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Aequationes Mathematicae

On Wendel's equality for intersections of balls

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Abstract. We study the analogue of Wendel's equality in random polytope models in which the hull of the random points is formed by intersections of congruent balls, called the spindle (or hyper-) convex hull. According to the classical identity of Wendel the probability that the origin is contained in the (linear) convex hull of n i.i.d. random points distributed according to an origin symmetric probability distribution in the d-dimensional Euclidean space $\mathbb{R}^d$ that assigns measure zero to hyperplanes is a constant depending only on n and d. While in the classical convex case one gets nonzero probabilities only for $n \geq d+1$ points in $\mathbb{R}^d$, for the spindle convex hull this happens for all $n \geq 2$. We study this question for the uniform and normally distributed random models.

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1. Introduction and results

Wendel's equality [10] is one of the classical results in geometric probability: it states that if $x_1, \ldots, x_n$ are i.i.d. random points in $\mathbb{R}^d$ whose distribution is (centrally) symmetric with respect to the origin o, and the probability measures of hyperplanes are 0, then the probability that o is not contained in the convex hull $[x_1, \ldots, x_n]$ is

$$\mathbb{P}(o \notin [x_1, \ldots, x_n]) = \frac{1}{2^{n-1}} \sum_{i=0}^{d-1} \binom{n-1}{i}. \tag{1.1}$$

One can find a simple proof of (1.1) in Bárány [1, pp. 94–95], which is independent of the distribution (under the above conditions).

It was proved by Wagner and Welzl [9], that o-symmetric distributions are extremal in this sense. For more information, see also [8, Section 8.1.2].

Recently, Kabluchko and Zaporozhets [3] investigated the related problem of finding the probability that the convex hull of n i.i.d. normally distributed random points in $\mathbb{R}^d$ contains a fixed points of space; they called these *absorption probabilities*. For a general introduction to random polytopes we refer to the recent survey paper by Schneider [7] and the book by Schneider and Weil [8].

We denote the d-dimensional origin centered unit radius closed ball by B^d and its boundary by S^{d-1}. The symbol κ_d denotes the volume (Lebesgue measure) of B^d, and ω_d is the surface volume of B^d. For general information on convex sets, see the monograph [6] by Schneider.

In this paper we study the following spindle convex variant of the above problems. Let $x, y \in \mathbb{R}^d$ be two points and $\varrho > 0$. If $|x-y| \leq 2\varrho$, then let the spindle $[x,y]_\varrho$ determined by x and y be the intersection of all radius ϱ closed balls that contain both x and y. If $|x-y| > 2\varrho$, then let $[x,y]_\varrho = \mathbb{R}^d$. We say that a convex body $K \subset \mathbb{R}^d$ (compact convex set with non-empty interior) is spindle convex with radius ϱ, or ϱ-spindle convex if together with any two points $x, y \in K$, it contains the spindle $[x,y]_\varrho$. It is known ([2]) that if a convex body $K \subset \mathbb{R}^d$ is spindle convex with radius ϱ, then K is the intersection of all radius ϱ closed balls that contain K. This latter property is called radius ϱ ball-convexity.

Let $X \subset \mathbb{R}^d$. If $X \subset \varrho B^d + v$ for some $v \in \mathbb{R}^d$, then the radius ϱ spindle convex hull $[X]_\varrho$ of K is defined as the intersection of all radius ϱ closed balls containing X. If $X \not\subset \varrho B^d + v$ for any $v \in \mathbb{R}^d$, then let $[X]_\varrho = \mathbb{R}^d$. If $K \subset \mathbb{R}^d$ is spindle convex with radius ϱ, and $X \subset K$, then $[X]_\varrho \subset K$. For more information on spindle convexity, see, for example, the paper [2] by Bezdek et al. and the book [4] by Martini, Montejano and Oliveros and the references therein.

First, we describe the ϱ-spindle convex uniform model. Let $\varrho > 0$, and let $K \subset \mathbb{R}^d$ be an o-symmetric convex body that is ϱ-spindle convex. Let $x_1, \ldots, x_n$ be i.i.d. uniform random points from K. We denote the radius ϱ spindle convex hull of $x_1, \ldots, x_n$ by $K^\varrho_{(n)} = [x_1, \ldots, x_n]_\varrho$. By the ϱ-spindle convexity of K, the random ball-polytope $K^\varrho_{(n)}$ is contained in K. We ask the same question as in the classical convex case: what is the probability that $o \in K^\varrho_{(n)}$? We note that in this model we may always achieve by scaling (simultaneously K and radius ϱ circles) that $\varrho = 1$. Henceforth, in the following two theorems we assume that $\varrho = 1$.

We study the special case when $K = rB^d$ with $0 < r \leq 1$. Then K is clearly spindle convex with radius $\varrho = 1$. We wish to determine the probability

$$P(d,r,n) := \mathbb{P}(o \in [x_1, \ldots, x_n]_1).$$

In Sect. 2 we prove the following theorem:

Theorem 1.1. *Let $K = rB^d$. Then*

$$P(d,r,2) = \frac{\omega_{d-1}\omega_d}{(r^d\kappa_d)^2}\int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_1^{d-1}r_2^{d-1}\sin^{d-2}\varphi\, d\varphi dr_2 dr_1,$$

where $\varphi(r_1,r_2) = \arcsin(r_1/2) + \arcsin(r_2/2)$. *In particular,*

$$P(2,1,2) = \frac{\sqrt{3}}{\pi} - \frac{1}{3} = 0.2179\ldots,$$
$$P(3,1,2) = \frac{1}{64}(23 + 12\sqrt{3}\pi - 8\pi^2) = 0.1459\ldots.$$

Furthermore, for the case of three points, we prove the following statement in Sect. 3.

Theorem 1.2. *Let $K = B^2$. Then*

$$P(2,1,3) = \frac{-84\pi^2 - 477 + 360\sqrt{3}\pi}{144\pi^2} = 0.4594\ldots.$$

Finally, in Sect. 4, we study the Gaussian ϱ-spindle convex model. Let $x_1,\ldots,x_n$ be i.i.d. random points from $\mathbb{R}^d$ distributed according to the standard normal distribution. The question is the same, what is the probability that $o \in K^{\varrho}_{(n)}$? We note that in this second case, it may, and does, happen that $K^{\varrho}_{(n)} = \mathbb{R}^d$. We give an integral formula for the probability that a Gaussian unit radius spindle contains the origin and evaluate it numerically in the plane.

2. Proof of Theorem 1.1

Note that it is the simplest case of the model when $n = 2$, and $K = rB^d$, where $0 < r \leq 1$ is a fixed number. This, of course, is of no interest in the classical version of Wendel's problem as $\mathbb{P}(o \in [x_1,x_2]) = 0$ since $[x_1,x_2]$ is a segment.

Let us examine what it means geometrically that $o \in [x_1,x_2]_1$. Let $M(x_1)$ denote the union of all open unit balls that contain o and x_1 on their boundary. Let $K(d,r,x_1)$ be the part of $rB^d\backslash M(x_1)$ that is in the closed half-space bounded by the hyperplane through o and orthogonal to x_1 which does not contain x_1. We depicted this region in Fig. 1 when $d = 2$. We will only use $K(2,r,x_1)$ in our calculations, so, in order to simplify notation, we will denote it by $K(r,x_1) = K(2,r,x_1)$.

In order to evaluate $P(d,r,2)$, we use the linear Blaschke-Petkantschin formula. Let $G(d,2)$ denote the Grassmannian manifold of 2-dimensional linear subspaces of $\mathbb{R}^d$, and ν_2 be the unique rotation invariant Haar probability measure on $G(d,2)$. The 2-dimensional special case of the linear Blaschke-Petkantschin formula (see, for example, [8, Theorem 7.2.1 on p. 271]) says the following: If $f : (\mathbb{R}^d)^2 \to \mathbb{R}$ is a non-negative measurable function, then

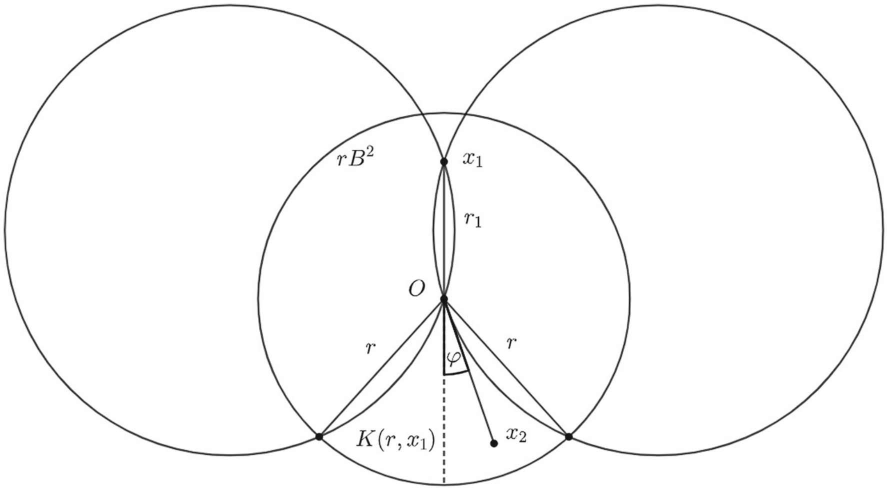

FIGURE 1. The region $K(r, x_1)$

$$\int_{(\mathbb{R}^d)^2} f\, d\lambda^2 = \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{L^2} f(x_1, x_2) \nabla_2^{d-2}(x_1, x_2)\, d\lambda_L^2\, \nu_2(dL), \tag{2.1}$$

where ∇_2 denotes the area of the parallelogram spanned by the vectors x_1, x_2 in L. The symbol λ denotes the Lebesgue measure in $\mathbb{R}^d$, and λ_L the (2-dimensional) Lebesgue measure in L.

Next, using polar coordinates for $x_1, x_2 \in L$, that is, $x_1 = r_1 u_1$, $x_2 = r_2 u_2$, where $u_1, u_2 \in S^1$, $r_1, r_2 \in \mathbb{R}_+$, we may write the right-hand-side of (2.1) as follows.

$$\begin{aligned}
&\frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{L^2} f(x_1, x_2) \nabla_2^{d-2}(x_1, x_2)\, d\lambda_L^2\, \nu_2(dL) \\
&\quad = \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{(S^1 \times \mathbb{R})^2} f(r_1 u_1, r_2 u_2) \\
&\qquad \times \nabla_2^{d-2}(r_1 u_1, r_2 u_2)\, r_1 r_2 dr_1 du_1 dr_2 du_2\, \nu_2(dL) \\
&\quad = \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{(S^1 \times \mathbb{R})^2} f(r_1 u_1, r_2 u_2) r_1^{d-1} r_2^{d-1} \times \\
&\qquad \times |u_1 \times u_2|^{d-2} dr_1 du_1 dr_2 du_2\, \nu_2(dL).
\end{aligned} \tag{2.2}$$

Now, from (2.2) we obtain that

$$\begin{aligned}
P(d, r, 2) &= \frac{1}{(r^d \kappa_d)^2} \int_{rB^d} \int_{rB^d} \mathbf{1}(o \in [x_1, x_2]_1)\, dx_1 dx_2 \\
&= \frac{1}{(r^d \kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{S^1} \int_0^r \int_{S^1} \int_0^r \mathbf{1}(o \in [r_1 u_1, r_2 u_2]_1) r_1^{d-1} r_2^{d-1}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times |u_1 \times u_2|^{d-2} dr_1 du_1 dr_2 du_2 \, \nu_2(dL)\\
&= \frac{1}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{S^1}\int_0^r\int_{S^1}\int_0^r \mathbf{1}(o \in [r_1u_1, r_2u_2]_1) r_1^{d-1} r_2^{d-1}\\
&\quad \times |u_1 \times u_2|^{d-2} dr_1 du_1 dr_2 du_2\\
&= \frac{1}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{S^1}\int_0^r\int_{S^1}\int_0^r \mathbf{1}(x_2 \in K(r, x_1)) r_1^{d-1} r_2^{d-1}\\
&\quad \times |u_1 \times u_2|^{d-2} dr_2 du_2 dr_1 du_1.
\end{aligned}
$$

By the rotational symmetry of rB^d, integration with respect to u_1 is a multiplication by 2π. Hence, from now on, we fix $u_1 = (0, 1)$. Let φ be the angle of u_2 and $-u_1$, as shown on Fig. 1, and let

$$
\varphi(r_1, r_2) = \arcsin(r_1/2) + \arcsin(r_2/2).
$$

Then

$$
\begin{aligned}
P(d, r, 2) &= \frac{2\pi}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_0^r\int_0^r\int_{-\varphi(r_1,r_2)}^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} |\sin\varphi|^{d-2} \, d\varphi dr_2 dr_1\\
&= \frac{4\pi}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} \sin^{d-2}\varphi \, d\varphi dr_2 dr_1\\
&= \frac{\omega_{d-1}\omega_d}{(r^d\kappa_d)^2} \int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} \sin^{d-2}\varphi \, d\varphi dr_2 dr_1.
\end{aligned}
$$

The above integral can be evaluated for any specific value of d using multiple integration by parts. In particular,

$$
\begin{aligned}
P(2, r, 2) &= \frac{4}{\pi r^4} \int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_2 r_1 d\varphi dr_2 dr_1\\
&= \frac{4}{\pi r^4} \int_0^r\int_0^r r_2 r_1 (\arcsin(r_1/2) + \arcsin(r_2/2)) \, dr_2 dr_1\\
&= \frac{4}{\pi r^4} \left(\frac{r^2}{4} (r\sqrt{4 - r^2} + 2(r^2 - 2)\arcsin(r/2)) \right)\\
&= \frac{1}{\pi r^2} \left(r\sqrt{4 - r^2} + 2(r^2 - 2)\arcsin(r/2) \right), \qquad (2.3)
\end{aligned}
$$

and

$$
\begin{aligned}
P(3, r, 2) &= \frac{9}{2r^6} \int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_2^2 r_1^2 \sin\varphi \, d\varphi dr_2 dr_1\\
&= \frac{9}{2r^6} \left(\frac{r^2}{288} (-72 + 90r^2 - 4r^4 + 9r^6) \right.\\
&\quad \left. + \frac{1}{4} \arcsin(r/2)(R\sqrt{4 - r^2}(r^2 - 2) + 4\arcsin(r/2)) \right).
\end{aligned}
$$

In particular,

$$P(2,1,2) = \frac{\sqrt{3}}{\pi} - \frac{1}{3} = 0.2179\ldots,$$
$$P(3,1,2) = \frac{1}{64}(23 + 12\sqrt{3}\pi - 8\pi^2) = 0.1459\ldots.$$

This finishes the proof of Theorem 1.1. □

We conclude this section with the following statements.

Corollary 2.1. *For any fixed $d \geq 2$, it holds that*

$$\lim_{r\to 0^+} P(d,r,2) = 0.$$

Furthermore, for any fixed $0 < r \leq 1$, it holds that

$$\lim_{d\to\infty} P(d,r,2) = 0.$$

Proof. Note that, using $\arcsin x \leq \pi x/2$ for $x \in [0,\pi/2]$ and $\sin x \leq x$ for $x \in [0,\pi/2]$, we get that

$$\begin{aligned} P(d,r,2) &\leq \frac{C(d)}{r^{2d}} \int_0^r \int_0^r \int_0^{r_1+r_2} r_1^{d-1} r_2^{d-1} (r_1+r_2)^{d-2}\, d\varphi dr_2 dr_1 \\ &\leq \frac{2^{d-1}C(d)}{r^{2d}} \int_0^r \int_{r_1}^r \int_0^{2r_2} r_2^{3d-4}\, d\varphi dr_2 dr_1 \\ &= \frac{2^d C(d)}{r^{2d}} \int_0^r \int_0^r r_2^{3d-3}\, dr_2 dr_1 \\ &= \frac{2^d C(d)}{r^{2d}} \frac{r^{3d-1}}{3d-2}, \end{aligned}$$

where the constant $C(d)$ depends only on the dimension d. From this it follows that

$$\lim_{r\to 0^+} P(d,r,2) = 0$$

for $d \geq 2$, as claimed.

In the proof of the second statement we use the fact that $\varphi(r_1,r_2) \leq \pi/3$. Thus

$$\begin{aligned} P(d,r,2) &\leq \frac{\omega_{d-1}\omega_d}{r^{2d}\kappa_d^2} \int_0^r \int_0^r r_1^{d-1} r_2^{d-1} \left(\frac{\sqrt{3}}{2}\right)^{d-1} dr_2 dr_1 \\ &= \frac{\omega_{d-1}\omega_d}{d^2\kappa_d^2} \left(\frac{\sqrt{3}}{2}\right)^{d-1} = \frac{d-1}{d} \frac{\kappa_{d-1}}{\kappa_d} \left(\frac{\sqrt{3}}{2}\right)^{d-1}. \end{aligned}$$

From $\kappa_{d-1}/\kappa_d \sim c \cdot \sqrt{d}$ as $d \to \infty$, it follows that $P(d,r,2) \to 0$ as $d \to \infty$. □

3. Proof of Theorem 1.2

The case when $n = 3$, can be treated, at least in the plane, as follows. We only consider when $r = 1$, that is, $K = B^2$. Let x_1, x_2, x_3 be i.i.d. uniform random points from B^2. Let

$$\begin{aligned} P(2,1,3) :&= \mathbb{P}(o \in [x_1, x_2, x_3]_1) \\ &= \mathbb{P}(o \in [x_1, x_2]_1) + \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1) \\ &= P(2,1,2) + \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1). \end{aligned}$$

Let

$$\overline{P}(2,1,3) := \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1).$$

Due to the rotational symmetry of B^2, we may assume that $x_1 = (0, r_1)$. Let $x_2 = r_2 u_2$, where φ is the angle of u_2 and the negative half of the y-axis. Making use of the previously introduced notation, we write $K(x_1) = K(1, x_1)$ and, similarly, $K(x_2) = K(1, x_2)$. The ray ox_i divides $K(x_i)$ into two congruent parts. The part that is on the positive side of ox_i is denoted by $K^+(x_i)$, and the negative part is $K^-(x_i)$, as shown in Fig. 2.

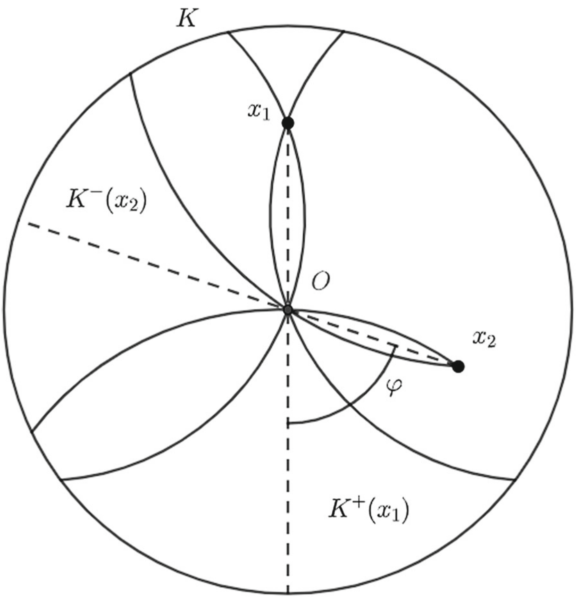

FIGURE 2. The regions $K^-(x_2)$ and $K^+(x_1)$

Let $V^+(x_i) = V_2(K^+(x_i))$ and $V^-(x_i) = V_2(K^-(x_i))$ for $i = 1, 2$. Then it holds that

$$V^+(x_i) = V^-(x_i) = \int_0^1 \int_0^{\varphi(r_i,r)} r\, d\varphi dr = \int_0^1 (\arcsin(r_i/2) + \arcsin(r/2)) r\, dr$$
$$= \frac{1}{12}\left(3\sqrt{3} - \pi + 6\arcsin(r_i/2)\right).$$

We distinguish four cases according to the relative position of x_1 and x_2.

Case 1. $r_2 \le r_1$ *and* $x_2 \notin [x_1, o]_1$.

In this case, $\varphi \in [\varphi(r_1, r_2), \pi - \arcsin(r_1/2) + \arcsin(r_2/2)]$. Then

$$\begin{aligned}
P_1 &:= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \notin [x_1, o]_1 \text{ and } r_1 \ge r_2) \\
&= \frac{2\pi}{\pi^3} \int_0^1 \int_0^{r_1} \int_{\varphi(r_1,r_2)}^{\pi-\arcsin(r_1/2)+\arcsin(r_2/2)} \\
&\quad \times \left(V^+(x_1) + V^-(x_2) + \frac{\pi - \varphi}{2}\right) r_1 r_2 d\varphi dr_2 dr_1 \\
&= \frac{1}{\pi^2} \int_0^1 \int_0^{r_1} \int_{\varphi(r_1,r_2)}^{\pi-\arcsin(r_1/2)+\arcsin(r_2/2)} \left(\sqrt{3} - \frac{\pi}{3} + \arcsin(r_1/2)\right. \\
&\quad \left. + \arcsin(r_2/2) + \frac{\pi - \varphi}{2}\right) r_1 r_2 d\varphi dr_2 dr_1 \\
&= -\frac{5}{72} - \frac{1}{\pi^2} + \frac{5}{4\sqrt{3}\pi}.
\end{aligned}$$

Case 2. $r_2 \ge r_1$ and $x_1 \notin [x_2, o]_1$. By the symmetry of x_1 and x_2,

$$\begin{aligned}
P_2 &:= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_1 \notin [x_2, o]_1 \text{ and } r_1 \le r_2) \\
&= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \notin [x_1, o]_1 \text{ and } r_1 \ge r_2) \\
&= -\frac{5}{72} - \frac{1}{\pi^2} + \frac{5}{4\sqrt{3}\pi}.
\end{aligned}$$

Case 3. $x_2 \in [x_1, o]_1$.

In this case $r_1 \ge r_2$ and $\varphi \in [\pi - \arcsin(r_1/2) + \arcsin(r_2/2), \pi]$. Then $K(x_2) \subset K(x_1)$, thus

$$\begin{aligned}
P_3 &:= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \in [x_1, o]_1) \\
&= \frac{2\pi}{\pi^3} \int_0^1 \int_0^{r_1} \int_{\pi-\arcsin(r_1/2)+\arcsin(r_2/2)}^{\pi} V(x_1) r_1 r_2 d\varphi dr_2 dr_1 \\
&= \frac{1}{\pi^2} \int_0^1 \int_0^{r_1} \int_{\pi-\arcsin(r_1/2)+\arcsin(r_2/2)}^{\pi} \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} + \arcsin(r_1/2)\right) r_1 r_2 d\varphi dr_2 dr_1 \\
&= \frac{99 - 24\sqrt{3}\pi + 4\pi^2}{576\pi^2}.
\end{aligned}$$

Case 4. $x_1 \in [x_2, o]_1$. Again, by the symmetry of x_1 and x_2,

$$\begin{aligned} P_4 &= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_1 \in [x_2, o]_1) \\ &= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \in [x_1, o]_1) \\ &= \frac{99 - 24\sqrt{3}\pi + 4\pi^2}{576\pi^2}. \end{aligned}$$

Thus, considering the symmetry with respect to the line ox_1, we obtain that

$$\overline{P}(2,1,3) = 2(P_1 + P_2 + P_3 + P_4) = \frac{-36\pi^2 - 477 + 216\sqrt{3}\pi}{144\pi^2}.$$

Thus,

$$P(2,1,3) = P(2,1,2) + \overline{P}(2,1,3) = \frac{-84\pi^2 - 477 + 360\sqrt{3}\pi}{144\pi^2} = 0.4594\ldots.$$

We note that the actual calculation can be carried out, at least numerically, for any $0 < r \leq 1$. Furthermore, the cases of $n = 4, 5, \ldots$ are essentially similar, although the case analysis grows significantly more complicated as n increases.

Finally, we note that according to Wendel's equality (1.1),

$$\mathbb{P}(0 \in [x_1, x_2, x_3]) = \frac{1}{4} < P(2,1,3).$$

4. The case of normally distributed random points

In this subsection we consider the model in which $\varrho = 1$ and $x_1, \ldots, x_n$ are i.i.d. random points in $\mathbb{R}^d$ that are distributed according to the standard normal distribution with density function

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2}},\ x \in \mathbb{R}^d.$$

Here we need to use the part of the definition of the spindle convex hull that normally does not come into play when the random points are chosen from a convex body that is spindle convex with radius less than or equal to 1. Namely, if $x, y \in \mathbb{R}^d$ are such that $|x - y| > 2$, then $[x, y]_1 := \mathbb{R}^d$.

We are interested in the following probability

$$P_N(d, 1, n) := \mathbb{P}(o \in [x_1, \ldots, x_n]_1).$$

It is clear that

$$\mathbb{P}(o \in [x_1, \ldots, x_n]) \leq \mathbb{P}(o \in [x_1, \ldots, x_n]_1)$$

as $[X] \subset [X]_1$ for any $X \subset \mathbb{R}^d$.

Let E be the event that $|x_1 - x_2| \leq 2$. Then

$$P_N(d, 1, 2) = \mathbb{P}(o \in [x_1, x_2]_1 \text{ and } E) + \mathbb{P}(E^c),$$

where E^c is the complement of E, as E^c automatically implies that $o \in [x_1, x_2]_1$.

Let l denote the length of the random segment $[x_1x_2]$. It is known (see [5, p. 438] and the historical references therein) that the density of $s := l^2/4$ is

$$g(s) = \frac{s^{\frac{d}{2}-1}e^{-s}}{\Gamma(d/2)},\ 0 < s < \infty. \tag{4.1}$$

Thus,

$$\mathbb{P}(E^c) = \int_1^\infty g(s)\, ds = \frac{\gamma(d/2, 1)}{\Gamma(d/2)},$$

where $\Gamma(\cdot)$ is Euler's gamma function, and $\gamma(d/2, x)$ denotes the lower incomplete gamma function.

Using the linear Blaschke–Petkantschin formula (2.2) and the rotational invariance of the standard normal distribution we obtain that

$$\begin{aligned}
&\mathbb{P}(o \in [x_1, x_2]_1 \text{ and } E)\\
&\quad= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}(o \in [x_1, x_2]_1 \text{ and } E)\, e^{-\frac{|x_1|^2+|x_2|^2}{2}}\, dx_1dx_2\\
&\quad= \frac{1}{(2\pi)^d}\frac{\omega_{d-1}\omega_d}{\omega_1\omega_2}\int_{G(d,2)}\int_{L^2} \mathbf{1}(o \in [x_1, x_2]_1 \text{ and } E)\\
&\qquad\times \Delta^{d-2}(x_1, x_2)\, e^{-\frac{|x_1|^2+|x_2|^2}{2}}\, dx_1dx_2\nu_2(dL)\\
&\quad= \frac{1}{(2\pi)^d}\frac{\omega_{d-1}\omega_d}{\omega_1\omega_2}\int_{L^2} \mathbf{1}(o \in [x_1, x_2]_1 \text{ and } E)\Delta^{d-2}(x_1, x_2)\, e^{-\frac{|x_1|^2+|x_2|^2}{2}}\, dx_1dx_2.
\end{aligned}$$

In order to evaluate the above integral, we use polar coordinates $x_1 = r_1u_1$ and $x_2 = r_2u_2$, $r_1, r_2 \geq 0$, $u_1, u_2 \in S^1$. Let φ be the angle of $-u_1$ and u_2, as before. For $2 - r_1 \leq r_2 \leq \sqrt{4 - r_1^2}$, let

$$\psi(r_1, r_2) = \pi - \arccos\left(\frac{r_1^2 + r_2^2 - 4}{2r_1r_2}\right).$$

We distinguish two cases according to r_2. When $0 \leq r_2 \leq 2 - r_1$, then $-\varphi(r_1, r_2) \leq \varphi \leq \varphi(r_1, r_2)$, and when $2-r_1 \leq r_2 \leq \sqrt{4 - r_1^2}$, then $-\varphi(r_1, r_2) \leq \varphi \leq -\psi(r_1, r_2)$ and $\psi(r_1, r_2) \leq \varphi(r_1, r_2)$, see Fig. 3.

By the rotational symmetry of the normal distribution, integration with respect to u_1 is just a multiplication by 2π. Then, w obtain that

$$\begin{aligned}
&\mathbb{P}(o \in [x_1, x_2]_1 \text{ and } E)\\
&\quad= \frac{2}{(2\pi)^{d-1}}\frac{\omega_{d-1}\omega_d}{\omega_1\omega_2}\int_0^2\int_0^{2-r_1}\int_0^{\varphi(r_1,r_2)} r_1^{d-1}r_2^{d-1}\sin^{d-2}(\varphi)\, e^{-\frac{r_1^2+r_2^2}{2}}\, d\varphi dr_2dr_1
\end{aligned}$$

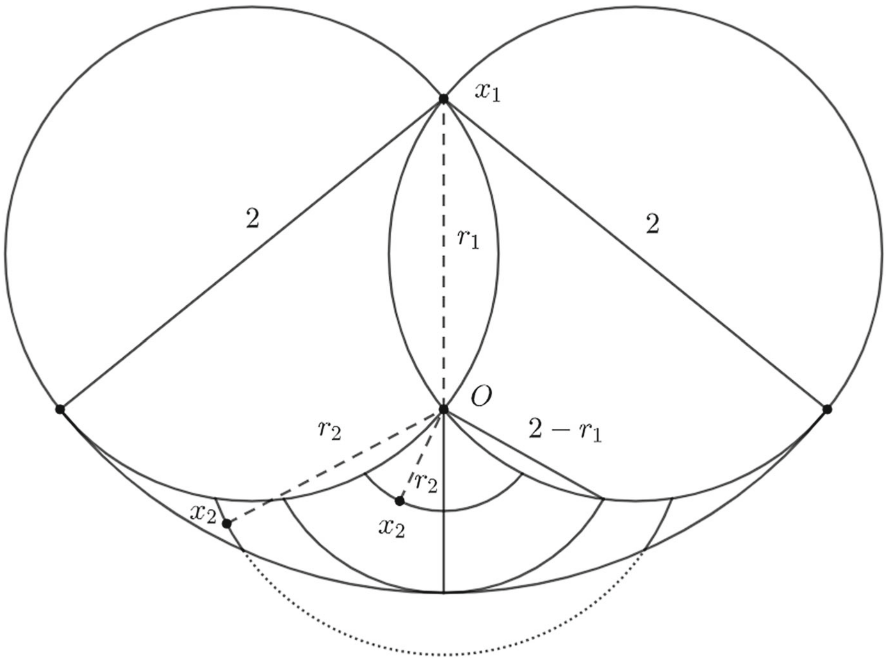

FIGURE 3. Integration bounds in φ according to r_2

$$+ \frac{2}{(2\pi)^{d-1}} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_0^2 \int_{2-r_1}^{\sqrt{4-r_1^2}} \int_{\psi(r_1,r_2)}^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1}$$
$$\times \sin^{d-2}(\varphi)\, e^{-\frac{r_1^2+r_2^2}{2}}\, d\varphi dr_2 dr_1.$$

The above integrals can be evaluated, at least numerically, for any specific value of d. In particular, for $d = 2$, we obtain for the first integral

$$\frac{1}{\pi} \int_0^2 \int_0^{2-r_1} \int_0^{\varphi(r_1,r_2)} r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}}\, d\varphi dr_2 dr_1$$
$$= \frac{1}{\pi} \int_0^2 \int_0^{2-r_1} (\arcsin(r_1/2) + \arcsin(r_2/2)) r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}}\, dr_2 dr_1$$
$$= 0.079214\ldots.$$

The second integral is

$$\frac{1}{\pi} \int_0^2 \int_{2-r_1}^{\sqrt{4-r_1^1}} \int_{\psi(r_1,r_2)}^{\varphi(r_1,r_2)} r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}}\, d\varphi dr_2 dr_1$$
$$= \frac{1}{\pi} \int_0^2 \int_{2-r_1}^{\sqrt{4-r_1^2}} \left(\arcsin(r_1/2) + \arcsin(r_2/2) - \pi + \arccos\left(\frac{r_1^2 + r_2^2 - 4}{2r_1 r_2} \right) \right)$$

$$\times\, r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}}\, dr_2 dr_1$$
$$= 0.01866\ldots.$$

For $d = 2$,

$$\mathbb{P}(E^c) = \frac{\gamma(1,1)}{\Gamma(1)} = \frac{1}{e} = 0.367879\ldots,$$

thus, in summary,

$$P_N(2,1,2) = 0.465753\ldots.$$

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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