \in \mathfrak{Q} \mid n \boxdot \alpha \le m \}, \qquad n, m \in M. \tag{3.1}$$

If we now compose the right action $\boxdot$ again with the symmetry of Sup, then $\boxdot$ induces a left action $\odot^{op}$ on the dual lattice of M as follows:

$$\alpha \odot^{op} m = \bigvee \{ n \in M \mid n \boxdot \alpha \le m \}, \qquad \alpha \in \mathfrak{Q},\ m \in M.$$

Since the underlying preorder of p coincides with the order $\le$ of the right $\mathfrak{Q}$-module M, p is a $\mathfrak{Q}$-enrichement of $\le$ and is called the *intrinsic* $\mathfrak{Q}$*-preorder* associated with M. It is not difficult to show that the intrinsic $\mathfrak{Q}$-preorder of M satisfies the following properties:

(i) $\bigwedge_{m \in A} p(m,n) = p\big(\bigvee A, n\big)$ and $\bigwedge_{n \in A} p(m,n) = p\big(m, \bigwedge A\big)$ for all $A \subseteq M$ and $m, n \in M$.
(ii) $\alpha \searrow p(m,n) = p((m \boxdot \alpha), n)$ and $p(m,n) \swarrow \alpha = p(m, (\alpha \odot^{op} n))$ for all $\alpha \in \mathfrak{Q}$ and $m, n \in M$.

It follows from (ii) that for every intrinsic $\mathfrak{Q}$-preorder p of a right $\mathfrak{Q}$-module M the $\mathfrak{Q}^{op}$*-enriched category* (M, p) is tensored and cotensored (cf. [3, Def. 6.5.1]). Moreover, (M, p) is cocomplete — i.e. all $\mathfrak{Q}$-joins $\mathsf{sup}(f)$ of $\mathfrak{Q}$-presheaves $f \in \mathfrak{Q}^M$ exist (cf. [32]):

$$\mathsf{sup}(f) = \bigvee_{m \in M} m \boxdot f(m), \qquad f \in \mathfrak{Q}^M. \tag{3.2}$$

(2) The right adjoint $\odot^{\vdash} : M \to \mathfrak{Q} \otimes M$ of a left action $\odot$ has the form

$$\odot^{\vdash}(m) = \bigvee \{ \alpha \otimes n \in \mathfrak{Q} \otimes M \mid \alpha \odot n \le m \}, \qquad m \in M,$$

and the evaluation at $\alpha \in \mathfrak{Q}$ determines a right action $\boxdot^{op}$ on the dual lattice of M as follows (cf. [7]):

$$m \boxdot^{op} \alpha = (\odot^{\vdash}(m))(\alpha) = \bigvee \{ n \in M \mid \alpha \odot n \le m \}, \quad \alpha \in \mathfrak{Q},\ m \in M. \tag{3.3}$$

If we now compose the left action $\odot$ with the symmetry of Sup, then $\odot$ induces always a $\mathfrak{Q}$-preorder q on M as follows:

$$q(m,n) = \bigvee \{ \alpha \in \mathfrak{Q} \mid \alpha \odot n \le m \}, \qquad m, n \in M. \tag{3.4}$$

Since the underlying preorder $\leq_q$ of q coincides with the dual order of the left $\mathfrak{Q}$-module M, q is consequently *not* a $\mathfrak{Q}$-enrichment of the order of M. With regard to (3.4) we note that in fact the $\mathfrak{Q}$-preorder q coincides with the *intrinsic* $\mathfrak{Q}$-preorder of the right $\mathfrak{Q}$-module given by the dual lattice of M and the right action $\boxdot^{op}$.

A $\mathfrak{Q}$-*submodule* of a right $\mathfrak{Q}$-module $(M, \boxdot)$ is a subobject of $(M, \boxdot)$ in the sense of $\mathsf{Mod}_r(\mathfrak{Q})$ — i.e. we identify each $\mathfrak{Q}$-submodule with a subset of S of M such that the *inclusion map* $S \hookrightarrow M$ is a *right* $\mathfrak{Q}$*-module morphism*, or, equivalently, if S is closed under arbitrary sups and $m \boxdot \alpha \in S$ for each $\alpha \in \mathfrak{Q}$ and $m \in S$.

Remark 3.1. For the convenience of the reader we recall that a $\mathfrak{Q}$-*coclosure operator* j on the $\mathfrak{Q}$-preordered set (M, p) is a $\mathfrak{Q}^{op}$-functor $(M, p) \xrightarrow{j} (M, p)$ and a traditional coclosure operator w.r.t. the underlying preorder $\leq_p$. If we now associate the intrinsic $\mathfrak{Q}$-preorder with every right $\mathfrak{Q}$-module, then it can be shown that $\mathfrak{Q}$-coclosure operators and $\mathfrak{Q}$-submodules of right $\mathfrak{Q}$-modules are equivalent concepts (cf. [13, p. 986]). In fact, if p is the intrinsic $\mathfrak{Q}$-preorder of a right $\mathfrak{Q}$-module M, then an isotone map $M \xrightarrow{j} M$ w.r.t. $\leq_p$ (which coincides with the order in M) is a $\mathfrak{Q}^{op}$-functor $(M, p) \xrightarrow{j} (M, p)$ if and only if j satisfies the property $j(m) \boxdot \alpha \leq j(m \boxdot \alpha)$ for all $\alpha \in \mathfrak{Q}$ and $m \in M$ (cf. [7, Prop. 3.3.23]). Hence, if j is a $\mathfrak{Q}$-coclosure operator on the associated $\mathfrak{Q}$-preordered set (M, p) of a right $\mathfrak{Q}$-module M, then

$$S = \{m \in M \mid m \leq j(m)\}$$

is a right $\mathfrak{Q}$-submodule of M, and vice verse, if S is a right $\mathfrak{Q}$-submodule of M, then $M \xrightarrow{j} M$ defined by

$$j(m) = \bigvee\{n \in S \mid n \leq m\}, \qquad m \in M \tag{3.5}$$

is a traditional coclosure operator. Since S is a right $\mathfrak{Q}$-submodule, the traditional coclosure operator j is also a $\mathfrak{Q}^{op}$-functor, and in particular S is induced by j.

3.1. The free right $\mathfrak{Q}$-module

Let X be a set and $P(X)$ be the power set of X — i.e. the *free complete lattice on* X in the sense of Sup. Further, let $\mathfrak{Q}$ be a unital quantale and $\mathfrak{Q}^X$ be provided with the pointwise order induced by the order on $\mathfrak{Q}$ and with the right multiplication on $\mathfrak{Q}^X$ as right action — i.e.

$$(f * \alpha)(x) = f(x) * \alpha, \qquad f \in \mathfrak{Q}^X,\ \alpha \in \mathfrak{Q},\ x \in X.$$

Then there exists a right $\mathfrak{Q}$-module isomorphism $\mathfrak{Q}^X \xrightarrow{\Phi} P(X) \otimes \mathfrak{Q}$ defined by:

$$\Phi(f) = \bigvee_{x \in X} \{x\} \otimes f(x), \qquad f \in \mathfrak{Q}^X.$$

In the following we will use the following notation for each $A \in P(X)$:

$$1_A(x) := \big(\Phi^{-1}(A \otimes e)\big)(x) = \begin{cases} e, & \text{if } x \in A, \\ \bot, & \text{if } x \notin A, \end{cases} \qquad x \in X. \tag{3.6}$$

Since $P(X) \otimes \mathfrak{Q}$ is the *free right* $\mathfrak{Q}$*-module on* $P(X)$ and the power set functor is left adjoint to the forgetful functor $\mathsf{Sup} \to \mathsf{Set}$, we can also understand $(\mathfrak{Q}^X, *)$ as the *free right* $\mathfrak{Q}$*-module on* X (cf. [20, p. 10]). The corresponding intrinsic $\mathfrak{Q}$-preorder d of $\big(\mathfrak{Q}^X, *\big)$ (see (3.1)) attains the form:

$$d(f_1, f_2) = \bigwedge_{x \in X} f_1(x) \searrow f_2(x), \qquad f_1, f_2 \in \mathfrak{Q}^X.$$

Finally, an isotone map $\mathfrak{Q}^X \xrightarrow{\mathcal{I}} \mathfrak{Q}^X$ is a $\mathfrak{Q}^{op}$-functor $\big(\mathfrak{Q}^X, d\big) \xrightarrow{\mathcal{I}} \big(\mathfrak{Q}^X, d\big)$ if and only if $\mathcal{I}$ satisfies the following condition:

(I0) If $f \in \mathfrak{Q}^X$ and $\alpha \in \mathfrak{Q}$, then $\mathcal{I}(f) * \alpha \leq \mathcal{I}(f * \alpha)$.

3.2. A right $\mathfrak{Q}$-module on the dual lattice of $\mathfrak{Q}^X$

Since $\mathfrak{Q}^X$ is also a left $\mathfrak{Q}$-module w.r.t. the left multiplication — i.e.

$$(\alpha * f)(x) = \alpha * f(x), \qquad f \in \mathfrak{Q}^X,\ \alpha \in \mathfrak{Q},\ x \in X,$$

the right action $\boxdot^{op}$ on the dual lattice of $\mathfrak{Q}^X$ (see (3.3)) is determined by:

$$(f \boxdot^{op} \alpha)(x) = \alpha \searrow f(x), \qquad f \in \mathfrak{Q}^X,\ \alpha \in \mathfrak{Q},\ x \in X.$$

Now let $\leq^{op}$ be the dual order of the pointwise order on $\mathfrak{Q}^X$. Then the intrinsic $\mathfrak{Q}$-preorder $d^\dagger$ of $((\mathfrak{Q}^X, \leq^{op}), \boxdot^{op})$ (see (3.4)) attains the form:

$$d^\dagger(f_1, f_2) = \bigwedge_{x \in X} f_1(x) \swarrow f_2(x), \qquad f_1, f_2 \in \mathfrak{Q}^X.$$

If $\mathfrak{Q}$ has a dualizing element δ, then the right action $\boxdot^{op}$ can also be expressed as follows (cf. Remark 2.2)[1]

$$f \boxdot^{op} \alpha = \alpha \searrow f = f *_\delta (\alpha \searrow \delta). \tag{3.7}$$

Hence, if δ is a dualizing element of $\mathfrak{Q}$, a subset $\mathcal{C}$ of $\mathfrak{Q}^X$ is a right $\mathfrak{Q}$-submodule of $((\mathfrak{Q}^X, \leq^{op}), \boxdot^{op})$ if it is closed w.r.t. arbitrary meets in the sense of $\mathfrak{Q}^X$ and satisfies the following condition:

(C0) If $f \in \mathfrak{Q}^X$ and $\alpha \in \mathfrak{Q}$, then $f *_\delta \alpha \in \mathcal{C}$.

Since for isotone maps $\mathfrak{Q}^X \xrightarrow{\mathcal{A}} \mathfrak{Q}^X$ with respect to the pointwise order on $\mathfrak{Q}^X$ the equivalence

$$\alpha \searrow \mathcal{A}(f) \leq^{op} \mathcal{A}(\alpha \searrow f) \iff \alpha * \mathcal{A}(f) \leq \mathcal{A}(\alpha * f)$$

holds — here the right side refers to the left action on $\mathfrak{Q}^X$, an isotone map $\mathfrak{Q}^X \xrightarrow{\mathcal{A}} \mathfrak{Q}^X$ is a $\mathfrak{Q}^{op}$-functor $(\mathfrak{Q}^X, d^\dagger) \xrightarrow{\mathcal{A}} (\mathfrak{Q}^X, d^\dagger)$ if and only if $\mathcal{A}$ satisfies the following condition:

(A0) If $f \in \mathfrak{Q}^X$ and $\alpha \in \mathfrak{Q}$, then $\alpha * \mathcal{A}(f) \leq \mathcal{A}(\alpha * f)$.

We have now the following results:

Lemma 3.2. *Every $\mathfrak{Q}^{op}$-functor $(\mathfrak{Q}^X, d) \xrightarrow{\mathcal{I}} (\mathfrak{Q}^X, d)$ induces a $\mathfrak{Q}^{op}$-functor $(\mathfrak{Q}^X, d^\dagger) \xrightarrow{\mathcal{A}_\mathcal{I}} (\mathfrak{Q}^X, d^\dagger)$ by:*

$$\mathcal{A}_\mathcal{I}(f)(x) = \bigwedge_{g \in \mathfrak{Q}^X} \Big(\big(\bigvee_{y \in X} (f(y) * g(y)) \big) \swarrow \mathcal{I}(g)(x) \Big), \quad f \in \mathfrak{Q}^X,\ x \in X, \tag{3.8}$$

and every $\mathfrak{Q}^{op}$-functor $(\mathfrak{Q}^X, d^\dagger) \xrightarrow{\mathcal{A}} (\mathfrak{Q}^X, d^\dagger)$ induces a $\mathfrak{Q}^{op}$-functor $(\mathfrak{Q}^X, d) \xrightarrow{\mathcal{I}_\mathcal{A}} (\mathfrak{Q}^X, d)$ by:

$$\mathcal{I}_\mathcal{A}(f)(x) = \bigwedge_{g \in \mathfrak{Q}^X} \Big(\mathcal{A}(g)(x) \searrow \big(\bigvee_{y \in X} (g(y) * f(y)) \big) \Big), \quad f \in \mathfrak{Q}^X,\ x \in X. \tag{3.9}$$

If $\mathcal{I}$ is a $\mathfrak{Q}$-coclosure operator, then $\mathcal{A}_\mathcal{I}$ is also a $\mathfrak{Q}$-coclosure operator.

Lemma 3.3. *Let $\mathfrak{Q}$ be a unital quantale with a dualizing element δ. Further, let $(\mathfrak{Q}^X, d) \xrightarrow{\mathcal{I}} (\mathfrak{Q}^X, d)$ and $(\mathfrak{Q}^X, d^\dagger) \xrightarrow{\mathcal{A}} (\mathfrak{Q}^X, d^\dagger)$ be $\mathfrak{Q}^{op}$-functors. Then:*

(1) *$\delta \swarrow \mathcal{I}(f \searrow \delta)(x) = \mathcal{A}_\mathcal{I}(f)(x)$ for each $f \in \mathfrak{Q}^X$ and $x \in X$.*
(2) *$\mathcal{A}(\delta \swarrow f)(x) \searrow \delta = \mathcal{I}_\mathcal{A}(f)(x)$ for each $f \in \mathfrak{Q}^X$ and $x \in X$.*

[1] Since the unital quantales $\mathfrak{Q}$ and $\mathfrak{Q}_\delta$ are isomorphic (see Remark 2.2 (3)), the formula (3.7) reflects the isomorphism between $\mathsf{Mod}_r(\mathfrak{Q})$ and $\mathsf{Mod}_r(\mathfrak{Q}_\delta)$. But here our intention is *not* to replace $\mathsf{Mod}_r(\mathfrak{Q})$ by $\mathsf{Mod}_r(\mathfrak{Q}_\delta)$.

Proof. Let δ be a dualizing element, $x \in X$ and $f \in \mathfrak{Q}^X$. Since $\mathcal{I}$ is a $\mathfrak{Q}^{op}$-functor, we obtain:

$$\begin{aligned}\mathcal{I}(f \searrow \delta)(x) &= \bigvee_{g\in\mathfrak{Q}^X} \mathcal{I}(g)(x) * d(g, f \searrow \delta)\\ &= \bigvee_{g\in\mathfrak{Q}^X} \mathcal{I}(g)(x) * \Big(\bigwedge_{y\in X} g(y) \searrow (f(y) \searrow \delta)\Big)\\ &= \bigvee_{g\in\mathfrak{Q}^X} \mathcal{I}(g)(x) * \Big(\Big(\bigvee_{y\in X} (f(y) * g(y))\Big) \searrow \delta\Big)\\ &= \bigvee_{g\in\mathfrak{Q}^X} \Big(\Big(\delta \swarrow \Big(\mathcal{I}(g)(x) * \Big(\Big(\bigvee_{y\in X} (f(y) * g(y))\Big) \searrow \delta\Big)\Big)\Big) \searrow \delta\Big)\\ &= \bigvee_{g\in\mathfrak{Q}^X} \Big(\Big(\delta \swarrow \Big(\Big(\bigvee_{y\in X} (f(y) * g(y))\Big) \searrow \delta\Big)\Big) \swarrow \mathcal{I}(g)(x)\Big) \searrow \delta\\ &= \bigvee_{g\in\mathfrak{Q}^X} \Big(\Big(\bigvee_{y\in X} (f(y) * g(y))\Big) \swarrow \mathcal{I}(g)(x)\Big) \searrow \delta = \mathcal{A}_{\mathcal{I}}(f)(x) \searrow \delta.\end{aligned}$$

Hence the relation (1) follows. If $\mathcal{A}$ is a $\mathfrak{Q}^{op}$-functor, then $\mathcal{A}(\delta \swarrow f)(x) = \delta \swarrow \mathcal{I}_{\mathcal{A}}(f)(x)$ holds analogously, and so also the relation (2) is verified. □

Comment. Note that the expressions on the left sides are independent from the chosen dualizing element δ.

We can summarize the previous results as follows. If the quantale $\mathfrak{Q}$ has a dualizing element, then we conclude from Lemma 3.3 that the correspondence $\mathcal{I} \longmapsto \mathcal{A}_{\mathcal{I}}$ is bijective and its inverse correspondence is given by $\mathcal{A} \longmapsto \mathcal{I}_{\mathcal{A}}$. Moreover, the restriction of this bijection to the respective $\mathfrak{Q}$-coclosure operators on $(\mathfrak{Q}^X, d)$ and $\big(\mathfrak{Q}^X, d^{\dagger}\big)$ remains a bijection. In particular, for any dualizing element δ of $\mathfrak{Q}$ and for all $f \in \mathfrak{Q}^X$ the following relations hold:

$$\mathcal{A}_{\mathcal{I}}(f) = \delta \swarrow \mathcal{I}(f \searrow \delta) \quad \text{and} \quad \mathcal{I}_{\mathcal{A}}(f) = \mathcal{A}(\delta \swarrow f) \searrow \delta. \tag{3.10}$$

4. $\mathfrak{Q}$-topological spaces and closed $\mathfrak{Q}$-presheaves

Notation. If $\alpha \in \mathfrak{Q}$, then $\underline{\alpha} \in \mathfrak{Q}^X$ is the constant map with value α.

Standing Assumption. In what follows $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ is a unital quantale and $(\mathfrak{Q}, \diamond)$ is always a strict quasi-magma.

Let X be a set and $\mathfrak{Q}^X$ be the free right $\mathfrak{Q}$-module on X (cf. Subsection 3.1). Then a $\mathfrak{Q}$-*topology* on X is a right $\mathfrak{Q}$-submodule $\mathcal{T}$ of $\mathfrak{Q}^X$ (i.e. a subset $\mathcal{T}$ of $\mathfrak{Q}^X$ closed under arbitrary sups and such that $f * \alpha \in \mathcal{T}$ for $\alpha \in \mathfrak{Q}$, $f \in \mathcal{T}$), which satisfies additionally the following topological axioms:

(T1) $\underline{\top}$ is an element of $\mathcal{T}$.
(T2) If $f_1, f_2 \in \mathcal{T}$, then $f_1 \diamond f_2 \in \mathcal{T}$, (where $f_1 \diamond f_2$ is defined pointwisely).

The pair $(X, \mathcal{T})$ is called a $\mathfrak{Q}$-*topological space* and a $\mathfrak{Q}$-presheaf $f \in \mathfrak{Q}^X$ is said to be *open* if $f \in \mathcal{T}$.

A subset $\mathcal{B}$ of a $\mathfrak{Q}$-topology on X is called a *base* of $\mathcal{T}$ if each open $\mathfrak{Q}$-presheaf of $\mathcal{T}$ is a join of elements of $\mathcal{B}$. Evidently the axioms of a $\mathfrak{Q}$-topology are preserved under arbitrary intersections. Hence a subset $\mathcal{S}$ of $\mathfrak{Q}^X$ is called a *subbase* of a $\mathfrak{Q}$-topology $\mathcal{T}$ on X if the following relation holds:

$$\mathcal{T} = \bigcap\{\, \mathcal{W} \mid \mathcal{S} \subseteq \mathcal{W} \subseteq \mathfrak{Q}^X \text{ and } \mathcal{W} \text{ is a } \mathfrak{Q}\text{-topology on } X \,\}.$$

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $\mathfrak{Q}$-topological spaces. A map $X \xrightarrow{\varphi} Y$ is $\mathfrak{Q}$-*continuous* if $f \circ \varphi \in \mathcal{T}_X$ for all $f \in \mathcal{T}_Y$. Note that if $\mathcal{S}$ is a subbase of $\mathcal{T}_Y$ then φ is $\mathfrak{Q}$-continuous if and only if $f \circ \varphi \in \mathcal{T}_X$ for all $f \in \mathcal{S}$. Obviously, $\mathfrak{Q}$-topological spaces and $\mathfrak{Q}$-continuous maps form a category $\mathsf{Top}(\mathfrak{Q}, \diamond)$ being topological over Set. Thus, if $(X, \mathcal{T}_X)$ is a $\mathfrak{Q}$-topological space and $Y \xrightarrow{\varphi} X$ a map, then $\mathcal{T}_{\text{initial}} = \{\, f \circ \varphi \mid f \in \mathcal{T}_X \,\}$ is the *initial* $\mathfrak{Q}$-*topology* on Y induced by φ, and otherwise, if $X \xrightarrow{\varphi} Y$, then $\mathcal{T}_{\text{final}} = \{\, f \in \mathfrak{Q}^Y \mid f \circ \varphi \in \mathcal{T}_X \,\}$ is the *final* $\mathfrak{Q}$-*topology* on Y induced by φ.

A $\mathfrak{Q}$-*interior operator* on a set X is a $\mathfrak{Q}$-coclosure operator $\mathcal{I}$ on $(\mathfrak{Q}^X, d)$ satisfying the following additional properties:

(I1) $\mathcal{I}(\underline{\top}) = \underline{\top}$,
(I2) If $f_1, f_2 \in \mathfrak{Q}^X$, then $\mathcal{I}(f_1) \diamond \mathcal{I}(f_2) \le \mathcal{I}(f_1 \diamond f_2)$.

An element $x \in X$ is an *interior point* of a $\mathfrak{Q}$-presheaf f on X (resp. f is a *neighborhood* of x), if $e \le \mathcal{I}(f)(x)$. The set of all neighborhoods of x is denoted by $\mathbb{U}_x$. A subset $\mathbb{B}_x$ of $\mathbb{U}_x$ is called a *neighborhood base* of x if for all $f \in \mathbb{U}_x$ the relation $e \le \bigvee_{b \in \mathbb{B}_x} d(b, f)$ holds. In particular the subset $\mathbb{B}_x = \{ b \in \mathcal{T} \mid e \le b(x) \}$ is always a neighborhood base of x.

Since $\mathfrak{Q}$-coclosure operators on $(\mathfrak{Q}^X, d)$ and right $\mathfrak{Q}$-submodules of $(\mathfrak{Q}^X, *)$ are equivalent concepts (cf. Remark 3.1), it is easily seen that every $\mathfrak{Q}$-interior operator $\mathcal{I}$ on X can be identified with a $\mathfrak{Q}$-topology $\mathcal{T}$ on X and vice verse. In this context the following relation holds (see (3.5)):

$$\mathcal{I}(f) = \bigvee \{ g \in \mathcal{T} \mid g \le f \}, \qquad f \in \mathfrak{Q}^X. \tag{4.1}$$

By Lemma 3.2, every $\mathfrak{Q}$-interior operator $\mathcal{I}$ on X induces a $\mathfrak{Q}$-coclosure operator $\mathcal{A}_{\mathcal{I}}$ (see (3.8)). We call $\mathcal{A}_{\mathcal{I}}$ the $\mathfrak{Q}$-*adherence operator* associated with $\mathcal{I}$. Let $\mathcal{T}$ be the $\mathfrak{Q}$-topology on X given by a $\mathfrak{Q}$-interior operator $\mathcal{I}$. Then for any base $\mathcal{B}$ of $\mathcal{T}$ the $\mathfrak{Q}$-adherence operator $\mathcal{A}_{\mathcal{I}}$ associated with $\mathcal{I}$ is already determined by:

$$\mathcal{A}_{\mathcal{I}}(f)(x) = \bigwedge_{g \in \mathcal{B}} \Big(\big(\bigvee_{y \in X} (f(y) * g(y)) \big) \swarrow g(x) \Big), \qquad x \in X,\ f \in \mathfrak{Q}^X. \tag{4.2}$$

Since the underlying preorder of the $\mathfrak{Q}$-preorder $d^{\dagger}$ is the dual order of $\mathfrak{Q}^X$, we introduce the following terminology:

Definition 4.1. Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space, and let $\mathcal{A}_{\mathcal{I}}$ be the $\mathfrak{Q}$-adherence operator associated with the $\mathfrak{Q}$-interior operator $\mathcal{I}$. Further, let $g \in \mathfrak{Q}^X$, $x \in X$ and $A \subseteq X$ (and we refer to (3.6) and identify A with the $\mathfrak{Q}$-presheaf 1_A on X). Then:

(1) g is a *closed* $\mathfrak{Q}$-presheaf on X if $\mathcal{A}_{\mathcal{I}}(g) \le g$ and x is an *adherent point* of g if $e \le \mathcal{A}_{\mathcal{I}}(g)(x)$.
(2) A is *dense* in $(X, \mathcal{T})$ if every $x \in X$ is an adherent point of 1_A — i.e. if $\underline{e} \le \mathcal{A}_{\mathcal{I}}(1_A)$.

If $\mathcal{B}$ is a base of $\mathcal{T}$, then it follows immediately from (4.2) that a subset A of X is dense in $(X, \mathcal{T})$ if and only if the following relation holds:

$$f(x) \le \bigvee_{a \in A} f(a), \qquad f \in \mathcal{B},\ x \in X. \tag{4.3}$$

Hence in this context the concept of density of subsets has its origin in Lowen's work on connectedness in many-valued topological spaces (cf. [26]).

Example 4.2. Let $\mathfrak{Q}$ be an integral quantale with the quasi-magma operation $\diamond = *$. We consider the following $\mathfrak{Q}$-presheaves f_1 and f_2 on $\mathfrak{Q}$:

$$f_1(\alpha) = \alpha \quad \text{and} \quad f_2(\alpha) = (\bot \swarrow \alpha) \wedge (\alpha \searrow \bot), \qquad \alpha \in \mathfrak{Q}.$$

Now let $\mathcal{T}$ be the $\mathfrak{Q}$-topology on $\mathfrak{Q}$ generated by $\{ f_1, f_2 \}$. Since $\mathfrak{Q}$ is integral and $f_1 * f_2 = f_2 * f_1 = \underline{\bot}$, we conclude that the subset $\{ \bot, \top \}$ is dense in $(\mathfrak{Q}, \mathcal{T})$.

Lemma 4.3. *Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space and $\mathcal{A}_{\mathcal{I}}$ be the $\mathfrak{Q}$-adherence operator associated with the $\mathfrak{Q}$-interior operator $\mathcal{I}$. If A is dense in $(X, \mathcal{T})$ and the quasi-magma operation is given by $\diamond = *$, then $\mathcal{A}_{\mathcal{I}}(f * 1_A) = \mathcal{A}_{\mathcal{I}}(f)$ for each $f \in \mathcal{T}$.*

Proof. The result follows immediately from (4.3) and (T2), since

$$\mathcal{A}_{\mathcal{I}}(f * 1_A) = \bigwedge_{g \in \mathcal{T}} \Big(\big(\bigvee_{y \in A} (f(y) * g(y)) \big) \swarrow g(x) \Big) = \bigwedge_{g \in \mathcal{T}} \Big(\big(\bigvee_{y \in X} (f(y) * g(y)) \big) \swarrow g(x) \Big) = \mathcal{A}_{\mathcal{I}}(f). \qquad \square$$

When the quantale has a dualizing element, a characterization of closed $\mathfrak{Q}$-presheaves can be given as follows.

Lemma 4.4. *Let $\mathfrak{Q}$ be a quantale with a dualizing element δ and $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space. A $\mathfrak{Q}$-presheaf g on X is closed if and only if $g \searrow \delta$ is open — i.e. $g \searrow \delta \in \mathcal{T}$.*

Proof. (Necessity). Let g be a closed $\mathfrak{Q}$-presheaf on X. Then we conclude from (4.1) and (3.8) that g satisfies the following condition:

$$g(x) = \bigwedge_{f \in \mathcal{T}} \Big(\big(\bigvee_{y \in X} (g(y) * f(y)) \big) \swarrow f(x) \Big), \qquad x \in X.$$

Now we apply a dualizing element δ of $\mathfrak{Q}$ and obtain $g(x) = \bigwedge_{f \in \mathcal{T}} \big(\delta \swarrow \big(f(x) * ((\bigvee_{y \in X} g(y) * f(y)) \searrow \delta)\big)\big)$. Hence the relation $g(x) \searrow \delta = \bigvee_{f \in \mathcal{T}} \big(f(x) * ((\bigvee_{y \in X} g(y) * f(y)) \searrow \delta)\big)$ follows. Since $\mathcal{T}$ is a right $\mathfrak{Q}$-submodule of $(\mathfrak{Q}^X, *)$, $g \searrow \delta$ is open.
(b) (Sufficiency) If $g \searrow \delta$ open and $x \in X$, then we observe:

$$\mathcal{A}_{\mathcal{I}}(g)(x) \le \big(\bigvee_{y \in X} g(y) * (g(y) \searrow \delta) \big) \swarrow (g(x) \searrow \delta) \le \delta \swarrow (g(x) \searrow \delta) = g(x).$$

Hence $\mathcal{A}_{\mathcal{I}}(g) \le g$ follows — i.e. g is closed. □

Proposition 4.5. *Let $\mathfrak{Q}$ be a unital quantale with a dualizing element and let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space. Then the $\mathfrak{Q}$-adherence operator $\mathcal{A}_{\mathcal{I}}$ associated with the $\mathfrak{Q}$-interior operator $\mathcal{I}$ satisfies the following conditions*:

(A1) $\mathcal{A}_{\mathcal{I}}(\underline{\perp}) = \underline{\perp}$.
(A2) *If $g_1, g_2 \in \mathfrak{Q}^X$, then $\mathcal{A}_{\mathcal{I}}(g_1 \diamond_\delta g_2) \le \mathcal{A}_{\mathcal{I}}(g_1) \diamond_\delta \mathcal{A}_{\mathcal{I}}(g_2)$ for every dualizing element $\delta \in \mathfrak{Q}$.*

Proof. The property (A1) is obvious. In order to verify (A2) we choose some dualizing element $\delta \in \mathfrak{Q}$, refer to (3.10) and obtain:

$$\begin{aligned} \mathcal{A}_{\mathcal{I}}(g_1 \diamond_\delta g_2) &= \delta \swarrow \mathcal{I}((g_1 \diamond_\delta g_2) \searrow \delta) = \delta \swarrow \mathcal{I}((g_1 \searrow \delta) \diamond (g_2 \searrow \delta)) \\ &\le \delta \swarrow \big(\mathcal{I}(g_1 \searrow \delta) \diamond \mathcal{I}(g_2 \searrow \delta)\big) = \delta \swarrow \big((\mathcal{A}_{\mathcal{I}}(g_1) \searrow \delta) \diamond \mathcal{A}_{\mathcal{I}}(g_2) \searrow \delta)\big) = \mathcal{A}_{\mathcal{I}}(g_1) \diamond_\delta \mathcal{A}_{\mathcal{I}}(g_2). \end{aligned}$$ □

As an immediate corollary from (3.10) and Proposition 4.5 we point out that in the case of quantales with a dualizing element $\mathfrak{Q}$-interior operators and $\mathfrak{Q}$-adherence operators satisfying (A1) and (A2) are equivalent concepts. If we now make use of the right $\mathfrak{Q}$-module $\big((\mathfrak{Q}^X, \le^{op}), \boxdot^{op}\big)$ on the dual lattice of $\mathfrak{Q}^X$ (cf. Subsection 3.2), then we can give a further characterization of closed $\mathfrak{Q}$-presheaves.

Proposition 4.6. *Let $\mathfrak{Q}$ be a unital quantale with a dualizing element. A right $\mathfrak{Q}$-submodule $\mathcal{C}$ of $\big((\mathfrak{Q}^X, \le^{op}), \boxdot^{op}\big)$ is a collection of closed $\mathfrak{Q}$-presheaves on a set X for some $\mathfrak{Q}$-topology $\mathcal{T}$ if and only if $\mathcal{C}$ satisfies the following conditions*:

(C1) *$\underline{\perp}$ is an element of $\mathcal{C}$.*
(C2) *If $f_1, f_2 \in \mathcal{C}$, then $f_1 \diamond_\delta f_2 \in \mathcal{C}$ for every dualizing element $\delta \in \mathfrak{Q}$.*

5. Lower separation axioms

Definition 5.1. A $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ is said to be

- *Kolmogorov separated* (or T_0), if for each $x, y \in X$ with $x \neq y$ there exists some $f \in \mathcal{T}$ such that $f(x) \not\le f(y)$ or $f(y) \not\le f(x)$.
- *Fréchet separated* (or T_1) if for each $x, y \in X$ with $x \neq y$ there exist $f_1 \in \mathcal{T}$ with $f_1(x) \not\le f_1(y)$ *and* $f_2 \in \mathcal{T}$ with $f_2(y) \not\le f_2(x)$.

– *Hausdorff separated* (or T_2) if for each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{T}$ with

$$f_1(x) \diamond f_2(y) \not\leq \bigvee_{z\in X} (f_1(z) \diamond f_2(z)) \quad \text{or} \quad f_2(y) \diamond f_1(x) \not\leq \bigvee_{z\in X} (f_2(z) \diamond f_1(z)). \tag{5.1}$$

Obviously, each Fréchet separated $\mathfrak{Q}$-topological space is also Kolmogorov separated, and it is easy to check that each Hausdorff separated $\mathfrak{Q}$-topological space is also Fréchet separated. Indeed, let $(X, \mathcal{T})$ be Hausdorff separated and $x, y \in X$ with $x \neq y$. Then there exist open $\mathfrak{Q}$-presheaves f_1, f_2 such that $f_1(x) \diamond f_2(y) \not\leq \bigvee_{z\in X}(f_1(z) \diamond f_2(z))$ or $f_2(y) \diamond f_1(x) \not\leq \bigvee_{z\in X}(f_2(z) \diamond f_1(z))$. Since $\mathfrak{Q} \times \mathfrak{Q} \xrightarrow{\diamond} \mathfrak{Q}$ is isotone, it follows that $f_1(x) \not\leq f_1(y)$ and $f_2(y) \not\leq f_2(x)$.

Remark 5.2. With regard to the categorical behavior, we point out that the full subcategories of Kolmogorov (resp. Fréchet, Hausdorff) separated $\mathfrak{Q}$-topological spaces are reflective in $\mathsf{Top}(\mathfrak{Q}, \diamond)$. In particular, they are all closed under products.

Indeed, one can construct the reflectors as follows. If $(X, \mathcal{T}_X)$ is a $\mathfrak{Q}$-topological space, we consider the equivalence relation on X given by $x \sim y$ if and only if $\varphi(x) = \varphi(y)$ for any $\mathfrak{Q}$-continuous $\varphi\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ such that $(Y, \mathcal{T}_Y)$ satisfies the Kolmogorov (resp. Fréchet, Hausdorff) separation axiom. Then we endow $X/\!\sim$ with the final $\mathfrak{Q}$-topology induced by the quotient map $p_X\colon X \to X/\!\sim$. Hence for every $\mathfrak{Q}$-continuous $\varphi\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ having a Kolmogorov (resp. Fréchet, Hausdorff) separated codomain there is a unique $\mathfrak{Q}$-continuous $\widehat{\varphi}\colon (X/\!\sim, \mathcal{T}_{\text{final}}) \to (Y, \mathcal{T}_Y)$ such that $\varphi = \widehat{\varphi} \circ p_X$. The only part remaining is to show that $(X/\!\sim, \mathcal{T}_{\text{final}})$ satisfies the Kolmogorov (resp. Fréchet, Hausdorff) axiom. We prove it for the Hausdorff case (the other cases follow similarly). In order to do so, assume that $p_X(x) \neq p_X(y)$ in $X/\!\sim$. Then there is a $\mathfrak{Q}$-continuous map $\varphi\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ with $\varphi(x) \neq \varphi(y)$. Since $(Y, \mathcal{T}_Y)$ is Hausdorff separated, there are $f_1, f_2 \in \mathcal{T}_Y$ with $f_1(\varphi(x)) \diamond f_2(\varphi(y)) \not\leq \bigvee_{z\in Y}(f_1(z) \diamond f_2(z))$ or $f_2(\varphi(y)) \diamond f_1(\varphi(x)) \not\leq \bigvee_{z\in Y}(f_2(z) \diamond f_1(z))$. Assume that the former holds. Then, in particular, one has

$$(f_1 \circ \hat{\varphi})(p_X(x)) \diamond (f_2 \circ \hat{\varphi})(p_X(y)) \not\leq \bigvee_{z\in X} ((f_1 \circ \hat{\varphi})(p_X(z)) \diamond (f_2 \circ \hat{\varphi})(p_X(z)))$$

and since $f_1 \circ \hat{\varphi}, f_2 \circ \hat{\varphi} \in \mathcal{T}_{\text{final}}$, it follows that $(X/\!\sim, \tau_{\text{final}})$ is Hausdorff separated.

With regard to the next lemmas we refer to the notation introduced in (3.6) and (3.8).

Lemma 5.3. *Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space and $\mathcal{A}_{\mathcal{I}}$ be the corresponding adherence operator. If $x, y \in X$, then the following assertions are equivalent*:

(i) $f(x) \leq f(y)$ *for all* $f \in \mathcal{T}$;
(ii) $\mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{x\}}) \leq \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{y\}})$ *for all* $\alpha \in \mathfrak{Q}$;
(iii) $\alpha * 1_{\{x\}} \leq \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{y\}})$ *for all* $\alpha \in \mathfrak{Q}$;
(iv) $1_{\{x\}} \leq \mathcal{A}_{\mathcal{I}}(1_{\{y\}})$;
(v) $\mathcal{A}_{\mathcal{I}}(1_{\{x\}}) \leq \mathcal{A}_{\mathcal{I}}(1_{\{y\}})$.

Proof. The assertion follows immediately from (3.8) and (4.1). □

Now we can give a characterization of the Kolmogorov and Fréchet separation axioms as follows.

Lemma 5.4. *Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space and $\mathcal{A}_{\mathcal{I}}$ be the corresponding adherence operator. Then the following assertions are equivalent*:

(i) *$(X, \mathcal{T})$ is Kolmogorov separated.*
(ii) *$\mathcal{A}_{\mathcal{I}}(1_{\{x\}}) \neq \mathcal{A}_{\mathcal{I}}(1_{\{y\}})$ for each $x, y \in X$ with $x \neq y$.*
(iii) *For each $x, y \in X$ with $x \neq y$ there exists $\alpha \in \mathfrak{Q}$ such that $\alpha * 1_{\{x\}} \not\leq \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{y\}})$ or $\alpha * 1_{\{y\}} \not\leq \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{x\}})$.*
(iv) *For each $x, y \in X$ with $x \neq y$ there exists $\alpha \in \mathfrak{Q}$ such that $\mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{x\}}) \neq \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{y\}})$.*
(v) *For each subbase $\mathcal{S}$ of $\mathcal{T}$ and for each $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{S}$ such that $f(x) \not\leq f(y)$ or $f(y) \not\leq f(x)$.*

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) follows immediately from Lemma 5.3, and (v) ⇒ (i) is obvious.
(iv) ⇒ (v): Let $\mathcal{S}$ be a subbase of $\mathcal{T}$ and $x, y \in X$ with $x \neq y$. Now we consider the $\mathfrak{Q}$-topology on X given by

$$\mathcal{W}_{x,y} = \{ f \in \mathfrak{Q}^X \mid f(x) = f(y) \}.$$

If we assume $\mathcal{S} \subseteq \mathcal{W}_{x,y}$, then the subbase property of $\mathcal{S}$ implies $\mathcal{T} \subseteq \mathcal{W}_{x,y}$ and so Lemma 5.3 implies $\mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{x\}}) = \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{y\}})$, contradicting (iv). Hence $\mathcal{S} \cap \big(\complement \mathcal{W}_{x,y}\big) \neq \varnothing$ — i.e. there exists $f \in \mathcal{S}$ such that $f(x) \neq f(y)$. □

Lemma 5.5. *Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space and $\mathcal{A}_{\mathcal{I}}$ be the corresponding adherence operator. Then the following assertions are equivalent*:

(i) *$(X, \mathcal{T})$ is Fréchet separated.*
(ii) *$1_{\{x\}} \not\leq \mathcal{A}_{\mathcal{I}}(1_{\{y\}})$ and $1_{\{y\}} \not\leq \mathcal{A}_{\mathcal{I}}(1_{\{x\}})$ for each $x, y \in X$ with $x \neq y$.*
(iii) *If $x, y \in X$ with $x \neq y$, then there exist $\alpha \in \mathfrak{Q}$ such that $\alpha * 1_{\{x\}} \not\leq \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{y\}})$ and $\beta \in \mathfrak{Q}$ such that $\beta * 1_{\{y\}} \not\leq \mathcal{A}_{\mathcal{I}}(\beta * 1_{\{x\}})$.*
(iv) *For each subbase $\mathcal{S}$ of $\mathcal{T}$ and for each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{S}$ such that $f_1(x) \not\leq f_1(y)$ and $f_2(y) \not\leq f_2(x)$.*

Proof. (i) ⇒ (ii) ⇒ (iii) follows immediately from Lemma 5.3, and (iv) ⇒ (i) is obvious.
(iii) ⇒ (iv): Let $\mathcal{S}$ be a subbase of $\mathcal{T}$ and $x, y \in X$ with $x \neq y$. Now we consider the family

$$\mathcal{T}_{x,y} = \{ f \in \mathfrak{Q}^X \mid f(x) \leq f(y) \},$$

and conclude from the isotonicity of $\diamond$ that $\mathcal{T}_{x,y}$ is a $\mathfrak{Q}$-topology on X. If we assume $\mathcal{S} \subseteq \mathcal{T}_{x,y}$, then the subbase property of $\mathcal{S}$ implies $\mathcal{T} \subseteq \mathcal{T}_{x,y}$, and so by Lemma 5.3 we have $\alpha * 1_{\{x\}} \leq \mathcal{A}_{\mathcal{I}}(\alpha * 1_{\{y\}})$ for all $\alpha \in \mathfrak{Q}$, contradicting (iii). Hence there exists $f_1 \in \mathcal{S}$ such that $f_1(x) \not\leq f_1(y)$. Analogously we proceed with the $\mathfrak{Q}$-topology $\mathcal{T}_{y,x}$ and conclude that there exists $f_2 \in \mathcal{S}$ such that $f_2(y) \not\leq f_2(x)$. □

In order to reveal the $\mathfrak{Q}$-enriched categorical background of the Kolmogorov and Fréchet separation axiom for $\mathfrak{Q}$-enriched topological spaces we briefly touch the specialization $\mathfrak{Q}$-preorder.

Let us recall that a quantale $\mathfrak{Q}$ is said to be *involutive* if it is provided with an order-preserving involution $'$ such that $(\alpha * \beta)' = \beta' * \alpha'$ for all $\alpha, \beta \in \mathfrak{Q}$.

Remark 5.6. Every unital quantale $\mathfrak{Q}$ can be embedded into an involutive unital quantale by means of the tensor product $\mathfrak{Q} \otimes \mathfrak{Q}^{op}$ (see the proof of [15, Cor. 2.13]). Hence the assumption of an involution on a unital quantale is not a restriction of generality.

Remark 5.7. Let $\mathfrak{Q} = (\mathfrak{Q}, *, e, ')$ be an involutive and unital quantale. The *specialization $\mathfrak{Q}$-preorder* p_s of a $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ is determined by

$$p_s(x, y) = \mathcal{A}_{\mathcal{I}}(1_{\{y\}})(x)' = \bigwedge_{f \in \mathcal{T}} (f'(x) \searrow f'(y)), \qquad x, y \in X.$$

Recall that a $\mathfrak{Q}$-preorder p is *antisymmetric* (i.e. the $\mathfrak{Q}^{op}$-enriched category determined by p is *skeletal*) if the underlying preorder $\leq_p$ of p is antisymmetric. Since $x \leq_{p_s} y$ iff $f(x) \leq f(y)$ for all $f \in \mathcal{T}$, we conclude from the Lemmas 5.3, 5.4 and 5.5:

(1) $(X, \mathcal{T})$ is Kolmogorov separated if and only if the specialization $\mathfrak{Q}$-preorder is antisymmetric.
(2) $(X, \mathcal{T})$ is Fréchet separated if and only if the underlying preorder $\leq_{p_s}$ of the specialization $\mathfrak{Q}$-preorder p_s is discrete.

Lemma 5.8. *Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space. If the operation of the quasi-magma $(\mathfrak{Q}, \diamond)$ is join-preserving in each variable separately, then the following assertions are equivalent*:

(i) *$(X, \mathcal{T})$ is Hausdorff separated.*

(ii) *For each base $\mathcal{B}$ of $\mathcal{T}$ and for each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{B}$ such that*

$$f_1(x) \diamond f_2(y) \not\leq \bigvee_{z\in X} (f_1(z) \diamond f_2(z)) \quad \text{or} \quad f_2(y) \diamond f_1(x) \not\leq \bigvee_{z\in X} (f_2(z) \diamond f_1(z)).$$

Proof. We only need to prove (i) $\Rightarrow$ (ii). Let $\mathcal{B}$ be a base of $\mathcal{T}$ and $x, y \in X$ with $x \neq y$. By hypothesis there exist $f_1, f_2 \in \mathcal{T}$ such that, for example, $f_1(x) \diamond f_2(y) \not\leq \bigvee_{z\in X}(f_1(z) \diamond f_2(z))$ holds. Now we consider $\{f_i^1\}_{i\in I}, \{f_j^2\}_{j\in J} \subseteq \mathcal{B}$ such that $f_1 = \bigvee_{i\in I} f_i^1$ and $f_2 = \bigvee_{j\in J} f_j^2$. Then we obtain $f_1(x) \diamond f_2(y) = \bigvee_{i\in I,\, j\in J} (f_i^1(x) \diamond f_j^2(y))$ and consequently there exist some $i \in I$ and $j \in J$ such that $f_i^1(x) \diamond f_j^2(y) \not\leq \bigvee_{z\in X}(f_i^1(z) \diamond f_j^2(z))$. The other case, when $f_2(y) \diamond f_1(x) \not\leq \bigvee_{z\in X}(f_2(z) \diamond f_1(z))$, can be treated analogously. Hence (ii) is verified. □

Lemma 5.9. *Let $\mathfrak{Q}$ be a quantale with a dualizing element δ and $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space. If the quasi-magma operation is given by $\diamond = *$, then the following assertions are equivalent*:

(i) *$(X, \mathcal{T})$ is Hausdorff separated.*
(ii) *For each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{T}$ such that*

$$\Big(f_1(x) * f_2(y) \not\leq \delta \text{ and } \bigvee_{z\in X} f_1(z) * f_2(z) \leq \delta\Big) \quad \text{or} \quad \Big(f_2(y) * f_1(x) \not\leq \delta \text{ and } \bigvee_{z\in X} f_2(z) * f_1(z) \leq \delta\Big).$$

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. In order to verify the necessity of (ii) we assume $f_1(x) * f_2(y) \not\leq \bigvee_{z\in X} f_1(z) * f_2(z) =: \varkappa$, i.e. since δ is dualizing, $f_1(x) * f_2(y) * (\varkappa \searrow \delta) \not\leq \delta$. Further, since $\mathcal{T}$ is a right $\mathfrak{Q}$-submodule of $\mathfrak{Q}^X$, $f_3 := f_2 * (\varkappa \searrow \delta) \in \mathcal{T}$, and so we have

$$f_1(x) * f_3(y) \not\leq \delta \quad \text{and} \quad \bigvee_{z\in X} f_1(z) * f_3(z) = \bigvee_{z\in X} f_1(z) * f_2(z) * (\varkappa \searrow \delta) \leq \delta.$$

Analogously we proceed in the case $f_2(y) * f_1(x) \not\leq \bigvee_{z\in X} f_2(z) * f_1(z)$. □

Remark 5.10. (1) The previous lemmas show that the Kolmogorov and Fréchet separation axiom are subbasic properties. Note also that this is not the case for the Hausdorff separation axiom. There exist simple examples of Hausdorff separated $\mathfrak{Q}$-topological spaces (e.g. the discrete $\mathfrak{Q}$-topological space) having subbases, which do not satisfy the Hausdorff separation axiom.
(2) Let $\mathfrak{Q}$ be a Girard quantale with the quasi-magma operation $\diamond = *$. Since every cyclic and dualizing element δ induces an order-reversing involution on $\mathfrak{Q}$, we conclude from Lemma 5.9 that the Hausdorff axiom T_2 implies Kubiak's Hausdorff axiom in [23, Def. 9.1], but not vice verse as the following counterexample shows.

On the 3-chain $C_3 = \{\bot, a, \top\}$ we consider the structure of the 3-valued MV-algebra $\mathfrak{Q}_1 = (C_3, *)$ (cf. Example 2.1). Now we consider the following maps $C_3 \xrightarrow{f_j} C_3$ ($j = 1, 2, 3$):

$$f_1(\bot) = a,\ f_1(a) = f_1(\top) = \bot,\ f_2(a) = a,\ f_2(\bot) = f_2(\top) = \bot,\ f_3(\top) = a,\ f_3(\bot) = f_3(a) = \bot.$$

Then $\mathcal{S} = \{f_1, f_2, f_3\}$ is a subbase of a $\mathfrak{Q}_1$-topology $\mathcal{T}$ on C_3. $(C_3, \mathcal{T})$ is clearly Fréchet separated, but since $f_1 \vee f_2 \vee f_3 = \underline{a}$ and $\mathcal{S} \cup \{f_1 \vee f_2, f_2 \vee f_3, f_1 \vee f_3, \underline{a}, \underline{\top}\}$ is a base of $\mathcal{T}$, we conclude from Lemma 5.9 that the Hausdorff axiom T_2 does not hold, but Kubiak's Hausdorff axiom is satisfied.

Before we proceed we briefly touch the effect created by the change of base (cf. [13, Sect. 6]) with regard to the lower separation axioms T_i ($i = 0, 1, 2$).

Remark 5.11. (a) Let $\mathfrak{Q}_1 \xrightarrow{\varphi} \mathfrak{Q}_2$ be a quantale homomorphism satisfying the condition $\varphi(\alpha) \leq \varphi(\beta) \iff \alpha \leq \beta$ for all $\alpha, \beta \in \mathfrak{Q}_1$ — i.e. φ is an extremal monomorphism in the category of quantales. Further, we assume that h preserves the respective quasi-magma operations. Then the lower separation axioms T_i ($i = 0, 1, 2$) are preserved under the change of base conveyed by φ.
(b) Now let us consider the special case given by the embedding $\mathbf{2} \overset{\varphi}{\hookrightarrow} \mathfrak{Q}$ with $\varphi(0) = \bot$ and $\varphi(1) = \top$ and a strict quasi-magma operation $\diamond$ on $\mathfrak{Q}$ being directed join-preserving in each variable separately. Then the change of base

of a traditional topological space (X, τ) conveyed by φ is $\mathcal{F}_\varphi(X, \tau) = (X, \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})))$, where $\mathbb{L}(\mathfrak{Q})$ is the subquantale of all left-sided elements of $\mathfrak{Q}$ and $\mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))$ is the right $\mathfrak{Q}$-submodule of $(\mathfrak{Q}^X, *)$ consisting of all lower semicontinuous maps $X \xrightarrow{f} \mathbb{L}(\mathfrak{Q})$ — i.e. f satisfies the property $f(x) \le \bigvee_{U\in\tau, x\in U} \big(\bigwedge_{y\in U} f(y)\big)$ for all $x \in X$. Further, we recall that we can identify every $U \in \tau$ with the elementary tensor $U \otimes \top$ and consequently with the $\mathfrak{Q}$-presheaf $1_U * \top$. Moreover, if $\mathcal{A}_\mathcal{I}$ denotes the adherence operator w.r.t. $(X, \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})))$, then for each $V \in \tau$ and $\alpha \in \mathbb{L}(\mathfrak{Q})$ the following relation holds:

$$\mathcal{A}_\mathcal{I}(1_V * \alpha) = 1_{\overline{V}} * \alpha * \top, \tag{5.2}$$

where $\overline{V}$ is the topological closure of V w.r.t. τ. Note also that every lower semicontinuous map $f \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))$ can be represented by

$$f = \bigvee_{i\in I} 1_{U_i} * \alpha, \qquad U_i \in \tau,\ \alpha_i \in \mathbb{L}(\mathfrak{Q}). \tag{5.3}$$

In particular for each subset A of X the following equivalence holds:

$$1_A * \top \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})) \iff A \in \tau. \tag{5.4}$$

We say that a property $\mathcal{P}$ of $\mathfrak{Q}$-topological spaces is a *direct extension* of its topological counterpart in Top if

(1) it coincides with the usual topological property when $\mathfrak{Q} = \mathbf{2}$ and
(2) a topological space (X, τ) satisfies property $\mathcal{P}$ if and only if $(X, \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})))$ does.

Now we have the following situation: Let $\diamond$ be a strict quasi-magma operation on $\mathfrak{Q}$ being directed join-preserving in each variable separately. The lower separations axioms T_i $(i = 0, 1, 2)$ coincide obviously with the usual topological properties when $\mathfrak{Q} = \mathbf{2}$. Since $\{\, 1_U * \alpha \mid U \in \tau,\ \alpha \in \mathbb{L}(\mathfrak{Q}) \,\}$ is a base of $\mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))$ (cf. (5.3)), it follows immediately from the Lemmas 5.4 and 5.5 that the lower separations axioms T_0 and T_1 are direct extensions of their topological counterparts, and by Lemma 5.8 it also follows that the T_2 axiom is a direct extension of its topological counterpart provided the quasi-magma operation $\diamond$ is join-preserving in each variable separately. With regard to (4.3) it is easily seen that the *density of subsets* is also a direct extension of its topological counterpart.

Based on the previous remark it is easy to find examples of $\mathfrak{Q}$-topological spaces satisfying these lower separations axioms. However, it is more interesting to present some examples originating *not* from traditional topology via change of base.

Examples 5.12. (1) Let $\mathfrak{Q}$ be an integral and involutive quantale viewed as right $\mathfrak{Q}$-module with respect to the right multiplication induced by $*$. Then the intrinsic $\mathfrak{Q}$-preorder p has the form

$$p(\alpha, \beta) = \alpha \searrow \beta, \qquad \alpha, \beta \in \mathfrak{Q},$$

and the right $\mathfrak{Q}$-module $\mathbb{P}(\mathfrak{Q}, p)$ of all contravariant $\mathfrak{Q}$-presheaves on $(\mathfrak{Q}, p)$ (cf. [7, p. 260]) is a $\mathfrak{Q}$-topology on $\mathfrak{Q}$ with respect to $\diamond = \wedge$. Then the specialization $\mathfrak{Q}$-preorder of $\big(\mathfrak{Q}, \mathbb{P}(\mathfrak{Q}, p)\big)$ coincides with the dual $\mathfrak{Q}$-preorder p^{op} of p — i.e. $p^{op}(\alpha, \beta) = p(\beta, \alpha)'$ for all $\alpha, \beta \in \mathfrak{Q}$. Since p is antisymmetric, the $\mathfrak{Q}$-topological space $\big(\mathfrak{Q}, \mathbb{P}(\mathfrak{Q}, p)\big)$ satisfies the Kolmogorov separation axiom, but since $f(\alpha) \le f(\bot)$ holds for all $\alpha \in \mathfrak{Q}$ and for all $f \in \mathbb{P}(\mathfrak{Q}, p)$, the $\mathfrak{Q}$-topological space $(\mathfrak{Q}, \mathbb{P}(\mathfrak{Q}, p))$ is *not* Fréchet separated.
(2) Referring to Figure 5 in [12] we consider the unital quantale $\mathfrak{Q} = \mathfrak{R}_4$ with the quasi-magma operation $\diamond = *$. Obviously, $\mathfrak{R}_4$ is involutive, where the involution $'$ is determined by:

$$a_\ell' = a_r, \quad a_r' = a_\ell, \quad \widetilde{a_\ell}' = \widetilde{a_r}, \quad \widetilde{a_r}' = \widetilde{a_\ell},$$

and the remaining 5 elements are hermitian. Let $\mathbb{M}_2$ be the C^*-algebra of all $(2,2)$-matrices with complex coefficients. Then the quantale $\mathfrak{A}$ of all left ideals of $\mathbb{M}_2$ is left-sided, idempotent and consequently semi-unital. Since the subquantale $\mathbb{I}(\mathbb{M}_2)$ of all two-sided ideals coincides with $\{\bot, \top\}$, the ideal multiplication has the following form:

$$a * b = \begin{cases} \bot, & \text{if } a = \bot, \\ b, & \text{if } a \neq \bot, \end{cases} \qquad a, b \in \mathfrak{A}.$$

Further, every pure state is a vector state, and every non-trivial left ideal is maximal and consequently prime (cf. [7, Thm. 2.3.21]). In particular, if $C = \{ x \in \mathbb{C}^2 \mid |x| = 1 \}$ is the unit circle in $\mathbb{C}^2$, then the relationship between pure states x and maximal left ideals is given by:

$$a_x = \{ \alpha \in \mathbb{M}_2 \mid \langle \alpha^* \alpha\, x\,,\, x \rangle = 0 \},$$

e.g. if $x = (0, 1)$, then the corresponding (maximal) left ideal a_x has the form:

$$a_x = \left\{ \begin{pmatrix} z_1 & 0 \\ z_2 & 0 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C} \right\}$$

Now with every non trivial left ideal a we associate a map $C \xrightarrow{\mathbb{A}_a} \mathfrak{R}_4$ defined by:

$$\mathbb{A}_a(x) = \begin{cases} \top, & \text{if } a \neq a_x, \\ a_\ell, & \text{if } a = a_x, \end{cases} \qquad x \in C.$$

In the case of the two-sided ideals $\perp$ and $\top$ we put $\mathbb{A}_\perp = \underline{\perp}$ and $\mathcal{A}_\top = \underline{\top}$. Then the non-commutative Gelfand topology $\mathcal{T}_{\mathbb{M}_2}$ on the unit circle attains the following form:

$$\mathcal{T}_{\mathbb{M}_2} = \{ \underline{a_\ell} \} \cup \{ \mathbb{A}_a \mid a \in \mathfrak{A} \}.$$

Since $a'_\ell = a_r$, the corresponding specialization $\mathfrak{R}_4$-preorder p_s is given by

$$p_s(x, y) = \begin{cases} \widetilde{a_r} = a_r \vee e, & \text{if } x = y, \\ \top \searrow a_r = \perp, & \text{if } x \neq y, \end{cases} \qquad x, y \in C.$$

Hence $(C, \mathcal{T}_{\mathbb{M}_2})$ is Fréchet separated, but *not* Hausdorff separated. It seems that this is the price for the non-commutativity.

In fact the Hausdorff reflection of the $\mathfrak{R}_4$-topological space $(C, \mathcal{T}_{\mathbb{M}_2})$ is a singleton space illustrating the fact that all pure states are unitary equivalent.

(3) Let $\mathfrak{Q}$ be the complete MV-algebra $([0, 1], *_Ł)$, where $*_Ł$ is the Łukasiewicz arithmetic conjunction — i.e. $\alpha *_Ł \beta = \max(\alpha + \beta - 1, 0)$ for each $\alpha, \beta \in [0, 1]$. Further, we consider the $\mathfrak{Q}$-topology $\mathcal{T}$ on $[0, 1]$ generated by the subbase $\{ \mathrm{id}_{[0,1]}, 1 - \mathrm{id}_{[0,1]} \}$. Now we choose $x, y \in [0, 1]$ with $x \neq y$ — e.g. $x < y$ and consider the following three cases:

(3a) If the quasi-magma operation is given by the quantale multiplication — i.e. $\diamond = *_Ł$, then

$$\mathrm{id}_{[0,1]}(y) *_Ł (1 - \mathrm{id}_{[0,1]})(x) = y - x \not\leq 0 = \bigvee_{z \in [0,1]} (\mathrm{id}_{[0,1]}(z) *_Ł (1 - \mathrm{id}_{[0,1]})(z)).$$

(3b) If the quasi-magma operation is given by the binary meet — i.e. $\diamond = \wedge$, then we choose $f_\alpha = \mathrm{id}_{[0,1]} *_Ł \alpha$ and $g_\beta = (1 - \mathrm{id}_{[0,1]}) *_Ł *\beta \in \mathcal{T}$ with $\alpha = 1 - \frac{x+y}{2}$ and $\beta = \frac{x+y}{2}$ and observe that

$$f_\alpha(y) \wedge g_\beta(x) = \frac{y - x}{2} \not\leq 0 = \bigvee_{z \in [0,1]} (f_\alpha(z) \wedge g_\beta(z)).$$

(3c) If the quasi-magma is given by the monoidal mean operator — i.e. $\circledast$ coincides with the binary arithmetic mean, then

$$\mathrm{id}_{[0,1]}(y) \circledast (1 - \mathrm{id}_{[0,1]})(x) = \tfrac{y-x+1}{2} \not\leq \tfrac{1}{2} = \bigvee_{z \in [0,1]} (\mathrm{id}_{[0,1]}(z) \circledast (1 - \mathrm{id}_{[0,1]})(z)).$$

Hence in all these cases the $\mathfrak{Q}$-topological space $([0, 1], \mathcal{T})$ is Hausdorff separated.

It is worthwhile to point out that the previous arguments in (3a) — (3c) remains valid for any complete MV-algebra with square roots such that $\perp^{1/2} \to \perp = \perp^{1/2}$ (cf. Example 2.7 (4)).

Under the assumption of the Hausdorff separation axiom the next theorem shows that $\mathfrak{Q}$-continuous maps are uniquely determined on dense subsets.

Theorem 5.13. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $\mathfrak{Q}$-topological spaces and A be a dense subset in $(X, \mathcal{T}_X)$. If $(Y, \mathcal{T}_Y)$ is Hausdorff separated and the maps $(X, \mathcal{T}_X) \xrightarrow{\psi,\varphi} (Y, \mathcal{T}_Y)$ are $\mathfrak{Q}$-continuous such that their restrictions to A coincide, then $\psi = \varphi$.*

Proof. Let $x \in X$ and $f_1, f_2 \in \mathcal{T}_Y$ be arbitrary open $\mathfrak{Q}$-presheaves. Then the $\mathfrak{Q}$-continuity implies $f_1 \circ \psi$, $f_2 \circ \varphi \in \mathcal{T}_X$, and hence $(f_1 \circ \psi) \diamond (f_2 \circ \varphi)$ and $(f_2 \circ \varphi) \diamond (f_1 \circ \psi)$ are open $\mathfrak{Q}$-presheaves on X. Now we invoke the density of A (see (4.3)) and conclude that

$$f_1(\psi(x)) \diamond f_2(\varphi(x)) \le \bigvee_{a \in A} (f_1(\psi(a)) \diamond f_2(\varphi(a))) \le \bigvee_{y \in Y} (f_1(y) \diamond f_2(y)) \quad \text{and}$$

$$f_2(\varphi(x)) \diamond f_1(\psi(x)) \le \bigvee_{a \in A} (f_2(\varphi(a)) \diamond f_1(\psi(a))) \le \bigvee_{y \in Y} (f_2(y) \diamond f_1(y))$$

and since $(Y, \mathcal{T}_Y)$ is Hausdorff separated, we infer from (5.1) that $\psi(x) = \varphi(x)$. □

Historical remark. Let $\mathfrak{Q}$ be an integral quantale and $(\mathfrak{Q}, \wedge)$ be the quasi-magma on $\mathfrak{Q}$. On this background the history of the Kolmogoroff and Fréchet separation axiom in Definition 5.1 can be traced back in the literature and especially in the context of many-valued topology as follows.

(a) The Kolmogorov separation axiom in its full generality goes back to Rodabaugh in the period of time from 1986 until 1995 ([29, Def. 2.5 and Disc. 3.1]), since Rodabaugh does not apply order-reversing involutions for the construction of closed $\mathfrak{Q}$-presheaves. Independently, Šostak invented the Kolmogorov separation axiom in 1989 under the name W_0-axiom (cf. [30]). In the case of commutative Girard quantales (cf. [23, Rem. 3.4]and see also [22, Rem. 7.7 (3)]), equivalent formulations appear in the work by Liu in 1983 (see the sub-T_0 axiom in [25]) and in the work by Wuyts and Lowen in 1986 (see the 0^*-T_0 axiom in [34]). The specialization $\mathfrak{Q}$-preorder in Remark 5.7 goes back to Lai and Zhang in 2006 in the case of $\mathfrak{Q} = [0, 1]$ viewed as an integral and commutative quantale (cf. [24, p. 1877]).

(b) The Fréchet separation axiom goes back to Kubiak in 1995 (cf. [23, Def. 9.1]).

If $\mathfrak{Q}$ is now a complete MV-algebra, then probabilistic topologies on X (cf. Remarks 6.14 (a)) are the same as cotensored $\mathfrak{Q}$-topologies on X. In this context the Hausdorff separation axiom goes back to Höhle in 1982 (cf. [16, Def. 3.3]).

6. Regularity and the continuous extension principle

We begin with a characterization of density in terms of $\mathfrak{Q}$-filters.

Definition 6.1. ([13, Sec. 5]) A $\mathfrak{Q}$-*enriched filter* ($\mathfrak{Q}$-*filter* for short) on a nonempty set X is a covariant $\mathfrak{Q}$-presheaf ω on $(\mathfrak{Q}^X, d)$ satisfying the following properties for all $f_1, f_2 \in \mathfrak{Q}^X$:

(F1) $\omega(\underline{\top}) = \top$,
(F2) $\omega(f) \diamond \omega(g) \le \omega(f \diamond g)$,
(F3) $\omega(f) \le \bigvee f(X)$.

Comments 6.2 *(Convergence of $\mathfrak{Q}$-filters (cf. [13, Sec. 4.2])).*

(1) Let ω_1 and ω_2 be $\mathfrak{Q}$-filters on X such that there exists another $\mathfrak{Q}$-filter ω on X with $\omega_1 \le \omega$ and $\omega_2 \le \omega$. Then the isotonicity of $\diamond$ and axioms (F2) and (F3) imply that the following holds:

$$\omega_1(f_1) \diamond \omega_2(f_2) \le \bigvee_{x \in X} (f_1(x) \diamond f_2(x)), \qquad f_1, f_2 \in \mathfrak{Q}^X.$$

(2) Given a $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ and $x \in X$, the covariant $\mathfrak{Q}$-presheaf ν_x on $(\mathfrak{Q}^X, d)$

$$\nu_x(f) = \mathcal{I}(f)(x), \qquad f \in \mathfrak{Q}^X \tag{6.1}$$

is a $\mathfrak{Q}$-filter, namely the $\mathfrak{Q}$-*neighborhood filter at* x (cf. [13, p. 986]). An element $x \in X$ is called a *limit point* of a $\mathfrak{Q}$-filter ω in $(X, \mathcal{T})$ if $\nu_x(f) \le \omega(f)$ for all $f \in \mathfrak{Q}^X$. We shall also say that ω *converges* to x in $(X, \mathcal{T})$.

(3) A $\mathfrak{Q}$-filter ω on X is called *convergent* if it has a limit point. Since every $\mathfrak{Q}$-topology on X viewed as $\mathfrak{Q}^{op}$-category is cocomplete, every convergent $\mathfrak{Q}$-filter ω satisfies the following (cf. (3.2)):

$$d\big(\underline{e}, \mathsf{sup}(F)\big) \le \bigvee_{f \in \mathcal{T}} \omega(f) * F(f), \qquad F \in \mathfrak{Q}^{\mathcal{T}},$$

— i.e. every convergent $\mathfrak{Q}$-filter on X *meets* every open cover of $\underline{e}$.
(4) If x and y are limit points of a $\mathfrak{Q}$-filter ω, then for all $f_1, f_2 \in \mathcal{T}$ the following relations hold:

$$f_1(x) \diamond f_2(y) = \nu_x(f_1) \diamond \nu_y(f_2) \le \omega(f_1 \diamond f_2) \le \bigvee_{z \in X} (f_1(z) \diamond f_2(z)) \quad \text{and}$$
$$f_2(y) \diamond f_1(x) = \nu_y(f_2) \diamond \nu_x(f_1) \le \omega(f_2 \diamond f_1) \le \bigvee_{z \in X} (f_2(z) \diamond f_1(z)).$$

Consequently, if $(X, \mathcal{T})$ is Hausdorff separated then every $\mathfrak{Q}$-filter has at most one limit point.
(5) A map $(X, \mathcal{T}_X) \xrightarrow{\varphi} (Y, \mathcal{T}_Y)$ between $\mathfrak{Q}$-topological spaces is $\mathfrak{Q}$-continuous if and only if for each $x \in X$ and each $\mathfrak{Q}$-filter ω on X converging to x the image $\mathfrak{Q}$-filter $\varphi(\omega)$ converges to $\varphi(x)$, where

$$\big(\varphi(\omega)\big)(g) = \omega(g \circ \varphi), \qquad g \in \mathfrak{Q}^Y.$$

Before we proceed we make a small digression on convergence being related to a $\mathfrak{Q}$-enriched version of [1, Lem. 1].

Remark 6.3. Let $\mathfrak{Q}$ be a quantale having a completely prime and dualizing element δ and $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space. If a $\mathfrak{Q}$-filter ω on X meets every open cover of $\underline{e}$, then ω is convergent. Indeed, let $F \in \mathfrak{Q}^{\mathcal{T}}$ be given by $F(f) = \omega(f) \searrow \delta$ for all $f \in \mathcal{T}$, then

$$d(\underline{e}, \mathsf{sup}(F)) = \bigwedge_{x \in X} \big(\bigvee_{f \in \mathcal{T}} f(x) * (\omega(f) \searrow \delta)\big) \le \bigvee_{f \in \mathcal{T}} \omega(f) * (\omega(f) \searrow \delta) \le \delta$$

and since δ is completely prime, we conclude that there exists some $x_0 \in X$ such that $\bigvee_{f \in \mathcal{T}} f(x_0) * (\omega(f) \searrow \delta) \le \delta$, hence $f(x_0) \le \omega(f)$ for all $f \in \mathcal{T}$.

The assumption that $\mathfrak{Q}$ has a completely prime and dualizing element cannot be dropped as the following counterexample shows.
Let $\mathfrak{Q} = \{\bot, a, b, \top\}$ be the Boolean algebra with 4 elements and $* = \diamond = \wedge$. Then on $X = \{a, b, \top\}$ we introduce a $\mathfrak{Q}$-topology $\mathcal{T}$ as follows:

$$\mathcal{T} = \{\underline{\top},\ \mathrm{id}_X \vee \underline{a},\ \mathrm{id}_X \vee \underline{b},\ \underline{a},\ \underline{b},\ \mathrm{id}_X,\ \mathrm{id}_X \wedge \underline{a},\ \mathrm{id}_X \wedge \underline{b},\ \underline{\bot}\}.$$

Further, we consider the coarsest $\mathfrak{Q}$-filter on X given by $\omega(g) = \bigwedge_{x \in X} g(x)$ for all $g \in \mathfrak{Q}^X$. Then:

$$\bigvee_{f \in \mathcal{T}} (f(a) \wedge (\omega(f) \to \bot)) = a \quad \text{and} \quad \bigvee_{f \in \mathcal{T}} (f(b) \wedge (\omega(f) \to \bot)) = b.$$

Since $a \wedge b = \bot$, ω meets every open cover of $\underline{\top}$, but ω is not convergent:

$$\omega(\mathrm{id}_X \vee \underline{b}) = b \not\ge (\mathrm{id}_X \vee \underline{b})(\top) = \top, \quad \omega(\mathrm{id}_X \vee \underline{b}) = b \not\ge (\mathrm{id}_X \vee \underline{b})(a) = \top,$$
$$\omega(\mathrm{id}_X \vee \underline{a}) = a \not\ge (\mathrm{id}_X \vee \underline{a})(b) = \top.$$

Since our convergence theory is different of that one in [1], we continue our train of thought and consider a nonempty and proper subset A of X. Then every $\mathfrak{Q}$-presheaf f on A can be identified with the unique $\mathfrak{Q}$-presheaf $\widehat{f}$ on X extending f and the constant $\mathfrak{Q}$-presheaf $\underline{\top}$ on $\complement A$ — i.e. $\widehat{f}$ is determined by:

$$\widehat{f}(x) = \begin{cases} f(x), & \text{if } x \in A, \\ \top, & \text{if } x \in \complement A. \end{cases}$$

Further, let ω be a $\mathfrak{Q}$-filter on X. The *trace* of ω on A is the covariant $\mathfrak{Q}$-presheaf ω_A on $\big(\mathfrak{Q}^A, d\big)$ defined by:

$$\omega_A(f) := \omega(\widehat{f}), \qquad f \in \mathfrak{Q}^A.$$

In fact, the relation $\omega_A(f) * \alpha \le \omega(\widehat{f} * \alpha) \le \omega\big(\widehat{f * \alpha}\big) = \omega_A(f * \alpha)$ holds. Since $\widehat{\top} = \top$ and $\widehat{f_1} \diamond \widehat{f_2} \le \widehat{f_1 \diamond f_2}$ for all $f_1, f_2 \in \mathfrak{Q}^A$, ω_A satisfies the filter axioms (F1) and (F2). But in general it is not clear whether ω_A satisfies (F3). Hence ω_A is not necessarily a $\mathfrak{Q}$-filter.

With regard to the $\mathfrak{Q}$-neighborhood filters of a $\mathfrak{Q}$-topology on X we have the following result.

Proposition 6.4. *Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space and $A \subseteq X$. Then*:

(1) *A is dense in $(X, \mathcal{T})$ if and only if for each $x \in X$ the trace $(\nu_x)_A$ of the $\mathfrak{Q}$-neighborhood filter ν_x on X is a $\mathfrak{Q}$-filter on A.*
(2) *Let A be provided with the initial $\mathfrak{Q}$-topology $\mathcal{T}_A$ induced by the inclusion map $A \overset{\iota}{\hookrightarrow} X$. If $x \in A$, then the $\mathfrak{Q}$-neighborhood filter at x with respect to $\mathcal{T}_A$ coincides with the trace $(\nu_x)_A$ of ν_x on A.*

Proof. (1) Let A be dense in $(X, \mathcal{T})$ and $\mathcal{I}$ be the $\mathfrak{Q}$-interior operator w.r.t. $\mathcal{T}$. Since $\mathcal{I}(\widehat{\underline{\alpha}})$ is an open $\mathfrak{Q}$-presheaf on X, we refer to (4.3) and obtain for each $x \in X$:

$$(\nu_x)_A(\underline{\alpha}) = \nu_x(\widehat{\underline{\alpha}}) = \mathcal{I}(\widehat{\underline{\alpha}})(x) \le \bigvee_{a \in A} \mathcal{I}(\widehat{\underline{\alpha}})(a) \le \alpha.$$

Hence $(\nu_x)_A$ is a $\mathfrak{Q}$-filter on A.

On the other hand, if for each $x \in X$ the trace $(\nu_x)_A$ is a $\mathfrak{Q}$-filter on A and f is an open $\mathfrak{Q}$-presheaf on X, then we put $\alpha_f = \bigvee_{a \in A} f(a)$ and obtain:

$$f(x) = \nu_x(f) \le \nu_x\big(\widehat{\underline{\alpha_f}}\big) = (\nu_x)_A\big(\underline{\alpha_f}\big) \le \alpha_f = \bigvee_{a \in A} f(a).$$

Hence A is dense in $(X, \mathcal{T})$.
(2) Let $x \in A$ and $\mathcal{I}_A$ be the $\mathfrak{Q}$-interior operator with respect to the initial $\mathfrak{Q}$-topology $\mathcal{T}_A$. Since for each $f \in \mathfrak{Q}^A$ and for each $g \in \mathcal{T}$ the equivalence

$$g \circ \iota \le f \iff g \le \widehat{f}$$

holds, we obtain $\mathcal{I}_A(f)(x) = \nu_x(\widehat{f}) = (\nu_x)_A(f)$. □

After these preparations we are now ready for the following definition. We will make use of the cocomepleteness of $\mathfrak{Q}$-topologies viewed as $\mathfrak{Q}^{op}$-enriched categories.

Definition 6.5. Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space and $\mathcal{A}_{\mathcal{I}}$ the corresponding $\mathfrak{Q}$-adherence operator. Then $(X, \mathcal{T})$ is said to be:

- *regular* if $f \le \bigvee_{g \in \mathcal{T}} g * d(\mathcal{A}_{\mathcal{I}}(g), f)$ for all $f \in \mathcal{T}$, where d is the intrinsic $\mathfrak{Q}$-preorder of the free right $\mathfrak{Q}$-module $\mathfrak{Q}^X$.
- T_3 if it is Hausdorff separated and regular.
- *weakly regular* if the set $\mathcal{S} = \big\{ f \in \mathcal{T} \mid f \le \bigvee_{g \in \mathcal{T}} g * d(\mathcal{A}_{\mathcal{I}}(g), f) \big\}$ is a subbase of $\mathcal{T}$.
- *weakly* T_3 if it is Hausdorff separated and weakly regular.

Comment. Regularity means that every open $\mathfrak{Q}$-presheaf $f \in \mathcal{T}$ is the $\mathfrak{Q}$-join of the $\mathfrak{Q}$-presheaf $F_f \in \mathfrak{Q}^{\mathcal{T}}$ defined by $F_f(g) = d(\mathcal{A}_{\mathcal{I}}(g), f)$ for all $g \in \mathcal{T}$ (cf. (3.2)).

It is easy to check that regularity (resp. weak regularity) coincides with (resp. is equivalent to) the usual regularity when $\mathfrak{Q} = \mathbf{2}$. Moreover, we have the following:

Proposition 6.6. *Let $(\mathfrak{Q}, \diamond)$ be a strict quasi-magma such that $\diamond$ is directed join-preserving in each variable separately. Then regularity is a direct extension of regularity in* Top.

Proof. Let (X, τ) be a topological space. We only have to prove that (X, τ) is regular if and only if $(X, \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})))$ is regular.

We assume the regularity of (X, τ) and choose $f = \bigvee_{i \in I} 1_{U_i} * \alpha_i \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))$ with $U_i \in \tau$ and $\alpha_i \in \mathbb{L}(\mathfrak{Q})$. Then for each $i \in I$ and each $V \in \tau$ with $\overline{V} \subset U_i$ the relation $\alpha_i = d(1_{\overline{V}} * \top, 1_{U_i} * \alpha_i)$ follows. Further, by (5.2) we have that $1_{\overline{V}} * \top = \mathcal{A}_{\mathcal{I}}(1_V * \top)$, and so we obtain:

$$\begin{aligned} 1_{U_i} * \alpha_i &= \bigvee_{V \in \tau,\, \overline{V} \subset U_i} 1_V * \top * \alpha_i = \bigvee_{V \in \tau,\, \overline{V} \subset U} 1_V * \top * d(1_{\overline{V}} * \top, 1_{U_i} * \alpha_i) \\ &= \bigvee_{V \in \tau,\, \overline{V} \subset U_i} 1_V * \top * d(\mathcal{A}_{\mathcal{I}}(1_V * \top), 1_{U_i} * \alpha_i) \le \bigvee_{g \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))} g * d(\mathcal{A}_{\mathcal{I}}(g), 1_{U_i} * \alpha_i) \\ &\le \bigvee_{g \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))} g * d(\mathcal{A}_{\mathcal{I}}(g), f). \end{aligned}$$

Hence $f \le \bigvee_{g \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})} g * d(\mathcal{A}_{\mathcal{I}}(g), f)$ and we conclude that $(X, \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})))$ is regular.

Conversely, let $(X, \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})))$ be regular, $U \in \tau$ and $x \in U$. We refer again to (5.2) and observe that

$$\begin{aligned} \top = 1_U(x) * \top &= \bigvee_{g \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))} g(x) * d(\mathcal{A}_{\mathcal{I}}(g), 1_U * \top) \\ &= \bigvee_{V \in \tau,\, \alpha \in \mathbb{L}(\mathfrak{Q})} (1_V * \alpha)(x) * d(\mathcal{A}_{\mathcal{I}}(1_V * \alpha), 1_U * \top) = \bigvee_{V \in \tau,\, x \in V,\, \alpha \in \mathbb{L}(\mathfrak{Q})} \alpha * d(1_{\overline{V}} * \alpha * \top, 1_U * \top). \end{aligned}$$

If $\overline{V} \nsubseteq U$, then $\alpha * d(1_{\overline{V}} * \alpha * \top, 1_U * \top) = \alpha * ((\alpha * \top) \searrow \bot) = \bot$ and so there exists $V \in \tau$ such that $x \in V$ and $\overline{V} \subseteq U$. Hence (X, τ) is regular. □

As an immediate corollary from Remark 5.11 and Proposition 6.6 we obtain the following result.

Corollary 6.7. *Let $(\mathfrak{Q}, \diamond)$ be a strict quasi-magma such that $\diamond$ is directed join-preserving in each variable separately. Then the T_3 axiom is a direct extension of the T_3 axiom in* Top.

The next examples shed some light on the concept of weak regularity.

Example 6.8. Let (X, τ) be a traditional T_3 space and A be a dense subset in (X, τ) such that $A \notin \tau$ — e.g. $X = \mathbb{R}$ with the Euclidean topology and $A = \mathbb{Q}$. Further, let τ^* be the *indiscrete extension* of τ (see [31, Example 66]) — i.e. the topology on X generated by $\tau \cup \{A\}$. It is well known that (X, τ^*) is T_2 but not regular, and A is still dense in (X, τ^*). Now we apply the change of base conveyed by the embedding $\mathbf{2} \hookrightarrow \mathfrak{Q}$ and consider the quasi-magma operation $\diamond = *$ on $\mathfrak{Q}$. Then the $\mathfrak{Q}$-topological space $(X, \mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q})))$ is also T_2 but not regular and A is dense in it (cf. Remark 5.11 and Proposition 6.6). We show now that $(X, \mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q})))$ is even *not weakly regular*. For this purpose let $\mathcal{A}_{\mathcal{I}}$ be the adherence operator of $(X, \mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q})))$. Since

$$\mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q})) = \{\, f_1 \vee (f_2 * 1_A) \mid f_1, f_2 \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})) \,\},$$

we refer to Lemma 4.3 and observe that for $f_1, f_2 \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))$ the relation

$$\mathcal{A}_{\mathcal{I}}((f_1 \vee f_2) * 1_A) = \mathcal{A}_{\mathcal{I}}(f_1 \vee (f_2 * 1_A)) = \mathcal{A}_{\mathcal{I}}(f_1 \vee f_2)$$

holds. Hence we obtain

$$\bigvee_{g \in \mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q}))} g * d(\mathcal{A}_{\mathcal{I}}(g), f) = \bigvee_{g \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))} g * d(\mathcal{A}_{\mathcal{I}}(g), f) \in \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q})).$$

Since $1_A * \top \notin \mathsf{LSC}(\tau, \mathbb{L}(\mathfrak{Q}))$ (cf. (5.4)), we conclude that

$$\Big\{\, f \in \mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q})) \mid f = \bigvee_{g \in \mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q}))} g * d(\mathcal{A}_{\mathcal{I}}(g), f) \,\Big\}$$

is not a subbase of $\mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q}))$. Thus $(X, \mathsf{LSC}(\tau^*, \mathbb{L}(\mathfrak{Q})))$ is T_2 but *not* weakly regular.

Since in the previous example both $\mathfrak{Q}$-topological spaces are induced by traditional topological spaces via change of base, we show that there exist also situations, in which this phenomenon does not occur. The next example is a modification of Example 5.12 (3a).

Example 6.9. Let $\mathfrak{Q}$ be the complete MV-algebra $([0,1], *_{Ł})$. We consider the quasi-magma operation $\diamond = *_{Ł}$ and the $\mathfrak{Q}$-topology $\mathcal{T}$ on $[0,1]$ generated by the subbase $\{ \mathrm{id}_{[0,1]}, 1 - \mathrm{id}_{[0,1]} \}$. Obviously every element of $\mathcal{T}$ is lower semi-continuous w.r.t. the usual topology on $[0,1]$. Moreover, the subset $A = \{0, 1\}$ is dense in $([0,1], \mathcal{T})$ (cf. Example 4.2) and since 1_A is not lower semi-continuous, we have $1_A \notin \mathcal{T}$. Then the *indiscrete extension of* $\mathcal{T}$, namely

$$\mathcal{T}^* = \{ f_1 \vee (f_2 *_{Ł} 1_A) \mid f_1, f_2 \in \mathcal{T} \}$$

is another $\mathfrak{Q}$-topology on $[0,1]$ being strictly finer than $\mathcal{T}$. Hence $([0,1], \mathcal{T}^*)$ is again Hausdorff separated (cf. Example 5.12 (3a)). Further, let $\mathcal{A}_{\mathcal{T}^*}$ be the adherence operator w.r.t. $([0,1], \mathcal{T}^*)$. Since A is still dense in $([0,1], \mathcal{T}^*)$, by analogy with Example 6.8 the following holds for $f_1, f_2 \in \mathcal{T}$:

$$\mathcal{A}_{\mathcal{T}^*}((f_1 \vee f_2) *_{Ł} 1_A) = \mathcal{A}_{\mathcal{T}^*}(f_1 \vee (f_2 *_{Ł} 1_A)) = \mathcal{A}_{\mathcal{T}^*}(f_1 \vee f_2),$$

$$\bigvee_{g \in \mathcal{T}^*} g *_{Ł} d(\mathcal{A}_{\mathcal{T}^*}(g), f) = \bigvee_{g \in \mathcal{T}} g *_{Ł} d(\mathcal{A}_{\mathcal{T}^*}(g), f) \in \mathcal{T}.$$

Now we use the fact $1_A \notin \mathcal{T}$ and conclude that

$$\mathcal{S} := \Big\{ f \in \mathcal{T}^* \mid f = \bigvee_{g \in \mathcal{T}^*} g * d(\mathcal{A}_{\mathcal{T}^*}(g), f) \Big\}$$

is not a subbase of $\mathcal{T}^*$. Thus $([0,1], \mathcal{T}^*)$ is *not* weakly regular. Moreover, it is easily seen that $([0,1], \mathcal{T}^*)$ is not induced by a traditional topological space via change of base by $\mathbf{2} \hookrightarrow \mathfrak{Q}$.

The next example presents a weakly regular $\mathfrak{Q}$-topological space being *not* regular.

Example 6.10. On the 3-chain $C_3 = \{ \bot, a, \top \}$ we consider the structure of the 3-valued MV-algebra $\mathfrak{Q}_1 = (C_3, *)$ (cf. Example 2.1). Further, we consider the quasi-magma operation $\diamond = *$ on $\mathfrak{Q}_1$ and the $\mathfrak{Q}_1$-topology $\mathcal{T}$ on C_3 generated by the subbase $\mathcal{S} = \{ \mathrm{id}_{C_3}, \mathrm{id}_{C_3} \to \bot \}$. An easy calculation shows that a $\mathfrak{Q}_1$-presheaf $\underline{\bot} \neq f \neq \underline{\top}$ is open if and only if $f(a) \leq a$, and hence it is closed if and only if $f(a) \geq a$. Since id_{C_3} and $\mathrm{id}_{C_3} \to \bot$ are closed in $(C_3, \mathcal{T})$ by Lemma 4.4, it follows that $(C_3, \mathcal{T})$ is weakly regular. However it is *not* regular. Indeed, if we select the open $\mathfrak{Q}_1$-presheaf $C_3 \xrightarrow{f_0} C_3$ given by $f_0(\bot) = f_0(a) = \bot$ and $f_0(\top) = \top$, then we have that $d(\mathcal{A}_{\mathcal{T}}(g), f_0) \leq \mathcal{A}_{\mathcal{T}}(g)(a) \to \bot \leq a$ for all $\underline{\bot} \neq g \in \mathcal{T}$ and so

$$\bigvee_{g \in \mathcal{T}} g(\top) * d(\mathcal{A}_{\mathcal{T}}(g), f_0) \leq \top * a = a < \top = f_0(\top).$$

Moreover, the $\mathfrak{Q}_1$-topological space $(C_3, \mathcal{T})$ is *not* induced by a traditional topological space via change of base determined by $\mathbf{2} \overset{h}{\hookrightarrow} \mathfrak{Q}_1$. But $(C_3, \mathcal{T})$ is Kolmogoroff separated. Anticipating Lemma 6.19 and Example 6.20 we already state here that $(C_3, \mathcal{T})$ is weakly T_3 but *not* T_3.

Since $(\mathfrak{Q}_1, *)$ is a projective right $\mathfrak{Q}_1$-module, we will present an expansion of this situation in Example 6.21.

Remark 6.11. It follows now from Examples 6.8, 6.9 and 6.10 that in general the implications $T_3 \Rightarrow$ (weak T_3) $\Rightarrow T_2$ cannot be reversed.

As a next step we note that in the case of two particular types of quasi-magmas weak regularity and regularity are equivalent concepts.

Proposition 6.12. *Let $\mathfrak{Q}$ be a frame (i.e. $* = \wedge$) and $(\mathfrak{Q}, \wedge)$ be the quasi-magma on $\mathfrak{Q}$. Then weak regularity implies regularity.*

Proof. We have to show that the family $\mathcal{U} = \{ f \in \mathfrak{Q}^X \mid f \le \bigvee_{g\in\mathcal{T}} g \wedge d(\mathcal{A}_{\mathcal{I}}(g), f) \}$ is a $\mathfrak{Q}$-topology. Obviously, $\mathcal{U}$ is a right $\mathfrak{Q}$-submodule of $\mathfrak{Q}^X$ and (T1) is evident. In order to verify (T2) we conclude from the isotonicity of the $\mathfrak{Q}$-adherence operator $\mathcal{A}_{\mathcal{I}}$ that for $f_1, f_2 \in \mathcal{U}$ the following relation

$$\begin{aligned} f_1 \wedge f_2 &\le \bigvee_{g_1,g_2\in\mathcal{T}} g_1 \wedge g_2 \wedge d(\mathcal{A}_{\mathcal{I}}(g_1), f_1) \wedge d(\mathcal{A}_{\mathcal{I}}(g_2), f_2) \\ &\le \bigvee_{g_1,g_2\in\mathcal{T}} g_1 \wedge g_2 \wedge d(\mathcal{A}_{\mathcal{I}}(g_1 \wedge g_2), (f_1 \wedge f_2)). \end{aligned}$$

holds. Since (T2) implies $g_1 \wedge g_2 \in \mathcal{T}$, we obtain $f_1 \wedge f_2 \in \mathcal{U}$. □

Proposition 6.13. *Let $\mathfrak{Q}$ be a complete MV-algebra with square roots and $(\mathfrak{Q}, \circledast)$ be the quasi-magma given by the monoidal mean operator on $\mathfrak{Q}$* (cf. Example 2.7 (4)). *If $\bot^{1/2} = \bot^{1/2} \to \bot$, then weak regularity implies regularity.*

Proof. The strategy of the proof is similar to that of Proposition 6.12, only the algebraic situation differs. We only have to show that the family $\mathcal{U} = \{ f \in \mathfrak{Q}^X \mid f \le \bigvee_{g\in\mathcal{T}} g * d(\mathcal{A}_{\mathcal{I}}(g), f) \}$ satisfies (T2). We first recall the following properties of strict MV-algebras (cf. [17, Prop. 2.11 and 2.17]):

$$(\alpha \to \beta)^{1/2} = \alpha^{1/2} \to \beta^{1/2}, \quad (\alpha * \beta)^{1/2} = (\alpha^{1/2} * \beta^{1/2}) \vee \bot^{1/2}.$$

Further, the formation of square roots does not only preserve arbitrary meets as right adjoint of the formation of squares, but also nonempty joins in the case of strict MV-algebras. On this basis we obtain now for $f_1, f_2 \in \mathcal{U}$:

$$\begin{aligned} f_1 \circledast f_2 &\le \big(\bigvee_{g\in\mathcal{T}} g * d(\mathcal{A}(g), f_1)\big)^{1/2} * \big(\bigvee_{g\in\mathcal{T}} g * d(\mathcal{A}(g), f_2)\big)^{1/2} \\ &= \bigvee_{g_1,g_2\in\mathcal{T}} \big(g_1 * d(\mathcal{A}(g_1), f_1)\big)^{1/2} * \big(g_2 * d(\mathcal{A}(g_2), f_1)\big)^{1/2} \\ &\le \bigvee_{g_1,g_2\in\mathcal{T}} \big((g_1 \circledast g_2) * d\big((\mathcal{A}(g_1) \circledast \mathcal{A}(g_2)), f_1 \circledast f_2\big)\big) \vee \\ &\qquad \vee \big((g_1 \circledast \bot) * d\big((\mathcal{A}(g_1) \circledast \bot), f_1 \circledast f_2\big)\big) \vee \big((\bot \circledast g_2) * d\big((\bot \circledast (\mathcal{A}(g_2)), f_1 \circledast f_2\big)\big). \end{aligned}$$

Since $\circledast = \circledast_\bot$ (cf. Example 2.7 (4)), we conclude from (T2), (A1) and (A2) that $f_1 \circledast f_2 \in \mathcal{U}$. □

As an immediate corollary of Proposition 6.13 we obtain that the $\mathfrak{Q}$-topological space in Example 5.12 (3c) is not only weakly regular, but even regular and consequently a T_3 space.

Since **2**-topological spaces are cotensored, we first recall the concept of cotensored $\mathfrak{Q}$-topological spaces and review subsequently the regularity axiom in this context.

Remarks 6.14. (a) A $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ is called *cotensored* if its $\mathfrak{Q}$-topology $\mathcal{T}$ is *cotensored* — i.e. $f \in \mathcal{T}$ and $\alpha \in \mathfrak{Q}$ imply $f \swarrow \alpha \in \mathcal{T}$. In terms of $\mathfrak{Q}$-interior operators $(X, \mathcal{T})$ is cotensored if and only if the associated $\mathfrak{Q}$-interior operator $\mathcal{I}$ satisfies the following condition:

$$\mathcal{I}(f) \swarrow \alpha \le \mathcal{I}(f \swarrow \alpha), \qquad \alpha \in \mathfrak{Q},\ f \in \mathfrak{Q}^X.$$

Hence, if $(X, \mathcal{T})$ is cotensored, then for all $f \in \mathfrak{Q}^X$ and $x \in X$ the $\mathfrak{Q}$-presheaf $f \swarrow \mathcal{I}(f)(x)$ is always a neighborhood of x. Now we observe that for any neighborhood base $\mathbb{B}_x$ of x (cf. Section 4) the $\mathfrak{Q}$-interior operator $\mathcal{I}$ and the $\mathfrak{Q}$-adherence operator $\mathcal{A}_{\mathcal{I}}$ of a cotensored $\mathfrak{Q}$-topology can be represented as follows:

$$\mathcal{I}(f)(x) = \bigvee_{b\in\mathbb{B}_x} \big(\bigwedge_{y\in X} b(y) \searrow f(y)\big) = \bigvee_{b\in\mathbb{B}_x} d(b, f), \tag{6.2}$$

$$\mathcal{A}_{\mathcal{I}}(f)(x) = \bigwedge_{b\in\mathbb{B}_x} \big(\bigvee_{y\in X} f(y) * b(y)\big). \tag{6.3}$$

(b) A cotensored $\mathfrak{Q}$-topological space is called *probabilistic regular* if for each $x \in X$ the set of all closed neighborhoods of x constitute a neighborhood base of x — i.e. if for all $f \in \mathbb{U}_x$ the relation $e \le \bigvee_{k\in\mathbb{U}_x} d(\mathcal{A}_{\mathcal{I}}(k), f)$ holds (cf.

(6.3) and [16, Def. 3.3]). Now we refer to (6.2) and observe that in any probabilistic regular, cotensored $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ with the associated $\mathfrak{Q}$-interior operator $\mathcal{I}$ the following relation holds:

$$f(x) = \mathcal{I}(f)(x) = \bigvee_{h\in\mathbb{U}_x} d(h, f) \le \bigvee_{h\in\mathbb{U}_x} \bigvee_{k\in\mathbb{U}_x} d(\mathcal{A}_{\mathcal{I}}(k), h) * d(h, f)$$
$$\le \bigvee_{k\in\mathbb{U}_x} d(\mathcal{A}_{\mathcal{I}}(k), f) \le \bigvee_{k\in\mathbb{U}_x} \mathcal{I}(k)(x) * d(\mathcal{A}_{\mathcal{I}}(\mathcal{I}(k)), f) \le \bigvee_{g\in\mathcal{T}} g(x) * d(\mathcal{A}_{\mathcal{I}}(g), f).$$

This means that probabilistic regularity implies regularity. If $\mathfrak{Q}$ is commutative, then

$$g(x) \le d\big(g \swarrow g(x), g\big) \le d\big(\mathcal{A}_{\mathcal{I}}(g \swarrow g(x)), \mathcal{A}_{\mathcal{I}}(g)\big)$$

follows. Hence in the commutative case regularity is equivalent to probabilistic regularity.

Before we proceed we make a small digression in order to provide a significant generalization of Proposition 6.6.

We start by introducing a special class of quasi-magmas: A quasi-magma $(\mathfrak{Q}, \diamond)$ is *dominating* (cf. [15]) if $\diamond$ satisfies the following conditions:

$$\alpha \le (\alpha \diamond e) \wedge (e \diamond \alpha), \quad \alpha \in \mathfrak{Q}$$
$$(\alpha_1 \diamond \beta_1) * (\alpha_2 \diamond \beta_2) \le (\alpha_1 * \alpha_2) \diamond (\beta_1 * \beta_2), \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathfrak{Q}.$$

Since $\alpha * \beta \le (\alpha \diamond e) * (e \diamond \beta) \le \alpha \diamond \beta$ for each $\alpha, \beta \in \mathfrak{Q}$, the terminology is justified. In particular, if

$$(\top \diamond \bot) \vee (\bot \diamond \top) \ne \top,$$

then a dominating quasi-magma $(\mathfrak{Q}, \diamond)$ is always strict.

Simple examples of dominating strict quasi-magma operations are commutative quantale multiplications or the binary meet on non-commutative and integral quantales.

The next result expresses the change of base for cotensored topologies.

Proposition 6.15. *Let $\mathfrak{Q}_1 = (\mathfrak{Q}_1, *_1, e_1)$ and $\mathfrak{Q}_2 = (\mathfrak{Q}_2, *_2, e_2)$ be unital quantales with the respective quasi-magma operations $\diamond_1$ and $\diamond_2$ such that $e_1 \le e_1 \diamond_1 e_1$ and $\diamond_2$ is dominating and non-empty join-preserving in each variable separately. Further, let $\mathfrak{Q}_1 \xrightarrow{\varphi} \mathfrak{Q}_2$ be a quantale homomorphism satisfying the properties*

(i) $e_2 \le \varphi(e_1)$.
(ii) $\varphi(\alpha \diamond_1 \beta) = \varphi(\alpha) \diamond_2 \varphi(\beta), \quad \alpha, \beta \in \mathfrak{Q}_1$.

If $(X, \mathcal{T})$ is a cotensored $\mathfrak{Q}_1$-topological space, then the $\mathfrak{Q}_2$-interior operator $\mathcal{I}_\varphi$ corresponding to the $\mathfrak{Q}_2$-topology $\mathcal{T}_\varphi$ generated by $\{\varphi \circ f \mid f \in \mathcal{T}\}$ has the following form:

$$\mathcal{I}_\varphi(f)(x) = \bigvee_{g\in\mathcal{T},\, e_1 \le g(x)} d_{\mathfrak{Q}_2}(\varphi \circ g, f), \qquad f \in \mathfrak{Q}_2^X,\ x \in X. \tag{6.4}$$

Proof. First let us denote the respective $\mathfrak{Q}_i$-preorders on $\mathfrak{Q}_i^X$ $(i = 1, 2)$ by:

$$d_{\mathfrak{Q}_i}(f, g) = \bigwedge_{x\in X} f(x) \searrow g(x), \qquad f, g \in \mathfrak{Q}_i^X.$$

Further, let $\mathbb{B}_x = \{\, b \in \mathcal{T} \mid e_1 \le b(x)\,\}$. Since $\mathcal{T}$ is cotensored, the $\mathfrak{Q}_1$-interior operator of $\mathcal{T}$ has the form (6.2):

$$\mathcal{I}(f)(x) = \bigvee_{b\in\mathbb{B}_x} d_{\mathfrak{Q}_1}\big(b, f\big), \qquad f \in \mathfrak{Q}_1^X,\ x \in X. \tag{6.5}$$

Now we introduce a $\mathfrak{Q}_2^{op}$-functor $\mathfrak{Q}_2^X \xrightarrow{\mathcal{I}_\varphi} \mathfrak{Q}_2^X$ defined by

$$\mathcal{I}_\varphi(g)(x) := \bigvee_{b\in\mathbb{B}_x} d_{\mathfrak{Q}_2}(\varphi \circ b, g), \qquad g \in \mathfrak{Q}_2^X,\ x \in X. \tag{6.6}$$

(a) We show that $\mathcal{I}_\varphi$ is a $\mathfrak{Q}_2$-interior operator. The property (I1) is evident and (I2) follows immediately from (T2), (ii) and the fact that $\diamond_2$ is dominating and non-empty join-preserving in each variable separately. Further, if $g \in \mathfrak{Q}_2^X$ and $x \in X$, then

$$\mathcal{I}_\varphi(g)(x) \le \bigvee_{b\in\mathbb{B}_x} \varphi(b(x)) \searrow g(x) \le \varphi(e_1) \searrow g(x) \le e_2 \searrow g(x) = g(x).$$

Hence we only have to prove the idempotency of $\mathcal{I}_\varphi$. We first note that if $b \in \mathbb{B}_x$ then

$$e_1 \le b(x) = \mathcal{I}(\mathcal{I}(b))(x) = \bigvee_{\widetilde{b}\in\mathbb{B}_x} d_{\mathfrak{Q}_1}\big(\widetilde{b}, \mathcal{I}(b)\big) = \bigvee_{\widetilde{b}\in\mathbb{B}_x} \Big(\bigwedge_{y\in X} \widetilde{b}(y) \searrow \Big(\bigvee_{c\in\mathbb{B}_y} d_{\mathfrak{Q}_1}(c,b)\Big)\Big).$$

Since φ is a quantale homomorphism satisfying $e_2 \le \varphi(e_1)$, we obtain:

$$\begin{aligned}\mathcal{I}_\varphi(g)(x) &= \bigvee_{b\in\mathbb{B}_x} e_2 *_2 d_{\mathfrak{Q}_2}(\varphi\circ b, g)\\ &\le \bigvee_{b\in\mathbb{B}_x}\Big(\bigvee_{\widetilde{b}\in\mathbb{B}_x}\Big(\bigwedge_{y\in X}\big(\varphi\circ\widetilde{b}\big)(y)\searrow\Big(\bigvee_{c\in\mathbb{B}_y} d_{\mathfrak{Q}_2}(\varphi\circ c,\varphi\circ b)\Big)\Big)\Big) *_2 d_{\mathfrak{Q}_2}(\varphi\circ b, g)\\ &\le \bigvee_{b\in\mathbb{B}_x}\Big(\bigvee_{\widetilde{b}\in\mathbb{B}_x}\Big(\bigwedge_{y\in X}\big(\varphi\circ\widetilde{b}\big)(y)\searrow\Big(\bigvee_{c\in\mathbb{B}_y} d_{\mathfrak{Q}_2}(\varphi\circ c,\varphi\circ b) *_2 d_{\mathfrak{Q}_2}(\varphi\circ b, g)\Big)\Big)\Big)\\ &\le \bigvee_{\widetilde{b}\in\mathbb{B}_x}\Big(\bigwedge_{y\in X}\big(\varphi\circ\widetilde{b}\big)(y)\searrow \mathcal{I}_\varphi(g)(y)\Big) = \mathcal{I}_\varphi(\mathcal{I}_\varphi(g))(x).\end{aligned}$$

(b) As an immediate corollary from (6.6) and the idempotency of $\mathcal{I}_\varphi$ we obtain:

$$\mathcal{I}_\varphi(g)(x) = \bigvee_{b\in\mathbb{B}_x} (\varphi\circ b)(x) *_2 d_{\mathfrak{Q}_2}(\varphi\circ b, \mathcal{I}_\varphi(g)) = \bigvee_{f\in\mathcal{T}} (\varphi\circ f)(x) *_2 d_{\mathfrak{Q}_2}(\varphi\circ f, \mathcal{I}_\varphi(g))$$

for each $g \in \mathfrak{Q}_2^X$ and $x \in X$. Hence $\{\mathcal{I}_\varphi(g) \mid g \in \mathfrak{Q}_2^X\} \subseteq \mathcal{T}_\varphi$ follows. On the other hand, we conclude from the idempotency of $\mathcal{I}$ and (6.5) that $\varphi\circ\mathcal{I}(f) = \mathcal{I}_\varphi(\varphi\circ\mathcal{I}(f))$ for all $f \in \mathfrak{Q}_1^X$ holds, and so the subbase $\{\varphi\circ f \mid f\in\mathcal{T}\}$ of $\mathcal{T}_\varphi$ is contained in $\{\mathcal{I}_\varphi(g) \mid f \in \mathfrak{Q}_2^X\}$ — i.e. $\{\mathcal{I}_\varphi(g) \mid g \in \mathfrak{Q}_2^X\} = \mathcal{T}_\varphi$. Thus $\mathcal{I}_\varphi$ is the $\mathfrak{Q}_2$-interior operator corresponding to $\mathcal{T}_\varphi$, and so (6.4) is verified. □

Finally we have the announced result extending Proposition 6.6.

Corollary 6.16. *Let $\mathfrak{Q}_1 = (\mathfrak{Q}_1, *_1, e_1)$ and $\mathfrak{Q}_2 = (\mathfrak{Q}_2, *_2, e_2)$ be unital quantales with the respective quasi-magma operations $\diamond_1$ and $\diamond_2$ such that $e_1 \le e_1 \diamond_1 e_1$ and $\diamond_2$ is dominating and non-empty join-preserving in each variable separately. Further, let $\mathfrak{Q}_1 \xrightarrow{\varphi} \mathfrak{Q}_2$ be a quantale homomorphism satisfying the properties*

(i) $e_2 \le \varphi(e_1)$.
(ii) $\varphi(\alpha \diamond_1 \beta) = \varphi(\alpha) \diamond_2 \varphi(\beta), \quad \alpha,\beta \in \mathfrak{Q}_1$.
(iii) *φ is nonempty meet-preserving.*

If $(X,\mathcal{T})$ is a regular and cotensored $\mathfrak{Q}_1$-topological space, then $(X,\mathcal{T}_\varphi)$ is a regular $\mathfrak{Q}_2$-topological space, where $\mathcal{T}_\varphi$ is generated by $\{\varphi\circ f \mid f\in\mathcal{T}\}$.

Proof. Let $x \in X$ and $\mathbb{B}_x = \{b \in \mathcal{T} \mid e_1 \le b(x)\}$. If $f \in \mathfrak{Q}_1^X$, since $\mathcal{T}$ is cotensored, we apply (6.3) and obtain that the relation

$$\big(\varphi\circ\mathcal{A}_\mathcal{I}(f)\big)(x) = \bigwedge_{b\in\mathbb{B}_x}\Big(\bigvee_{y\in X} (\varphi\circ f)(y) *_2 (\varphi\circ b)(y)\Big) \ge \mathcal{A}_{\mathcal{I}_\varphi}(\varphi\circ f)(x). \tag{6.7}$$

holds, since $\varphi\circ b$ is a neighborhood of x w.r.t. $\mathcal{T}_\varphi$ for all $b \in \mathbb{B}_x$. Now we fix $g \in \mathcal{T}_\varphi$. We conclude from Proposition 6.15 that the following relation holds:

$$g(x) = \bigvee_{f\in\mathcal{T},\, e_1\le f(x)} (\varphi\circ f)(x) *_2 d_{\mathfrak{Q}_2}(\varphi\circ f, g), \qquad x \in X. \tag{6.8}$$

Since $\mathcal{T}$ is regular, for each $f \in \mathcal{T}$ one has

$$f(x) \le \bigvee_{h\in\mathcal{T}} h(x) *_1 d_{\mathfrak{Q}_1}(\mathcal{A}_{\mathcal{I}}(h), f), \qquad x \in X.$$

Now we apply the quantale homomorphism φ and (6.7) and obtain for each $f \in \mathcal{T}$ and $x \in X$:

$$(\varphi\circ f)(x) \le \bigvee_{h\in\mathcal{T}} (\varphi\circ h)(x) *_2 d_{\mathfrak{Q}_2}((\varphi\circ\mathcal{A}_{\mathcal{I}})(h), \varphi\circ f) \le \bigvee_{h\in\mathcal{T}} (\varphi\circ h)(x) *_2 d_{\mathfrak{Q}_2}(\mathcal{A}_{\mathcal{I}_\varphi}(\varphi\circ h), \varphi\circ f).$$

Combining this with (6.8) it follows that

$$\begin{aligned} g(x) &\le \bigvee_{f,h\in\mathcal{T},\, e_1\le f(x)} (\varphi\circ h)(x) *_2 d_{\mathfrak{Q}_2}(\mathcal{A}_{\mathcal{I}_\varphi}(\varphi\circ h), \varphi\circ f) *_2 d_{\mathfrak{Q}_2}(\varphi\circ f, g) \\ &\le \bigvee_{h\in\mathcal{T}} (\varphi\circ h)(x) *_2 d_{\mathfrak{Q}_2}(\mathcal{A}_{\mathcal{I}_\varphi}(\varphi\circ h), g). \end{aligned}$$

Hence $(X, \mathcal{T}_\varphi)$ is regular. □

We continue with a discuss on further concepts of regularity having appeared in the literature.

Remarks 6.17. (1) A $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ is H-R-*regular* if $f \le \bigvee\{ g \in \mathcal{T} \mid \mathcal{A}_{\mathcal{I}}(g) \le f \}$ for all $f \in \mathcal{T}$ (cf. [19, Def. (R,T_3)]). Hence H-R-regularity implies regularity. If $\mathcal{T}$ is cotensored and $\mathfrak{Q}$ is a commutative Girard quantale, then $\mathcal{A}_{\mathcal{I}}(g) * \alpha$ is a closed $\mathfrak{Q}$-presheaf for any $\alpha \in \mathfrak{Q}$, and consequently regularity is equivalent to H-R-regularity. If $\mathfrak{Q}$ is not commutative, then H-R-regularity is strictly stronger than regularity as the next counterexample shows. Let $\mathfrak{Q} = [C_3, C_3]$ be the unital quantale of all join-preserving self-maps of the 3-chain (cf. [14, Subsec. 5.1]) and $\diamond$ be a directed join-preserving quasi-magma operation on $\mathfrak{Q}$ in each variable separately. Obviously, $\mathfrak{Q}$ is a non-integral, non-commutative Girard quantale, and the subquantale of all left-sided elements of $\mathfrak{Q}$ is the left-sided, idempotent and non-commutative quantale C_3^ℓ on the 3-chain. Since $\mathfrak{Q}$ is finite, the change of base by $\mathbf{2} \overset{h}{\hookrightarrow} \mathfrak{Q}$ leads to a cotensored $\mathfrak{Q}$-topological space $(X, \mathsf{LSC}(\tau, C_3^\ell))$. Hence, if (X, τ) is regular, then $(X, \mathsf{LSC}(\tau, C_3^\ell))$ is regular (cf. Proposition 6.6) but not H-R-regular, because there exists a *left-sided* element of $\mathfrak{Q} = [C_3, C_3]$ being not *right-sided*.
(2) It is well known that the monadic framework for $\mathfrak{Q}$-topological spaces is given by the monad of $\mathfrak{Q}$-filters (cf. [13, Sec. 5]). On the other hand the regularity axiom can also be based on the *closure operator* derived from a *partially ordered monad* on Set (cf. [9] and [18, Sec. 3.4]). Unfortunately, the relationship between this concept of monadic regularity going back to Gähler (see [18, p. 73]) and regularity in the sense of our paper is an open question at the moment. One reason for this is the situation that we do not know much about $\mathfrak{Q}$-filter extension theorems and maximal $\mathfrak{Q}$-filters.

In the next example we present a regular $\mathfrak{Q}$-topological space, which is not H-R-regular and not induced via change of base conveyed by $\mathbf{2} \hookrightarrow \mathfrak{Q}$.

Example 6.18. Let $\mathfrak{Q}_2 = (C_3, *)$ be the commutative, idempotent and non-integral Girard quantale on $C_3 = \{\bot, a, \top\}$ (cf. Example 2.1). Further, we consider the quasi-magma operation $\diamond = *$ on $\mathfrak{Q}$ and the $\mathfrak{Q}_2$-topology $\mathcal{T}$ on C_3 generated by the subbase $\mathcal{S} = \{\mathrm{id}_{C_3}, \mathrm{id}_{C_3} \to a\}$ — i.e.

$$\mathcal{T} = \{ f_1, f_2, f_3, f_4, f_1 * \top, f_2 * \top, f_3 * \top, f_4 * \top \} \cup \{\underline{\bot}\}$$

where $f_1 = \mathrm{id}_{C_3}$, $f_2 = \mathrm{id}_{C_3} \to a$, $f_3 = \mathrm{id}_{C_3} * (\mathrm{id}_{C_3} \to a)$ and $f_4 = \mathrm{id}_{C_3} \vee (\mathrm{id}_{C_3} \to a)$. Since f_1, f_2, f_3 and f_4 are open and closed in $(C_3, \mathcal{T})$ by Lemma 4.4, it follows that

$$f_i = f_i * d(f_i, f_i) = f_i * d(\mathcal{A}_{\mathcal{I}}(f_i), f_i) \le \bigvee_{g\in\mathcal{T}} g * d(\mathcal{A}_{\mathcal{I}}(g), f_i)$$

and

$$f_i * \top = f_i * d(f_i, f_i * \top) = f_i * d(\mathcal{A}_{\mathcal{I}}(f_i), f_i * \top) \le \bigvee_{g\in\mathcal{T}} g * d(\mathcal{A}_{\mathcal{I}}(g), f_i * \top),$$

for each $i = 1, 2, 3, 4$, hence $(C_3, \mathcal{T})$ is regular. However it is *not* H-R-regular. Indeed, since $\mathcal{A}_{\mathcal{I}}(f_i * \top) = \underline{\top}$ for each $i = 1, 2, 3, 4$, we have that $f_1 * \top \not\leq f_1 = \bigvee\{ g \in \mathcal{T} \mid \mathcal{A}_{\mathcal{I}}(g) \leq f_1 * \top \}$.

Moreover, with regard to Remark 6.17 (1) we point out that $(C_3, \mathcal{T})$ is *not* induced by a traditional topological space via change of base conveyed by $\mathbf{2} \overset{h}{\hookrightarrow} \mathfrak{Q}_2$.

The previous example and remarks show that in general regularity and a fortiori weak regularity are strictly weaker than well known regularity notions in the literature with the exception of monadic regularity. But now we will show that even weak regularity (more precisely the weak T_3 axiom) will suffice to establish the principle of continuous extension for $\mathfrak{Q}$-topological spaces.

Standing Assumption. For the remaining part of this section we are working in the framework of the quasi-magma $(\mathfrak{Q}, *)$ — i.e. $\diamond = *$.
In this context we would like to emphasize that the expression of the intersection axiom by the quantale multiplication goes already back to Goguen in 1973 (cf. [10]).

It follows from Examples 5.12 (1) and (2) and Examples 6.8 and 6.9 that in general the implications

$$(\text{weak } T_3) \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

cannot be reversed. But under the assumption of weak regularity the situation changes fundamentally as the next lemma shows.

Lemma 6.19. *Let $\mathfrak{Q}$ be quantale with a dualizing element. Then any weakly regular and Kolmogorov separated $\mathfrak{Q}$-topological space is weakly T_3.*

Proof. Let δ be a dualizing element of $\mathfrak{Q}$, $(X, \mathcal{T})$ a regular and Kolmogorov separated $\mathfrak{Q}$-topological space and $\mathcal{A}_{\mathcal{I}}$ the corresponding $\mathfrak{Q}$-adherence operator. In order to prove that $(X, \mathcal{T})$ is Hausdorff separated we fix $x, y \in X$ with $x \neq y$. Since the set

$$\mathcal{S} = \Big\{ f \in \mathcal{T} \mid f \leq \bigvee_{g \in \mathcal{T}} g * d(\mathcal{A}_{\mathcal{I}}(g), f) \Big\}$$

is a subbase of $\mathcal{T}$, Lemma 5.4implies that there exists $f \in \mathcal{S}$ such that $f(x) \not\leq f(y)$ or $f(y) \not\leq f(x)$. Let us assume, w.l.o.g., that $f(x) \not\leq f(y)$. For each $g \in \mathcal{T}$ the $\mathfrak{Q}$-presheaf $\mathcal{A}_{\mathcal{I}}(g)$ is closed, and consequently we have $\mathcal{A}(g) \searrow \delta \in \mathcal{T}$ (cf. Lemma 4.4). Since $f(x) \leq \bigvee_{g \in \mathcal{T}} g(x) * d(\mathcal{A}_{\mathcal{I}}(g), f)$ and $\bigvee_{g \in \mathcal{T}} \mathcal{A}_{\mathcal{I}}(g)(y) * d\big(\mathcal{A}_{\mathcal{I}}(g), f\big) \leq f(y)$ it follows that

$$\bigvee_{g \in \mathcal{T}} g(x) * d(\mathcal{A}_{\mathcal{I}}(g), f) \not\leq \bigvee_{g \in \mathcal{T}} \mathcal{A}_{\mathcal{I}}(g)(y) * d\big(\mathcal{A}_{\mathcal{I}}(g), f\big).$$

Hence there exists $g \in \mathcal{T}$ such that $g(x) \not\leq \mathcal{A}_{\mathcal{I}}(g)(y) = \delta \swarrow (\mathcal{A}_{\mathcal{I}}(g)(y) \searrow \delta)$ — this means $g(x) * (\mathcal{A}_{\mathcal{I}}(g)(y) \searrow \delta) \not\leq \delta$. Since $\bigvee_{y \in X} g(y) * (\mathcal{A}_{\mathcal{I}}(g)(y) \searrow \delta) \leq \delta$, Lemma 5.9 implies that $(X, \mathcal{T})$ is Hausdorff separated. □

As an illustration of Lemma 6.19 we insert the following examples.

Example 6.20. Let $\mathfrak{Q}$ be a Girard quantale. Then we choose a cyclic and dualizing element $\delta \in Q$ and consider the $\mathfrak{Q}$-topology $\mathcal{T}$ on $\mathfrak{Q}$ generated by $\{ \mathrm{id}_{\mathfrak{Q}}, (\mathrm{id}_{\mathfrak{Q}} \searrow \delta) \}$. Since $\mathrm{id}_{\mathfrak{Q}}$ and $\mathrm{id}_{\mathfrak{Q}} \searrow \delta$ are also closed in $(\mathfrak{Q}, \mathcal{T})$ by Lemma 4.4, it is easily seen that $(\mathfrak{Q}, \mathcal{T})$ is Kolmogorov separated and weakly regular. Hence $(\mathfrak{Q}, \mathcal{T})$ is weakly T_3.

The next example describes a class of $\mathfrak{Q}$-topological spaces satisfying the weak T_3 axiom.

Example 6.21. Let $\mathfrak{Q}$ be a commutative Girard quantale and M be a projective $\mathfrak{Q}$-module in Sup with the right action $\boxdot$. We recall the intrinsic $\mathfrak{Q}$-preorder p (see (3.1)) and note that (M, p) is cocomplete (cf. (3.2)) and the totally below relation $\lhd$ is approximating (for more details see [14, Sec. 3] and [33]). Now we choose a dualizing element $\delta \in \mathfrak{Q}$ and consider the *interval* $\mathfrak{Q}$*-topology* $\mathcal{T}_I$ on M generated by the following subbase[2]:

$$\{ p(\_, m) \searrow \delta \mid m \in M \} \cup \{ p(n, \_) \searrow \delta \mid n \in M \}.$$

Obviously, $(M, \mathcal{T}_I)$ is Kolmogorov separated (see Lemma 5.4). Further, for each $n \in M$ we put $s_n := \mathsf{sup}(p(n, \_) \searrow \delta)$. Since $m \longmapsto \lhd(\_, m)$ is left adjoint to sup, we obtain:

$$\begin{aligned} p(m, s_n) &= d\big(\lhd(\_, m), p(n, \_) \searrow \delta\big) = \bigwedge_{o \in M} \big(\lhd(o, m) \searrow (p(n, o) \searrow \delta)\big) \\ &= \big(\bigvee_{o \in M} p(n, o) * \lhd(o, m)\big) \searrow \delta = \lhd(n, m) \searrow \delta. \end{aligned}$$

Hence $p(m, s_n) \searrow \delta = \lhd(n, m)$ — i.e. $\lhd(n, \_) \in \mathcal{T}_I$. Using again the left adjointness of $m \longmapsto \lhd(\_, m)$ to sup we note $\bigwedge_{o \in M} \lhd(o, m) \searrow p(o, n) = p(m, n)$, and consequently we have $p(n, m) \searrow \delta = \bigvee_{o \in M} \lhd(o, m) * (p(o, n) \searrow \delta)$ — this means:

$$p(m, n) \searrow \delta = \bigvee_{o \in M} \lhd(o, m) * d\big(p(o, \_), p(\_, n) \searrow \delta\big). \tag{6.9}$$

On the other hand, we conclude from [14, Thm. 4.1] that the dual $\mathfrak{Q}$-module M^{op} of M is also projective. Hence the totally below relation $\lhd^{op}$ w.r.t. the dual $\mathfrak{Q}$-preorder p^{op} (i.e. $p^{op}(x, y) = p(y, x)$) is also approximating. By analogy to (6.9) we obtain:

$$p(n, m) \searrow \delta = \bigvee_{o \in M} \lhd^{op}(o, m) * d\big(p(\_, o), p(n, \_) \searrow \delta\big). \tag{6.10}$$

Since $\lhd(o, \_) \le p(o, \_)$ and $\lhd^{op}(o, \_) \le p^{op}(o, \_) = p(\_, o)$ and the $\mathfrak{Q}$-presheaves $p(o, \_)$ and $p(o, \_)$ are closed w.r.t. the interval $\mathfrak{Q}$-topology, we conclude from (6.9) and (6.10) that $(M, \mathcal{T}_I)$ is weakly regular. To sum up M provided with the interval $\mathfrak{Q}$-topology satisfies the *weak* T_3 *axiom*, but is in general not regular (cf. Example 6.10).

In a next step we prepare the principle of $\mathfrak{Q}$-continuous extension.

Lemma 6.22. *Let $\mathfrak{Q}$ be quantale with a dualizing element, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $\mathfrak{Q}$-topological spaces and A be a dense subset in $(X, \mathcal{T}_X)$. Further, let $(Y, \mathcal{T}_Y)$ be weakly regular and $A \xrightarrow{\psi} Y$ be a map. If $X \xrightarrow{\varphi} Y$ is a map such that for all $x \in X$ the point $\varphi(x)$ is a limit point of the image $\mathfrak{Q}$-filter $\psi((\nu_x)_A)$, then φ is $\mathfrak{Q}$-continuous.*

Proof. (a) Let us fix $x \in X$ and consider the $\mathfrak{Q}$-neighborhood filter ν_x at x w.r.t. $\mathcal{T}_X$ and the $\mathfrak{Q}$-neighborhood filter $\nu_{\varphi(x)}$ at $\varphi(x)$ w.r.t. $\mathcal{T}_Y$. Since A is dense, $(\nu_x)_A$ is a $\mathfrak{Q}$-filter on A (cf. Proposition 6.4 (1)). Further, we choose a dualizing element $\delta \in \mathfrak{Q}$. Since $\varphi(x)$ is a limit point of the $\mathfrak{Q}$-filter $\psi\big((\nu_x)_A\big)$, we obtain for all $h \in \mathfrak{Q}^Y$:

$$\psi\big((\nu_x)_A\big)(h) * \nu_{\varphi(x)}(h \searrow \delta) \le \psi\big((\nu_x)_A\big)(h) * \psi\big((\nu_x)_A\big)(h \searrow \delta) \le \psi\big((\nu_x)_A\big)(\underline{\delta}) \le \delta.$$

Consequently, if $\mathcal{A}_Y$ is the $\mathfrak{Q}$-adherence operator of $(Y, \mathcal{T}_Y)$, then by (6.1), (3.10) and the definition of the image $\mathfrak{Q}$-filter $\psi\big((\nu_x)_A\big)$ it follows that

$$\mathcal{I}_X(\widehat{h \circ \psi})(x) = \nu_x(\widehat{h \circ \psi}) = (\nu_x)_A(h \circ \psi) = \psi\big((\nu_x)_A\big)(h) \le \delta \swarrow (\nu_{\varphi(x)}(h \searrow \delta)) = \big(\mathcal{A}_Y(h)\big)(\varphi(x)).$$

Consequently $\mathcal{I}_X(\widehat{h \circ \psi}) \le \mathcal{A}_Y(h) \circ \varphi$. Since $\mathcal{I}_X$ is idempotent, the relation

$$\psi\big((\nu_x)_A\big)(h) = \mathcal{I}_X(\widehat{h \circ \psi}) \le \mathcal{I}_X(\mathcal{A}_Y(h) \circ \varphi)$$

holds. Using again the fact that $\varphi(x)$ is a limit point of $\psi\big((\nu_x)_A\big)$ we finally have:

$$\nu_{\varphi(x)}(h) \le \psi\big((\nu_x)_A\big)(h) \le \big(\mathcal{I}_X(\mathcal{A}_Y(h) \circ \varphi)\big)(x). \tag{6.11}$$

[2] If a unital quantale $\mathfrak{Q}$ is viewed as a projective right $\mathfrak{Q}$- module with respect to its right multiplication, then it is easily seen that the $\mathfrak{Q}$- topologies in Examples 6.10 and 6.18 are necessarily interval $\mathfrak{Q}$- topologies.

(b) Now we distinguish between the respective intrinsic $\mathfrak{Q}$-preorders on $\mathfrak{Q}^X$ and $\mathfrak{Q}^Y$ and denote them by d_X and d_Y. Since $(Y, \mathcal{T}_Y)$ is weakly regular, the set

$$\mathcal{S} = \{ g \in \mathcal{T}_Y \mid g \le \bigvee_{h \in \mathcal{T}_Y} h * d_Y\big(\mathcal{A}_Y(h), g\big) \}$$

is a subbase of $\mathcal{T}_Y$. Let $g \in \mathcal{S}$ and $x \in X$. We conclude from (6.11) and (I0):

$$\begin{aligned}(g \circ \varphi)(x) &\le \bigvee_{h \in \mathcal{T}_Y} h(\varphi(x)) * d_Y(\mathcal{A}_Y(h), g) = \bigvee_{h \in \mathcal{T}_Y} \nu_{\varphi(x)}(h) * d_Y(\mathcal{A}_Y(h), g)\\ &\le \bigvee_{h \in \mathcal{T}_Y} \big(\mathcal{I}_X(\mathcal{A}_Y(h) \circ \varphi)\big)(x) * d_X(\mathcal{A}_Y(h) \circ \varphi, g \circ \varphi) \le \big(\mathcal{I}_X(g \circ \varphi)\big)(x).\end{aligned}$$

Hence $g \circ \varphi \in \mathcal{T}_X$. Since $\mathcal{S}$ is a subbase, the $\mathfrak{Q}$-continuity of φ follows. □

The next theorem is the $\mathfrak{Q}$-enriched version of the principle of continuous extension whose set-theoretical version goes back to Bourbaki and Dieudonné in 1939 (see [4, p. 180], see also [5, Thm. 1 on p. TG I.57]).

Theorem 6.23 *(Principle of $\mathfrak{Q}$-continuous extension). Let $\mathfrak{Q}$ be quantale with a dualizing element, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $\mathfrak{Q}$-topological spaces, A be a dense subset in $(X, \mathcal{T}_X)$ and $A \overset{\iota}{\hookrightarrow} X$ be the inclusion map. Further, let $(Y, \mathcal{T}_Y)$ satisfy the weak T_3 axiom and $A \xrightarrow{\psi} Y$ be a $\mathfrak{Q}$-continuous map w.r.t. the initial $\mathfrak{Q}$-topology on A induced by $\mathcal{T}_X$. Then the following assertions are equivalent:*

(1) *ψ has a unique $\mathfrak{Q}$-continuous extension to X — i.e. there exists a unique $\mathfrak{Q}$-continuous map $X \xrightarrow{\varphi} Y$ making the following diagram commutative:*

$$\begin{array}{ccc} A & \overset{\iota}{\hookrightarrow} & X \\ & {\scriptstyle\psi}\searrow & \downarrow{\scriptstyle\varphi} \\ & & Y \end{array}$$

(2) *There exists a map $X \xrightarrow{\varphi} Y$ such that for all $x \in X$ the point $\varphi(x)$ is a limit point of $\psi\big((\nu_x)_A\big)$, where $\psi\big((\nu_x)_A\big)$ denotes the image of the trace of the $\mathfrak{Q}$-neighborhood filter ν_x on A under ψ.*

Proof. (1) $\Rightarrow$ (2): Let $X \xrightarrow{\varphi} Y$ be the $\mathfrak{Q}$-continuous extension of ψ. Since $g \circ \varphi \le \widehat{g \circ \varphi \circ \iota}$, for each $g \in \mathfrak{Q}^Y$ and $x \in X$, the $\mathfrak{Q}$-continuity of φ implies:

$$\nu_{\varphi(x)}(g) \le \nu_x(g \circ \varphi) \le \nu_x\big(\widehat{g \circ \varphi \circ \iota}\big) = (\nu_x)_A(g \circ \psi) = \psi\big((\nu_x)_A\big)(g),$$

where ν_x and $\nu_{\varphi(x)}$ are the respective $\mathfrak{Q}$-neighborhood filters. Hence $\varphi(x)$ is a limit point of the image $\mathfrak{Q}$-filter $\psi((\nu_x)_A)$.

(2) $\Rightarrow$ (1): Let $X \xrightarrow{\varphi} Y$ be such that for all $x \in X$ the point $\varphi(x)$ is a limit point of $\psi((\nu_x)_A)$. Since $(Y, \mathcal{T}_Y)$ is Hausdorff separated, we infer from Proposition 6.4 (2) and Comments 6.2 (4) and (5) that $\varphi(x)$ and $\psi(x)$ coincide for each $x \in A$ — i.e. the map φ is an extension of ψ. The $\mathfrak{Q}$-continuity of φ is guaranteed by Lemma 6.22. Finally the unicity follows from Proposition 5.13. □

Historical remark. If $\mathfrak{Q}$ is a complete MV-algebra and the quasi-magma is given by the binary meet operation, then Theorem 6.23 has already been established for cotensored $\mathfrak{Q}$-topological (resp. probabilistic topological) spaces in 1982 (cf. [16, Thm. 3.5]).

Since Lemma 6.22 depends essentially on the assumption $\diamond = *$ and the property that $\mathfrak{Q}$ has a dualizing element, we ask therefore the following:

Open Question. Let $\mathfrak{Q}$ be an non-commutative, integral quantale. Does the principle of $\mathfrak{Q}$-continuous extension hold in the case of the quasi-magma $(\mathfrak{Q}, \wedge)$?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] B. Banaschewski, S.S. Hong, Extension by continuity in pointfree topology, Appl. Categ. Struct. 8 (2000) 475–486.
[2] G. Birkhoff, Lattice Theory, 3rd ed., AMS Coll. Publ., vol. 25, Providence, R.I., 1973.
[3] F. Borceux, Handbook of Categorical Algebra Vol. 2, Categories and Structures, Cambridge University Press, 1994.
[4] N. Bourbaki, J. Dieudonné, Note de tératopologie. II, Rev. Sci. (Revue Rose) 77 (1939) 180–181.
[5] N. Bourbaki, Éléments de Mathématique, Topologie Générale Chap. I–IV, Springer, 2007.
[6] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, first edition, Cambridge Univ. Press, Cambridge, 1990, second edition, 2002.
[7] P. Eklund, J. Gutiérrez García, U. Höhle, J. Kortelainen, Semigroups in Complete Lattices: Quantales, Modules and Related Topics, Developments in Mathematics, vol. 54, Springer International Publishing, 2018.
[8] M. Fisch, A. Turquette, Peirce's triadic logic, Trans. Charles S. Peirce Soc. 2 (2) (1966) 71–85.
[9] W. Gähler, Monadic topology — a new concept of generalized topology, in: W. Gähler, H. Herrlich, G. Preuß (Eds.), Recent Developments of General Topology and Its Applications, Akademie Verlag, Berlin, 1992.
[10] J.A. Goguen, The fuzzy Tychonoff theorem, J. Math. Anal. 43 (1973) 734–742.
[11] J. Gutiérrez García, U. Höhle, Enriched topologies and topological representation of semi-unital and semi-integral quantales, Topol. Appl. 273 (2020) 106967.
[12] J. Gutiérrez García, U. Höhle, Right algebras in Sup and the topological representation of semi-unital and semi-integral quantales, revisited, submitted for publication.
[13] J. Gutiérrez García, U. Höhle, T. Kubiak, Basic concepts of quantale-enriched topologies, Appl. Categ. Struct. 29 (2021) 983–1003.
[14] J. Gutiérrez García, U. Höhle, T. Kubiak, Invariance of projective modules in Sup under self-duality, Algebra Univers. 82 (2021) 9.
[15] J. Gutiérrez García, U. Höhle, T. Kubiak, A theory of quantale-enriched dcpos and their topologization, Fuzzy Sets Syst. 444 (2022) 103–130.
[16] U. Höhle, Probabilistic topologies induced by L-fuzzy uniformities, Manuscr. Math. 38 (1982) 289–323.
[17] U. Höhle, Commutative residuated ℓ-monoids, in: U. Höhle, E.P. Klement (Eds.), Nonclassical Logics and Their Applications to Fuzzy Subsets, Kluwer Academic Publishers, 1995, pp. 53–106.
[18] U. Höhle, Many Valued Topology and Its Applications, Kluwer Academic Publishers, Dordrecht, 2001.
[19] B. Hutton, I. Reilly, Separation axioms in fuzzy topological spaces, Fuzzy Sets Syst. 3 (1980) 93–104.
[20] A. Joyal, M. Tierney, An extension of the Galois theory of Grothendieck, Mem. Am. Math. Soc. 51 (309) (1984).
[21] G.M. Kelly, Basic Concepts of Enriched Category Theory, Lecture Notes in Math., vol. 64, Cambridge University Press, Cambridge, 1982.
[22] T. Kubiak, The topological modification of the L-fuzzy unit interval, in: S.E. Rodabaugh, et al. (Eds.), Applications of Category Theory to Fuzzy Subsets, Kluwer Academic Publishers, 1992, pp. 275–305.
[23] T. Kubiak, On L-Tychonoff spaces, Fuzzy Sets Syst. 73 (1995) 25–53.
[24] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, Fuzzy Sets Syst. 157 (14) (2006) 1865–1885.
[25] Y.-M. Liu, Pointwise characterization of complete regularity and imbedding theorem in fuzzy topological spaces, Sci. Sin., Ser. A 26 (1983) 138–147.
[26] R. Lowen, Connectedness in fuzzy topological spaces, Rocky Mt. J. Math. 11 (1981) 427–433.
[27] J. Łukasiewicz, On three-valued logic, in: L. Borkowski (Ed.), Jan Łukasiewicz Selected Works, North-Holland, 1920, pp. 87–88, First published as *O logice ťrójwartościowej* in Ruch Filoz. 5 (1920) 170–171.
[28] S. Mac Lane, Categories for the Working Mathematician, 2nd. edn., Graduate Texts in Mathematics, vol. 5, Springer, Berlin, Heidelberg, New York, 1998.
[29] S.E. Rodabaugh, Applications of localic separation axioms, compactness axioms, representations, and compactifications to poslat topological spaces, Fuzzy Sets Syst. 73 (1995) 55–87.
[30] A. Šostak, On the concept of E-regularity for fuzzy topology, Mat. Vesn. 41 (1989) 199–209.
[31] L.A. Steen, J.A. Seebach Jr., Counterexamples in Topology, Dover Publications Inc., Mineola, NY, 1995, reprint of the second (1978) edition.
[32] I. Stubbe, Categorical structures enriched in a quantaloid: tensored and cotensored categories, Theory Appl. Categ. 16 (2006) 283–306.
[33] I. Stubbe, Towards "dynamic domains". Totally continuous cocomplete $\mathcal{Q}$-categories, Theor. Comput. Sci. 373 (2007) 142–160.
[34] P. Wuyts, R. Lowen, On local and global measures of separation in fuzzy topological spaces, Fuzzy Sets Syst. 19 (1986) 51–80.