1-2024

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Citation Details

Published as: Caughman, J., & Terada, T. (2024). Proof of the Kresch-Tamvakis conjecture. Proceedings of the American Mathematical Society, 152(03), 1265-1277.

Proof of the Kresch-Tamvakis Conjecture

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September 19, 2023

Abstract

In this paper we resolve a conjecture of Kresch and Tamvakis [5]. Our result is the following.

Theorem: For any positive integer D and any integers i, j $(0 \leq i, j \leq D)$, the absolute value of the following hypergeometric series is at most 1:

$${}_4F_3 \left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ;1 \right].$$

To prove this theorem, we use the Biedenharn-Elliott identity, the theory of Leonard pairs, and the Perron-Frobenius theorem.

Keywords. Racah polynomial; Biedenharn-Elliott identity; Leonard pair; 6–j symbols.
2020 Mathematics Subject Classification. Primary 33C45; Secondary 26D15.

1 Introduction

In 2001, Kresch and Tamvakis conjectured an inequality involving certain terminating ${}_4F_3$ hypergeometric series [5, Conjecture 2]. In this paper, we prove the conjecture.

To describe the conjecture, we bring in some notation. For any real number a and nonnegative integer n, define

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1). \tag{1}$$

Let z denote an indeterminate. Given real numbers $\{a_i\}_{i=1}^4$ and $\{b_i\}_{i=1}^3$, the corresponding ${}_4F_3$ hypergeometric series is defined by

$${}_4F_3 \left[\begin{matrix} a_1,\ a_2,\ a_3,\ a_4 \\ b_1,\ b_2,\ b_3 \end{matrix} ;z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(b_1)_n(b_2)_n(b_3)_n} \frac{z^n}{n!}. \tag{2}$$

We now state the conjecture of Kresch and Tamvakis.

Conjecture 1.1. [5, Conjecture 2] *For any positive integer D and any integers i, j $(0 \leq i, j \leq D)$, the absolute value of the following hypergeometric series is at most 1:*

$${}_4F_3 \left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ;1 \right]. \tag{3}$$

Note 1.2. Conjecture 1.1 is taken from [5, Conjecture 2] with

$$n = i, \qquad s = j, \qquad T = D+1.$$

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Next we discuss the evidence for Conjecture 1.1 offered by Kresch, Tamvakis, and others.

In [5, Proposition 2], Kresch and Tamvakis prove that the absolute value of (3) is at most 1, provided that $i \le 3$ or $i = D$. In [4, p. 863], Ismail and Simeonov prove that the absolute value of (3) is at most 1, provided that $i = D - 1$ and $D \ge 6$. They also give asymptotic estimates to further support the conjecture. In [7], Mishev obtains several relations satisfied by the ${}_4F_3$ hypergeometric series in question.

In this paper, we prove Conjecture 1.1 from scratch, without invoking the above partial results. The following is a statement of our result.

Theorem 1.3. *For any positive integer D and any integers i, j $(0 \le i, j \le D)$, the absolute value of the following hypergeometric series is at most 1:*

$$ {}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix}\ ;1\right]. $$

To prove Theorem 1.3 we use the following approach. For $0 \le i \le D$ we define a certain matrix $B_i \in \mathrm{Mat}_{D+1}(\mathbb{R})$. Using the Biedenharn-Elliott identity [1, p. 356], we show that the entries of B_i are nonnegative. Using the theory of Leonard pairs [8–12], we show that the eigenvalues of B_i are $2i + 1$ times

$$ {}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix}\ ;1\right] \qquad (0 \le j \le D). $$

We also show that the all 1's vector in $\mathbb{R}^{D+1}$ is an eigenvector for B_i with eigenvalue $2i + 1$. Applying the Perron-Frobenius theorem [3, p. 529], we show that the eigenvalues of B_i have absolute value at most $2i+1$. Using these results, we obtain the proof of Theorem 1.3.

This paper is organized as follows. In Section 2, we recall the definition of a Leonard pair and give an example relevant to our work. In Section 3, we use the Leonard pair in Section 2 to define a sequence of orthogonal polynomials. In Section 4, we use these orthogonal polynomials to define the matrices $\{B_i\}_{i=0}^D$. We then compute the eigenvalues of $\{B_i\}_{i=0}^D$. In Section 5, we show that the entries of B_i are nonnegative for $0 \le i \le D$. In Section 6, we use the Perron-Frobenius theorem to prove Theorem 1.3. In the appendix, we give some details about a key formula in our proof.

Throughout this paper, the square root of a nonnegative real number is understood to be nonnegative.

2 Leonard pairs

Throughout this paper, D denotes a positive integer. Let $\mathrm{Mat}_{D+1}(\mathbb{R})$ denote the $\mathbb{R}$-algebra of all $(D+1) \times (D+1)$ matrices that have all entries in $\mathbb{R}$. We index the rows and columns by $0, 1, 2, \ldots, D$. Let $\mathbb{R}^{D+1}$ denote the vector space over $\mathbb{R}$ consisting of $(D+1) \times 1$ matrices that have all entries in $\mathbb{R}$. We index the rows by $0, 1, 2, \ldots, D$. The algebra $\mathrm{Mat}_{D+1}(\mathbb{R})$ acts on $\mathbb{R}^{D+1}$ by left multiplication.

A matrix $B \in \mathrm{Mat}_{D+1}(\mathbb{R})$ is called *tridiagonal* whenever each nonzero entry lies on the diagonal, the subdiagonal, or the superdiagonal. Assume that B is tridiagonal. Then B is called *irreducible* whenever each entry on the subdiagonal is nonzero, and each entry on the superdiagonal is nonzero.

We now recall the definition of a Leonard pair. Let V denote a vector space over $\mathbb{R}$ with dimension $D + 1$.

Definition 2.1. [10] By a *Leonard pair on V*, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

The above Leonard pair A, A^* is said to be *over* $\mathbb{R}$.

Note 2.2. According to a common notational convention, A^* denotes the conjugate-transpose of A. We are not using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i), (ii) above.

Our next goal is to give an example of a Leonard pair. To do so, we give two definitions.

Definition 2.3. Define

$$c_i = \frac{3(D-i+1)i(D+i+1)}{D(D+2)(2i+1)} \qquad (1 \le i \le D), \tag{4}$$

$$a_i = \frac{3i(i+1)}{D(D+2)} \qquad (0 \le i \le D), \tag{5}$$

$$b_i = \frac{3(D-i)(i+1)(D+i+2)}{D(D+2)(2i+1)} \qquad (0 \le i \le D-1), \tag{6}$$

$$\theta_i = 3 - 2a_i \qquad (0 \le i \le D). \tag{7}$$

We remark that the scalars $\{\theta_i\}_{i=0}^D$ are mutually distinct.

Let A, A^* denote the following matrices in $\mathrm{Mat}_{D+1}(\mathbb{R})$:

$$A = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{D-1} & a_{D-1} & b_{D-1} \\ \mathbf{0} & & & c_D & a_D \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \ddots & & \\ & & & \theta_{D-1} & \\ \mathbf{0} & & & & \theta_D \end{pmatrix}. \tag{8}$$

Definition 2.4. We define a matrix $P \in \mathrm{Mat}_{D+1}(\mathbb{R})$ with the following entries:

$$P_{i,j} = (2j+1)\, {}_4F_3 \left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ; 1 \right] \qquad (0 \le i, j \le D). \tag{9}$$

Lemma 2.5. ([11, Ex. 5.10] and [12, Thm. 4.9]) *The following hold:*

(i) $P^2 = (D+1)^2 I$;

(ii) $PA = A^*P$;

(iii) $PA^* = AP$;

(iv) the pair A, A^* *is a Leonard pair over* $\mathbb{R}$.

Proof. Calculations *(i)–(iii)* are the following special case of [11, Ex. 5.10] and [12, Thm. 4.9]:

$$d = D, \qquad \theta_0 = \theta_0^* = 3, \qquad s = s^* = r_1 = 0, \qquad r_2 = D+1, \qquad h = h^* = \frac{-6}{D(D+2)}.$$

Item *(iv)* follows from items *(i)–(iii)*. □

The Leonard pairs from [11, Ex. 5.10] are said to have Racah type. So the Leonard pair A, A^* in Lemma 2.5 has Racah type. This Leonard pair is self-dual in the sense of [9, p. 5].

3 Some orthogonal polynomials

In this section we interpret Conjecture 1.1 in terms of orthogonal polynomials.

Let λ denote an indeterminate. Let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra of polynomials in λ that have all coefficients in $\mathbb{R}$.

Definition 3.1. With reference to Definition 2.3, let $u_0(\lambda), u_1(\lambda), \ldots, u_D(\lambda)$ denote the polynomials in $\mathbb{R}[\lambda]$ that satisfy:

$$u_0(\lambda) = 1, \qquad u_1(\lambda) = \lambda/3,$$

$$\lambda u_i(\lambda) = b_i u_{i+1}(\lambda) + a_i u_i(\lambda) + c_i u_{i-1}(\lambda) \qquad (1 \le i \le D-1). \tag{10}$$

Note that the polynomial $u_i(\lambda)$ has degree exactly i for $0 \le i \le D$.

By [11, Ex. 5.10], the polynomials $\{u_i(\lambda)\}_{i=0}^D$ are a special case of the Racah polynomials. Also by [11, Ex. 5.10],

$$u_i(\theta_j) = {}_4F_3 \left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ; 1 \right] \qquad (0 \le i, j \le D). \tag{11}$$

Lemma 3.2. *The following hold:*

(i) $u_i(\theta_j) = u_j(\theta_i) \qquad (0 \le i, j \le D)$;

(ii) $u_i(\theta_0) = 1 \qquad (0 \le i \le D)$;

(iii) $u_0(\theta_j) = 1 \qquad (0 \le j \le D)$.

Proof. Each of (i)–(iii) is immediate from (11). □

In light of Equation (11), Conjecture 1.1 asserts that

$$|u_i(\theta_j)| \le 1 \qquad (0 \le i, j \le D). \tag{12}$$

To prove (12) it will be useful to adjust the normalization of the polynomials $u_i(\lambda)$.

Define

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (0 \le i \le D). \tag{13}$$

One checks that

$$k_i = 2i + 1 \qquad (0 \le i \le D). \tag{14}$$

Definition 3.3. With reference to Definition 3.1, let

$$v_i(\lambda) = k_i u_i(\lambda) \qquad (0 \le i \le D). \tag{15}$$

By construction,

$$v_i(\theta_j) = k_i u_i(\theta_j) \qquad (0 \le i, j \le D). \tag{16}$$

The polynomials $v_i(\lambda)$ satisfy the following three-term recurrence.

Lemma 3.4. [12, Lem. 3.11] *We have*

$$v_0(\lambda) = 1, \qquad v_1(\lambda) = \lambda,$$

$$\lambda v_i(\lambda) = c_{i+1} v_{i+1}(\lambda) + a_i v_i(\lambda) + b_{i-1} v_{i-1}(\lambda) \qquad (1 \le i \le D-1). \tag{17}$$

Lemma 3.5. *For* $0 \le i, j \le D$ *we have*

$$P_{i,j} = v_j(\theta_i). \tag{18}$$

Proof. Immediate by (9),(11),(14), and (16). □

We emphasize two special cases of (18).

Lemma 3.6. *The following hold:*

(i) $P_{i,0} = 1 \qquad (0 \leq i \leq D)$;

(ii) $P_{0,j} = k_j \qquad (0 \leq j \leq D)$.

Proof. Immediate from (16) and (18). □

We have some comments about the parameters (13). For notational convenience, define

$$\nu = (D+1)^2. \tag{19}$$

By (14),

$$\sum_{i=0}^{D} k_i = \nu.$$

Next, we state the orthogonality relations for the polynomials $\{u_i(\lambda)\}_{i=0}^{D}$.

Lemma 3.7. [12, p. 282] *For integers* $0 \leq n, m \leq D$ *we have*

$$\sum_{j=0}^{D} k_j u_n(\theta_j) u_m(\theta_j) = \nu k_n^{-1} \delta_{n,m}; \tag{20}$$

$$\sum_{j=0}^{D} k_j u_j(\theta_n) u_j(\theta_m) = \nu k_n^{-1} \delta_{n,m}. \tag{21}$$

Next, we state the orthogonality relations for the polynomials $\{v_i(\lambda)\}_{i=0}^{D}$.

Lemma 3.8. [12, p. 281] *For integers* $0 \leq n, m \leq D$ *we have*

$$\sum_{j=0}^{D} k_j v_n(\theta_j) v_m(\theta_j) = \nu k_n \delta_{n,m}; \tag{22}$$

$$\sum_{j=0}^{D} k_j^{-1} v_j(\theta_n) v_j(\theta_m) = \nu k_n^{-1} \delta_{n,m}. \tag{23}$$

4 Two commutative subalgebras of $\mathrm{Mat}_{D+1}(\mathbb{R})$

We continue to discuss the Leonard pair A, A^* from Definition 2.3.

Definition 4.1. Let M denote the subalgebra of $\mathrm{Mat}_{D+1}(\mathbb{R})$ generated by A. Let M^* denote the subalgebra of $\mathrm{Mat}_{D+1}(\mathbb{R})$ generated by A^*.

In this section, we describe a basis for M and a basis for M^*.

Definition 4.2. For $0 \leq i \leq D$ define

$$B_i = v_i(A), \qquad B_i^* = v_i(A^*),$$

where $v_i(\lambda)$ is from (15).

Lemma 4.3. *For $0 \leq i \leq D$ we have*

$$PB_i = B_i^* P, \qquad PB_i^* = B_i P.$$

Proof. By Lemma 2.5, Definition 4.2, and linear algebra. □

Lemma 4.3 tells us that for integers $0 \leq i, j \leq D$, column j of P is an eigenvector of B_i with eigenvalue $v_i(\theta_j)$. We emphasize one special case. Let $\mathbb{1}$ denote the vector in $\mathbb{R}^{D+1}$ that has all entries 1.

Lemma 4.4. *For $0 \leq i \leq D$ the vector $\mathbb{1}$ is an eigenvector for B_i with eigenvalue k_i.*

Proof. Immediate from Lemma 3.6 and Lemma 4.3. □

Lemma 4.5. *The matrices $\{B_i\}_{i=0}^D$ form a basis for M. The matrices $\{B_i^*\}_{i=0}^D$ form a basis for M^*.*

Proof. By Definition 2.3, the matrix A^* has $D+1$ distinct eigenvalues, so M^* has dimension $D+1$. By Definition 4.2, the matrices $\{B_i^*\}_{i=0}^D$ belong to M^*. By these comments, the matrices $\{B_i^*\}_{i=0}^D$ form a basis for M^*. We have now verified the second assertion. The first assertion follows from this and Lemma 4.3. □

Next we discuss the entries of the matrices $\{B_i\}_{i=0}^D$. The following definition will be convenient.

Definition 4.6. For $0 \leq h, i, j \leq D$ let $p_{i,j}^h$ denote the (h, j)-entry of B_i. In other words,

$$p_{i,j}^h = (B_i)_{h,j}. \tag{24}$$

We have a comment about the scalars $p_{i,j}^h$.

Lemma 4.7. [9, Lem. 4.19] *For $0 \leq i, j \leq D$ we have*

$$B_i B_j = \sum_{h=0}^D p_{i,j}^h B_h, \qquad B_i^* B_j^* = \sum_{h=0}^D p_{i,j}^h B_h^*. \tag{25}$$

The scalars $p_{i,j}^h$ can be computed using the following result. This result is from [8]; we include a proof for the sake of completeness.

Proposition 4.8. [8, Lem. 12.12] *For $0 \leq h, i, j \leq D$ we have*

$$p_{i,j}^h = \frac{k_i k_j}{\nu} \sum_{t=0}^D k_t u_t(\theta_i) u_t(\theta_j) u_t(\theta_h). \tag{26}$$

Proof. We invoke Equation (24). By Lemma 2.5(i) and Lemma 4.3 we have that $B_i = \nu^{-1} P B_i^* P$. Recall that the matrix P has entries $P_{i,j} = k_j u_j(\theta_i)$. We also have $B_i^* = v_i(A^*)$ and $A^* = \mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_D)$. Evaluating (24) using these comments, we obtain the result. □

We have a comment about Proposition 4.8.

Lemma 4.9. *For $0 \leq h, i, j \leq D$ we have*

$$p_{i,j}^h = p_{j,i}^h, \qquad k_h p_{i,j}^h = k_j p_{h,i}^j = k_i p_{j,h}^i. \tag{27}$$

Proof. Immediate from (26). □

5 The nonnegativity of the $p^h_{i,j}$

Our next goal is to show that $p^h_{i,j} \geq 0$ for $0 \leq h, i, j \leq D$. To obtain this inequality, we use the Biedenharn-Elliott identity [1, p. 356].

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Note that $\frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$.

Definition 5.1. Given $a, b, c \in \frac{1}{2}\mathbb{N}$, we say that the triple (a, b, c) is *admissible* whenever $a + b + c \in \mathbb{N}$ and

$$a \leq b + c, \qquad b \leq c + a, \qquad c \leq a + b. \tag{28}$$

Definition 5.2. Referring to Definition 5.1, assume that (a, b, c) is admissible. Define

$$\Delta(a, b, c) = \left(\frac{(a + b - c)!(b + c - a)!(c + a - b)!}{(a + b + c + 1)!} \right)^{\frac{1}{2}}. \tag{29}$$

Next, we recall the Racah coefficients.

Definition 5.3. ([1, Eq. 5.11.4] and [6, p. 1063]) For $a, b, c, d, e, f \in \frac{1}{2}\mathbb{N}$, we define a real number $W(a, b, c, d; e, f)$ as follows.

First assume that each of (a, b, e), (c, d, e), (a, c, f), (b, d, f) is admissible. Then

$$\begin{aligned} W(a, b, c, d; e, f) = {} & \frac{\Delta(a, b, e)\Delta(c, d, e)\Delta(a, c, f)\Delta(b, d, f)(\beta_1 + 1)!(-1)^{\beta_1 - (a+b+c+d)}}{(\beta_2 - \beta_1)!(\beta_3 - \beta_1)!(\beta_1 - \alpha_1)!(\beta_1 - \alpha_2)!(\beta_1 - \alpha_3)!(\beta_1 - \alpha_4)!} \\ & \times {}_4F_3 \left[\begin{matrix} \alpha_1 - \beta_1, \; \alpha_2 - \beta_1, \; \alpha_3 - \beta_1, \; \alpha_4 - \beta_1 \\ -\beta_1 - 1, \; \beta_2 - \beta_1 + 1, \; \beta_3 - \beta_1 + 1 \end{matrix} ; 1 \right], \end{aligned} \tag{30}$$

where

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \text{any permutation of } \; (a + b + e, \; c + d + e, \; a + c + f, \; b + d + f),$$

and where

$$\beta_1 = \min(a + b + c + d, \; a + d + e + f, \; b + c + e + f),$$

and β_2, β_3 are the other two values in the triple $(a + b + c + d, a + d + e + f, b + c + e + f)$ in either order.

Next assume that (a, b, e), (c, d, e), (a, c, f), (b, d, f), are not all admissible. Then

$$W(a, b, c, d; e, f) = 0. \tag{31}$$

We call $W(a, b, c, d; e, f)$ the *Racah coefficient* associated with a, b, c, d, e, f.

Let $0 \leq h, i, j \leq D$. In order to show that $p^h_{i,j} \geq 0$, we will show that

$$p^h_{i,j} = (2i + 1)(2j + 1)(D + 1)\Big(W\left(\tfrac{D}{2}, \tfrac{D}{2}, i, h; j, \tfrac{D}{2}\right)\Big)^2.$$

We will use the Biedenharn-Elliott identity.

Proposition 5.4. (Biedenharn-Elliott identity [1, p. 356]) *Let $a, a', b, b', c, c', e, f, g \in \frac{1}{2}\mathbb{N}$. Then*

$$\begin{aligned} \sum_{d \in \frac{1}{2}\mathbb{N}} & (-1)^{c+c'-d}(2d + 1)W(b, b', c, c'; d, e)W(a, a', c, c'; d, f)W(a, a', b, b'; d, g) \\ & = (-1)^{e+f-g}W(a, b, f, e; g, c)W(a', b', f, e; g, c'). \end{aligned} \tag{32}$$

In order to evaluate the Racah coefficients in the Biedenharn-Elliott identity, we will use the following transformation formula of Whipple.

Proposition 5.5. (Whipple transformation [2, p. 49]) *For integers* $p, q, a_1, a_2, r, b_1, b_2$ *we have*

$$ {}_4F_3\left[\begin{matrix}-p,\, q,\, a_1,\, a_2\\ r,\, b_1,\, b_2\end{matrix};1\right] = \frac{(b_1-q)_p(b_2-q)_p}{(b_1)_p(b_2)_p}\,{}_4F_3\left[\begin{matrix}-p,\, q,\, r-a_1,\, r-a_2\\ r,\, 1+q-b_1-p,\, 1+q-b_2-p\end{matrix};1\right], \tag{33} $$

provided that $p \geq 0$ *and* $q + a_1 + a_2 + 1 = r + b_1 + b_2 + p$.

We are interested in the following Racah coefficient. For $0 \leq i, j \leq D$ consider

$$ W\left(\tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}; i, j\right). $$

Evaluating this Racah coefficient using Definition 5.3 we get a scalar multiple of a certain ${}_4F_3$ hypergeometric series. Applying several Whipple transformations to this hypergeometric series, we get the following result as we will see.

Proposition 5.6. *For integers* $0 \leq i, j \leq D$ *we have*

$$ W\left(\tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}; i, j\right) = \frac{(-1)^{i+j-D}}{D+1}\,{}_4F_3\left[\begin{matrix}-i,\, i+1,\, -j,\, j+1\\ 1,\, D+2,\, -D\end{matrix};1\right]. \tag{34} $$

Proof. To evaluate $W\left(\tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}; i, j\right)$, we will consider two cases: $i + j \leq D$ and $i + j > D$.

Case $i + j \leq D$. In this case, from (30) we get $\beta_1 = D + i + j$, $\beta_2 = 2D$, $\beta_3 = D + i + j$, $\alpha_1 = \alpha_2 = D + i$, $\alpha_3 = \alpha_4 = D + j$. The hypergeometric term in (30), after rearranging the upper indices, becomes

$$ {}_4F_3\left[\begin{matrix}-i,\, -i,\, -j,\, -j\\ -D-i-j-1,\, D-i-j+1,\, 1\end{matrix};1\right]. \tag{35} $$

The coefficient in (30) is

$$ \frac{\left(\Delta\left(\tfrac{D}{2}, \tfrac{D}{2}, i\right)\right)^2\left(\Delta\left(\tfrac{D}{2}, \tfrac{D}{2}, j\right)\right)^2 (D+i+j+1)!(-1)^{i+j-D}}{(D-i-j)!(j!)^2(i!)^2} $$
$$ = \frac{(D-i)!(i!)^2(D-j)!(j!)^2(D+i+j+1)!(-1)^{i+j-D}}{(D+i+1)!(D+j+1)!(D-i-j)!(j!)^2(i!)^2}. \tag{36} $$

The expression (36) is equal to

$$ \frac{(D-i)!(D-j)!(D+i+j+1)!(-1)^{i+j-D}}{(D+i+1)!(D+j+1)!(D-i-j)!}. \tag{37} $$

Performing a Whipple transformation (33) with the substitutions $-p = -i$, $q = -j$, $a_1 = -i$, $a_2 = -j$, $r = 1$, $b_1 = -D - i - j - 1$, $b_2 = D - i - j + 1$, the hypergeometric component in (35), after rearranging lower indices, becomes

$$ {}_4F_3\left[\begin{matrix}-i,\, i+1,\, -j,\, j+1\\ 1,\, D+2,\, -D\end{matrix};1\right]. \tag{38} $$

The coefficient contribution from the Whipple transformation is

$$ \frac{(-D-i-1)_i(D-i+1)_i}{(-D-i-j-1)_i(D-i-j+1)_i} = \frac{(-1)^i(D+i+1)!}{(D+1)!}\,\frac{D!}{(D-i)!}\,\frac{(D+j+1)!}{(-1)^i(D+i+j+1)!}\,\frac{(D-i-j)!}{(D-j)!}. \tag{39} $$

We see that coefficients (37) and (39) multiply to $\frac{(-1)^{i+j-D}}{D+1}$, as desired.

Case $i + j > D$. In this case, from (30) we get $\beta_1 = 2D$, $\beta_2 = D + i + j$, $\beta_3 = D + i + j$, $\alpha_1 = \alpha_2 = D + i$, $\alpha_3 = \alpha_4 = D + j$. The hypergeometric term in (30) becomes

$$ {}_4F_3\left[\begin{matrix}i-D,\, i-D,\, j-D,\, j-D\\ -2D-1,\, i+j-D+1,\, i+j-D+1\end{matrix};1\right]. \tag{40} $$

The coefficient in (30) is

$$\frac{\left(\Delta\left(\frac{D}{2},\frac{D}{2},i\right)\right)^2\left(\Delta\left(\frac{D}{2},\frac{D}{2},j\right)\right)^2(2D+1)!}{\big((i+j-D)!\big)^2\big((D-i)!\big)^2\big((D-j)!\big)^2}=\frac{(D-i)!(i!)^2(D-j)!(j!)^2(2D+1)!}{(D+i+1)!(D+j+1)!\big((i+j-D)!(D-i)!(D-j)!\big)^2}. \quad (41)$$

The expression (41) is equal to

$$C_0=\frac{(i!)^2(j!)^2(2D+1)!}{(D+i+1)!(D+j+1)!\big((i+j-D)!\big)^2(D-i)!(D-j)!}. \quad (42)$$

Now we will perform three Whipple transformations. For each one we list the indices chosen $-p$, q, a_1, a_2, r, b_1, b_2, the resulting hypergeometric term (with possible rearranging of some upper indices), and the coefficient contribution, C_i, from the corresponding Whipple transformation.

1. Using $-p=i-D$, $q=j-D$, $a_1=i-D$, $a_2=j-D$, $r=i+j-D+1$, $b_1=-2D-1$, $b_2=i+j-D+1$:

$$ {}_4F_3\left[\begin{matrix} i-D,\, i+1,\, j-D,\, j+1 \\ i+j+2,\, -D,\, i+j-D+1 \end{matrix};1\right], \quad (43)$$

$$\begin{aligned} C_1 &= \frac{(-D-j-1)_{D-i}(i+1)_{D-i}}{(-2D-1)_{D-i}(i+j-D+1)_{D-i}} \\ &= \frac{(-1)^{D-i}(D+j+1)!}{(i+j+1)!}\frac{D!}{i!}\frac{(D+i+1)!}{(-1)^{D-i}(2D+1)!}\frac{(i+j-D)!}{j!}. \end{aligned} \quad (44)$$

2. Using $-p=i-D$, $q=j+1$, $a_1=i+1$, $a_2=j-D$, $r=-D$, $b_1=i+j+2$, $b_2=i+j-D+1$:

$$ {}_4F_3\left[\begin{matrix} i-D,\, -D-i-1,\, -j,\, j+1 \\ -D,\, -D,\, 1 \end{matrix};1\right], \quad (45)$$

$$\begin{aligned} C_2 &= \frac{(i+1)_{D-i}(i-D)_{D-i}}{(i+j+2)_{D-i}(i+j-D+1)_{D-i}} \\ &= \frac{D!}{i!}(-1)^{D-i}(D-i)!\frac{(i+j+1)!}{(D+j+1)!}\frac{(i+j-D)!}{j!}. \end{aligned} \quad (46)$$

3. Using $-p=-j$, $q=j+1$, $a_1=i-D$, $a_2=-D-i-1$, $r=-D$, $b_1=-D$, $b_2=1$:

$$ {}_4F_3\left[\begin{matrix} -i,\, i+1,\, -j,\, j+1 \\ -D,\, D+2,\, 1 \end{matrix};1\right] = {}_4F_3\left[\begin{matrix} -i,\, i+1,\, -j,\, j+1 \\ 1,\, D+2,\, -D \end{matrix};1\right], \quad (47)$$

$$\begin{aligned} C_3 &= \frac{(-D-j-1)_j(-j)_j}{(-D)_j(1)_j} \\ &= \frac{(-1)^j(D+j+1)!}{(D+1)!}(-1)^j j!\frac{(D-j)!}{(-1)^jD!}\frac{1}{j!}. \end{aligned} \quad (48)$$

Combining coefficients we see that $C_0C_1C_2C_3=\frac{(-1)^{D-i+j}}{D+1}=\frac{(-1)^{i+j-D}}{D+1}$, since i,j,D are integers. □

We now evaluate the Biedenharn-Elliott identity using Proposition 5.6.

Proposition 5.7. *For integers $0\le h,i,j\le D$ we have*

$$\sum_{t=0}^{D}(2t+1)u_t(\theta_h)u_t(\theta_i)u_t(\theta_j)=(D+1)^3\Big(W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right)\Big)^2. \quad (49)$$

Proof. First we apply Proposition 5.4 with $a = a' = b = b' = c = c' = \frac{D}{2}$, $e = h$, $f = i$, $g = j$, and $d = t$ to obtain

$$\sum_{t\in\frac{1}{2}\mathbb{N}} (-1)^{D-t}(2t+1)W(\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2};t,h)W(\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2};t,i)W(\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2};t,j)$$
$$= (-1)^{h+i-j} W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right) W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right). \quad (50)$$

Note that $\frac{D}{2} + \frac{D}{2} + t$ is an integer if and only if t is an integer. So by (31), the terms of the sum vanish in which t is not an integer or $t > D$. By Proposition 5.6 and (11), the left hand side of (50) becomes

$$\sum_{t=0}^{D}(-1)^{D-t}(2t+1)\frac{(-1)^{t+h-D}u_t(\theta_h)}{D+1}\,\frac{(-1)^{t+i-D}u_t(\theta_i)}{D+1}\,\frac{(-1)^{t+j-D}u_t(\theta_j)}{D+1},$$

which simplifies to

$$\frac{(-1)^{i+j+h}}{(D+1)^3}\sum_{t=0}^{D}(2t+1)u_t(\theta_h)u_t(\theta_i)u_t(\theta_j). \quad (51)$$

Setting (51) equal to the right hand side of (50) and dividing by the coefficients completes the proof. □

Corollary 5.8. *For $0 \le h,i,j \le D$ we have*

$$p^h_{i,j} = (2i+1)(2j+1)(D+1)\Big(W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right)\Big)^2. \quad (52)$$

Proof. Using Propositions 4.8, 5.7 and substituting (14),(19) we have

$$\begin{aligned} p^h_{i,j} &= \frac{k_ik_j}{\nu}\sum_{t=0}^{D} k_t u_t(\theta_i)u_t(\theta_j)u_t(\theta_h) \\ &= \frac{(2i+1)(2j+1)}{(D+1)^2}\left((D+1)^3\Big(W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right)\Big)^2\right) \\ &= (2i+1)(2j+1)(D+1)\Big(W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right)\Big)^2. \end{aligned}$$

□

Corollary 5.9. *For $0 \le h,i,j \le D$ we have*

$$p^h_{i,j} \ge 0.$$

Proof. Immediate from Corollary 5.8. □

6 Proof of the Kresch-Tamvakis conjecture

We are now ready to prove our main result. We will use the Perron-Frobenius theorem [3, p. 529].

Proposition 6.1. *For $0 \le i,j \le D$ we have*

$$|u_i(\theta_j)| \le 1.$$

Proof. By Lemma 4.4, the vector $\mathbb{1}$ is an eigenvector for B_i with eigenvalue k_i. By Corollary 5.9, the entries of B_i are all nonnegative. By Lemma 4.3 the scalar $v_i(\theta_j)$ is an eigenvalue of B_i. By the Perron-Frobenius theorem [3, p. 529], we have $|v_i(\theta_j)| \le k_i$. The result follows from this and (16). □

Equation (11) and Proposition 6.1 imply Theorem 1.3.

7 Appendix

In this appendix we give more detail about the formula for $p^h_{i,j}$ in Corollary 5.8. By Lemma 4.9, without loss of generality we assume $i \le j \le h$. Also, in order to avoid trivialities we assume that h, i, j satisfy the triangle inequalities; which in this case become $h \le i + j$. As we evaluate $p^h_{i,j}$ in line (52) we consider the last factor. We evaluate that factor using Definition 5.3 with

$$a = \frac{D}{2}, \qquad b = \frac{D}{2}, \qquad c = i, \qquad d = h, \qquad e = j, \qquad f = \frac{D}{2}.$$

For these values,

$$\alpha_1 = D + i, \qquad \alpha_2 = D + j, \qquad \alpha_3 = D + h, \qquad \alpha_4 = h + i + j,$$

$$\beta_1 = D + i + j, \qquad \beta_2 = D + h + i, \qquad \beta_3 = D + h + j.$$

Note that

$$\alpha_1 - \beta_1 = -j, \qquad \alpha_2 - \beta_1 = -i, \qquad \alpha_3 - \beta_1 = h - i - j, \qquad \alpha_4 - \beta_1 = h - D$$

$$-\beta_1 - 1 = -D - i - j - 1, \qquad \beta_2 - \beta_1 + 1 = h - j + 1, \qquad \beta_3 - \beta_1 + 1 = h - i + 1.$$

For the above data, (52) becomes

$$p^h_{i,j} = C^h_{i,j}(2i+1)(2j+1)(D+1)\left({}_4F_3\left[\begin{matrix} -j,\ -i,\ h-i-j,\ h-D \\ -D-i-j-1,\ h-j+1,\ h-i+1 \end{matrix} ;1\right]\right)^2,$$

where

$$\begin{aligned} C^h_{i,j} &= \left(\frac{\Delta(\frac{D}{2},\frac{D}{2},i)\Delta(\frac{D}{2},\frac{D}{2},j)\Delta(\frac{D}{2},\frac{D}{2},h)\Delta(i,j,h))(D+i+j+1)!}{(h-i)!(h-j)!i!j!(i+j-h)!(D-h)!}\right)^2 \\ &= \frac{(D-i)!(D-j)!(D-h)!(j+h-i)!(h+i-j)!}{(D+i+1)!(D+j+1)!(D+h+1)!(i+j+h+1)!(i+j-h)!}\left(\frac{h!(D+i+j+1)!}{(h-i)!(h-j)!(D-h)!}\right)^2. \end{aligned}$$

Acknowledgement. We would like to express our gratitude to Professor Paul Terwilliger, whose careful feedback greatly enhanced the clarity of the exposition.

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