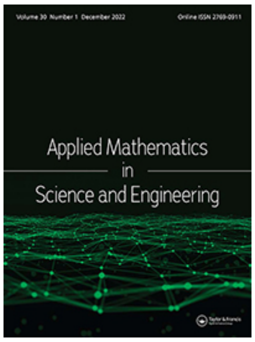

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Applications of q-derivative operator to subclasses of bi-univalent functions involving Gegenbauer polynomials

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ABSTRACT
In recent years, using the idea of analytic and bi-univalent functions, many ideas have been developed by different well-known authors, but the using Gegenbauer polynomials along with certain bi-univalent functions is very rare in the literature. We are essentially motivated by this recent research going on, here in our present investigation, we make use of certain q-derivative operator and Gegenbauer polynomials and define a new subclass of analytic and bi-univalent functions. We then obtain certain coefficient bounds, the Fekete–Szegö inequalities and upper bounds for the second-order Hankel determinant for the defined functions class.

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1. Introduction and preliminaries

Geometric Functions Theory is a fascinating area of research in Complex Analysis, with applications in a variety of mathematical areas, including Mathematical Physics. Researchers in the field of Complex Analysis have been looking into holomorphic functions because of their various applications in analytical solutions to problems like electrostatics and fluid mechanics.

Analytic functions such as $\vartheta(z)$ can be stated in Taylor's series expansion about the origin z_0 as

$$\vartheta(z) = S_0 + S_1 z + S_2 z^2 + S_3 z^3 + S_4 z^4 + \cdots, \quad |z| < 1,$$

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which can be normalized in the following way:

$$f(z) = \frac{\vartheta(z) - S_0}{S_1} = z + \sum_{j=2}^{\infty} b_j z^j, \tag{1}$$

where $S_1 \neq 0$, $b_j = S_j/S_1$, $z \in \mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$ and $f(z)$ is convergent for $|z| < 1$. Let $\mathcal{A}$ indicate a class of functions $f(z)$ that are holomorphic in $\mathcal{U}$, having form (1), and normalized by the constraints $f'(0) - 1 = f(0) = 0$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \tfrac{1}{4}),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent function in $\mathcal{U}$ given by (2). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad \log\frac{1}{1-z} \quad \text{and} \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is a member of class Σ. Other common examples of functions in $\mathcal{U}$ such as

$$\frac{2z - z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ.

Lewin [1] investigated a bi-univalent functions class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [3], on the other hand, showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. The coefficient for each of the Taylor–Maclaurin coefficients $|a_2|$ $(n \geq 3, n \in \mathbb{N})$ is presumably still an open problem.

Similar to the familiar subclass $\mathcal{S}^*(\zeta)$ and $\mathcal{K}(\zeta)$ of starlike and convex functions of order $\zeta (0 \leq \zeta < 1)$, respectively. Brannan and Taha [?] introduced certain subclasses of the bi-univalent function class $\Sigma, \mathcal{S}_\Sigma^*(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ of bi-starlike functions and bi-convex functions of order $\zeta (0 \leq \zeta < 1)$, respectively. For each of the function classes $\mathcal{S}_\Sigma^*(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ they found non-sharp bounds on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$.

Let $s_1(z)$ and $s_2(z)$ are analytic functions in open unit disc $\mathcal{U}$, then the function s_1 is subordinated to s_2 symbolically denoted as $s_1(z) \prec s_2(z), z \in \mathcal{U}$, if there occur an analytic function $w(z)$ with properties that $w(0) = 0$ and $|w(z)| < 1$, such that suppose ω holomorphic in $\mathcal{U}$, such that $s_1(z) = s_2(w(z))$. If the function $s_2(z)$ is univalent in $\mathcal{U}$, then above condition is equivalent to $s_1(z) \prec s_2(z) \Leftrightarrow s_1(0) = s_2(0)$ and $s_1(\mathcal{U}) \subset s_2(\mathcal{U})$.

Jackson [5, 6] introduced and studied the q-derivative operator $\mathfrak{D}_q$ of a function as follows:

$$\mathfrak{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} = \frac{1}{z}\left\{z + \sum_{j=2}^{\infty}[j]_q a_j z^j\right\} \tag{3}$$

and $(\mathfrak{D}_q f)(0) = f'(0)$. In case $f(z) = z^j$ for j is a positive integer, the q-derivative of $f(z)$ is given by

$$\mathfrak{D}_q z^j = \frac{z^j - (zq)^j}{z(1-q)} = [j]_q z^{j-1}, \tag{4}$$

$$\lim_{q \longrightarrow 1-} [j]_q = \lim_{q \longrightarrow 1-} \frac{1-q^j}{1-q} = j, \tag{5}$$

where $(z \neq 0,\ q \neq 0)$, for more details on the concepts of q-derivative (see [7]).

The quantum (or q-) calculus is an essential tool for studying diverse families of analytic functions, and its applications in mathematics and related fields have inspired researchers. Srivastava [8] was the first person to apply it in the context of univalent functions. Numerous scholars conducted substantial work on q-calculus and examined its various applications due to the applicability of q-analysis in mathematics and other domains. More importantly, the convolution theory enable us to investigate various properties of analytic functions. Due to the large range of applications of q-calculus and the importance of q-operators instead of regular operators, many researchers have explored q-calculus in depth. In addition, Srivastava [9] recently published survey-cum-expository review paper which might be useful for researchers and scholars working on these subjects. Also, Srivastava's recent survey-cum-expository review article [9] further motivates the use of the q-analysis in Geometric Function Theory, as well as commenting on the triviality of the so-called $(\mathfrak{p}, q)$-analysis involving an insignificant and redundant parameter $(\mathfrak{p}, q)$ (see especially [9, p.340]). For some recent investigation about q-calculus, we may refer the readers to [10–15]

The class of functions φ that is holomorphic in $\mathcal{U}$ and has the form

$$\varphi(z) = 1 + r_1 z + r_2 z^2 + \cdots, \quad z \in \mathcal{U},$$

with

$$\varphi(0) = 1 \quad \text{and} \quad \Re(\varphi(z)) > 0$$

is denoted by $\mathcal{P}$.

The nth coefficient of a class $\mathcal{S}$ function is well known to be restricted by n, and the coefficient limits give information about the functions' geometric characteristics. The famous problem solved by Fekete–Szegö [16] is to determine the greatest value of the coefficient functional $\Omega_\sigma(f) := |b_3 - \sigma b_2^2|$ over the class $\mathcal{S}$ for each $\sigma \in [0, 1]$, which was demonstrated using the Loewner technique.

Noonan and Thomas [17] introduced and investigated the mth Hankel determinant of f for $m \geq 1$ and $n \geq 1$ as

$$\mathcal{H}_m(j) = \begin{vmatrix} b_j & b_{j+1} & b_{j+2} & \cdots & \cdots & b_{j+m-1} \\ b_{j+1} & b_{j+2} & b_{j+3} & \cdots & \cdots & b_{j+m} \\ b_{j+2} & b_{j+3} & b_{j+4} & \cdots & \cdots & b_{j+m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{j+m-1} & b_{j+m} & b_{j+m+1} & \cdots & \cdots & b_{j+2(m-1)} \end{vmatrix} \quad (m, j \in \mathcal{N}). \tag{6}$$

Several writers, notably Noor [18], have investigated this determinant, with topics ranging from the rate of development of $\mathcal{H}_m(j)$ (as $j \longrightarrow \infty$) to the determinant of exact limits for particular subclasses of analytic functions on the unit disk $\mathcal{U}$ with specified values of j and m. When $m = 2, j = 1$, and $b_1 = 1$, the Hankel determinant is $\mathcal{H}_2(1) = |b_3 - b_2^2|$. The Hankel determinant simplifies to $\mathcal{H}_2(2) = |b_2 b_4 - b_3^2|$ when $j = m = 2$. Fekete and Szegö [19] consider the Hankel determinant $\mathcal{H}_2(1)$ and refer to $H_2(2)$ as the second Hankel determinant. If f is univalent in $\mathcal{U}$, then the sharp upper inequality $\mathcal{H}_2(1) = |b_3 - b_2^2| \leq 1$ is known (see [16]). Janteng et al. [20] obtained sharp bounds for the functional $\mathcal{H}_2(2)$ for the function f in the subclass $\mathcal{RT}$ of $\mathcal{S}$, which was introduced by Mac Gregor [21] and consists of functions whose derivative has a positive real part. They demonstrated that $\mathcal{H}_2(2) = |b_2 b_4 - b_3^2| \leq 4/9$ for each $f \in \mathcal{RT}$. They also discovered the sharp second Hankel determinant for the classical subclass of $\mathcal{S}$, namely the S^* and $\mathcal{K}$ which are the class of starlike and convex functions (see [20]). These two classes have bounds of $|b_2 b_4 - b_3^2| \leq 1/8$ and $|b_2 b_4 - b_3^2| \leq 1$. The Hankel determinants for starlike and convex functions with respect to symmetric points were recently discovered by Ready and Krishna [22]. For functions belonging to subclasses of Ma–Minda starlike and convex functions, Lee et al. [23] found the second Hankel determinant. Mishra and Gochhayat [24] found the sharp bound to the nonlinear functional $|b_2 b_4 - b_3^2|$ for the subclass of analytic functions.

Deniz et al. [25] discussed the upper bounds of $\mathcal{H}_2(2)$ for the classes S^* and $\mathcal{K}$ lately. Later, Altinkaya and Yalcin [26], Caglar et al. [27], Kanas et al. [28], and Orhan et al. [29] determined the upper bounds of $\mathcal{H}_2(2)$ for several subclasses of Σ.

Gegenbauer polynomials, also known as ultraspherical polynomials $G_j^{(\upsilon)}(t)$, are orthogonal polynomials with regard to the weight function $(1 - t^2)^{\upsilon - 1/2}$ on the interval $[1, 1]$. They are particular instances of Jacobi polynomials and generalize Legendre and Chebyshev polynomials. They were given the name Leopold Gegenbauer. The following generating function of polynomials can be used to define them.

$$H(t, z) = \frac{1}{(1 - 2tz + z^2)^{\upsilon}} = \sum_{j=0}^{\infty} G_j^{(\upsilon)}(t) z^j. \tag{7}$$

The recurrence relation is satisfied by the polynomials.

$$G_0^{(\upsilon)}(t) = 1,$$
$$G_1^{(\upsilon)}(t) = 2\upsilon t,$$
$$jG_j^{(\upsilon)}(t) = 2t(j + \upsilon - 1)G_{j-1}^{(\upsilon)}(t) - (j + 2\upsilon - 2)G_{j-2}^{(\upsilon)}(t).$$

Gegenbauer polynomials are specific solutions to

$$(1+t^2)y'' - (2\upsilon+1)ty' + j(j+2\upsilon)y = 0$$

differential equation. The equation becomes the Legendre equation when $\upsilon = 1/2$, and the Gegenbauer polynomials become Legendre polynomials. When $\upsilon = 1$, the equation becomes a Chebyshev differential equation and the Gegenbauer polynomials become second-order Chebyshev polynomials.

The Gegenbauer polynomials naturally emerge as extensions of Legendre polynomials in the context of potential theory and harmonic analysis. The Gegenbauer polynomial looks to be fascinating and significant in the subject of mathematical physics. Gegenbauer polynomials have lately been studied in the setting of mathematical physics by a number of authors (see [30–35]).

Many scholars have recently started investigating bi-univalent functions related to orthogonal polynomials, with a few to name [36–38]. As far as we know, there is minimal work-related to bi-univalent functions in the literatures for the Gegenbauer polynomial. The major objective of this work is to begin an investigation into the characteristics of bi-univalent functions linked with Gegenbauer polynomials.

Definition 1.1: Let $H(t,z)$ be defined as follows:

$$H(t,z) = 1 + \sum_{j=1}^{\infty} G_j^{(\upsilon)}(t) z^j.$$

A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{N}_q^{\Sigma}(\beta,\gamma,t)$ if the following subordination conditions are fulfilled:

$$1 + \frac{1}{\gamma}[\mathfrak{D}_q f(z) + \beta z \mathfrak{D}_q(\mathfrak{D}_q f(z)) - 1] \prec H(z,t) \tag{8}$$

and

$$1 + \frac{1}{\gamma}[\mathfrak{D}_q g(\omega) + \beta \omega \mathfrak{D}_q(\mathfrak{D}_q g(\omega)) - 1] \prec H(\omega,t), \tag{9}$$

where $\gamma \in \mathbb{R} \setminus \{0\}$, $0 \le \beta \le 1$, $0 < q < 1$ and the function g is given by (2).

We use the Gegenbauer polynomials expansions to determine the initial coefficient estimates, Fekete Szegö problem and estimate of $|\mathcal{H}_2(2)|$ Hankel determinant for a subclass of analytic and bi-univalent functions in this work.

Lemma 1.1 ([19]): *Let $\varphi(z) \in \mathcal{P}$, then*

$$|p_j| \le 2 \quad (j \in \mathcal{N}).$$

Lemma 1.2 ([39]): *Let $\varphi(z) \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2),$$
$$4p_3 = p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z$$

for some complex number satisfying x, z, $|x| \le 1$ and $|z| \le 1$.

2. Coefficient estimates for the class $\mathcal{N}_q^{\Sigma}(\beta,\gamma,t)$

Theorem 2.1: *Let $f \in \mathcal{N}_q^{\Sigma}(\beta,\gamma,t)$, $\gamma \in \mathbb{R}\setminus\{0\}$, $0 \le \beta \le 1$, $0<q<1$, $t \in (1/2,1]$. Then*

$$|b_2| \le \sqrt{\frac{8\upsilon^3 t^3\gamma^3}{|4\upsilon^2 t^2\gamma^2[3]_q(1+\beta[2]_q)+[2]_q^2(1+\beta)^2(2\upsilon t-2\upsilon t^2(1+\upsilon)+\upsilon)|}}, \tag{10}$$

$$|a_3| \le \frac{4\upsilon^2 t^2\gamma^2}{[2]_q^2(1+\beta)^2}+\frac{2\upsilon t\gamma}{[3]_q(1+\beta[2]_q)}, \tag{11}$$

$$\begin{aligned}|b_4| \le{}& \frac{10[4]_q\upsilon^2 t^2(1+\beta[3]_q)\gamma^2}{[2]_q[3]_q[4]_q(1+\beta[2]_q)(1+\beta[3]_q)(1+\beta)}\\ &+\frac{2(2\upsilon t^2(1+\upsilon)-\upsilon-2\upsilon t)\gamma}{(1+\beta[3]_q)[4]_q}+\frac{2\upsilon t\gamma}{(1+\beta[3]_q)[4]_q}\\ &+\frac{[6\upsilon(1+t)-12\upsilon t^2(1+\upsilon)+2t\upsilon(2+\upsilon)(2t^2(1+\upsilon)-1)-2\upsilon t(1+2\upsilon)]\gamma}{3(1+\lambda[2]_q)[4]_q}\end{aligned} \tag{12}$$

and for some $\delta \in \mathbb{R}$,

$$|b_3-\delta b_2^2| \le \begin{cases} 2|1-\delta|\Lambda_1 t(q,v,t) & \left(|1-\delta|\Lambda_1 t(q,v,t) \ge \dfrac{2\upsilon t}{(1+\beta[2]_q)[3]_q}\right),\\ \dfrac{4\upsilon t}{(1+\beta[2]_q)[3]_q} & \left(|1-\delta|\Lambda_1 t(q,v,t) \le \dfrac{2\upsilon t}{(1+\beta[2]_q)[3]_q}\right),\end{cases}$$

where

$$\Lambda_1 t(q,v,t) = \frac{8\gamma^3\upsilon^3 t^3}{|4\cdot[3]_q\upsilon^2 t^2\gamma^2(1+\beta[2]_q)+[2]_q^2\gamma(1+\beta)^2(2\upsilon t-2\upsilon t^2(1+\upsilon)+\upsilon)|}. \tag{13}$$

Proof: Let $f \in \Sigma$ given by (1) be in the class $\mathcal{N}_q^{\Sigma}(\beta,\gamma,t)$. Then

$$1+\frac{1}{\gamma}[\mathfrak{D}_q f(z)+\beta z\mathfrak{D}_q(\mathfrak{D}_q f(z))-1] = H(\omega(z),t) \tag{14}$$

and

$$1+\frac{1}{\gamma}[\mathfrak{D}_q g(\omega)+\beta\omega\mathfrak{D}_q(\mathfrak{D}_q g(\omega))-1] = H(\varpi(\omega),t), \tag{15}$$

where $p, y \in \mathcal{P}$ and let $p, y \in \mathcal{P}$ be define as

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)} = 1+p_1(z)+p_2z^2+p_3z^3+\cdots \Rightarrow \omega(z)=\frac{p(z)-1}{p(z)+1}, \quad (z \in U) \tag{16}$$

and

$$y(\omega)=\frac{1+\varpi(\omega)}{1-\varpi(\omega)} = 1+y_1(\omega)+y_2\omega^2+y_3\omega^3+\cdots \Rightarrow \varpi(\omega)=\frac{y(\omega)-1}{y(\omega)+1}, \quad (\omega \in U). \tag{17}$$

It follows from (16) and (17) that

$$\omega(z) = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)z^3 + \cdots\right] \tag{18}$$

and

$$\varpi(\omega) = \frac{1}{2}\left[y_1 \omega + \left(y_2 - \frac{y_1^2}{2}\right)\omega^2 + \left(y_3 - y_1y_2 + \frac{y_1^3}{4}\right)\omega^3 + \cdots\right]. \tag{19}$$

From (18) and (19), applying $H(t,z)$ as given in (7), we see that

$$\begin{aligned} H(\omega(z),t) = 1 &+ \frac{G_1^{(\upsilon)}(t)}{2}p_1 z + \left[\frac{G_1^{(\upsilon)}(t)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{G_2^{(\upsilon)}(t)}{4}p_1^2\right]z^2 \\ &+ \left[\frac{G_1^{(\upsilon)}(t)}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)\right. \\ &+ \left.\frac{G_2^{(\upsilon)}(t)}{2}p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{G_3^{(\upsilon)}(t)}{8}p_1^3\right]z^3 + \cdots \end{aligned}$$

and

$$\begin{aligned} H(\varpi(\omega),t) = 1 &+ \frac{G_1^{(\upsilon)}(t)}{2}y_1 \omega + \left[\frac{G_1^{(\upsilon)}(t)}{2}\left(y_2 - \frac{y_1^2}{2}\right) + \frac{G_2^{(\upsilon)}(t)}{4}y_1^2\right]\omega^2 \\ &+ \left[\frac{G_1^{(\upsilon)}(t)}{2}\left(y_3 - y_1y_2 + \frac{y_1^3}{4}\right)\right. \\ &+ + \left.\frac{G_2^{(\upsilon)}(t)}{2}y_1\left(y_2 - \frac{y_1^2}{2}\right) + \frac{G_3^{(\upsilon)}(t)}{8}y_1^3\right]\omega^3 + \cdots. \end{aligned} \tag{20}$$

It the following follows from (14), (20) and (15) that

$$\frac{(1+\beta)[2]_q}{\gamma}b_2 = \frac{G_1^{(\upsilon)}(t)}{2}p_1, \tag{21}$$

$$\frac{(1+\beta[2]_q)[3]_q}{\gamma}b_3 = \frac{G_1^{(\upsilon)}(t)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{G_2^{(\upsilon)}(t)}{4}p_1^2, \tag{22}$$

$$\begin{aligned} \frac{(1+\beta[3]_q)[4]_q}{\gamma}b_4 &= \frac{G_1^{(\upsilon)}(t)}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) \\ &+ \frac{G_2^{(\upsilon)}(t)}{2}p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{G_3^{(\upsilon)}(t)}{8}p_1^3, \end{aligned} \tag{23}$$

$$-\frac{(1+\beta)[2]_q}{\gamma}b_2 = \frac{G_1^{(\upsilon)}(t)}{2}y_1, \tag{24}$$

$$\frac{[3]_q(1+\beta[2]_q)(2a_2^2 - a_3)}{\gamma}b_3 = \frac{G_1^{(\upsilon)}(t)}{2}\left(y_2 - \frac{y_1^2}{2}\right) + \frac{G_2^{(\upsilon)}(t)}{4}y_1^2, \tag{25}$$

$$-\frac{[4]_q(1+\beta[3]_q)(5b_2^3-5b_2b_3+b_4)}{\gamma}b_4=\frac{G_1^{(\upsilon)}(t)}{2}\left(y_3-y_1y_2+\frac{y_1^3}{4}\right)$$
$$+\frac{G_2^{(\upsilon)}(t)}{2}y_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{G_3^{(\upsilon)}(t)}{8}y_1^3. \quad (26)$$

Adding (21) and (24), we have

$$p_1=-y_1,\quad p_1^2=y_1^2 \quad \text{and} \quad p_1^3=-y_1^3 \quad (27)$$

and

$$b_2^2=\frac{(G_1^{(\upsilon)}(t))^2(p_1^2+y_1^2)\gamma}{8[2]_q^2(1+\beta)^2}. \quad (28)$$

Also, adding (22), (25) and applying (27) yields

$$\frac{4[3]_q(1+\beta[2]_q)b_2^2}{\gamma}=G_1^{(\upsilon)}(t)(p_2+y_2)-y_1^2(G_1^{(\upsilon)}(t)-G_2^{(\upsilon)}(t)). \quad (29)$$

Applying (27) in (28) gives

$$y_1^2=\frac{4[2]_q^2(1+\beta)^2b_2^2}{(G_1^{(\upsilon)}(t))^2\gamma^2}. \quad (30)$$

Putting (30) into (29) and with some calculations, we have

$$|b_2|^2=\left|\frac{(G_1^{(\upsilon)}(t))^3\gamma^3(p_2+y_2)}{4[3]_q(G_1^{(\upsilon)}(t))^2\gamma^2(1+\beta[2]_q)+4[2]_q^2\gamma(1+\beta)^2(G_1^{(\upsilon)}(t)-G_2^{(\upsilon)}(t))}\right|.$$

Applying triangular inequality and Lemma 1.1, we have

$$|b_2|\leq\sqrt{\Lambda_1t(q,\nu,t)}. \quad (31)$$

Subtracting (25) from (22) and with some calculations, we have

$$b_3=b_2^2+\frac{G_1^{(\upsilon)}(t)\gamma[p_2-y_2]}{4[3]_q(1+\beta[2]_q)} \quad (32)$$

and

$$b_3=\frac{(G_1^{(\upsilon)}(t))^2\gamma^2p_1^2}{4[2]_q^2(1+\beta)^2}+\frac{G_1^{(\upsilon)}(t)\gamma[p_2-y_2]}{4[3]_q(1+\beta[2]_q)}. \quad (33)$$

Applying triangular inequality, and Lemma 1.1, we have

$$|b_3|\leq\frac{4\upsilon^2t^2\gamma^2}{[2]_q^2(1+\beta)^2}+\frac{2\upsilon t\gamma}{[3]_q(1+\beta[2]_q)}. \quad (34)$$

Subtracting (26) from (23), we have

$$\frac{2[4]_q(1+\beta[3]_q)}{\gamma}b_4=\frac{5[4]_q(G_1^{(\upsilon)}(t))^2(1+\beta[3]_q)\gamma^2p_1(p_2-y_2)}{8[2]_q[3]_q(1+\beta[2]_q)(1+\beta)\gamma}$$

$$+\frac{G_1^{(\upsilon)}(t)(p_3-y_3)}{2}+\frac{[G_2^{(\upsilon)}(t)-G_1^{(\upsilon)}(t)]p_1(p_2+y_2)}{2}$$
$$+\frac{(G_1^{(\upsilon)}(t)-2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t))p_1^3}{4}. \tag{35}$$

Applying triangular inequality and Lemma 1.1, we have

$$|b_4|\le\frac{10[4]_q\upsilon^2t^2(1+\beta[3]_q)\gamma^2}{[2]_q[3]_q[4]_q(1+\beta[2]_q)(1+\beta[3]_q)(1+\beta)}+\frac{2\upsilon t\gamma}{(1+\beta[3]_q)[4]_q}$$
$$+\frac{2(2\upsilon t^2(1+\upsilon)-\upsilon-2\upsilon t)\gamma}{(1+\beta[3]_q)[4]_q}$$
$$+\frac{[6\upsilon(1+t)-12\upsilon t^2(1+\upsilon)+2t\upsilon(2+\upsilon)(2t^2(1+\upsilon)-1)-2\upsilon t(1+2\upsilon)]\gamma}{3(1+\lambda[2]_q)[4]_q}.$$

From (32), we have

$$b_3-\delta b_2^2=b_2^2+\frac{G_1^{(\upsilon)}(t)\gamma[p_2-y_2]}{4[3]_q(1+\beta[2]_q)}-\delta b_2^2$$
$$=\frac{\upsilon t(p_2-y_2)}{2(1+\beta[2]_q)[3]_q}+(1-\delta)$$
$$\times\left[\frac{2\gamma^3(p_2+y_2)\upsilon^3t^3}{4\cdot[3]_q\upsilon^2t^2\gamma^2(1+\beta[2]_q)+[2]_q^2\gamma(1+\beta)^2(2\upsilon t-2\upsilon t^2(1+\upsilon)+\upsilon)}\right].$$

By triangular inequality, we have

$$|b_3-\delta b_2^2|\le\frac{2\upsilon t}{(1+\beta[2]_q)[3]_q}+|1-\delta|\Lambda_1t(q,\nu,t). \tag{36}$$

Suppose

$$|1-\delta|\Lambda_1t(q,\nu,t)\ge\frac{2\upsilon t}{(1+\beta[2]_q)[3]_q}$$

then, we have

$$|b_3-\delta b_2^2|\le 2|1-\delta|\Lambda_1t(q,\nu,t), \tag{37}$$

where

$$|1-\delta|\ge\Lambda_1t(q,\nu,t)$$

and suppose

$$|1-\delta|\Lambda_1t(q,\nu,t)\le\frac{2\upsilon t}{(1+\beta[2]_q)[3]_q},$$

then, we have

$$|b_3-\delta b_2^2|\le\frac{4\upsilon t}{(1+\beta[2]_q)[3]_q},$$

where

$$|1-\delta| \leq \frac{2\upsilon t}{(1+\beta[2]_q)[3]_q \Lambda_1 t(q, \nu, t)}$$

and $\Lambda_1(q, \nu, t)$ is given in (13). ■

Remark 2.1: If we let $\lim_{q\to 1-}$ in the above result, we can get the same bounds for the function class $\mathcal{N}^{\Sigma}(\beta, \gamma, t)$ of analytic and bi-univalent functions, involving the Gegenbauer polynomials.

3. Second Hankel determinant for the class $\mathcal{N}_q^{\Sigma}(\beta, \gamma, t)$

Theorem 3.1: *Let the function* $f(z)$ *given by* (1) *be in the class* $\mathcal{N}_q^{\Sigma}(\beta, \gamma, t)$, $\gamma \in \mathbb{R} \setminus \{0\}$, $0 \leq \beta \leq 1$, $0 < q < 1$, $t \in (1/2, 1]$. *Then*

$$H_2(2) = |b_2 b_4 - b_3^2| \leq \begin{cases} T(2,t) & (B_1 \geq 0 \text{ and } B_2 \geq 0), \\ \max\left\{\dfrac{4\upsilon^2 t^2 \gamma^2}{(1+\beta[2]_q)^2[3]_q^2}, T(2,t)\right\} & (B_1 > 0 \text{ and } B_2 < 0), \\ \dfrac{4\upsilon^2 t^2 \gamma^2}{(1+\beta[2]_q)^2[3]_q^2} & (B_1 \leq 0 \text{ and } B_2 \leq 0), \\ \max\{T(m_0,t), T(2,t)\} & (B_1 < 0 \text{ and } B_2 > 0). \end{cases}$$

Where

$$T(2,t) = \frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t) - 2G_2^{(\upsilon)}(t) + G_3^{(\upsilon)}(t)]\gamma^2}{[2]_q[4]_q(1+\beta[2]_q)(1+\beta)} + \frac{2G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t) + G_1^{(\upsilon)}(t)]\gamma^2}{[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}$$
$$+ \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} + \frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{16[2]_q^4(1+\beta)^4},$$

$$T(m_0,t) = \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{(1+\beta[2]_q)^2[3]_q} + \frac{B_2^2}{8[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta[2]_q)^2 B_1}$$
$$- \frac{B_2^2}{4[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta[2]_q)^2 B_1},$$

$$\begin{aligned} B_1 = {} & G_1^{(\upsilon)}(t)[2[G_1^{(\upsilon)}(t) - 2G_2^{(\upsilon)}(t) \\ & + G_3^{(\upsilon)}(t)]\gamma^2(1+\beta[2]_q)[2]_q^3(1+\beta)^3(1+\beta[3]_q)[3]_q^3 \\ & + 2(G_1^{(\upsilon)}(t))^3(1+\beta[2]_q)^2[4]_q(1+\beta[3]_q)[3]_q^3 \\ & - 4G_1^{(\upsilon)}(t)\gamma^2(1+\beta[2]_q)^2[2]_q^3(1+\beta)^3[3]_q^3 \\ & + 2G_1^{(\upsilon)}(t)\gamma^2[2]_q^4[4]_q(1+\beta)^4(1+\beta[3]_q) \\ & - (G_1^{(\upsilon)}(t))^2(1+\beta[2]_q)^2[2]_q^2[3]_q(1+\beta)^2[4]_q(1+\beta[3]_q)], \end{aligned}$$

and

$$B_2 = G_1^{(\upsilon)}(t)[4[G_2^{(\upsilon)}(t) - G_1^{(\upsilon)}(t)]\gamma^2[2]_q(1+\beta)(1+\beta[2]_q)^2[3]_q^2$$

$$- 4G_1^{(\upsilon)}(t)\gamma^2[2]_q^2(1+\beta[3]_q)[4]_q + 6G_1^{(\upsilon)}(t)\gamma^2[2]_q[3]_q^2(1+\beta)(1+\beta[2]_q)^2$$
$$+ (G_1^{(\upsilon)}(t))^2[3]_q(1+\beta[2]_q)(1+\beta[3]_q)[4]_q].$$

Proof: From (21) and (35), we have

$$\begin{aligned} b_2b_4 = &\frac{5(G_1^{(\upsilon)}(t))^3\gamma^3[4]_q(1+\beta[3]_q)(p_2-y_2)}{32[2]_q^2[3]_q[4]_q(1+\beta[2]_q)(1+\beta[3]_q)(1+\beta)^2}p_1^2 \\ &+ \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(p_3-y_3)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1 \\ &+ \frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t)+G_1^{(\upsilon)}(t)]\gamma^2[4]_q(p_2+y_2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1^2 \\ &+ \frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t)-2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t)]\gamma^2}{16[2]_q[4]_q(1+\beta[2]_q)(1+\beta)}p_1^4. \end{aligned}$$

With some calculations, we have

$$\begin{aligned} b_2b_4 - b_3^2 = &\frac{(G_1^{(\upsilon)}(t))^3\gamma^3(p_2-y_2)}{32[2]_q^2[3]_q(1+\beta[2]_q)(1+\beta)^2}p_1^2 + \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(p_3-y_3)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1 \\ &+ \frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t)+G_1^{(\upsilon)}(t)]\gamma^2(p_2+y_2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1^2 \\ &+ \frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t)-2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t)]\gamma^2}{16[2]_q[4]_q(1+\beta[2]_q)(1+\beta)}p_1^4 \\ &- \frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{16[2]_q^4(1+\beta)^4}p_1^4 - \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(p_2-y_2)^2}{16[3]_q^2(1+\beta[2]_q)^2}. \end{aligned} \tag{38}$$

By using Lemma 1.2,

$$p_2 - y_2 = \frac{4-p_1^2}{2}(x-h), \tag{39}$$

$$p_2 + y_2 = p_1^2 + \frac{4-p_1^2}{2}(x+h), \tag{40}$$

and

$$\begin{aligned} p_3 - y_3 = &\frac{p_1^3}{2} + \frac{4-p_1^2}{2}p_1(x+h) - \frac{4-p_1^2}{4}p_1(x^2+h^2) \\ &+ \frac{4-p_1^2}{2}[(1-|x|^2z)-(1-|h|^2)w] \end{aligned} \tag{41}$$

for some x, h, z, w with $|x| \le 1, |h| \le 1, |z| \le 1, |w| \le 1,\ |p_1| \in [0,2]$ and substituting $(p_2+y_2), (p_2-y_2)$ and (p_3-y_3), and after some straightforward simplifications, we have

$$b_2b_4 - b_3^2 = \frac{(G_1^{(\upsilon)}(t))^3\gamma^3(4-p_1^2)(x-h)}{64[2]_q^2[3]_q(1+\beta[2]_q)(1+\beta)^2}p_1^2$$

$$+\frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t)+G_1^{(\upsilon)}(t)]\gamma^2}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1^4+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1^4$$
$$+\frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t)+G_1^{(\upsilon)}(t)]\gamma^2(4-p_1^2)(x+h)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1^2$$
$$+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-p_1^2)(x+h)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1^2-\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-p_1^2)(x^2+h^2)}{32[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1^2$$
$$+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-p_1^2)[(1-|x|^2)z-(1-|h|^2)w]}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}p_1$$
$$-\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-p_1^2)^2(x-h)^2}{64(1+\beta[2]_q)^2[3]_q^2}$$
$$-\frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{16[2]_q^4(1+\beta)^4}p_1^4+\frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t)-2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t)]\gamma^2}{16[2]_q[4]_q(1+\beta[2]_q)(1+\beta)}p_1^4.$$

Let $m=p_1$, assume without any restriction that $m\in[0,2]$, $\lambda_1=|x|\le 1$, $\lambda_2=|h|\le 1$ and applying triangular inequality, we have

$$|b_2b_4-b_3^2|\le\left\{\frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t)-2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t)]\gamma^2}{16[2]_q[4]_q(1+\beta[2]_q)(1+\beta)}m^4\right.$$
$$+\frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t)+G_1^{(\upsilon)}(t)]\gamma^2}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^4+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^4$$
$$\left.+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m+\frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{16[2]_q^4(1+\beta)^4}m^4\right\}$$
$$+\left\{\frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t)+G_1^{(\upsilon)}(t)]\gamma^2(4-m^2)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^2\right.$$
$$+\frac{(G_1^{(\upsilon)}(t))^3\gamma^3(4-m^2)}{64[2]_q^2[3]_q(1+\beta[2]_q)(1+\beta)^2}m^2$$
$$\left.+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^2\right\}(\lambda_1+\lambda_2)$$
$$+\left\{\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{32[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^2\right.$$
$$\left.-\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m\right\}(\lambda_1^2+\lambda_2^2)$$
$$+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)^2(\lambda_1+\lambda_2)^2}{64(1+\beta[2]_q)^2[3]_q^2}$$

and equivalently, we have

$$|b_2b_4 - b_3^2| \leq N_1(t) + N_2(t,m)(\lambda_1+\lambda_2) + N_3(t,m)(\lambda_1^2+\lambda_2^2) + N_4(t,m)(\lambda_1+\lambda_2)^2 = Z(\lambda_1,\lambda_2), \quad (42)$$

where

$$N_1(t,m) = \left\{ \frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t) - 2G_2^{(\upsilon)}(t) + G_3^{(\upsilon)}(t)]\gamma^2}{16[2]_q[4]_q(1+\beta[2]_q)(1+\beta)} m^4 + \frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t) + G_1^{(\upsilon)}(t)]\gamma^2}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m^4 + \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m^4 + \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m + \frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{16[2]_q^4(1+\beta)^4} m^4 \right\} \geq 0,$$

$$N_2(t,m) = \left\{ \frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t) + G_1^{(\upsilon)}(t)]\gamma^2(4-m^2)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m^2 + \frac{(G_1^{(\upsilon)}(t))^3\gamma^3(4-m^2)}{64[2]_q^2[3]_q(1+\beta[2]_q)(1+\beta)^2} m^2 + \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m^2 \right\} \geq 0,$$

$$N_3(t,m) = \left\{ \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{32[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m^2 - \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m \right\} \leq 0,$$

$$N_4(t,m) = \frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)^2}{64(1+\beta[2]_q)^2[3]_q^2} \geq 0,$$

where $0 \leq m \leq 2$. Now, we maximize the function $Z(\lambda_1,\lambda_2)$ in the closed square

$$\Delta = \{(\lambda_1,\lambda_2) : \lambda_1 \in [0,1], \lambda_2 \in [0,1]\} \quad \text{for } m \in [0,2].$$

For a fixed value of t, the coefficients of the function $Z(\lambda_1,\lambda_2)$ in (42) are dependent on m, thus the maximum of $Z(\lambda_1,\lambda_2)$ with regard to m must be investigated, taking into account the cases when $m=0$, $r=2$ and $m \in (0,2)$.

First Case: When $m=0$,

$$Z(\lambda_1,\lambda_2) = N_4(t,0) = \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{4(1+\beta[2]_q)^2[3]_q^2}(\lambda_1+\lambda_2)^2.$$

It is obvious that the function $Z(\lambda_1, \lambda_2)$ reaches its maximum at (λ_1, λ_2) and

$$\max\{Z(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in [0,1]\} = Z(1,1) = \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{(1+\beta[2]_q)^2[3]_q^2}. \tag{43}$$

Second Case: When $m = 2$, $Z(\lambda_1, \lambda_2)$ is expressed as a constant function with respect to m, we have

$$\begin{aligned} Z(\lambda_1, \lambda_2) = N_1(t,2) = \Bigg\{ & \frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t) - 2G_2^{(\upsilon)}(t) + G_3^{(\upsilon)}(t)]\gamma^2}{[2]_q[4]_q(1+\beta[2]_q)(1+\beta)} \\ & + \frac{2G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t) + G_1^{(\upsilon)}(t)]\gamma^2}{[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} + \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} \\ & + \frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{[2]_q^4(1+\beta)^4} \Bigg\}. \end{aligned}$$

Third Case: When $m \in (0,2)$, let $\lambda_1 + \lambda_2 = s$ and $\lambda_1 \cdot \lambda_2 = l$ in this case, then (42) can be of the form

$$Z(\lambda_1, \lambda_2) = N_1(t,m) + N_2(t,m)s + (N_3(t,m) + N_4(t,m))s^2 - 2N_3(t,m)l = V(s,l), \tag{44}$$

where $s \in [0,2]$ and $l \in [0,1]$. Now, we need to investigate the maximum of

$$V(s,l) \in \Lambda = \{(s,l) : s \in [0,2], l \in [0,1]\}. \tag{45}$$

By differentiating $V(s,l)$ partially, we have

$$\begin{aligned} \frac{\partial V}{\partial s} &= N_2(t,m) + 2(N_3(t,m) + N_4(t,m))s = 0, \\ \frac{\partial V}{\partial l} &= -2N_3(t,m) = 0. \end{aligned}$$

These results reveal that $V(s,l)$ does not have a critical point in Λ, and so $Z(\lambda_1, \lambda_2)$ does not have a critical point in the square Δ.

As a result, the function $Z(\lambda_1, \lambda_2)$ cannot have its maximum value in the interior of Δ. The maximum of $Z(\lambda_1, \lambda_2)$ on the boundary of the square Δ will be investigated next.

For $\lambda_1 = 0, \lambda_2 \in [0,1]$ (also, for $\lambda_2 = 0, \lambda_1 \in [0,1]$) and

$$Z(0, \lambda_2) = N_1(t,m) + N_2\beta_2 + (N_3(t,m) + N_4(t,m))\lambda_2^2 = Q(\lambda_2). \tag{46}$$

Now, since $N_3(t,m) + N_4(t,m) \geq 0$, then we have

$$Q'(\lambda_2) = N_2(t,m) + 2[N_3(t,m) + N_4(t,m)]\lambda_2 > 0,$$

which implies that $Q(\beta_2)$ is an increasing function. Therefore, for a fixed $m \in [0,2)$ and $t \in (1/2, 1]$, the maximum occurs at $\lambda_2 = 1$. Thus, from (46),

$$\max\{G(0, \lambda_2) : \lambda_2 \in [0,1]\} = Z(0,1)$$

$$= N_1(t,m) + N_2(t,m) + N_3(t,m) + N_4(t,m). \tag{47}$$

For $\lambda_1 = 1, \lambda_2 \in [0,1]$ (also, for $\lambda_2 = 1, \lambda_1 \in [0,1]$) and

$$Z(1,\lambda_2) = N_1(t,m) + N_2(t,m) + N_3(t,m) + N_4(t,m) + [N_2(t,m) + 2N_4(t,m)]\lambda_2 + [N_3(t,m) + N_4(t,m)]\lambda_2^2 = D(\lambda_2), \tag{48}$$

$$D'(\lambda_2) = [N_2(t) + 2N_4(t)] + 2[N_3 + N_4]\lambda_2. \tag{49}$$

We know that $N_3(t) + N_4 \geq 0$, then

$$D'(\lambda_2) = [N_2(t) + 2N_4(t)] + 2[N_3 + N_4]\lambda_2 > 0.$$

Therefore, the function $D(\lambda_2)$ is an increasing function and the maximum occurs at $\lambda_2 = 1$. From (48), we have

$$\max\{Z(1,\lambda_2) : \lambda_2 \in [0,1]\} = Z(1,1) = N_1(t,m) + 2[N_2(t,m) + N_3(t,m)] + 4N_4(t,m). \tag{50}$$

Hence, for every $m \in (0,2)$, taking it from (47) and (50), we have

$$N_1(t,m) + 2[N_2(t,m) + N_3(t,m)] + 4N_4(t,m) > N_1(t,m) + N_2(t,m) + N_3(t,m) + N_4(t,m).$$

Therefore,

$$\max\{Z(\lambda_1,\lambda_2) : \lambda_1 \in [0,1], \beta\lambda_2 \in [0,1]\} = N_1(t,m) + 2[N_2(t,m) + N_3(t,m)] + 4N_4(t,m).$$

Since,

$$Q(1) \leq D(1) \quad \text{for } m \in [0,2] \text{ and } t \in [1,1],$$

then

$$\max\{Z(\lambda_1,\lambda_2)\} = Z(1,1)$$

occurs on the boundary of square Δ.

Let $T : (0,2) \to \mathbb{R}$ defined by

$$T(m,t) = \max\{Z(\lambda_1,\lambda_2)\} = Z(1,1) = N_1(t,m) + 2N_2(t,m) + 2N_3(t,m) + 4N_4(t,m). \tag{51}$$

Now, inserting the values of $N_1(t,m), N_2(t,m), N_3(t,m)$ and $N_4(t,m)$ into (51) and with some calculations, we have

$$T(m,t) = \left\{ \frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t) - 2G_2^{(\upsilon)}(t) + G_3^{(\upsilon)}(t)]\gamma^2}{16[2]_q[4]_q(1+\beta[2]_q)(1+\beta)} m^4 \right. + \frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t) + G_1^{(\upsilon)}(t)]\gamma^2}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m^4 + \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} m^4$$

$$+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m+\frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{16[2]_q^4(1+\beta)^4}m^4\Bigg\}$$

$$+\Bigg\{\frac{G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t)+G_1^{(\upsilon)}(t)]\gamma^2(4-m^2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^2$$

$$+\frac{(G_1^{(\upsilon)}(t))^3\gamma^3(4-m^2)}{32[2]_q^2[3]_q(1+\beta[2]_q)(1+\beta)^2}m^2+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^2\Bigg\}$$

$$+\Bigg\{\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{16[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m^2$$

$$-\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)}{8[2]_q[4]_q(1+\beta[3]_q)(1+\beta)}m\Bigg\}+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2(4-m^2)^2}{16(1+\beta[2]_q)^2[3]_q^2}.$$

By simplifying, we have

$$T(m,t)=\frac{B_1}{32[2]_q^4[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta)^4(1+\beta[2]_q)^2}m^4 \tag{52}$$

$$+\frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{(1+\beta[2]_q)^2[3]_q}+\frac{B_2}{8[2]_q^2[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta)^2(1+\beta[2]_q)^2}m^2, \tag{53}$$

where

$$B_1=G_1^{(\upsilon)}(t)\Big[2[G_1^{(\upsilon)}(t)-2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t)]\gamma^2(1+\beta[2]_q)[2]_q^3$$
$$\cdot(1+\beta)^3(1+\beta[3]_q)[3]_q^3+2(G_1^{(\upsilon)}(t))^3(1+\beta[2]_q)^2[4]_q(1+\beta[3]_q)[3]_q^3$$
$$-4G_1^{(\upsilon)}(t)\gamma^2(1+\beta[2]_q)^2[2]_q^3(1+\beta)^3[3]_q^3+2G_1^{(\upsilon)}(t)\gamma^2$$
$$[2]_q^4[4]_q(1+\beta)^4(1+\beta[3]_q)$$
$$-(G_1^{(\upsilon)}(t))^2(1+\beta[2]_q)^2[2]_q^2[3]_q(1+\beta)^2[4]_q(1+\beta[3]_q)\Big],$$

$$B_2=G_1^{(\upsilon)}(t)\Big[4[G_2^{(\upsilon)}(t)-G_1^{(\upsilon)}(t)]\gamma^2[2]_q(1+\beta)(1+\beta[2]_q)^2[3]_q^2$$
$$-4G_1^{(\upsilon)}(t)\gamma^2[2]_q^2(1+\beta[3]_q)[4]_q+6G_1^{(\upsilon)}(t)\gamma^2[2]_q[3]_q^2(1+\beta)(1+\beta[2]_q)^2$$
$$+(G_1^{(\upsilon)}(t))^2[3]_q(1+\beta[2]_q)(1+\beta[3]_q)[4]_q\Big].$$

If $T(m,t)$ has a maximum value in the interior of $m\in[0,2]$ and by applying some elementary calculus, we have

$$T'(m,t)=\frac{B_1}{8[2]_q^4[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta)^4(1+\beta[2]_q)^2}m^3$$
$$+\frac{B_2}{4[2]_q^2[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta)^2(1+\beta[2]_q)^2}m.$$

Now, we need to examine the sign of the function $T'(m,t)$ depending on the signs of B_1 and B_2 as follows:

First Result: Suppose $B_1 \geq 0$ and $B_2 \geq 0$ then,

$T'(m,t) \geq 0$. This shows that $T(m,t)$ is an increasing function on the boundary of $m \in [0,2]$ that is $m = 2$. Therefore,

$$\begin{aligned}\max\{T(m,t) : m \in (0,2)\} &= \frac{G_1^{(\upsilon)}(t)[G_1^{(\upsilon)}(t) - 2G_2^{(\upsilon)}(t) + G_3^{(\upsilon)}(t)]\gamma^2}{[2]_q[4]_q(1+\beta[2]_q)(1+\beta)} \\ &+ \frac{2G_1^{(\upsilon)}(t)[G_2^{(\upsilon)}(t) + G_1^{(\upsilon)}(t)]\gamma^2}{[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} \\ &+ \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{[2]_q[4]_q(1+\beta[3]_q)(1+\beta)} + \frac{(G_1^{(\upsilon)}(t))^4\gamma^4}{16[2]_q^4(1+\beta)^4}.\end{aligned}$$

Second Result: If $B_1 > 0$ and $B_2 < 0$, then

$$T'(m,t) = \frac{B_1 m^3 + 2[2]_q^2(1+\beta)^2 m B_2}{8[2]_q^4[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta)^4(1+\beta[2]_q)^2} m^3 = 0 \tag{54}$$

at critical point

$$m_0 = \sqrt{\frac{-2[2]_q^2(1+\beta)^2 B_2}{B_2}} \tag{55}$$

is a critical point of the function $T(m,t)$. Now,

$$\begin{aligned}T''(m_0) &= \frac{-3B_2}{4[2]_q^2[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta)^2(1+\beta[2]_q)^2} m^3 \\ &+ \frac{B_2}{4[2]_q^2[4]_q[3]_q(1+\beta[3]_q)(1+\beta)^2(1+\beta[2]_q)^2} m^3.\end{aligned}$$

Therefore, m_0 is the minimum point of the function $T(m,t)$. Hence, $T(m,t)$ cannot have a maximum.

Third Result: If $B_1 \leq 0$ and $B_2 \leq 0$, then

$$T'(m,t) \leq 0.$$

Therefore, $T(m,t)$ is a decreasing function on the interval $(0,2)$. Hence,

$$\max\{T(m,t) : m \in (0,2)\} = T(0) = \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{(1+\beta[2]_q)^2[3]_q^2}. \tag{56}$$

Fourth Result: If $B_1 < 0$ and $B_2 > 0$

$$\begin{aligned}T''(m_0,t) &= \frac{-3B_2}{4[2]_q^2[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta)^2(1+\beta[2]_q)^2} m^3 \\ &+ \frac{B_2}{4[2]_q^2[4]_q[3]_q(1+\beta[3]_q)(1+\beta)^2(1+\beta[2]_q)^2} m^3\end{aligned}$$

$$< 0.$$

Therefore, $T''(m,t) < 0$. Hence, m_0 is the maximum point of the function $T(m,t)$ and the maximum value occurs at $m = m_0$. Thus,

$$\max\{T(m,t) : m \in (0,2)\} = T(m_0,t),$$

$$T(m_0,t) = \frac{(G_1^{(\upsilon)}(t))^2\gamma^2}{(1+\beta[2]_q)^2[3]_q} + \frac{B_2^2}{8[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta[2]_q)^2 B_1} - \frac{B_2^2}{4[4]_q[3]_q^2(1+\beta[3]_q)(1+\beta[2]_q)^2 B_1}. \quad \blacksquare$$

4. Conclusion

Many researchers have recently started investigating bi-univalent functions related to orthogonal polynomials as described in the introduction section. But as far as we know, there is minimal work-related with bi-univalent functions in the literatures for the Gegenbauer polynomial. In our present study make used of certain q-derivative operator and Gegenbauer polynomials, we have first defined a new subclass of analytic and bi-univalent functions. We have then obtained certain coefficients bounds, the Fekete–Szegö inequalities and upper bounds for the second-order Hankel determinant for our defined functions class.

Disclosure statement

The authors declare that they have no competing interest(s).

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Author’s contributions

All authors jointly worked on the results, and they read and approved the final manuscript

Data availability statement

No data were used to support this study.

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