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An energy stable discontinuous Galerkin time-domain finite element method in optics and photonics

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ABSTRACT

In this paper, a time-domain discontinuous Galerkin (TDdG) finite element method for the full system of Maxwell's equations in optics and photonics is investigated, including a complete proof of a semi-discrete error estimate. The new capabilities of methods of this type are to efficiently model linear and nonlinear effects, for example of Kerr nonlinearities. Energy stable discretizations both at the semi-discrete and the fully discrete levels are presented. In particular, the proposed semi-discrete scheme is optimally convergent in the spatial variable on Cartesian meshes with Q_k-type elements, and the fully discrete scheme is conditionally stable with respect to a specially defined nonlinear electromagnetic energy.

1. Introduction

There is great interest in developing time-domain discontinuous Galerkin (TDdG) methods for the full system of Maxwell's equations in optics and photonics, for instance to design optical devices with higher complexity. One of the most famous and general problems is the third order Kerr-type nonlinear model. A few excellent and efficient schemes are available on developing the finite difference time-domain (FDTD) methods, to solve the nonlinear Maxwell's equations in Kerr media (e.g., [1–5]). There are also many studies of TDFEMs for Maxwell's equations for considering the flexibility of finite element methods for complex domains and materials. Recent advancements and more references on TDFEMs and TDdG for Maxwell's equations with Kerr-type nonlinearity can be found in some recent reviews such as [6–13].

In the past few years, TDdG have gotten considerable attention and are being employed to a wide range of problems in optics and photonics. To the authors' knowledge, few mathematical proofs for the convergence of the discontinuous Galerkin method when applied to Maxwell's equations were given in the papers [11,14]. These methods allowed a comparatively easy handling of elements of various types and shapes, irregular non-conforming meshes and even locally varying polynomial degrees. There are still accessibly few analysis, error estimates, and simulation results by employing TDdG available for the system of Maxwell's equations with Kerr-type nonlinearity. Moreover, the results were given mostly for the 1D case using TDdG schemes.

Our prime object is to develop an energy stable numerical scheme in this paper that can preserve the stability relation at the semi-discrete and fully discrete levels. Energy preserving schemes are robust since they are able to maintain and preserve the shape and phase of the waves accurately after long-term numerical simulations. Moreover, error estimates are also presented at the semi-discrete and fully discrete levels. In this paper, we extend our results about semi-discrete

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conforming mixed finite element methods [15,16] and fully discrete conforming finite element methods [8,17,18] for the Maxwell's equations with nonlinearities by a method, which combines a locally discontinuous Galerkin discretization in space and a time discretization of leap-frog type. For the sake of simplicity, we will present the result in 2D, and analogous result are obtained for 3D. At the end of this brief and by no means complete overview on the related literature it should be mentioned that, in the course of preparing this work, the paper [19] was published with the same intention and comparable results, the authors of which probably were unaware of the first author's PhD thesis [20]. An essential part of the presented paper is an revised outcome of [20], in which the ideas and results about the proposed method were formulated for the first time.

In the spatial discretization we use Q_k-type elements on Cartesian meshes, therefore there are restrictions on the geometry of the domain and a higher computational effort compared to P_k-type elements. However, for P_k elements there are indications that such dG methods do not achieve the optimal order of accuracy, and it is also known that some of the required properties of the L^2-projectors (e.g. property (A2) in [19, Lemma A1]) do not generally hold in the multidimensional case, especially not for non-tensor product meshes [21].

Let $\Omega := (r,s) \times (p,q)$, $r < s$, $p < q$, be a rectangular domain in $\mathbb{R}^2$ with boundary Γ and unit outward normal $\mathbf{n}$. As usual, $\mathbf{D} = \mathbf{D}(\mathbf{x},t)$, $\mathbf{B} = \mathbf{B}(\mathbf{x},t)$, $\mathbf{E} = \mathbf{E}(\mathbf{x},t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{x},t)$ represent the electric displacement field, magnetic induction, electric and magnetic field intensities, resp., where $\mathbf{x} \in \Omega$ and the time variable t ranges in some interval $(0,T]$, $T > 0$. Given an electric current density $\mathbf{J} = \mathbf{J}(\mathbf{x},t)$, we write the transient Maxwell's equations as

$$\partial_t \mathbf{D} - \nabla \times \mathbf{H} = \mathbf{J} \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} = 0 \quad \text{in } \Omega \times (0,T), \tag{2}$$

where

$$\mathbf{D} = \varepsilon_0 \Big((1 + \chi^{(1)}) \mathbf{E} + \chi^{(3)} |\mathbf{E}|^2 \, \mathbf{E} \Big).$$

Here $\varepsilon_0 > 0$ denotes the constant vacuum permittivity, $\mu_0 : \Omega \to (0,\infty)$ is the permeability, and $\chi^{(1)}, \chi^{(3)} : \Omega \to (0,\infty)$ are the media susceptibility coefficients. We assume that the coefficient functions are bounded almost everywhere, i.e. $\mu_0, \chi^{(1)}, \chi^{(3)} \in L_\infty(\Omega)$.

We will consider the derivative

$$\partial_t \mathbf{D} = \varepsilon_0 \Big((1 + \chi^{(1)}) \partial_t \mathbf{E} + \chi^{(3)} [|\mathbf{E}|^2 + 2\mathbf{E}\mathbf{E}^T] \partial_t \mathbf{E} \Big). \tag{3}$$

as an additional equation.

A perfect electric conductor (PEC) boundary condition on Γ is assumed, that is

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma \times (0,T). \tag{4}$$

In addition, initial conditions have to be specified:

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega,$$

where $\mathbf{E}_0, \mathbf{H}_0 : \Omega \to \mathbb{R}^2$ are given functions, and $\mathbf{H}_0$ satisfies

$$\nabla \cdot (\mu_0 \mathbf{H}_0) = 0 \quad \text{in } \Omega, \quad \mathbf{H}_0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{5}$$

The divergence-free condition in (5) together with (2) implies that

$$\nabla \cdot (\mu_0 \mathbf{H}) = 0 \quad \text{in } \Omega \times (0,T).$$

Here the Transverse Electric mode is considered, where – for simplicity – the direction of propagation coincides with the direction of the z-axis, i.e. essentially we deal with a two-dimensional problem in space (TE_z-mode). This restriction is only used to simplify the presentation technically; analogous results for the three-dimensional case can be obtained, too. The fields reduce to $\mathbf{D} = (D_x, D_y)$, $\mathbf{E} = (E_x, E_y)$, $\nabla \times \mathbf{E} = \partial_x E_y - \partial_y E_x$, $\mathbf{H} = H_z$, $\nabla \times \mathbf{H} = (\partial_y H_z, -\partial_x H_z)^T$, and $\mathbf{J} = (J_x, J_y)$, where the subscripts x, y and z denote the x-component, y-component, and z-component of the vector field, respectively. In addition we write $\mathbf{x} := (x,y)^T$ and $|\mathbf{E}|^2 := E_x^2 + E_y^2$. Then

$$\mathbf{E}\mathbf{E}^T \partial_t \mathbf{E} = \begin{pmatrix} E_x^2 \partial_t E_x + E_x E_y \partial_t E_y \\ E_x E_y \partial_t E_x + E_y^2 \partial_t E_y \end{pmatrix},$$

and the Eqs. (1)–(3) take the form

$$\begin{aligned}
\partial_t D_x &= \partial_y H_z + J_x, \\
\partial_t D_y &= -\partial_x H_z + J_y, \\
\mu_0 \partial_t H_z &= -\partial_x E_y + \partial_y E_x, \\
\partial_t D_x &= \varepsilon_0 \Big((1 + \chi^{(1)}) \partial_t E_x + \chi^{(3)} [|\mathbf{E}|^2 \partial_t E_x + 2\big(E_x^2 \partial_t E_x + E_x E_y \partial_t E_y\big)] \Big), \\
\partial_t D_y &= \varepsilon_0 \Big((1 + \chi^{(1)}) \partial_t E_y + \chi^{(3)} [|\mathbf{E}|^2 \partial_t E_y + 2\big(E_x E_y \partial_t E_x + E_y^2 \partial_t E_y\big)] \Big).
\end{aligned} \tag{6}$$

The corresponding initial conditions are

$$E_x(\mathbf{x},0)=E_x^0(\mathbf{x}),\quad E_y(\mathbf{x},0)=E_y^0(\mathbf{x})\quad \text{and}\quad H_z(\mathbf{x},0)=E_z^0(\mathbf{x}).$$

The PEC condition (4) reads as

$$E_x(\mathbf{x},t)|_{y=p,q}=E_y(\mathbf{x},t)|_{x=r,s}=0. \tag{7}$$

2. The nonlinear electromagnetic energy at the continuous level

According to the particular structure of the nonlinearity, a "nonlinear" electromagnetic energy of the system (6) can be defined by

$$\mathcal{E}(t):=\|\mathbf{E}(t)\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|H_z(t)\|^2_{\mu_0}+\frac{3}{2}\left\||\mathbf{E}(t)|^2\right\|^2_{\varepsilon_0\chi^{(3)}},$$

$t\in[0,T)$, where we have used the notation

$$\|\mathbf{E}(t)\|_\omega:=\left(\int_\Omega|\mathbf{E}(\mathbf{x},t)|^2\omega(\mathbf{x})d\mathbf{x}\right)^{1/2}$$

for a given weight function $\omega:\ \Omega\to(0,\infty)$. In the case $\omega=1$, the subscript is omitted.

The following theorem demonstrates that the nonlinear electromagnetic energy is a conservative quantity.

Theorem 2.1. *If $(E_x,E_y,H_z)^T$ is the weak solution of the system (6) in the case of no sources, i.e. $\mathbf{J}=0$, then the nonlinear electromagnetic energy of the system (6) at any time $t\in[0,T)$ satisfies*

$$\mathcal{E}(t)=\mathcal{E}(0)=\|\mathbf{E}_0\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|H_z^0\|^2_{\mu_0}+\frac{3}{2}\left\||\mathbf{E}_0|^2\right\|^2_{\varepsilon_0\chi^{(3)}}.$$

We skip the proof since the details are similar to the more complicated proof of the semi-discrete energy law (Theorem 4.1).

The domain Ω is partitioned into rectangular cells $K_{ij}:=I_i\times J_j$ with $I_i:=(x_{i-\frac12},x_{i+\frac12})$, $i=1,2,3,\ldots,N_x$, and $J_j:=(y_{j-\frac12},y_{j+\frac12})$, $j=1,2,3,\ldots,N_y$, where

$$r=:x_{\frac12}<x_{\frac32}<\cdots<x_{N_x+\frac12}:=s,\quad p=:y_{\frac12}<y_{\frac32}<\cdots<y_{N_y+\frac12}:=q.$$

The mesh sizes are denoted by $h_i^x:=x_{i+\frac12}-x_{i-\frac12}$ and $h_j^y:=y_{j+\frac12}-y_{i-\frac12}$ with $h_x^{\max}:=\max_{1\le i\le N_x}h_i^x$ and $h_y^{\max}:=\max_{1\le j\le N_y}h_j^y$. The maximal mesh size is defined by $h:=\max\{h_x^{\max},h_y^{\max}\}$. We assume that the mesh is shape-regular, i.e., if $\varrho_{K_{ij}}$ denotes the radius of the biggest circle contained in K_{ij}, we have $h_i^xh_j^y\le C_{sr}\varrho_{K_{ij}}$ for all K_{ij} with a positive constant C_{sr}. The family of cells is denoted by $\mathcal{T}_h:=\{K_{ij}\}_{\substack{i=1,2,3,\ldots,N_x\\ j=1,2,3,\ldots,N_y}}$.

The finite element space U_h^k is the space of tensor products of piecewise polynomials of degree at most $k\in\mathbb{N}$ in each variable on every element K_{ij}:

$$U_h^k:=\{u:\ u|_K\in Q_k(K)\quad\text{for all } K\in\mathcal{T}_h\},$$

where the local space $Q_k(K)$ consists of tensor products of univariate polynomials of degree up to k on a cell K. Note that $U_h^k\not\subset C(\overline{\Omega})$ in general.

The numerical approximation of a function $u:\ \overline{\Omega}\to\mathbb{R}$ is denoted by $u_h\in U_h^k$. The limiting value of u_h at $x_{i+\frac12}$ from the right cell $K_{i+1,j}$ is denoted by $u_h(x^+_{i+\frac12},y)$, $(u_h)^+_{i+\frac12,y}$ or $u_h^+(x_{i+\frac12},y)$, and from the left cell K_{ij} by $u_h(x^-_{i+\frac12},y)$, $(u_h)^-_{i+\frac12,y}$ or $u_h^-(x_{i+\frac12},y)$, respectively. An analogous convention is used in the y-direction.

The numerical flux densities are obtained by means of integration by parts. They should be considered and designed carefully to ensure conservation of energy, numerical stability and optimal accuracy of the approximate solution. The numerical flux densities are functions that are defined on the cell boundaries. The alternating flux densities are defined in a simple and elegant way like in LDG (local discontinuous Galerkin) methods for diffusion equations, second order wave equation and Maxwell's equations [22–25]. Fixing a constant $c_0>0$ independent of h, the alternating flux densities are:

$$\begin{aligned}
\hat{E}_{x,h}(x,y_{j+\frac12})&:=E^+_{x,h}(x,y_{j+\frac12})\quad\text{for all } j=1,2,3,\ldots,N_y-1,\\
\hat{E}_{x,h}(x,y_{\frac12})&:=\hat{E}_{x,h}(x,y_{N_y+\frac12}):=0,\\
\hat{E}_{y,h}(x_{i+\frac12},y)&:=E^+_{y,h}(x_{i+\frac12},y)\quad\text{for all } i=1,2,3,\ldots,N_x-1,\\
\hat{E}_{y,h}(x_{\frac12},y)&:=\hat{E}_{y,h}(x_{N_x+\frac12},y):=0,\\
\hat{H}_{z,h}(x,y_{j+\frac12})&:=H^-_{z,h}(x,y_{j+\frac12})\quad\text{for all } j=1,2,3,\ldots,N_y,
\end{aligned}$$

$$\hat{H}_{z,h}(x, y_{\frac{1}{2}}) := H^{+}_{z,h}(x, y_{\frac{1}{2}}) + c_0[\![E_{x,h}(x, y_{\frac{1}{2}})]\!], \tag{8}$$

$$\hat{H}_{z,h}(x_{i+\frac{1}{2}}, y) := H^{-}_{z,h}(x_{i+\frac{1}{2}}, y) \quad \text{for all } i = 1, 2, 3, \ldots, N_x,$$

$$\hat{H}_{z,h}(x_{\frac{1}{2}}, y) := H^{+}_{z,h}(x_{\frac{1}{2}}, y) - c_0[\![E_{y,h}(x_{\frac{1}{2}}, y)]\!], \tag{9}$$

where the jump terms in (8), (9) are defined as

$$[\![E_{x,h}(x, y_{\frac{1}{2}})]\!] := E^{+}_{x,h}(x, y_{\frac{1}{2}}) - 0, \qquad [\![E_{y,h}(x_{\frac{1}{2}}, y)]\!] := E^{+}_{y,h}(x_{\frac{1}{2}}, y) - 0.$$

On interior cell boundaries, the jumps are denoted by $[\![\Psi]\!] := \Psi^{+} - \Psi^{-}$. For $c_0 = \frac{1}{2}$, the flux densities (8), (9) match with the standard upwind flux densities

$$\hat{H}_{z,h}(x, y_{\frac{1}{2}}) := \frac{1}{2}[H^{+}_{z,h}(x, y_{\frac{1}{2}}) + H^{-}_{z,h}(x, y_{\frac{1}{2}})] + \frac{1}{2}[\![E_{x,h}(x, y_{\frac{1}{2}})]\!],$$

$$\hat{H}_{z,h}(x_{\frac{1}{2}}, y) := \frac{1}{2}[H^{+}_{z,h}(x_{\frac{1}{2}}, y) + H^{-}_{z,h}(x_{\frac{1}{2}}, y)] - \frac{1}{2}[\![E_{y,h}(x_{\frac{1}{2}}, y)]\!],$$

where the undefined $H^{-}_{z,h}(x, y_{\frac{1}{2}})$ and $H^{-}_{z,h}(x_{\frac{1}{2}}, y)$ are replaced by $H^{+}_{z,h}(x, y_{\frac{1}{2}})$ and $H^{+}_{z,h}(x_{\frac{1}{2}}, y)$, respectively.

3. Spatial discretization by a discontinuous Galerkin method

For the test functions $(\Phi_{1h}, \Phi_{2h}, \Phi_{3h})^T \in (U_h^k)^3$, the discontinuous Galerkin formulation for the Eqs. (6) with respect to the semi-discrete solution $(E_{x,h}, E_{y,h}, H_{z,h})^T \in C^1(0, T, U_h^k)^3$ reads as follows (for shortness, we omit the formal differentials $d\mathbf{x}$ in the double integrals):

$$\int_{K_{ij}} \partial_t D_{x,h}\, \Phi_{1h} - \int_{I_i} [(\hat{H}_{z,h}\Phi^{-}_{1h})_{x,j+\frac{1}{2}} - (\hat{H}_{z,h}\Phi^{+}_{1h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} H_{z,h}\partial_y \Phi_{1h} - \int_{K_{ij}} J_{x,h}\Phi_{1h} = 0, \tag{10}$$

$$\int_{K_{ij}} \partial_t D_{y,h}\, \Phi_{2h} + \int_{J_j} [(\hat{H}_{z,h}\Phi^{-}_{2h})_{i+\frac{1}{2},y} - (\hat{H}_{z,h}\Phi^{+}_{2h})_{i-\frac{1}{2},y}]dy - \int_{K_{ij}} H_{z,h}\partial_x \Phi_{2h} - \int_{K_{ij}} J_{y,h}\Phi_{2h} = 0, \tag{11}$$

$$\begin{aligned} &\int_{K_{ij}} \mu_0 \partial_t H_{z,h}\, \Phi_{3h} + \int_{J_j} [(\hat{E}_{y,h}\Phi^{-}_{3h})_{i+\frac{1}{2},y} - (\hat{E}_{y,h}\Phi^{+}_{3h})_{i-\frac{1}{2},y}]dy - \int_{K_{ij}} E_{y,h}\partial_x \Phi_{3h} \\ &\quad - \int_{I_i} [(\hat{E}_{x,h}\Phi^{-}_{3h})_{x,j+\frac{1}{2}} - (\hat{E}_{x,h}\Phi^{+}_{3h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} E_{x,h}\partial_y \Phi_{3h} = 0, \end{aligned} \tag{12}$$

$$\int_{K_{ij}} \partial_t D_{x,h}\, \Phi_{1h} = \int_{K_{ij}} \varepsilon_0(1 + \chi^{(1)})\partial_t E_{x,h}\, \Phi_{1h} + \int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[|\mathbf{E}_h|^2 \partial_t E_{x,h}\, \Phi_{1h} + 2\big(E^2_{x,h}\partial_t E_{x,h}\, \Phi_{1h} + E_{x,h}E_{y,h}\partial_t E_{y,h}\, \Phi_{1h}\big)\Big], \tag{13}$$

$$\int_{K_{ij}} \partial_t D_{y,h}\, \Phi_{2h} = \int_{K_{ij}} \varepsilon_0(1 + \chi^{(1)})\partial_t E_{y,h}\, \Phi_{2h} + \int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[|\mathbf{E}_h|^2 \partial_t E_{y,h}\, \Phi_{2h} + 2\big(E^2_{y,h}\partial_t E_{y,h}\, \Phi_{2h} + E_{x,h}E_{y,h}\partial_t E_{x,h}\, \Phi_{2h}\big)\Big]. \tag{14}$$

The initial conditions are defined as

$$E_{x,h}(\mathbf{x}, 0) = E^0_{x,h}(\mathbf{x}), \quad E_{y,h}(\mathbf{x}, 0) = E^0_{y,h}(\mathbf{x}) \quad \text{and} \quad H_{z,h}(\mathbf{x}, 0) = E^0_{z,h}(\mathbf{x}),$$

where the concrete choice of the discrete initial data $(E^0_{x,h}, E^0_{y,h}, H^0_{z,h})^T \in (U_h^k)^3$ will be given later.

4. The nonlinear electromagnetic energy of the semi-discrete discontinuous Galerkin discretization

The nonlinear electromagnetic energy of the semi-discrete discontinuous Galerkin discretization (10)–(14) is defined by

$$\mathcal{E}_h(t) := \|\mathbf{E}_h(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}(t)\|^2_{\mu_0} + \frac{3}{2}\big\||\mathbf{E}_h(t)|^2\big\|^2_{\varepsilon_0\chi^{(3)}} + \frac{c_0}{2}\int_0^t \Big[\int_r^s (E^{+}_{x,h}(t))^2_{x,\frac{1}{2}}\, dx + \int_p^q (E^{+}_{y,h}(t))^2_{\frac{1}{2},y}\, dy\Big],$$

$t \in [0, T)$. In the next, we will show that the nonlinear electromagnetic energy at the semi-discrete level of the system (10)–(14) at time t is conserved and bounded.

Theorem 4.1. *Let $(E_{x,h}, E_{y,h}, H_{z,h})^T \in C^1(0, T, U_h^k)^3$ be the semi-discrete solution of the system (10)–(14) for given $\mathbf{J}_h \in C(0, T, U_h^k)^2$, then the nonlinear electromagnetic energy of the system (10)–(14) for the vanishing current density at any time $t \in [0, T)$ satisfies*

$$\mathcal{E}_h(t) = \mathcal{E}_h(0) = \|\mathbf{E}^0_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H^0_{z,h}\|^2_{\mu_0} + \frac{3}{2}\big\||\mathbf{E}^0_h|^2\big\|^2_{\varepsilon_0\chi^{(3)}}, \tag{15}$$

and for non-zero current density

$$\mathcal{E}_h(t) \le 2\mathcal{E}_h(0) + 8\left(\int_0^t \|\mathbf{J}_h(s)\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} ds\right)^2. \tag{16}$$

Proof. Taking $\Phi_{1h} := E_{x,h}$ in the Eqs. (10) and (13), and substituting Eq. (13) into Eq. (10), we have

$$\begin{aligned}&\int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})\partial_t E_{x,h}\, E_{x,h} + \int_{K_{ij}} \varepsilon_0\chi^{(3)} \Big[\, |\mathbf{E}_h|^2 \partial_t E_{x,h}\, E_{x,h} + 2\big(E_{x,h}^2 \partial_t E_{x,h}\, E_{x,h} + E_{x,h}E_{y,h}\partial_t E_{y,h}\, E_{x,h}\big)\, \Big]\\ &- \int_{I_i} [(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac12} - (\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac12}]dx + \int_{K_{ij}} H_{z,h}\partial_y E_{x,h} - \int_{K_{ij}} J_{x,h}E_{x,h} = 0.\end{aligned} \tag{17}$$

Taking $\Phi_{2h} := E_{y,h}$ in the Eqs. (11) and (14), and substituting Eq. (14) into Eq. (11), we have

$$\begin{aligned}&\int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})\partial_t E_{y,h}\, E_{y,h} + \int_{K_{ij}} \varepsilon_0\chi^{(3)} \Big[\, |\mathbf{E}_h|^2 \partial_t E_{y,h}\, E_{y,h} + 2\big(E_{y,h}^2 \partial_t E_{y,h}\, E_{y,h} + E_{x,h}E_{y,h}\partial_t E_{x,h}\, E_{y,h}\big)\, \Big]\\ &+ \int_{J_j} [(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac12,y} - (\hat{H}_{z,h}E_{y,h}^+)_{i-\frac12,y}]dy - \int_{K_{ij}} H_{z,h}\partial_x E_{y,h} - \int_{K_{ij}} J_{y,h}E_{y,h} = 0.\end{aligned} \tag{18}$$

Adding the Eqs. (17) and (18), we get

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})|\mathbf{E}_h|^2 + \int_{K_{ij}} \varepsilon_0\chi^{(3)}|\mathbf{E}_h|^2\frac12\partial_t|\mathbf{E}_h|^2\\ &+ \int_{K_{ij}} \varepsilon_0\chi^{(3)}2\Big[E_{x,h}^2\frac12\partial_t E_{x,h}^2 + E_{y,h}^2\frac12\partial_t E_{y,h}^2\Big] + \int_{K_{ij}} \varepsilon_0\chi^{(3)}2\Big[E_{x,h}E_{y,h}[\partial_t E_{y,h}\, E_{x,h} + \partial_t E_{x,h}\, E_{y,h}]\Big]\\ &- \int_{I_i} [(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac12} - (\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac12}]dx + \int_{J_j} [(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac12,y} - (\hat{H}_{z,h}E_{y,h}^+)_{i-\frac12,y}]dy\\ &+ \int_{K_{ij}} H_{z,h}\partial_y E_{x,h} - \int_{K_{ij}} H_{z,h}\partial_x E_{y,h} - \int_{K_{ij}} J_{x,h}E_{x,h} - \int_{K_{ij}} J_{y,h}E_{y,h} = 0.\end{aligned}$$

The integrands corresponding to the cubic nonlinearity can be rewritten as follows:

$$\begin{aligned}&|\mathbf{E}_h|^2\frac12\partial_t|\mathbf{E}_h|^2 + 2\Big[E_{x,h}^2\frac12\partial_t E_{x,h}^2 + E_{y,h}^2\frac12\partial_t E_{y,h}^2\Big] + 2\Big[E_{x,h}E_{y,h}[\partial_t E_{y,h}\, E_{x,h} + \partial_t E_{x,h}\, E_{y,h}]\Big]\\ &= \frac14\partial_t|\mathbf{E}_h|^4 + \frac12\Big[\partial_t E_{x,h}^4 + \partial_t E_{y,h}^4\Big] + 2\Big[E_{x,h}E_{y,h}\partial_t(E_{y,h}E_{x,h})\Big]\\ &= \frac14\partial_t|\mathbf{E}_h|^4 + \frac12\partial_t(E_{x,h}^4 + E_{y,h}^4) + \partial_t(E_{y,h}E_{x,h})^2\\ &= \frac14\partial_t|\mathbf{E}_h|^4 + \frac12\partial_t|\mathbf{E}_h|^4 = \frac34\partial_t|\mathbf{E}_h|^4.\end{aligned}$$

Thus we arrive at

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})|\mathbf{E}_h|^2 + \frac34\frac{d}{dt}\int_{K_{ij}} \varepsilon_0\chi^{(3)}|\mathbf{E}_h|^4\\ &- \int_{I_i} [(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac12} - (\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac12}]dx + \int_{J_j} [(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac12,y} - (\hat{H}_{z,h}E_{y,h}^+)_{i-\frac12,y}]dy\\ &+ \int_{K_{ij}} H_{z,h}\partial_y E_{x,h} - \int_{K_{ij}} H_{z,h}\partial_x E_{y,h} - \int_{K_{ij}} J_{x,h}E_{x,h} - \int_{K_{ij}} J_{y,h}E_{y,h} = 0.\end{aligned} \tag{19}$$

Taking $\Phi_{3h} := H_{z,h}$ in the Eqs. (12), we have

$$\begin{aligned}&\int_{K_{ij}} \mu_0\partial_t H_{z,h}\, H_{z,h} + \int_{J_j} [(\hat{E}_{y,h}H_{z,h}^-)_{i+\frac12,y} - (\hat{E}_{y,h}H_{z,h}^+)_{i-\frac12,y}]dy - \int_{K_{ij}} E_{y,h}\partial_x H_{z,h}\\ &- \int_{I_i} [(\hat{E}_{x,h}H_{z,h}^-)_{x,j+\frac12} - (\hat{E}_{x,h}H_{z,h}^+)_{x,j-\frac12}]dx + \int_{K_{ij}} E_{x,h}\partial_y H_{z,h} = 0.\end{aligned} \tag{20}$$

Adding the Eqs. (19) and (20), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})|\mathbf{E}_h|^2 + \frac{1}{2}\frac{d}{dt}\int_{K_{ij}} \mu_0 H_{z,h}^2 + \frac{3}{4}\frac{d}{dt}\int_{K_{ij}} \varepsilon_0\chi^{(3)}|\mathbf{E}_h|^4 \\
&- \int_{I_i}[(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac{1}{2}} - (\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac{1}{2}}]dx + \int_{J_j}[(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac{1}{2},y} - (\hat{H}_{z,h}E_{y,h}^+)_{i-\frac{1}{2},y}]dy \\
&+ \int_{J_j}[(\hat{E}_{y,h}H_{z,h}^-)_{i+\frac{1}{2},y} - (\hat{E}_{y,h}H_{z,h}^+)_{i-\frac{1}{2},y}]dy - \int_{I_i}[(\hat{E}_{x,h}H_{z,h}^-)_{x,j+\frac{1}{2}} - (\hat{E}_{x,h}H_{z,h}^+)_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{ij}} H_{z,h}\partial_y E_{x,h} - \int_{K_{ij}} H_{z,h}\partial_x E_{y,h} - \int_{K_{ij}} E_{y,h}\partial_x H_{z,h} + \int_{K_{ij}} E_{x,h}\partial_y H_{z,h} - \int_{K_{ij}} J_{x,h}E_{x,h} - \int_{K_{ij}} J_{y,h}E_{y,h} = 0.
\end{aligned} \tag{21}$$

In the next step, the Eqs. (21) are summed up with respect to both indices $1 \le i \le N_x$ and $1 \le j \le N_y$. The sums resulting from the terms on the second to fourth lines allow the following simplification, see [25, eqs. (3.18)–(3.19)]:

$$\begin{aligned}
&\sum_{j=1}^{N_y}\Big[- \int_{I_i}[(\hat{H}_{z,h}E_{x,h}^-)_{x,j+\frac{1}{2}} - (\hat{H}_{z,h}E_{x,h}^+)_{x,j-\frac{1}{2}}]dx - \int_{I_i}[(\hat{E}_{x,h}H_{z,h}^-)_{x,j+\frac{1}{2}} - (\hat{E}_{x,h}H_{z,h}^+)_{x,j-\frac{1}{2}}]dx \\
&+ \int_{K_{ij}} H_{z,h}\partial_y E_{x,h} + \int_{K_{ij}} E_{x,h}\partial_y H_{z,h}\Big] = c_0\int_{I_i}(E_{x,h}^+)^2_{x,\frac{1}{2}}dx, \\
&\sum_{i=1}^{N_x}\Big[\int_{J_j}[(\hat{H}_{z,h}E_{y,h}^-)_{i+\frac{1}{2},y} - (\hat{H}_{z,h}E_{y,h}^+)_{i-\frac{1}{2},y}]dy + \int_{J_j}[(\hat{E}_{y,h}H_{z,h}^-)_{i+\frac{1}{2},y} - (\hat{E}_{y,h}H_{z,h}^+)_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{ij}} H_{z,h}\partial_x E_{y,h} - \int_{K_{ij}} E_{y,h}\partial_x H_{z,h}\Big] = c_0\int_{J_j}(E_{y,h}^+)^2_{\frac{1}{2},y}dy.
\end{aligned} \tag{22}$$

Using these relationships, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\frac{d}{dt}\|H_{z,h}\|^2_{\mu_0} + \frac{3}{4}\frac{d}{dt}\big\||\mathbf{E}_h|^2\big\|^2_{\varepsilon_0\chi^{(3)}} \\
&+ c_0\int_r^s (E_{x,h}^+)^2_{x,\frac{1}{2}}dx + c_0\int_p^q (E_{y,h}^+)^2_{\frac{1}{2},y}dy = \int_\Omega [J_{x,h}E_{x,h} + J_{y,h}E_{y,h}].
\end{aligned} \tag{23}$$

The right-hand side of Eq. (23) is estimated by means of Cauchy–Schwarz inequalities as follows:

$$\int_\Omega [J_{x,h}E_{x,h} + J_{y,h}E_{y,h}] \le \int_\Omega |\mathbf{J}_h||\mathbf{E}_h| = \int_\Omega |\mathbf{E}_h|\sqrt{\varepsilon_0(1+\chi^{(1)})}|\mathbf{J}_h|\sqrt{(\varepsilon_0(1+\chi^{(1)}))^{-1}} \le \|\mathbf{E}_h\|_{\varepsilon_0(1+\chi^{(1)})}\|\mathbf{J}_h\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}.$$

Then we obtain from Eq. (23)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{E}_h\|^2_{\varepsilon_0(1+\chi^{(1)})} + \frac{1}{2}\frac{d}{dt}\|H_{z,h}\|^2_{\mu_0} + \frac{3}{4}\frac{d}{dt}\big\||\mathbf{E}_h|^2\big\|^2_{\varepsilon_0\chi^{(3)}} + c_0\int_r^s (E_{x,h}^+)^2_{x,\frac{1}{2}}dx + c_0\int_p^q (E_{y,h}^+)^2_{\frac{1}{2},y}dy \\
&\le \|\mathbf{E}_h\|_{\varepsilon_0(1+\chi^{(1)})}\|\mathbf{J}_h\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}.
\end{aligned}$$

Integration of both sides from 0 to t yields

$$\frac{1}{2}\mathcal{E}_h(t) - \frac{1}{2}\mathcal{E}_h(0) \le \int_0^t \|\mathbf{E}_h(s)\|_{\varepsilon_0(1+\chi^{(1)})}\|\mathbf{J}_h(s)\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds,$$

hence

$$\mathcal{E}_h(t) \le \mathcal{E}_h(0) + 2\int_0^t \sqrt{\mathcal{E}_h(s)}\|\mathbf{J}_h(s)\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds.$$

Then the Gronwall–Ou-Iang's inequality [26] implies that

$$\sqrt{\mathcal{E}_h(t)} \le \sqrt{\mathcal{E}_h(0)} + 2\int_0^t \|\mathbf{J}_h(s)\|_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds.$$

Squaring this estimate together with an elementary inequality completes the proof of (16). The relationship (15) immediately follows from integration of (23) for the case $\mathbf{J}_h = 0$. ◂

5. Error estimates for the semi-discrete discontinuous Galerkin discretization

Projection operators play an important role in the error analysis, and we will begin with defining 1D projectors that are frequently used in discontinuous Galerkin methods [27,28]. In this presentation, we closely follow [25]. Let $\mathcal{P}_k(I_i)$ denote the kth degree polynomial space over the interval I_i, $k \in \mathbb{N}$. For any function $u \in H^1(I_i)$, we define

$$P_x^{\pm} : H^1(I_i) \to \mathcal{P}_k(I_i)$$

by

$$\int_{I_i} (P_x^+ u) w dx = \int_{I_i} u\, w dx \quad \text{for all } w \in P_{k-1}(I_i) \text{ and } P_x^+ u\left(x_{i-\frac{1}{2}}^+\right) := u\left(x_{i-\frac{1}{2}}^+\right),$$
$$\int_{I_i} (P_x^- u) w dx = \int_{I_i} u\, w dx \quad \text{for all } w \in P_{k-1}(I_i) \text{ and } P_x^- u\left(x_{i+\frac{1}{2}}^-\right) := u\left(x_{i+\frac{1}{2}}^-\right).$$

Analogously, for any function $u \in H^1(J_j)$, the projection operators in y-direction

$$P_y^{\pm} : H^1(J_j) \to P_k(J_j)$$

are defined by

$$\int_{J_j} (P_y^+ u) w dy = \int_{J_j} u\, w dy \quad \text{for all } w \in P_{k-1}(J_j) \text{ and } P_y^+ u\left(y_{j-\frac{1}{2}}^+\right) := u\left(y_{j-\frac{1}{2}}^+\right),$$
$$\int_{J_j} (P_y^- u) w dy = \int_{J_j} u\, w dy \quad \text{for all } w \in P_{k-1}(J_j) \text{ and } P_y^- u\left(y_{j+\frac{1}{2}}^-\right) := u\left(y_{j+\frac{1}{2}}^-\right).$$

The standard local L_2-projection operators in 1D are denoted by

$$P_x : H^1(I_i) \to \mathcal{P}_k(I_i) \quad \text{and} \quad P_y : H^1(J_j) \to P_k(J_j).$$

The 2D projection operators for the rectangular elements $K_{ij} = I_i \times J_j$ are defined as tensor products of the 1D projectors. We define

$$\Pi_1 := P_x \otimes P_y^+ : H^2(K_{ij}) \to Q_k(K_{ij}), \tag{24}$$

which satisfies

$$\int_{K_{ij}} [\Pi_1 w(x,y) \partial_y u_h(x,y)] = \int_{K_{ij}} [w(x,y) \partial_y u_h(x,y)],$$
$$\int_{I_i} \Pi_1 w\left(x, y_{j-\frac{1}{2}}^+\right) u_h\left(x, y_{j-\frac{1}{2}}^+\right) dx = \int_{I_i} w\left(x, y_{j-\frac{1}{2}}^+\right) u_h\left(x, y_{j-\frac{1}{2}}^+\right) dx$$

for all $w \in H^2(K_{ij})$ and $u_h \in Q_k(K_{ij})$ [25,29]. The projection Π_2 is defined as

$$\Pi_2 := P_x^+ \otimes P_y : H^2(K_{ij}) \to Q_k(K_{ij}) \tag{25}$$

and satisfies

$$\int_{K_{ij}} [\Pi_2 w(x,y) \partial_x u_h(x,y)] = \int_{K_{ij}} [w(x,y) \partial_x u_h(x,y)],$$
$$\int_{J_j} \Pi_2 w\left(x_{i-\frac{1}{2}}^+, y\right) u_h\left(x_{i-\frac{1}{2}}^+, y\right) dy = \int_{J_j} w\left(x_{i-\frac{1}{2}}^+, y\right) u_h\left(x_{i-\frac{1}{2}}^+, y\right) dy$$

for all $w \in H^2(K_{ij})$ and $u_h \in Q_k(K_{ij})$. The projection Π_3 is defined as

$$\Pi_3 := P_x^- \otimes P_y^- : H^2(K_{ij}) \to Q_k(K_{ij}). \tag{26}$$

It satisfies

$$\int_{K_{ij}} [\Pi_3 w(x,y) u_h(x,y)] = \int_{K_{ij}} [w(x,y) u_h(x,y)],$$
$$\int_{I_i} \Pi_3 w\left(x, y_{j+\frac{1}{2}}^-\right) u_h\left(x, y_{j+\frac{1}{2}}^-\right) dx = \int_{I_i} w\left(x, y_{j+\frac{1}{2}}^-\right) u_h\left(x, y_{j+\frac{1}{2}}^-\right) dx,$$
$$\int_{J_j} \Pi_3 w\left(x_{i+\frac{1}{2}}^-, y\right) u_h\left(x_{i+\frac{1}{2}}^-, y\right) dy = \int_{J_j} w\left(x_{i+\frac{1}{2}}^-, y\right) u_h\left(x_{i+\frac{1}{2}}^-, y\right) dy,$$
$$\Pi_3 w\left(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-\right) = w\left(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-\right)$$

for all $w \in H^2(K_{ij})$ and $u_h \in Q_{k-1}(K_{ij})$. The use of the H^2-spaces for the point values makes sense due to the Sobolev embedding $H^2 \subset C^0$ in 2D. The 2D L_2-projection operator is usually defined by

$$\Pi_4 := P_x \otimes P_y : H^2(K_{ij}) \to Q_k(K_{ij}), \tag{27}$$

for their properties see [27,28], [25, eqs. (3.33)–(3.42)].

Lemma 5.1. *If w is a product of 1D functions, i.e. $w(x,y)=f(x)g(y)$, where $f\in H^1(I_i)$ and $g\in H^1(J_j)$, then*

$$\Pi_1 w(x,y)=P_x f(x)P_y^+ g(y),\quad \Pi_2 w(x,y)=P_x^+ f(x)P_y g(y),$$

$$\Pi_3 w(x,y)=P_x^- f(x)P_y^- g(y),\ \Pi_4 w(x,y)=P_x f(x)P_y g(y).$$

These results demonstrate that the 2D projections are tensor products of 1D projections, for details see [27,28].

Lemma 5.2. *The projection operators $\Pi_1,\dots,\Pi_4$, defined in (24)–(27), have the following property: For $k\in\mathbb{N}$, there exists a constant $C>0$ independent of h such that*

$$\|\Pi_i u-u\|\le Ch^{k+1}\|u\|_{H^{k+1}(\Omega)}$$

for all $u\in H^{k+1}(\Omega)$, $i=1,\dots,4$.

Now we are prepared to derive an error estimate. Let $(E_x,E_y,H_z)^T$ be the weak solution of (6) and $(E_{x,h},E_{y,h},H_{z,h})^T$ be corresponding numerical solution of the semi-discrete scheme (10)–(14). We denote the error terms for later use by

$$\zeta_x:=E_x-E_{x,h}=\eta_x-\eta_{x,h}, \tag{28}$$

where

$$\eta_x:=E_x-\Pi_1 E_x,\quad \eta_{x,h}:=E_{x,h}-\Pi_1 E_x. \tag{29}$$

Similarly for the y-component of the electric field we set

$$\zeta_y:=E_y-E_{y,h}=\eta_y-\eta_{y,h}, \tag{30}$$

where

$$\eta_y:=E_y-\Pi_2 E_y,\quad \eta_{y,h}:=E_{y,h}-\Pi_2 E_y. \tag{31}$$

The error terms for the magnetic field are defined by:

$$\xi_z:=H_z-H_{z,h}=\theta_z-\theta_{z,h}, \tag{32}$$

where

$$\theta_z:=H_z-\Pi_3 H_z,\quad \theta_{z,h}:=H_{z,h}-\Pi_3 H_z. \tag{33}$$

Lemma 5.3. *There exists a constant $C>0$ independent of h such that*

$$\sum_{i=1}^{N_x}\Big[-\int_{I_i}[(\hat\theta_z\eta_{x,h}^-)_{x,j+\frac12}-(\hat\theta_z\eta_{x,h}^+)_{x,j-\frac12}]dx+\int_{K_{ij}}\theta_z\partial_y\eta_{x,h}\Big]\le Ch^{2k+2}+\|\eta_{x,h}\|^2,$$

$$\sum_{j=1}^{N_y}\Big[\int_{J_j}[(\hat\theta_z\eta_{y,h}^-)_{i+\frac12,y}-(\hat\theta_z\eta_{y,h}^+)_{i-\frac12,y}]dy-\int_{K_{ij}}\theta_z\partial_x\eta_{y,h}\Big]\le Ch^{2k+2}+\|\eta_{y,h}\|^2.$$

Proof. See [25, Lemma 3.4]. ◂

Lemma 5.4. *There exists a constant $C>0$ independent of h such that*

$$\sum_{i=1}^{N_x}\Big[-\int_{I_i}[(\hat\theta_z\eta_{x,h}^-)_{x,j+\frac12}-(\hat\theta_z\eta_{x,h}^+)_{x,j-\frac12}]dx+\int_{K_{ij}}\theta_z\partial_y\eta_{x,h}\Big]-\sum_{i=1}^{N_x}c_0\int_{I_i}[\eta_{x,h}^+(x,y_{\frac12})]^2dx\le Ch^{2k+2}+\|\eta_{x,h}\|^2,$$

$$\sum_{j=1}^{N_y}\Big[\int_{J_j}[(\hat\theta_z\eta_{y,h}^-)_{i+\frac12,y}-(\hat\theta_z\eta_{y,h}^+)_{i-\frac12,y}]dy-\int_{K_{ij}}\theta_z\partial_x\eta_{y,h}\Big]-\sum_{j=1}^{N_y}c_0\int_{J_j}[\eta_{y,h}^+(x_{\frac12},y)]^2dy\le Ch^{2k+2}+\|\eta_{y,h}\|^2.$$

Proof. See [25, Lemma 3.5]. ◂

Remark 5.5. When $c_0=0$, we obtain PEC boundary condition without the jump terms in (8) and (9). In this case, we can only control the term $\sum_{1\le i\le N_x}\int_{I_i}(\theta_z^+,\eta_{z,h}^+)(x,c)$ as follows

$$\sum_{1\le i\le N_x}\int_{I_i}(\theta_z^+,\eta_{z,h}^+)(x,c)\le h^{-1}\int_r^s(\theta_z^+)^2(x,c)+h\int_r^s(\eta_{z,h}^+)^2(x,c)\le Ch^{2k+1}+h\|\eta_{z,h}\|^2,$$

by an inverse inequality. Therefore we lose half an order.

The following result formulates the announced error estimate for the semi-discrete problem. As in many cases of qualitative estimates, higher regularity requirements are placed on the weak solution, which of course do not have to be met in all real world situations. In particular, we assume that the semi-discrete solution is uniformly bounded. In some special cases, this assumption can be removed at the expense of additional conditions, mainly a smallness condition to the nonlinearity [30, Thm. 4.1], [19, Thm. 3].

Theorem 5.6. *Suppose that a weak solution $(E_x, E_y, H_z)^T \in C^1(0,T,H^{k+1}(\Omega))^3$, $k \in \mathbb{N}$, of the system (6), and a finite element solution $(E_{x,h}, E_{y,h}, H_{z,h})^T \in C^1(0,T,U_h^k \cap L_\infty(\Omega))^3$ of the system (10)–(14) with the initial data $E^0_{x,h} := \Pi_1 E^0_x$, $E^0_{y,h} := \Pi_2 E^0_y$, $H^0_{z,h} := \Pi_3 H^0_z$, respectively exist, where the L_∞-boundedness of the finite element solution is uniform w.r.t. h. Then, if $h > 0$ is sufficiently small, the following error estimate holds with a coefficient $C(t) > 0$ independent of h:*

$$\|E_x(t) - E_{x,h}(t)\|_{\varepsilon_0(1+\chi^{(1)})} + \|E_y(t) - E_{y,h}(t)\|_{\varepsilon_0(1+\chi^{(1)})} + \|H_z(t) - H_{z,h}(t)\|_{\mu_0} \le C(t)h^{k+1}, \quad t \in (0,T).$$

(The concrete structure of $C(t)$ will become apparent from the proof.)

Proof. Subtracting the Eqs. (10)–(14) from the weak formulations of the Eqs. (6), using the error identities (28), (30), and (32), for all test functions $\Phi_{1h}, \Phi_{2h}, \Phi_{3h} \in Q_k(K_{ij})$, we obtain

$$\int_{K_{ij}} \partial_t (D_x - D_{x,h})\Phi_{1h} - \int_{I_i} [(\hat{\xi}_z \Phi^-_{1h})_{x,j+\frac{1}{2}} - (\hat{\xi}_z \Phi^+_{1h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} \xi_z \partial_y \Phi_{1h} = 0, \tag{34}$$

$$\int_{K_{ij}} \partial_t (D_y - D_{y,h})\Phi_{2h} + \int_{J_j} [(\hat{\xi}_z \Phi^-_{2h})_{i+\frac{1}{2},y} - (\hat{\xi}_z \Phi^+_{2h})_{i-\frac{1}{2},y}]dy - \int_{K_{ij}} \xi_z \partial_x \Phi_{2h} = 0, \tag{35}$$

$$\begin{aligned}&\int_{K_{ij}} \mu_0 \partial_t \xi_z \Phi_{3h} + \int_{J_j} [(\hat{\zeta}_y \Phi^-_{3h})_{i+\frac{1}{2},y} - (\hat{\zeta}_y \Phi^+_{3h})_{i-\frac{1}{2},y}]dy \\ &- \int_{K_{ij}} \zeta_y \partial_x \Phi_{3h} - \int_{I_i} [(\hat{\zeta}_x \Phi^-_{3h})_{x,j+\frac{1}{2}} - (\hat{\zeta}_x \Phi^+_{3h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} \zeta_x \partial_y \Phi_{3h} = 0,\end{aligned}$$

$$\begin{aligned}&\int_{K_{ij}} \partial_t (D_x - D_{x,h})\Phi_{1h} = \int_{K_{ij}} \varepsilon_0 (1+\chi^{(1)})\partial_t \zeta_x \Phi_{1h} + \int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[[|\mathbf{E}|^2 - |\mathbf{E}_h|^2]\partial_t E_x \Phi_{1h} \\ &+ |\mathbf{E}_h|^2 \partial_t [E_x - E_{x,h}]\Phi_{1h} + 2\big([E_x^2 - E_{x,h}^2]\partial_t E_x \Phi_{1h} + [E_x E_y - E_{x,h}E_{y,h}]\partial_t E_y \Phi_{1h}\big) \\ &+ 2\big(E^2_{x,h}\partial_t [E_x - E_{x,h}]\Phi_{1h} + E_{x,h}E_{y,h}\partial_t [E_y - E_{y,h}]\Phi_{1h}\big)\Big],\end{aligned} \tag{36}$$

$$\begin{aligned}&\int_{K_{ij}} \partial_t (D_y - D_{y,h})\Phi_{2h} = \int_{K_{ij}} \varepsilon_0 (1+\chi^{(1)})\partial_t \zeta_y \Phi_{2h} + \int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[[|\mathbf{E}|^2 - |\mathbf{E}_h|^2]\partial_t E_y \Phi_{2h} \\ &+ |\mathbf{E}_h|^2 \partial_t [E_y - E_{y,h}]\Phi_{2h} + 2\big([E_y^2 - E_{y,h}^2]\partial_t E_y \Phi_{2h} + [E_x E_y - E_{x,h}E_{y,h}]\partial_t E_x \Phi_{2h}\big) \\ &+ 2\big(E^2_{y,h}\partial_t [E_y - E_{y,h}]\Phi_{2h} + E_{x,h}E_{y,h}\partial_t [E_x - E_{x,h}]\Phi_{2h}\big)\Big].\end{aligned} \tag{37}$$

First we substitute the Eqs. (36)–(37) into the Eqs. (34)–(35), respectively. Further decomposing the terms in the resulting equations using (29), (31) and (33) and taking $\Phi_{1h} := \eta_{x,h}$, $\Phi_{2h} := \eta_{y,h}$ and $\Phi_{3h} := \theta_{z,h}$, we obtain, after a few slight rearrangements,

$$\begin{aligned}&\int_{K_{ij}} \varepsilon_0 (1+\chi^{(1)})\partial_t \eta_{x,h}\, \eta_{x,h} + \int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[|\mathbf{E}_h|^2 \partial_t \eta_{x,h}\, \eta_{x,h} + 2E^2_{x,h}\partial_t \eta_{x,h}\, \eta_{x,h} + 2E_{x,h}E_{y,h}\partial_t \eta_{y,h}\, \eta_{x,h}\Big] \\ &- \int_{I_i} [(\hat{\theta}_{z,h}\eta^-_{x,h})_{x,j+\frac{1}{2}} - (\hat{\theta}_{z,h}\eta^+_{x,h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} \theta_{z,h}\partial_y \eta_{x,h} \\ &= \int_{K_{ij}} \varepsilon_0 (1+\chi^{(1)})\partial_t \eta_x\, \eta_{x,h} + \int_{K_{ij}} \varepsilon_0 \chi^{(3)} |\mathbf{E}_h|^2 \partial_t \eta_x\, \eta_{x,h} - \int_{I_i} [(\hat{\theta}_z \eta^-_{x,h})_{x,j+\frac{1}{2}} - (\hat{\theta}_z \eta^+_{x,h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} \theta_z \partial_y \eta_{x,h} \\ &+ \int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[[|\mathbf{E}|^2 - |\mathbf{E}_h|^2]\partial_t E_x\, \eta_{x,h} \\ &+ 2[E_x^2 - E^2_{x,h}]\partial_t E_x\, \eta_{x,h} + 2[E_x E_y - E_{x,h}E_{y,h}]\partial_t E_y\, \eta_{x,h} + 2E^2_{x,h}\partial_t \eta_x\, \eta_{x,h} + 2E_{x,h}E_{y,h}\partial_t \eta_y\, \eta_{x,h} \Big],\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&\int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})\partial_t\eta_{y,h}\,\eta_{y,h} + \int_{K_{ij}} \varepsilon_0\chi^{(3)}\Big[|\mathbf{E}_h|^2\partial_t\eta_{y,h}\,\eta_{y,h} + 2E_{y,h}^2\partial_t\eta_{y,h}\,\eta_{y,h} + 2E_{x,h}E_{y,h}\partial_t\eta_{x,h}\,\eta_{y,h}\Big] \\
&+ \int_{J_j}[(\hat{\theta}_{z,h}\eta_{y,h}^-)_{i+\frac12,y} - (\hat{\theta}_{z,h}\eta_{y,h}^+)_{i-\frac12,y}]dy - \int_{K_{ij}} \theta_{z,h}\partial_x\eta_{y,h} \\
&= \int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})\partial_t\eta_y\,\eta_{y,h} + \int_{K_{ij}} \varepsilon_0\chi^{(3)}|\mathbf{E}_h|^2\partial_t\eta_y\,\eta_{y,h} + \int_{J_j}[(\hat{\theta}_z\eta_{y,h}^-)_{i+\frac12,y} - (\hat{\theta}_z\eta_{y,h}^+)_{i-\frac12,y}]dy - \int_{K_{ij}} \theta_z\partial_x\eta_{y,h} \\
&+ \int_{K_{ij}} \varepsilon_0\chi^{(3)}\Big[[|\mathbf{E}|^2 - |\mathbf{E}_h|^2]\partial_t E_y\,\eta_{y,h} \\
&+ 2[E_y^2 - E_{y,h}^2]\partial_t E_y\,\eta_{y,h} + 2[E_xE_y - E_{x,h}E_{y,h}]\partial_t E_x\,\eta_{y,h} + 2E_{y,h}^2\partial_t\eta_y\,\eta_{y,h} + 2E_{x,h}E_{y,h}\partial_t\eta_x\,\eta_{y,h}\Big].
\end{aligned} \tag{39}$$

For the magnetic field we have that

$$\begin{aligned}
&\int_{K_{ij}} \mu_0\partial_t\theta_{z,h}\,\theta_{z,h} + \int_{J_j}[(\hat{\eta}_{y,h}\theta_{z,h}^-)_{i+\frac12,y} - (\hat{\eta}_{y,h}\theta_{z,h}^+)_{i-\frac12,y}]dy \\
&- \int_{I_i}[(\hat{\eta}_{x,h}\theta_{z,h}^-)_{x,j+\frac12} - (\hat{\eta}_{x,h}\theta_{z,h}^+)_{x,j-\frac12}]dx - \int_{K_{ij}} \eta_{y,h}\partial_x\theta_{z,h} + \int_{K_{ij}} \eta_{x,h}\partial_y\theta_{z,h} \\
&= \int_{K_{ij}} \mu_0\partial_t\theta_z\,\theta_{z,h} + \int_{J_j}[(\hat{\eta}_y\theta_{z,h}^-)_{i+\frac12,y} - (\hat{\eta}_y\theta_{z,h}^+)_{i-\frac12,y}]dy \\
&- \int_{K_{ij}} \eta_y\partial_x\theta_{z,h} - \int_{I_i}[(\hat{\eta}_x\theta_{z,h}^-)_{x,j+\frac12} - (\hat{\eta}_x\theta_{z,h}^+)_{x,j-\frac12}]dx + \int_{K_{ij}} \eta_x\partial_y\theta_{z,h}.
\end{aligned} \tag{40}$$

Now we apply similar arguments as in the proof of Theorem 4.1. Adding the Eqs. (38)–(40), summing over the indices $1 \le i \le N_x$ and $1 \le j \le N_y$ and making use of the identities (22), we obtain the left-hand side of the result as

$$LHS = LHSL + LHSN$$

with

$$\begin{aligned}
LHSL &:= \frac{1}{2}\frac{d}{dt}\left[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}\|^2_{\mu_0}\right] \\
&+ \sum_{i=1}^{N_x}\int_{I_i}((\hat{\theta}_{z,h} - \theta_{z,h}^+)\eta_{x,h}^+)(x, y_{\frac12})dx + \sum_{j=1}^{N_y}\int_{J_j}((\theta_{z,h}^+ - \hat{\theta}_{z,h})\eta_{y,h}^+)(x_{\frac12}, y)dy \\
&= \frac{1}{2}\frac{d}{dt}\left[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}\|^2_{\mu_0}\right] \\
&+ \sum_{i=1}^{N_x} c_0\int_{I_i}[\eta_{x,h}^+(x, y_{\frac12})]^2dx + \sum_{j=1}^{N_y} c_0\int_{J_j}[\eta_{y,h}^+(x_{\frac12}, y)]^2dy,
\end{aligned} \tag{41}$$

where the last equation follows from the definition of the boundary flux densities (8), (9) (cf. (22)). Furthermore,

$$\begin{aligned}
LHSN = \int_\Omega \varepsilon_0\chi^{(3)}\Bigg[\ &\frac12\partial_t\Big[|\mathbf{E}_h|^2\big(\eta_{x,h}^2 + \eta_{y,h}^2\big)\Big] + \partial_t\big(E_{x,h}\eta_{x,h} + E_{y,h}\eta_{y,h}\big)^2 \\
&- \frac12\big(\eta_{x,h}^2 + \eta_{y,h}^2\big)\partial_t|\mathbf{E}_h|^2 - \partial_t E_{x,h}^2\,\eta_{x,h}^2 - \partial_t E_{y,h}^2\,\eta_{y,h}^2 - 2\partial_t\big(E_{x,h}E_{y,h}\big)\eta_{x,h}\eta_{y,h}\ \Bigg].
\end{aligned} \tag{42}$$

The right-hand side gets the form

$$RHS = RHSL + RHSN,$$

where

$$\begin{aligned}
RHSL &:= \int_\Omega\Big[\varepsilon_0(1+\chi^{(1)})[\partial_t\eta_x\,\eta_{x,h} + \partial_t\eta_y\,\eta_{y,h}] + \mu_0\partial_t\theta_z\,\theta_{z,h}\Big] \\
&+ \sum_{i=1}^{N_x}\Big[-\int_{I_i}[(\hat{\theta}_z\eta_{x,h}^-)_{x,j+\frac12} - (\hat{\theta}_z\eta_{x,h}^+)_{x,j-\frac12}]dx + \int_{K_{ij}} \theta_z\partial_y\eta_{x,h}\Big]
\end{aligned}$$

$$+\sum_{j=1}^{N_y}\Big[\int_{J_j}[(\hat{\theta}_z\eta_{y,h}^-)_{i+\frac{1}{2},y}-(\hat{\theta}_z\eta_{y,h}^+)_{i-\frac{1}{2},y}]dy-\int_{K_{ij}}\theta_z\partial_x\eta_{y,h}\Big]$$

and

$$\begin{aligned}RHSN:=\int_{\Omega}\varepsilon_0\chi^{(3)}\Big[\,&\big[|\mathbf{E}|^2-|\mathbf{E}_h|^2\big]\big[\partial_tE_x\,\eta_{x,h}+\partial_tE_y\,\eta_{y,h}\big]\\&+2[E_x^2-E_{x,h}^2]\partial_tE_x\,\eta_{x,h}+2[E_y^2-E_{y,h}^2]\partial_tE_y\,\eta_{y,h}+2[E_xE_y-E_{x,h}E_{y,h}]\big[\partial_tE_y\,\eta_{x,h}+\partial_tE_x\,\eta_{y,h}\big]\\&+2E_{x,h}^2\partial_t\eta_x\,\eta_{x,h}+2E_{y,h}^2\partial_t\eta_y\,\eta_{y,h}+2E_{x,h}E_{y,h}\big[\partial_t\eta_y\,\eta_{x,h}+\partial_t\eta_x\,\eta_{y,h}\big]+|\mathbf{E}_h|^2\big[\partial_t\eta_x\,\eta_{x,h}+\partial_t\eta_y\,\eta_{y,h}\big]\,\Big].\end{aligned}\tag{43}$$

Next, using $\partial_tE_x=\partial_t\eta_x+\partial_t(\Pi_1E_x)$ and $\partial_tE_y=\partial_t\eta_y+\partial_t(\Pi_2E_y)$ (see (29), (31)) in the nonlinear terms (43), we obtain

$$\begin{aligned}RHSN=\int_{\Omega}\varepsilon_0\chi^{(3)}\Big[&\Big[\big(E_x+E_{x,h}\big)\big(E_x-E_{x,h}\big)+\big(E_y+E_{y,h}\big)\big(E_y-E_{y,h}\big)\Big]\big[\partial_t(\Pi_1E_x)\,\eta_{x,h}+\partial_t(\Pi_2E_y)\,\eta_{y,h}\big]\\&+|\mathbf{E}|^2\big[\partial_t\eta_x\,\eta_{x,h}+\partial_t\eta_y\,\eta_{y,h}\big]+2[\big(E_x+E_{x,h}\big)\big(E_x-E_{x,h}\big)]\partial_t(\Pi_1E_x)\,\eta_{x,h}\\&+2E_x^2\partial_t\eta_x\,\eta_{x,h}+2[\big(E_y+E_{y,h}\big)\big(E_y-E_{y,h}\big)]\partial_t(\Pi_2E_y)\,\eta_{y,h}+2E_y^2\partial_t\eta_y\,\eta_{y,h}+2[E_y\big(E_x-E_{x,h}\big)\\&+E_{x,h}\big(E_y-E_{y,h}\big)]\big[\partial_t(\Pi_2E_y)\,\eta_{x,h}+\partial_t(\Pi_1E_x)\,\eta_{y,h}\big]+2E_xE_y\big[\partial_t\eta_y\,\eta_{x,h}+\partial_t\eta_x\,\eta_{y,h}\big]\Big].\end{aligned}$$

Furthermore, since $E_x-E_{x,h}=\eta_x-\eta_{x,h}$ and $E_y-E_{y,h}=\eta_y-\eta_{y,h}$, we have, after some rearrangement,

$$\begin{aligned}RHSN=\int_{\Omega}\varepsilon_0\chi^{(3)}\Big[&\Big[E_x\eta_x\partial_t(\Pi_1E_x)+E_{x,h}\eta_x\partial_t(\Pi_1E_x)+E_y\eta_y\partial_t(\Pi_1E_x)+E_{y,h}\eta_y\partial_t(\Pi_1E_x)\\&+|\mathbf{E}|^2\partial_t\eta_x+2E_x\eta_x\partial_t(\Pi_1E_x)+2E_{x,h}\eta_x\partial_t(\Pi_1E_x)\\&+2E_x^2\partial_t\eta_x+2E_y\eta_x\partial_t(\Pi_2E_y)+2E_{x,h}\eta_y\partial_t(\Pi_2E_y)+2E_xE_y\partial_t\eta_y\Big]\eta_{x,h}\\&+\Big[E_x\eta_x\partial_t(\Pi_2E_y)+E_{x,h}\eta_x\partial_t(\Pi_2E_y)+E_y\eta_y\partial_t(\Pi_2E_y)+E_{y,h}\eta_y\partial_t(\Pi_2E_y)\\&+|\mathbf{E}|^2\partial_t\eta_y+2E_y\eta_y\partial_t(\Pi_2E_y)+2E_{y,h}\eta_y\partial_t(\Pi_2E_y)\\&+2E_y^2\partial_t\eta_y+2E_y\eta_x\partial_t(\Pi_1E_x)+2E_{x,h}\eta_y\partial_t(\Pi_1E_x)+2E_xE_y\partial_t\eta_x\Big]\eta_{y,h}\\&+\Big[-E_x\partial_t(\Pi_1E_x)-E_{x,h}\partial_t(\Pi_1E_x)-2E_x\partial_t(\Pi_1E_x)-2E_{x,h}\partial_t(\Pi_1E_x)-2E_y\partial_t(\Pi_2E_y)\Big]\eta_{x,h}^2\\&+\Big[-E_y\partial_t(\Pi_2E_y)-E_{y,h}\partial_t(\Pi_2E_y)-2E_y\partial_t(\Pi_2E_y)-2E_{y,h}\partial_t(\Pi_2E_y)-2E_{x,h}\partial_t(\Pi_1E_x)\Big]\eta_{y,h}^2\\&+\Big[-E_y\partial_t(\Pi_1E_x)-E_{y,h}\partial_t(\Pi_1E_x)-E_x\partial_t(\Pi_2E_y)-E_{x,h}\partial_t(\Pi_2E_y)-2E_y\partial_t(\Pi_1E_x)-2E_{x,h}\partial_t(\Pi_2E_y)\Big]\eta_{x,h}\eta_{y,h}\Big].\end{aligned}$$

In a next step, we shift the last two terms of (41) to *RHSL* and the last four terms of (42) to *RHSN*. Then the new left-hand side is

$$\begin{aligned}LHS':=&\frac{1}{2}\frac{d}{dt}\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\theta_{z,h}\|^2_{\mu_0}\big]\\&+\int_{\Omega}\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\partial_t\big[|\mathbf{E}_h|^2\big(\eta_{x,h}^2+\eta_{y,h}^2\big)\big]+\partial_t\big(E_{x,h}\eta_{x,h}+E_{y,h}\eta_{y,h}\big)^2\Big],\end{aligned}\tag{44}$$

while the new right-hand side is

$$RHS':=RHSL'+RHSN'$$

with

$$RHSL':=RHSL-\sum_{i=1}^{N_x}c_0\int_{I_i}[\eta_{x,h}^+(x,y_{\frac{1}{2}})]^2dx-\sum_{j=1}^{N_y}c_0\int_{J_j}[\eta_{y,h}^+(x_{\frac{1}{2}},y)]^2dy,$$

$$\begin{aligned}RHSN':=RHSN+\int_{\Omega}\varepsilon_0\chi^{(3)}\Big[&\frac{1}{2}\big(\eta_{x,h}^2+\eta_{y,h}^2\big)\partial_t|\mathbf{E}_h|^2\\&+\partial_tE_{x,h}^2\,\eta_{x,h}^2+\partial_tE_{y,h}^2\,\eta_{y,h}^2+2\partial_t\big(E_{x,h}E_{y,h}\big)\eta_{x,h}\eta_{y,h}\Big].\end{aligned}$$

The first three terms from $RHSL'$ are estimated using the Cauchy–Schwarz inequality and Lemma 5.2:

$$\begin{aligned}
&\int_\Omega \left[\varepsilon_0(1+\chi^{(1)})[\partial_t\eta_x\,\eta_{x,h}+\partial_t\eta_y\,\eta_{y,h}]+\mu_0\partial_t\theta_z\,\theta_{z,h}\right]\\
&\le \|\partial_t\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}+\|\partial_t\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}+\|\partial_t\theta_z\|_{\mu_0}\|\theta_{z,h}\|_{\mu_0}\\
&\le C_1h^{k+1}[\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}+\|\theta_{z,h}\|_{\mu_0}],
\end{aligned}$$

where the constant $C_1>0$ depends on $\|\varepsilon_0(1+\chi^{(1)})\|_{L_\infty(\Omega)}$, $\|\mu_0\|_{L_\infty(\Omega)}$, $\|\partial_tE_x\|_{H^{k+1}(\Omega)}$, $\|\partial_tE_y\|_{H^{k+1}(\Omega)}$, and $\|\partial_tH_z\|_{H^{k+1}(\Omega)}$, as can be seen be the following exemplary argument:

$$\|\partial_t\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\le\|\varepsilon_0(1+\chi^{(1)})\|_{L_\infty(\Omega)}\|\partial_t\eta_x\|\le C_1h^{k+1}\|\partial_tE_x\|_{H^{k+1}(\Omega)}.$$

The remaining terms from $RHSL'$ are estimated by means of the Lemmata 5.3, 5.4:

$$\begin{aligned}
&\sum_{i=1}^{N_x}\Big[-\int_{I_i}[(\hat\theta_z\eta^-_{x,h})_{x,j+\frac12}-(\hat\theta_z\eta^+_{x,h})_{x,j-\frac12}]dx+\int_{K_{ij}}\theta_z\partial_y\eta_{x,h}\Big]\\
&+\sum_{j=1}^{N_y}\Big[\int_{J_j}[(\hat\theta_z\eta^-_{y,h})_{i+\frac12,y}-(\hat\theta_z\eta^+_{y,h})_{i-\frac12,y}]dy-\int_{K_{ij}}\theta_z\partial_x\eta_{y,h}\Big]\\
&-\sum_{i=1}^{N_x}c_0\int_{I_i}[\eta^+_{x,h}(x,y_{\frac12})]^2dx-\sum_{j=1}^{N_y}c_0\int_{J_j}[\eta^+_{y,h}(x_{\frac12},y)]^2dy\\
&\le Ch^{2k+2}+\|\eta_{x,h}\|^2+\|\eta_{y,h}\|^2\\
&\le Ch^{2k+2}+\|(\varepsilon_0(1+\chi^{(1)}))^{-1}\|_{L_\infty(\Omega)}\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\big].
\end{aligned}$$

The terms from the right-hand side part $RHSN$ can be estimated as follows:

$$\begin{aligned}
RHSN\le{}&\|\chi^{(3)}(1+\chi^{(1)})^{-1}\|_{L_\infty(\Omega)}\Bigg[\Big[\|E_x\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\\
&+\|E_{x,h}\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}+\|E_y\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}\\
&+\|E_{y,h}\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}+\||\mathbf{E}|^2\|_{L_\infty(\Omega)}\|\partial_t\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\\
&+2\|E_x\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}+2\|E_{x,h}\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}\\
&+2\|E_x^2\|_{L_\infty(\Omega)}\|\partial_t\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}+2\|E_y\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}\\
&+2\|E_{x,h}\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}+2\|E_x\|_{L_\infty(\Omega)}\|E_y\|_{L_\infty(\Omega)}\|\partial_t\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\Big]\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}\\
&+\Big[\|E_x\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}\\
&+\|E_{x,h}\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}+\|E_y\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}\\
&+\|E_{y,h}\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}+\||\mathbf{E}|^2\|_{L_\infty(\Omega)}\|\partial_t\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\\
&+2\|E_y\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}+2\|E_{y,h}\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}\\
&+2\|E_y^2\|_{L_\infty(\Omega)}\|\partial_t\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}+\|E_y\|_{L_\infty(\Omega)}\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}\\
&+2\|E_{x,h}\|_{L_\infty(\Omega)}\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}+2\|E_x\|_{L_\infty(\Omega)}\|E_y\|_{L_\infty(\Omega)}\|\partial_t\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}\Big]\|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\\
&+\Big[\|E_x\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}+\|E_{x,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}\\
&+2\|E_x\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}+2\|E_{x,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1E_x)\|_{L_\infty(\Omega)}\\
&+2\|E_y\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}\Big]\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\\
&+\Big[\|E_y\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}+\|E_{y,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}\\
&+2\|E_y\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}+2\|E_{y,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2E_y)\|_{L_\infty(\Omega)}
\end{aligned}$$

$$+ \ 2\|E_{x,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1 E_x)\|_{L_\infty(\Omega)}\Big]\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}$$
$$+ \Big[\|E_y\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1 E_x)\|_{L_\infty(\Omega)} + \|E_{y,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1 E_x)\|_{L_\infty(\Omega)}$$
$$+ \ \|E_x\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2 E_y)\|_{L_\infty(\Omega)} + \|E_{x,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2 E_y)\|_{L_\infty(\Omega)}$$
$$+ \ 2\|E_y\|_{L_\infty(\Omega)}\|\partial_t(\Pi_1 E_x)\|_{L_\infty(\Omega)} + 2\|E_{x,h}\|_{L_\infty(\Omega)}\|\partial_t(\Pi_2 E_y)\|_{L_\infty(\Omega)}\Big]\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\Big].$$

This estimate shows that we have to discuss upper bounds for the terms $\|E_x\|_{L_\infty(\Omega)}$, $\|E_y\|_{L_\infty(\Omega)}$, $\||\mathbf{E}|^2\|_{L_\infty(\Omega)}$, $\|E_x^2\|_{L_\infty(\Omega)}$, $\|E_y^2\|_{L_\infty(\Omega)}$, $\|\partial_t(\Pi_1 E_x)\|_{L_\infty(\Omega)}$, $\|\partial_t(\Pi_2 E_y)\|_{L_\infty(\Omega)}$, $\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}$, $\|\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}$, $\|\partial_t\eta_x\|_{\varepsilon_0(1+\chi^{(1)})}$, $\|\partial_t\eta_y\|_{\varepsilon_0(1+\chi^{(1)})}$, $\|E_{x,h}\|_{L_\infty(\Omega)}$, and $\|E_{y,h}\|_{L_\infty(\Omega)}$. The first five terms are bounded thanks to the assumption w.r.t. the weak solution and the continuous embedding $H^{k+1}(\Omega) \subset L_\infty(\Omega)$ for $k \in \mathbb{N}$, see, e.g., [31, (3.1.4)] (this embedding remains valid for $d = 3$, too). The eighth to eleventh terms are estimated by means of Lemma 5.2, for instance:

$$\|\eta_x\|_{\varepsilon_0(1+\chi^{(1)})} \le \|\varepsilon_0(1+\chi^{(1)})\|^{1/2}_{L_\infty(\Omega)}\|\eta_x\| \le Ch^{k+1}\|E_x\|_{H^{k+1}(\Omega)},$$

where here the constant $C > 0$ depends on $\|\varepsilon_0(1+\chi^{(1)})\|_{L_\infty(\Omega)}$. The last two terms are bounded thanks to the assumption w.r.t. the numerical solution.

So it remains to investigate the sixth and seventh terms. Taking into account the commutation property $\partial_t(\Pi_1 E_x) = \Pi_1(\partial_t E_x)$, we first observe that there exist at least one element K_{ij} such that

$$\|\partial_t(\Pi_1 E_x)\|_{L_\infty(\Omega)} = \|\Pi_1(\partial_t E_x)\|_{L_\infty(\Omega)} = \|\Pi_1(\partial_t E_x)\|_{L_\infty(K_{ij})}.$$

The latter norm can be estimated by an inverse inequality [31, Thm. 3.2.6]:

$$\|\Pi_1(\partial_t E_x)\|_{L_\infty(K_{ij})} \le |K_{ij}|^{-1/2}\|\Pi_1(\partial_t E_x)\|_{L_2(K_{ij})}$$

(note that we need only a local variant, i.e. we may omit the inverse assumption [31, (3.2.28)]). Using the triangle inequality, we get

$$\begin{aligned}\|\partial_t(\Pi_1 E_x)\|_{L_\infty(\Omega)} &\le |K_{ij}|^{-1/2}\big[\|\partial_t E_x\|_{L_2(K_{ij})} + \|\Pi_1(\partial_t E_x) - \partial_t E_x\|_{L_2(K_{ij})}\big]\\ &\le |K_{ij}|^{-1/2}\big[|K_{ij}|^{1/2}\|\partial_t E_x\|_{L_\infty(K_{ij})} + C|K_{ij}|^{(k+1)/2}\|\partial_t E_x\|_{H^{k+1}(K_{ij})}\big].\end{aligned}$$

The estimate of the first term in the square brackets results from Hölder's inequality, whereas the second term is estimated by means of a local variant of Lemma 5.2, see [27, Lemma 3.2]. So if the mesh size h is sufficiently small, we get

$$\|\Pi_1(\partial_t E_x)\|_{L_\infty(\Omega)} \le \|\partial_t E_x\|_{L_\infty(K_{ij})} + C|K_{ij}|^{k/2}\|\partial_t E_x\|_{H^{k+1}(K_{ij})} \le C\|\partial_t E_x\|_{H^{k+1}(\Omega)},$$

where we have used the continuous embedding $H^{k+1}(\Omega) \subset L_\infty(\Omega)$ in the last step again. An analogous argument applies to the seventh term.

So in summary we arrive at the estimate

$$\begin{aligned}RHSN &\le C_2 h^{k+1}\big[\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\big]\\ &\quad + C_3\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + C_4\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + C_5\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\\ &\le C_2 h^{k+1}\big[\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\big] + C_6\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\big],\end{aligned}$$

where the constants C_2, C_6 depend on $\|\chi^{(3)}(1+\chi^{(1)})^{-1}\|_{L_\infty(\Omega)}$, the $C^1(0, T, H^{k+1}(\Omega))$-norms of E_x, E_y and the $C^1(0, T, U_h^k \cap L_\infty(\Omega))$-norms of $E_{x,h}, E_{y,h}$.

Furthermore the remaining terms from $RHSN'$ can be bounded from above by

$$\begin{aligned}\|\ \chi^{(3)}(1+\chi^{(1)})^{-1}\ \|_{L_\infty(\Omega)}\Bigg[\ &\frac{1}{2}\|\partial_t|\mathbf{E}_h|^2\|_{L_\infty(\Omega)}\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\big]\\ &+ \|\partial_t E^2_{x,h}\|_{L_\infty(\Omega)}\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\partial_t E^2_{y,h}\|_{L_\infty(\Omega)}\|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\\ &+ 2\|\partial_t(E_{x,h}E_{y,h})\|_{L_\infty(\Omega)}\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})}\|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\Bigg].\end{aligned}$$

Here we have to take care of $\|\partial_t|\mathbf{E}_h|^2\|_{L_\infty(\Omega)}$, $\|\partial_t E^2_{x,h}\|_{L_\infty(\Omega)}$, $\|\partial_t E^2_{y,h}\|_{L_\infty(\Omega)}$, and $\|\partial_t(E_{x,h}E_{y,h})\|_{L_\infty(\Omega)}$, but all these terms can be bounded from above by the $C^1(0, T, U_h^k \cap L_\infty(\Omega))$-norms of $E_{x,h}, E_{y,h}$. Therefore we get the upper bound

$$C\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\big],$$

where the constant C depends on $\|\chi^{(3)}(1+\chi^{(1)})^{-1}\|_{L_\infty(\Omega)}$ and the $C^1(0,T,U_h^k \cap L_\infty(\Omega))$-norms of $E_{x,h}, E_{y,h}$. Since such a term already occurs in the upper bound of $RHSN$, we modify the constant C_6 correspondingly and conclude

$$RHSN' \le C_2 h^{k+1}\big[\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\big] + C_6\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\big]. \tag{45}$$

Combining the right-hand side estimate (45) with the left the-hand side (44), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}\|^2_{\mu_0}\big] \\
&\quad + \int_\Omega \varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\partial_t\big[|\mathbf{E}_h|^2\big(\eta_{x,h}^2 + \eta_{y,h}^2\big)\big] + \partial_t\big(E_{x,h}\eta_{x,h} + E_{y,h}\eta_{y,h}\big)^2\Big] \\
&\le C_2 h^{k+1}\big[\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\big] + C_6\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\big].
\end{aligned}$$

Setting

$$\begin{aligned}
\mathcal{D}_h^2(t) &:= \|\eta_{x,h}(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}(t)\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\theta_{z,h}(t)\|^2_{\mu_0} \\
&\quad + \int_\Omega \varepsilon_0\chi^{(3)}\Big[|\mathbf{E}_h(t)|^2\big(\eta_{x,h}^2(t) + \eta_{y,h}^2(t)\big) + 2\big(E_{x,h}(t)\eta_{x,h}(t) + E_{y,h}(t)\eta_{y,h}(t)\big)^2\Big],
\end{aligned}$$

we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\mathcal{D}_h^2(t) &\le C_2 h^{k+1}\big[\|\eta_{x,h}\|_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|_{\varepsilon_0(1+\chi^{(1)})}\big] + C_6\big[\|\eta_{x,h}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|\eta_{y,h}\|^2_{\varepsilon_0(1+\chi^{(1)})}\big] \\
&\le C_2\sqrt{2}h^{k+1}\mathcal{D}_h(t) + C_6\mathcal{D}_h^2(t).
\end{aligned}$$

Integrating this inequality with respect to time, we obtain

$$\mathcal{D}_h^2(t) \le \mathcal{D}_h(0) + 2\int_0^t \big[C_2\sqrt{2}h^{k+1}\mathcal{D}_h(s) + C_6\mathcal{D}_h^2(s)\big]ds.$$

Now we apply a Gronwall-type lemma [32, Lemma 4.1] and obtain

$$\mathcal{D}_h(t) \le \mathcal{D}_h(0)e^{C_6 t} + C_2\sqrt{2}h^{k+1}te^{C_6 t}.$$

From this and the triangle inequality in conjunction with Lemma 5.2 the statement follows. ◂

6. The fully discrete scheme

The energy stable semi-discrete method (10)–(14) can now serve as a starting point for full discretization. We have chosen a relatively simple method as an example, which in the end only leads to a conditional stability, as is natural in leap-frog methods.

We divide the time interval $(0,T)$ into $N \in \mathbb{N}$ equally spaced subintervals by using the nodal points $t^n := n\Delta t$, $n = 0,1,2,\ldots,N$, and $\Delta t := \frac{T}{N}$. Given initial values $(E^0_{x,h}, E^0_{y,h}, H^0_{z,h})^T \in (U_h^k)^3$ of the electric and magnetic field intensities, the fully discrete scheme w.r.t. the electric and magnetic field intensities $(E^{n+1}_{x,h}, E^{n+1}_{y,h}, H^{n+\frac{3}{2}}_{z,h})^T \in (U_h^k)^3$, $n = 1,2,\ldots,N-1$, reads as

$$\int_{K_{ij}} \frac{D^{n+1}_{x,h} - D^n_{x,h}}{\Delta t}\Phi_{1h} - \int_{I_i}[(\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^-_{1h})_{x,j+\frac{1}{2}} - (\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^+_{1h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} H^{n+\frac{1}{2}}_{z,h}\partial_y\Phi_{1h} - \int_{K_{ij}} J^{n+\frac{1}{2}}_{x,h}\Phi_{1h} = 0, \tag{46}$$

$$\int_{K_{ij}} \frac{D^{n+1}_{y,h} - D^n_{y,h}}{\Delta t} + \int_{J_j}[(\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^-_{2h})_{i+\frac{1}{2},y} - (\hat{H}^{n+\frac{1}{2}}_{z,h}\Phi^+_{2h})_{i-\frac{1}{2},y}]dy - \int_{K_{ij}} H^{n+\frac{1}{2}}_{z,h}\partial_x\Phi_{2h} - \int_{K_{ij}} J^{n+\frac{1}{2}}_{y,h}\Phi_{2h} = 0, \tag{47}$$

$$\begin{aligned}
&\int_{K_{ij}} \mu_0\frac{H^{n+\frac{3}{2}}_{z,h} - H^{n+\frac{1}{2}}_{z,h}}{\Delta t}\Phi_{3h} + \int_{J_j}[(\hat{E}^{n+1}_{y,h}\Phi^-_{3h})_{i+\frac{1}{2},y} - (\hat{E}^{n+1}_{y,h}\Phi^+_{3h})_{i-\frac{1}{2},y}]dy \\
&- \int_{K_{ij}} E^{n+1}_{y,h}\partial_x\Phi_{3h} - \int_{I_i}[(\hat{E}^{n+1}_{x,h}\Phi^-_{3h})_{x,j+\frac{1}{2}} - (\hat{E}^{n+1}_{x,h}\Phi^+_{3h})_{x,j-\frac{1}{2}}]dx + \int_{K_{ij}} E^{n+1}_{x,h}\partial_y\Phi_{3h} = 0,
\end{aligned} \tag{48}$$

$$\begin{aligned}
&\int_{K_{ij}} (D^{n+1}_{x,h} - D^n_{x,h})\Phi_{1h} = \int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})(E^{n+1}_{x,h} - E^n_{x,h})\Phi_{1h} \\
&+ \int_{K_{ij}} \varepsilon_0\chi^{(3)}\Big[\frac{1}{2}[(E^{n+1}_{x,h})^2 + (E^n_{x,h})^2 + (E^{n+1}_{y,h})^2 + (E^n_{y,h})^2](E^{n+1}_{x,h} - E^n_{x,h})\Phi_{1h}
\end{aligned}$$

$$+ \Big([(E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2](E_{x,h}^{n+1} - E_{x,h}^{n})\Phi_{1h} + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}](E_{y,h}^{n+1} - E_{y,h}^{n})\Phi_{1h} \Big)\Big], \quad (49)$$

$$\int_{K_{ij}} (D_{y,h}^{n+1} - D_{y,h}^{n})\Phi_{2h} = \int_{K_{ij}} \varepsilon_0(1+\chi^{(1)})(E_{y,h}^{n+1} - E_{y,h}^{n})\Phi_{2h}$$

$$+ \int_{K_{ij}} \varepsilon_0\chi^{(3)} \Big[\frac{1}{2}\big((E_{x,h}^{n+1})^2 + (E_{x,h}^{n})^2 + (E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2\big)(E_{y,h}^{n+1} - E_{y,h}^{n})\Phi_{2h}$$

$$+ \Big([(E_{y,h}^{n+1})^2 + (E_{y,h}^{n})^2](E_{y,h}^{n+1} - E_{y,h}^{n})\Phi_{2h} + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n}E_{y,h}^{n}](E_{x,h}^{n+1} - E_{x,h}^{n})\Phi_{2h} \Big)\Big] \quad (50)$$

for all test functions $(\Phi_{1h}, \Phi_{2h}, \Phi_{3h})^T \in (U_h^k)^3$. The differences $D_{x,h}^{n+1} - D_{x,h}^{n}$ and $D_{y,h}^{n+1} - D_{y,h}^{n}$ play the role of auxiliary variables, and the flux densities are defined by

$$\hat{E}_{x,h}^{n+1}(x, y_{j+\frac{1}{2}}) := E_{x,h}^{n+1,+}(x, y_{j+\frac{1}{2}}) \quad \text{for all } j = 1, 2, 3, \ldots, N_y - 1, \quad (51)$$

$$\hat{E}_{x,h}^{n+1}(x, y_{\frac{1}{2}}) := \hat{E}_{x,h}^{n+1}(x, y_{N_y+\frac{1}{2}}) := 0, \quad (52)$$

$$\hat{E}_{y,h}^{n+1}(x_{i+\frac{1}{2}}, y) := E_{y,h}^{n+1,+}(x_{i+\frac{1}{2}}, y) \quad \text{for all } i = 1, 2, 3, \ldots, N_x - 1, \quad (53)$$

$$\hat{E}_{y,h}^{n+1}(x_{\frac{1}{2}}, y) := \hat{E}_{y,h}^{n+1}(x_{N_x+\frac{1}{2}}, y) := 0, \quad (54)$$

$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x, y_{j+\frac{1}{2}}) := H_{z,h}^{n+\frac{1}{2},-}(x, y_{j+\frac{1}{2}}) \quad \text{for all } j = 1, 2, 3, \ldots, N_y, \quad (55)$$

$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x, y_{\frac{1}{2}}) := H_{z,h}^{n+\frac{1}{2},+}(x, y_{\frac{1}{2}}) + \frac{c_0}{2}\Big[E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{+}) + E_{x,h}^{n}(x, y_{\frac{1}{2}}^{+})\Big], \quad (56)$$

$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x_{i+\frac{1}{2}}, y) := H_{z,h}^{n+\frac{1}{2},-}(x_{i+\frac{1}{2}}, y) \quad \text{for all } j = 1, 2, 3, \ldots, N_x, \quad (57)$$

$$\hat{H}_{z,h}^{n+\frac{1}{2}}(x_{\frac{1}{2}}, y) := H_{z,h}^{n+\frac{1}{2},+}(x_{\frac{1}{2}}, y) - \frac{c_0}{2}\Big[E_{y,h}^{n+1}(x_{\frac{1}{2}}^{+}, y) + E_{x,h}^{n}(x_{\frac{1}{2}}^{+}, y)\Big]. \quad (58)$$

Due to the PEC condition (7) we have $E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{+}) = E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{+}) - E_{x,h}^{n+1}(x, y_{\frac{1}{2}}^{-}) = [\![E_{x,h}^{n+1}(x, y_{\frac{1}{2}})]\!]$ in Eq. (56), and the analogous one for the other artificial viscosity in Eq. (58). The boundary terms are defined as follows:

$$\sigma_{Ih} := -\int_{I_i} [(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^{n})^{-})_{x,j+\frac{1}{2}} - (\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^{n})^{+})_{x,j-\frac{1}{2}}]dx$$

$$- \int_{I_i} [(\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{-})_{x,j+\frac{1}{2}} - (\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{+})_{x,j-\frac{1}{2}}]dx,$$

$$\sigma_{Jh} := \int_{J_j} [(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^{n})^{-})_{i+\frac{1}{2},y} - (\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^{n})^{+})_{i-\frac{1}{2},y}]dy$$

$$+ \int_{J_j} [(\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{-})_{i+\frac{1}{2},y} - (\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^{+})_{i-\frac{1}{2},y}]dy.$$

It should be noted that a nonlinear system of equations remains to be solved in each time step. An investigation of nonlinear solvers, especially under the aspect of energy conservation also for the approximations obtained with them, is still pending. However, we have had very positive experiences in the application of Newton (or Newton-like) methods in solving such similar nonlinear problems that arise when applying conforming methods [8].

The proof of the energy relation in the subsequent section is based on the following lemmas.

Lemma 6.1. *For $n = 1, 2, \ldots, N$, with the flux densities (51)–(58), we have*

$$\sum_{n=0}^{N}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \int_{K_{ij}} \Big[H_{z,h}^{n+\frac{1}{2}}\partial_y(E_{x,h}^{n+1} + E_{x,h}^{n}) + E_{x,h}^{n+1}\partial_y(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})\Big] + \sum_{n=0}^{N}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sigma_{Ih}$$

$$= \sum_{j=1}^{N_y-1}\Big[\int_r^s \Big(E_{x,h}^{N+1,+}[\![H_{z,h}^{N+\frac{3}{2}}]\!]\Big)_{x,j+\frac{1}{2}} - \int_r^s \Big(E_{x,h}^{0,+}[\![H_{z,h}^{\frac{1}{2}}]\!]\Big)_{x,j+\frac{1}{2}}\Big]$$

$$+ \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\int_{K_{ij}}\Big[E_{x,h}^{N+1}\partial_y H_{z,h}^{N+\frac{3}{2}} - E_{x,h}^{0}\partial_y H_{z,h}^{\frac{1}{2}}\Big] + \frac{c_0}{2}\sum_{n=0}^{N}\int_r^s (E_{x,h}^{n+1,+} + E_{x,h}^{n,+})^2_{x,\frac{1}{2}}.$$

Proof. For details see [25, Lemma 4.1]. ◂

Lemma 6.2. *For $n = 1, 2, \dots, N$, with the flux densities (51)–(58), we have*

$$-\sum_{n=0}^{N}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\int_{K_{ij}}\Big[H_{z,h}^{n+\frac{1}{2}}\partial_x(E_{y,h}^{n+1}+E_{y,h}^{n})+E_{y,h}^{n+1}\partial_x(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})\Big]+\sum_{n=0}^{N}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sigma_{Jh}$$
$$=\sum_{i=1}^{N_x-1}\Big[-\int_p^q\Big(E_{y,h}^{N+1,+}[\![H_{z,h}^{N+\frac{3}{2}}]\!]\Big)_{i+\frac{1}{2},y}+\int_p^q\Big(E_{y,h}^{0,+}[\![H_{z,h}^{\frac{1}{2}}]\!]\Big)_{i+\frac{1}{2},y}\Big]$$
$$+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\int_{K_{ij}}\Big[-E_{y,h}^{N+1}\partial_x H_{z,h}^{N+\frac{3}{2}}+E_{y,h}^{0}\partial_x H_{z,h}^{\frac{1}{2}}\Big]+\frac{c_0}{2}\sum_{n=0}^{N}\int_p^q(E_{y,h}^{n+1,+}+E_{y,h}^{n,+})^2_{\frac{1}{2},y}.$$

Proof. For details see [25, Lemma 4.2]. ◂

7. The nonlinear electromagnetic energy of the full discretization

The nonlinear electromagnetic energy for the fully discrete approximation (i.e. both in space and time) of the system (46)–(50) at t^n, $n = 0, 1, 2, \dots, N$, is defined by

$$\mathcal{E}_h^n := \|\mathbf{E}_h^n\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|H_{z,h}^{n+\frac{1}{2}}\|^2_{\mu_0}+\big\||\mathbf{E}_h^n|^2\big\|^2_{\varepsilon_0\chi^{(3)}}.$$

In analogy to the conservativity and boundedness results for the continuous and semi-discrete nonlinear electromagnetic energy (Theorems 2.1, 4.1), in this section we demonstrate a stability result for the fully discrete nonlinear electromagnetic energy of the system (46)–(50).

Theorem 7.1. *Let $(E_{x,h}^n, E_{y,h}^n, H_{z,h}^{n+\frac{1}{2}})^T \in (U_h^k)^3$, $n \in \mathbb{N}$, be the fully discrete solution of (46)–(50) for given $\mathbf{J}_h \in C(0,T,U_h^k)^2$. Then, if $\Delta t > 0$, $h > 0$ are sufficiently small and if $\Delta t/h$ is bounded by some constant, the fully discrete nonlinear electromagnetic energy satisfies*

$$\mathcal{E}_h^N \le 3\mathcal{E}_h^0 = 3\Big[\|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}+\big\||\mathbf{E}_h^0|^2\big\|^2_{\varepsilon_0\chi^{(3)}}\Big]$$

for vanishing current density and

$$\mathcal{E}_h^N \le \exp(8T+1)\Big[3\mathcal{E}_h^0+\Delta t\sum_{n=0}^{N-1}\|\mathbf{J}_h^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}\Big]$$

for non-zero current density.

Proof. Taking $\Phi_{1h} := (E_{x,h}^{n+1}+E_{x,h}^n)$ in Eq. (49), we have

$$\int_{K_{ij}}(D_{x,h}^{n+1}-D_{x,h}^n)(E_{x,h}^{n+1}+E_{x,h}^n)$$
$$=\int_{K_{ij}}\varepsilon_0(1+\chi^{(1)})(E_{x,h}^{n+1}-E_{x,h}^n)(E_{x,h}^{n+1}+E_{x,h}^n)$$
$$+\int_{K_{ij}}\varepsilon_0\chi^{(3)}\Big[\frac{1}{2}\big[(E_{x,h}^{n+1})^2+(E_{x,h}^n)^2+(E_{y,h}^{n+1})^2+(E_{y,h}^n)^2\big](E_{x,h}^{n+1}-E_{x,h}^n)(E_{x,h}^{n+1}+E_{x,h}^n)$$
$$+\Big([(E_{x,h}^{n+1})^2+(E_{x,h}^n)^2](E_{x,h}^{n+1}-E_{x,h}^n)(E_{x,h}^{n+1}+E_{x,h}^n)+[E_{x,h}^{n+1}E_{y,h}^{n+1}+E_{x,h}^nE_{y,h}^n](E_{y,h}^{n+1}-E_{y,h}^n)(E_{x,h}^{n+1}+E_{x,h}^n)\Big)\Big]. \quad (59)$$

Taking $\Phi_{2h} := (E_{y,h}^{n+1}+E_{y,h}^n)$ in Eq. (50), we have

$$\int_{K_{ij}}(D_{y,h}^{n+1}-D_{y,h}^n)(E_{y,h}^{n+1}+E_{y,h}^n)$$
$$=\int_{K_{ij}}\varepsilon_0(1+\chi^{(1)})(E_{y,h}^{n+1}-E_{y,h}^n)(E_{y,h}^{n+1}+E_{y,h}^n)$$

$$+\int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[\frac{1}{2}\big[(E_{x,h}^{n+1})^2 + (E_{x,h}^n)^2 + (E_{y,h}^{n+1})^2 + (E_{y,h}^n)^2\big](E_{y,h}^{n+1} - E_{y,h}^n)(E_{y,h}^{n+1} + E_{y,h}^n)$$
$$+ \Big([(E_{y,h}^{n+1})^2 + (E_{y,h}^n)^2](E_{y,h}^{n+1} - E_{y,h}^n)(E_{y,h}^{n+1} + E_{y,h}^n) + [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^n E_{y,h}^n](E_{x,h}^{n+1} - E_{x,h}^n)(E_{y,h}^{n+1} + E_{y,h}^n) \Big) \Big]. \quad (60)$$

Adding the Eqs. (59) and (60), we see that

$$\int_{K_{ij}} (D_{x,h}^{n+1} - D_{x,h}^n)(E_{x,h}^{n+1} + E_{x,h}^n) + \int_{K_{i,j}} (D_{y,h}^{n+1} - D_{y,h}^n)(E_{y,h}^{n+1} + E_{y,h}^n)$$
$$= \int_{K_{ij}} \varepsilon_0 (1+\chi^{(1)}) \big[|\mathbf{E}_h^{n+1}|^2 - |\mathbf{E}_h^n|^2 \big] + \int_{K_{ij}} \varepsilon_0 \chi^{(3)} \Big[\frac{1}{2}\big[|\mathbf{E}_h^{n+1}|^2 + |\mathbf{E}_h^n|^2\big]\big[|\mathbf{E}_h^{n+1}|^2 - |\mathbf{E}_h^n|^2\big]$$
$$+ [(E_{x,h}^{n+1})^2 + (E_{x,h}^n)^2][(E_{x,h}^{n+1})^2 - (E_{x,h}^n)^2] + [(E_{y,h}^{n+1})^2 + (E_{y,h}^n)^2][(E_{y,h}^{n+1})^2 - (E_{y,h}^n)^2]$$
$$+ [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^n E_{y,h}^n][E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^n E_{y,h}^{n+1} - E_{x,h}^{n+1}E_{y,h}^n - E_{y,h}^n E_{x,h}^n]$$
$$+ [E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^n E_{y,h}^n][E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^{n+1} E_{y,h}^{n} - E_{x,h}^{n}E_{y,h}^{n+1} - E_{x,h}^n E_{y,h}^n] \Big].$$

The term in square brackets in the second integral of the right-hand side can be simplified as follows:

$$\big[\ldots\big] = \frac{1}{2}\big[|\mathbf{E}_h^{n+1}|^2 + |\mathbf{E}_h^n|^2\big]\big[|\mathbf{E}_h^{n+1}|^2 - |\mathbf{E}_h^n|^2\big] + |E_{x,h}^{n+1}|^4 - |E_{x,h}^n|^4 + |E_{y,h}^{n+1}|^4 - |E_{y,h}^n|^4$$
$$+ 2\big[E_{x,h}^{n+1}E_{y,h}^{n+1} + E_{x,h}^n E_{y,h}^n\big]\big[E_{x,h}^{n+1}E_{y,h}^{n+1} - E_{x,h}^n E_{y,h}^n\big]$$
$$= \frac{1}{2}\big[|\mathbf{E}_h^{n+1}|^4 - |\mathbf{E}_h^n|^4\big] + |E_{x,h}^{n+1}|^4 + |E_{y,h}^{n+1}|^4 - |E_{x,h}^n|^4 - |E_{y,h}^n|^4 + 2|E_{x,h}^{n+1}|^2|E_{y,h}^{n+1}|^2 - 2|E_{x,h}^n|^2|E_{y,h}^n|^2$$
$$= \frac{3}{2}\big[|\mathbf{E}_h^{n+1}|^4 - |\mathbf{E}_h^n|^4\big].$$

Hence we get

$$\int_{K_{ij}} (D_{x,h}^{n+1} - D_{x,h}^n)(E_{x,h}^{n+1} + E_{x,h}^n) + \int_{K_{ij}} (D_{y,h}^{n+1} - D_{y,h}^n)(E_{y,h}^{n+1} + E_{y,h}^n)$$
$$= \int_{K_{ij}} \varepsilon_0 (1+\chi^{(1)}) \big[|\mathbf{E}_h^{n+1}|^2 - |\mathbf{E}_h^n|^2 \big] + \frac{3}{2}\int_{K_{ij}} \varepsilon_0 \chi^{(3)} \big[|\mathbf{E}_h^{n+1}|^4 - |\mathbf{E}_h^n|^4 \big]. \quad (61)$$

Taking $\Phi_{1h} := 2\Delta t(E_{x,h}^{n+1} + E_{x,h}^n)$ in the Eq. (46), we have

$$2\int_{K_{ij}} (D_{x,h}^{n+1} - D_{x,h}^n)(E_{x,h}^{n+1} + E_{x,h}^n) - 2\Delta t \int_{I_i} [(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^n)^-)_{x,j+\frac{1}{2}} - (\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1} + E_{x,h}^n)^+)_{x,j-\frac{1}{2}}]dx$$
$$+ 2\Delta t \int_{K_{ij}} H_{z,h}^{n+\frac{1}{2}} \partial_y (E_{x,h}^{n+1} + E_{x,h}^n) - 2\Delta t \int_{K_{ij}} J_{x,h}^{n+\frac{1}{2}} (E_{x,h}^{n+1} + E_{x,h}^n) = 0. \quad (62)$$

Taking $\Phi_{2h} := 2\Delta t(E_{y,h}^{n+1} + E_{y,h}^n)$ in the Eq. (47), we have

$$2\int_{K_{ij}} (D_{y,h}^{n+1} - D_{y,h})(E_{y,h}^{n+1} + E_{y,h}^n) + 2\Delta t \int_{J_j} [(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^n)^-)_{i+\frac{1}{2},y} - (\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1} + E_{y,h}^n)^+)_{i-\frac{1}{2},y}]dy$$
$$- 2\Delta t \int_{K_{ij}} H_{z,h}^{n+\frac{1}{2}} \partial_x (E_{y,h}^{n+1} + E_{y,h}^n) - 2\Delta t \int_{K_{ij}} J_{y,h}^{n+\frac{1}{2}} (E_{y,h}^{n+1} + E_{y,h}^n) = 0. \quad (63)$$

Taking $\Phi_{3h} := 2\Delta t(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})$ in Eq. (48), we have

$$2\int_{K_{ij}} \mu_0 (H_{z,h}^{n+\frac{3}{2}} - H_{z,h}^{n+\frac{1}{2}})(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})$$
$$+ 2\Delta t \int_{J_j} [(\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^-)_{i+\frac{1}{2},y} - (\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^+)_{i-\frac{1}{2},y}]dy - 2\Delta t \int_{K_{ij}} E_{y,h}^{n+1} \partial_x (H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})$$
$$- 2\Delta t \int_{I_i} [(\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^-)_{x,j+\frac{1}{2}} - (\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}})^+)_{x,j-\frac{1}{2}}]dx + 2\Delta t \int_{K_{ij}} E_{x,h}^{n+1} \partial_y (H_{z,h}^{n+\frac{3}{2}} + H_{z,h}^{n+\frac{1}{2}}) = 0.$$

Adding the Eqs. (62) and (63), substituting the result in Eq. (61), we obtain

$$\begin{aligned}
&2\int_{K_{ij}}\varepsilon_0(1+\chi^{(1)})\Big[|\mathbf{E}_h^{n+1}|^2-|\mathbf{E}_h^{n}|^2\Big]+2\int_{K_{ij}}\mu_0[(H_{z,h}^{n+\frac{3}{2}})^2-(H_{z,h}^{n+\frac{1}{2}})^2]+3\int_{K_{ij}}\varepsilon_0\chi^{(3)}\Big[|\mathbf{E}_h^{n+1}|^4-|\mathbf{E}_h^{n}|^4\Big]\\
&-2\Delta t\int_{I_i}[(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n})^-)_{x,j+\frac{1}{2}}-(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n})^+)_{x,j-\frac{1}{2}}]dx+2\Delta t\int_{K_{ij}}H_{z,h}^{n+\frac{1}{2}}\partial_y(E_{x,h}^{n+1}+E_{x,h}^{n})\\
&+2\Delta t\int_{J_j}[(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1}+E_{y,h}^{n})^-)_{i+\frac{1}{2},y}-(\hat{H}_{z,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1}+E_{y,h}^{n})^+)_{i-\frac{1}{2},y}]dy-2\Delta t\int_{K_{ij}}H_{z,h}^{n+\frac{1}{2}}\partial_x(E_{y,h}^{n+1}+E_{y,h}^{n})\\
&+2\Delta t\int_{J_j}[(\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^-)_{i+\frac{1}{2},y}-(\hat{E}_{y,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^+)_{i-\frac{1}{2},y}]dy-2\Delta t\int_{K_{ij}}E_{y,h}^{n+1}\partial_x(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})\\
&-2\Delta t\int_{I_i}[(\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^-)_{x,j+\frac{1}{2}}-(\hat{E}_{x,h}^{n+1}(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})^+)_{x,j-\frac{1}{2}}]dx+2\Delta t\int_{K_{ij}}E_{x,h}^{n+1}\partial_y(H_{z,h}^{n+\frac{3}{2}}+H_{z,h}^{n+\frac{1}{2}})\\
&=2\Delta t\int_{K_{ij}}J_{x,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n})+2\Delta t\int_{K_{ij}}J_{y,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1}+E_{y,h}^{n}).
\end{aligned}$$

Summing up over the $1\le i\le N_x$, $1\le i\le N_y$, and with respect to time from $n=1$ to N, and using the Lemmas 6.1–6.2 we arrive at

$$\begin{aligned}
&2\Big[\|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}-\|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0}-\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big]+3\Big[\||\mathbf{E}_h^{N+1}|^2\|^2_{\varepsilon_0\chi^{(3)}}-\||\mathbf{E}_h^{0}|^2\|^2_{\varepsilon_0\chi^{(3)}}\Big]\\
&=2\Delta t\sum_{n=0}^{N}\int_\Omega J_{y,h}^{n+\frac{1}{2}}(E_{y,h}^{n+1}+E_{y,h}^{n})+2\Delta t\sum_{n=0}^{N}\int_\Omega J_{x,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n})\\
&\quad-2B_y\Big(E_{x,h}^{N+1},H_{z,h}^{N+\frac{3}{2}}\Big)+2B_y\Big(E_{x,h}^{0},H_{z,h}^{\frac{1}{2}}\Big)+2B_x\Big(E_{y,h}^{N+1},H_{z,h}^{N+\frac{3}{2}}\Big)-2B_x\Big(E_{y,h}^{0},H_{z,h}^{\frac{1}{2}}\Big),
\end{aligned}\tag{64}$$

where the bilinear forms are defined as

$$B_x\Big(E_{y,h}^{n+1},H_{z,h}^{n+\frac{3}{2}}\Big):=\Delta t\Big[\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\int_{K_{ij}}E_{y,h}^{n+1}\partial_x H_{z,h}^{n+\frac{3}{2}}+\sum_{i=1}^{N_x-1}\int_p^q(E_{y,h})^+_{i+\frac{1}{2}}[\![H_{z,h}^{n+\frac{3}{2}}]\!]_{i+\frac{1}{2}}dy\Big],$$

$$B_y\Big(E_{x,h}^{n+1},H_{z,h}^{n+\frac{3}{2}}\Big):=\Delta t\Big[\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\int_{K_{ij}}E_{x,h}^{n+1}\partial_y H_{z,h}^{n+\frac{3}{2}}+\sum_{j=1}^{N_y-1}\int_r^s(E_{x,h})^+_{j+\frac{1}{2}}[\![H_{z,h}^{n+\frac{3}{2}}]\!]_{j+\frac{1}{2}}dx\Big]$$

(cf. [25, Proof of Thm. 4.1] or [33, eq. (4.1)]). Using an inverse estimate (cf. [25, Proof of Thm. 4.1] or [33, Lemma 4.1]), we have that

$$B_y\Big(E_{x,h}^{n+1},H_{z,h}^{n+\frac{3}{2}}\Big)\le 2\Delta tC_{INV}\frac{C_{\varepsilon\mu}}{h}\|E_{x,h}^{n+1}\|_{\varepsilon_0(1+\chi^{(1)})}\|H_{z,h}^{n+\frac{3}{2}}\|_{\mu_0},$$

where C_{INV} is a positive constant that is independent of h and Δt, and $C_{\varepsilon\mu}:=\left\|(\varepsilon_0\mu_0(1+\chi^{(1)}))^{-1/2}\right\|_{L_\infty(\Omega)}$. The right-hand side is estimated by means of Young's inequality with ε (see, e.g., [18, Lemma 1, 2)]), where the parameter called here $\alpha>0$ will be determined later:

$$B_y\Big(E_{x,h}^{n+1},H_{z,h}^{n+\frac{3}{2}}\Big)\le\alpha\|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\Big(\Delta tC_{INV}\frac{C_{\varepsilon\mu}}{\alpha h}\Big)^2\|H_{z,h}^{n+\frac{3}{2}}\|^2_{\mu_0}.\tag{65}$$

Similarly we get (with the same parameter α)

$$B_x\Big(E_{x,y}^{n+1},H_{z,h}^{n+\frac{3}{2}}\Big)\le\alpha\|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\Big(\Delta tC_{INV}\frac{C_{\varepsilon\mu}}{\alpha h}\Big)^2\|H_{z,h}^{n+\frac{3}{2}}\|^2_{\mu_0},\tag{66}$$

and

$$B_y\Big(E_{x,h}^{0},H_{z,h}^{\frac{1}{2}}\Big)\le\Delta tC_{INV}\frac{C_{\varepsilon\mu}}{h}\Big[\|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big],\tag{67}$$

$$B_x\Big(E_{y,h}^{0},H_{z,h}^{\frac{1}{2}}\Big)\le\Delta tC_{INV}\frac{C_{\varepsilon\mu}}{h}\Big[\|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})}+\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\Big].\tag{68}$$

The first two terms from the right-hand side of Eq. (64) are estimated by means of Young's inequality, too. This gives

$$2\Delta t\sum_{n=0}^{N}\int_\Omega J_{x,h}^{n+\frac{1}{2}}(E_{x,h}^{n+1}+E_{x,h}^{n})$$

$$
\begin{aligned}
&= 2\Delta t \sum_{n=0}^{N} \int_{\Omega} (\varepsilon_0(1+\chi^{(1)}))^{-1/2} J_{x,h}^{n+\frac{1}{2}} (\varepsilon_0(1+\chi^{(1)}))^{1/2} (E_{x,h}^{n+1} + E_{x,h}^{n}) \\
&\leq \Delta t \sum_{n=0}^{N} \|J_{x,h}^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} + \Delta t \sum_{n=0}^{N} \|E_{x,h}^{n+1} + E_{x,h}^{n}\|^2_{\varepsilon_0(1+\chi^{(1)})} \\
&\leq \Delta t \sum_{n=0}^{N} \|J_{x,h}^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} + 2\Delta t \sum_{n=0}^{N} \Big[\|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|E_{x,h}^{n}\|^2_{\varepsilon_0(1+\chi^{(1)})} \Big] \\
&\leq \Delta t \sum_{n=0}^{N} \|J_{x,h}^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} + 4\Delta t \sum_{n=0}^{N} \|E_{x,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\Delta t\, \|E_{x,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})}
\end{aligned} \tag{69}
$$

and

$$
2\Delta t \sum_{n=0}^{N} \int_{\Omega} J_{y,h}^{n+\frac{1}{2}} (E_{y,h}^{n+1} + E_{y,h}^{n}) \leq \Delta t \sum_{n=0}^{N} \|J_{y,h}^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} + 4\Delta t \sum_{n=0}^{N} \|E_{y,h}^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\Delta t\, \|E_{y,h}^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})}. \tag{70}
$$

Finally, using the estimates (69), (70), and (65)–(68) in (64), we obtain

$$
\begin{aligned}
&2\Big[\|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \Big] + \||\mathbf{E}_h^{N+1}|^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\leq 4\Delta t \sum_{n=0}^{N} \|\mathbf{E}_h^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\Delta t\, \|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\alpha \|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 4\Big(\Delta t\, C_{INV} \frac{C_{\varepsilon\mu}}{\alpha h} \Big)^2 \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} \\
&+ \Delta t \sum_{n=0}^{N} \|\mathbf{J}_h^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} \\
&+ 2\Delta t\, C_{INV} \frac{C_{\varepsilon\mu}}{h} \Big[\|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} \Big] + 2\Big[\|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} \Big] + 3\||\mathbf{E}_h^{0}|^2\|^2_{\varepsilon_0\chi^{(3)}}.
\end{aligned}
$$

Now we chose $\alpha := 1/2$ and move the corresponding term to the left-hand side. If the condition

$$
\frac{\Delta t}{h} \leq \min\left\{ \frac{1}{4C_{INV} C_{\varepsilon\mu}}; \frac{1}{4h} \right\} \tag{71}
$$

is satisfied, we obtain

$$
\begin{aligned}
&\|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} + \||\mathbf{E}_h^{N+1}|^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\leq 4\Delta t \sum_{n=0}^{N} \|\mathbf{E}_h^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\Delta t\, \|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \Delta t \sum_{n=0}^{N} \|\mathbf{J}_h^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} \\
&\quad + \frac{1}{2}\Big[\|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} \Big] + 2\Big[\|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} \Big] + 3\||\mathbf{E}_h^{0}|^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\leq 4\Delta t \sum_{n=0}^{N} \|\mathbf{E}_h^{n+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \Delta t \sum_{n=0}^{N} \|\mathbf{J}_h^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} + 3\Big[\|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} + \||\mathbf{E}_h^{0}|^2\|^2_{\varepsilon_0\chi^{(3)}} \Big].
\end{aligned}
$$

If we strengthen the condition (71) to

$$
\frac{\Delta t}{h} < \min\left\{ \frac{1}{4C_{INV} C_{\varepsilon\mu}}; \frac{1}{4h} \right\},
$$

then we may apply a discrete Gronwall's inequality [34, Lemma 5.1] (also cited in [18, Lemma 2]) to obtain

$$
\begin{aligned}
&\|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} + \||\mathbf{E}_h^{N+1}|^2\|^2_{\varepsilon_0\chi^{(3)}} \\
&\leq \exp\Big(4\Delta t \sum_{n=0}^{N} (1 - 4\Delta t)^{-1} \Big) \Big[\Delta t \sum_{n=0}^{N} \|\mathbf{J}_h^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} + 3\Big[\|\mathbf{E}_h^{0}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} + \||\mathbf{E}_h^{0}|^2\|^2_{\varepsilon_0\chi^{(3)}} \Big] \Big].
\end{aligned}
$$

If even

$$\frac{\Delta t}{h} \leq \min\left\{\frac{1}{4C_{INV}C_{\varepsilon\mu}}; \frac{1}{8h}\right\},$$

holds, then

$$\|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0} + \||\mathbf{E}_h^{N+1}|^2\|^2_{\varepsilon_0\chi^{(3)}}$$
$$\leq \exp(8T+1)\left[\Delta t\sum_{n=0}^{N}\|\mathbf{J}_h^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}} + 3\left[\|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0} + \||\mathbf{E}_h^0|^2\|^2_{\varepsilon_0\chi^{(3)}}\right]\right].$$

Since the term $\Delta t\sum_{n=0}^{N}\|\mathbf{J}_h^{n+\frac{1}{2}}\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}$ can be interpreted as an approximation to $\int_0^T\|\mathbf{J}(s)\|^2_{(\varepsilon_0(1+\chi^{(1)}))^{-1}}ds$ it can be regarded as being bounded independently of h.

To prove the first statement, the estimates (69), (70) are not needed, and we immediately get from (64) the relation

$$2\left[\|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0}\right] + \||\mathbf{E}_h^{N+1}|^2\|^2_{\varepsilon_0\chi^{(3)}}$$
$$\leq 2\alpha\|\mathbf{E}_h^{N+1}\|^2_{\varepsilon_0(1+\chi^{(1)})} + 4\left(\Delta t\, C_{INV}\frac{C_{\varepsilon\mu}}{\alpha h}\right)^2\|H_{z,h}^{N+\frac{3}{2}}\|^2_{\mu_0}$$
$$+\, 2\Delta t\, C_{INV}\frac{C_{\varepsilon\mu}}{h}\left[\|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + 2\|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\right] + 2\left[\|\mathbf{E}_h^0\|^2_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{\frac{1}{2}}\|^2_{\mu_0}\right] + 3\||\mathbf{E}_h^0|^2\|^2_{\varepsilon_0\chi^{(3)}}.$$

Now the condition

$$\frac{\Delta t}{h} \leq \frac{1}{4C_{INV}C_{\varepsilon\mu}}$$

already leads to the statement. ◂

8. Error behavior of the fully discrete solution

If the assumptions of Theorem 5.6 and Theorem 7.1 are combined with the additional requirements that the weak solution $(E_x, E_y, H_z)^T$ of the system (6) belongs to $C^2(0,T,H^{k+1}(\Omega))^3$, $k\in\mathbb{N}$, the fully discrete solution $(E_{x,h}^n, E_{y,h}^n, H_{z,h}^{n+\frac{1}{2}})^T \in (U_h^k)^3$ of (46)–(50) is uniformly bounded w.r.t. h and $n\in\mathbb{N}$ and the initial values are chosen such that $\mathcal{E}_h^0 \leq Ch^{2(k+1)}$ is satisfied, then it is possible to prove a bound for the norm

$$\|\mathbf{E}_h^N - \mathbf{E}(T)\|_{\varepsilon_0(1+\chi^{(1)})} + \|H_{z,h}^{N+\frac{1}{2}} - H_z(T)\|_{\mu_0}$$

of the error of optimal order, i.e. of the type $C(h^{k+1}+(\Delta t)^2)$.

The proof is based on the stability result Theorem 7.1 and runs structurally like the proof of Theorem 5.6, whereby on the one hand the assumed boundedness of the fully discrete solution (similar to the proof of Theorem 5.6) and on the other hand standard estimates for time discretizations (cf. [35, Sect. 9.8]) are used.

We do not want to describe the proof in detail, not only because it is quite technical (and therefore very lengthy), but above all because we see a conceptual discrepancy between the fact that on the one hand the introduced family of spatial dG discretizations can be shown to be energy stable (see Theorem 2.1), while on the other hand – as far as known to the authors – (nonlinear) results analogous to Theorem 7.1 are only available for a few selected temporal discretization methods of first and second order. Although there is active research on methods that are aimed at establishing or improving certain conservation properties (for instance implicit Runge–Kutta methods [36], implicit-explicit Runge–Kutta (IMEX-RK) methods [37] with an appropriately chosen IMEX strategy, or symplectic methods [38]), most of the theoretical results (if any) are related to the classical (linear) Maxwell's system. To carry over these results to a nonlinear situation like the one above, however, nontrivial modifications are required, which lead to challenging additional theoretical end experimental investigations.

9. Summary

In this paper, a TDdG has been developed for a system of Maxwell's equations with a cubic nonlinearity. The new capabilities of the proposed method permit that linear and nonlinear effects of the electric polarization are modeled in an efficient manner that conserves the energy or is energy stable. The novel approach allows energy stability both at the semi-discrete and fully discrete levels, which were not yet available for the full system of nonlinear Maxwell's equations. Although the fully discrete method is only conditionally stable, the semi-discrete method naturally offers the potential to also use other discretization methods, whose unconditional stability would of course also has to be shown. A detailed error estimate is provided for the semi-discrete problem. The approach is almost completely general and could replace the electric field formulation, magnetic field formulation, and A-formulation.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: No relevant relationships.

Data availability

No data was used for the research described in the article.

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