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The *n*-player Hirshleifer contest ☆

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ABSTRACT

While the game-theoretic analysis of conflict is often based on the assumption of multiplicative noise, additive noise such as considered by Hirshleifer (1989) may be equally plausible depending on the application. In this paper, we examine the equilibrium set of the *n*-player difference-form contest with heterogeneous valuations. For high and intermediate levels of noise, the equilibrium is in pure strategies, with at most one player being active. For small levels of noise, however, we find a variety of equilibria in which some but not necessarily all players randomize. In the case of homogeneous valuations, we obtain a partial uniqueness result for symmetric equilibria. As the contest becomes increasingly decisive, at least two contestants bid up to the valuation of the second-ranked contestant, while any others ultimately drop out. Thus, in the limit, equilibria of the Hirshleifer contest share important properties of equilibria of the corresponding all-pay auction.

1. Introduction

Recent years have witnessed a tremendous surge in interest in the game-theoretic analysis of conflict.[1] Much of this interest has focused on specific classes of contest technologies that admit both a plausible axiomatic characterization and a stochastic foundation. Among those, the technologies introduced by Tullock (1980) and Hirshleifer (1989) figure most prominently as the canonical representatives of the respective classes of ratio-form and difference-form contests.[2]

The Tullock contest is analytically convenient because its technology is homogeneous of degree zero, i.e., what matters for the probability of winning is the ratio of efforts. Moreover, at least two agents are active in any equilibrium, which is certainly appealing in the analysis of conflict. An undesirable implication of that assumption, however, is that a zero bid never wins against any bid that is only slightly above zero. The difference-form contest, on the other hand, has often been criticized on the grounds that its technology is, by definition, unresponsive if all efforts are raised by the same amount, even if that brings efforts on a similar level in

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[1] For an introduction to the theory of contests, see Konrad (2009). A more recent survey is by Corchón and Serena (2018).

[2] While the definitions may vary somewhat across papers, the basic idea is that, in a *ratio-form* (*difference-form*) contest, the odds of winning depend on the pairwise ratios (differences) of inputs alone. See Skaperdas (1996) and Ewerhart (2015a) for axiom systems applying to a population of varying and constant size, respectively. Stochastic foundations of the Tullock contest in terms of multiplicative noise are due to Hirshleifer and Riley (1992) and Jia (2008). See also Jia and Skaperdas (2012) and Jia et al. (2013). An analogous foundation of the Hirshleifer contest in terms of additive noise is known as the multinomial discrete choice model (with i.i.d. Gumbel distributed error terms). See, e.g., Anderson et al. (1992).

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relative terms. Notwithstanding, Hirshleifer's technology is arguably more suitable than the ratio form in some applications, e.g., for modeling military conflict. This is due to two remarkable features of the difference-form model, viz. a positive probability of winning despite exerting zero effort and increasing returns to marginal effort up to the inflection point where the probability of winning is just one half. Indeed, these features constituted the major motivation for Hirshleifer (1989, pp. 103-104) to develop an alternative to the ratio-form model.[3]

This paper examines the equilibrium set of the n-player Hirshleifer contest with heterogeneous valuations. To discuss the findings, it proves useful to organize the analysis along the decisiveness of the contest, i.e., along the level of noise in the contest technology. For *high and intermediate levels of noise*, we find pure-strategy Nash equilibria (PSNE) of two types. In one type of equilibrium, all players exert zero effort. We will refer to this type of equilibrium as "multilateral peace." Indeed, when the contest technology is not very sharp, incentives for the exertion of effort are weak, making inactivity the optimal choice. In another type of PSNE, one contestant chooses a positive level of effort, while all others remain inactive. That type of equilibrium will be referred to as "one-sided dominance." The intuition here is as follows. As the technology becomes more decisive, contestants with relatively high valuations (or, equivalently, relatively low marginal cost of effort) have an incentive to become active. However, once a single contestant grasps this opportunity, the incentives for the other contestants are often weakened, so that they optimally choose to remain inactive. Notably, however, the identity of the dominating player need not be determined by the ranking of the valuations alone, which implies the possibility of multiple, payoff-inequivalent equilibria when $n \geq 3$. These results, which amount to a fairly complete characterization of the set of PSNE, confirm and extend the intuitions derived in prior analyses of the case of two contestants in the literature.

Once the noise in the contest technology reaches a *low level*, bidding slightly more than a competitor may ensure a win with high probability. Then, additional contestants may choose to become active, overbidding any previously dominating player. However, in contrast to the case of multiplicative noise, at most one contestant may be active with probability one. Therefore, there is no equilibrium in pure strategies anymore. Instead, either one player randomizes, or several players randomize. That is, we obtain mixed-strategy Nash equilibria (MSNE). Moreover, given the smoothness properties of Hirshleifer's technology, contestants randomize over a finite set of strategies, where the bids and the probabilities with which those are used are jointly determined by a system of marginal and indifference conditions. The following results are obtained. In the case of homogeneous valuations, considering symmetric equilibria, the number of bids over which contestants randomize is bound to increase, and the equilibrium payoff is bound to decline, as the noise becomes smaller. We also obtain a partial uniqueness result in that case, by combining ideas from the literature on zero-sum games with concepts from the theory of monotone comparative statics. Allowing for asymmetric equilibria, however, the equilibrium set becomes convoluted even in the case of homogeneous valuations. For example, over a non-degenerate set of the parameter space, some players may remain inactive while others randomize. We present a selection of numerical examples that illustrate the large variety of MSNE that the model admits. In the case of heterogeneous valuations, we derive a general inequality that relates the cardinalities of equilibrium supports across contestants and apply it to obtain a structural result in the case where all contestants but one use a pure strategy.

Next, we study the case of *vanishing noise*, in which the contest technology approaches that of the standard all-pay auction. It is shown that, as the technology becomes increasingly deterministic, at least two contestants become heavily engaged in the sense that they bid, with positive probability, arbitrarily close to the valuation of the second-ranked contestant. Therefore, the undissipated rent goes to zero for all but the single contestant of the highest valuation (if any). Moreover, contestants that are not heavily engaged ultimately become inactive in the limit. Finally, if the two strongest contestants have equal valuations, then they both become active with probability arbitrarily close to one. This collection of necessary properties shows that MSNE in n-player Hirshleifer contests with arbitrarily small noise share important properties with MSNE of the corresponding all-pay auction.

The analysis is complemented by an investigation of the case of large populations. Specifically, keeping the decisiveness parameter fixed, we find that, if the number of contestants n grows sufficiently large while valuations remain bounded, then the unique MSNE in the n-player Hirshleifer contest is multilateral peace. This observation contrasts, in particular, with the case of the symmetric n-player Tullock contest where, regardless of n, all contestants are active in the unique, symmetric PSNE (Corcoran, 1988).

General classes of difference-form contests, which include the Hirshleifer contest as a special case, have been analyzed for somewhat more than two decades (Baik, 1998). Assuming a uniform distribution of noise, Che and Gale (2000) were the first in comprehensively characterizing MSNE for a class of two-player contests of the difference form. More recently, Cubel and Sanchez-Pages (2020) have generalized that analysis by allowing for more than two contestants and more flexible difference-form contests.[4] However, none of those papers touches upon the questions addressed in the present study. In his seminal contribution, Hirshleifer (1989) identified two main types of PSNE between two contestants, viz. bilateral peace and one-sided dominance. He also offered an informal discussion of MSNE for the case of two contestants. Finally, when introducing the n-player generalization, he noted the equivalence of representations (1) and (2) below. However, his analysis is silent on equilibria in the n-player case. In earlier work (Ewerhart and Sun, 2018), we showed that the two-player Hirshleifer contest with homogeneous valuations generally admits a unique MSNE. We also provided an explicit characterization of the equilibrium, which is necessarily symmetric. The case of heterogeneous valuations, still with two players, has been considered by Ewerhart (2021), who showed, in particular, that the MSNE is unique in

[3] For further discussion of the relative merits of ratio-form and difference-form contests, see Hirshleifer (1989, 1995, 2000), Mueller (2003, p. 342), and Garfinkel and Skaperdas (2007, pp. 655-656).

[4] See also Cubel and Sanchez-Pages (2016), who axiomatized difference-form technologies more generally.

that case. However, as far as we know, the equilibrium set of the Hirshleifer contest with more than two contestants has not been studied so far.

A technically important dimension in which the Hirshleifer contest differs from the Tullock contest is that equilibria for two contestants and homogeneous valuations cannot be used for constructing equilibria in the case of n contestants and heterogeneous valuations. As noted by Alcalde and Dahm (2010) for the ratio-form model, having one of two active contestants exert zero effort with positive probability is equivalent to lowering the valuation of the other active player. Based on this observation, so-called *all-pay auction equilibria* may be constructed in generalized Tullock models. That trick, however, does not work in models with additive noise. Indeed, while the marginal return from raising a positive bid against a profile of zero bids vanishes in a Tullock contest, this is never the case in a Hirshleifer contest.

The remainder of the paper is structured as follows. Section 2 contains preliminaries. Section 3 concerns high and intermediate levels of noise, while Section 4 concerns low levels of noise. The limit case of vanishing noise is dealt with in Section 5. Section 6 discusses the case of large populations. Section 7 concludes. All technical proofs are relegated to an Appendix.

2. Preliminaries

2.1. Set-up and notation

There are $n \geq 2$ contestants (or players), collected in a set $N = \{1, \ldots, n\}$, that exert effort to win a single indivisible prize. Contestant i's valuation of the prize is denoted by $V_i > 0$. Thus, we allow for heterogeneous valuations of the prize. Without loss of generality, valuations will be ordered by magnitude, i.e., we assume throughout that $V_1 \geq \ldots \geq V_n > 0$. Contestant i's expected payoff in the *n-player Hirshleifer contest* is given as

$$\Pi_i(x_1, \ldots, x_n) = \frac{V_i}{\sum_{j=1}^{n} \exp(\alpha(x_j - x_i))} - x_i, \tag{1}$$

where $x_j \geq 0$, for $j \in N$, denotes contestant j's effort (or bid), and the parameter $\alpha > 0$ measures the *decisiveness* of the contest technology. It is easy to see that, as $\alpha \to 0$, the contest converges to the limit case of a pure lottery, where decisions about expenses do not matter and the winner is determined by chance alone. As $\alpha \to \infty$, however, the vector of payoffs approximates that of the standard all-pay auction, where the highest bidder wins with certainty (Baye et al., 1990, 1996). Thus, chance plays a larger (smaller) role in the determination of the winner when α is smaller (larger).

Rewriting relationship (1), one obtains the equivalent logit representation of contestant i's expected payoff as

$$\Pi_i(x_1, \ldots, x_n) = \frac{\exp(\alpha x_i) V_i}{\sum_{j=1}^{n} \exp(\alpha x_j)} - x_i. \tag{2}$$

It is noteworthy that the impact function $x_i \mapsto X_i \equiv \exp(\alpha x_i)$ exhibits strictly increasing returns, i.e., it is strictly convex, for all values of α.[5]

As usual, a *pure-strategy Nash equilibrium (PSNE)* is a vector of bids, $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n_{\geq 0}$, such that $\Pi_i^* \equiv \Pi_i(x_i^*, x_{-i}^*) \geq \Pi_i(x_i, x_{-i}^*)$ holds for any $i \in N$ and $x_i \in \mathbb{R}_{\geq 0}$, where we adhere to the convention that $x^* = (x_i^*, x_{-i}^*)$, etc.

We will also allow for equilibria in randomized strategies. By a *mixed strategy* for contestant i, we mean a probability measure μ_i on (the Borel subsets of) the interval $[0, V_i]$. Note that the upper bound is introduced without loss of generality because any effort level weakly exceeding a player's valuation is strictly dominated by the zero bid. Let M_i denote the set of mixed strategies for contestant i, where pure strategies $x_i \in [0, V_i]$ correspond to Dirac probability measures, as usual. Contestant i's expected payoff from a mixed-strategy profile $\mu = (\mu_i, \mu_{-i}) \in M \equiv M_1 \times \ldots \times M_n$ will be written as $E_{(\mu_i, \mu_{-i})}[\Pi_i(x_i, x_{-i})]$. A *mixed-strategy Nash equilibrium (MSNE)* is then a tuple $\mu^* = (\mu_1^*, \ldots, \mu_n^*) \in M$ such that $E_{(\mu_i^*, \mu_{-i}^*)}[\Pi_i(x_i, x_{-i})] \geq E_{(\mu_i, \mu_{-i}^*)}[\Pi_i(x_i, x_{-i})]$ holds true for any $i \in N$ and $\mu_i \in M_i$. The existence of an equilibrium in randomized strategies is always guaranteed.

Lemma 1 *(Existence). For any $n \geq 2$, the n-player Hirshleifer contest with parameter $\alpha > 0$ and valuations $V_1 \geq \ldots \geq V_n > 0$ admits a MSNE μ^*.*

Proof. See the Appendix. □

2.2. An initial observation

We prepare the main analysis by making a general observation. We say that contestant $i \in N$ is *active* (*always active, inactive*) in a mixed-strategy profile $\mu \in M$ if her strategy μ_i employs positive bids with positive probability (with probability one, with probability zero). Thus, contestant i is active (always active, inactive) if and only if $\mu_i^*((0, V_i]) > 0$ $(= 1, = 0)$.

[5] In contests between more than two parties, increasing returns suggest a motive to form alliances. However, as discussed by Garfinkel and Skaperdas (2007, Sec. 7), the prospect of free-riding and intragroup conflict might render such alliances instable. Further, if alliances are stable, then parties might be able to avoid the contest altogether (by forming a grand coalition).

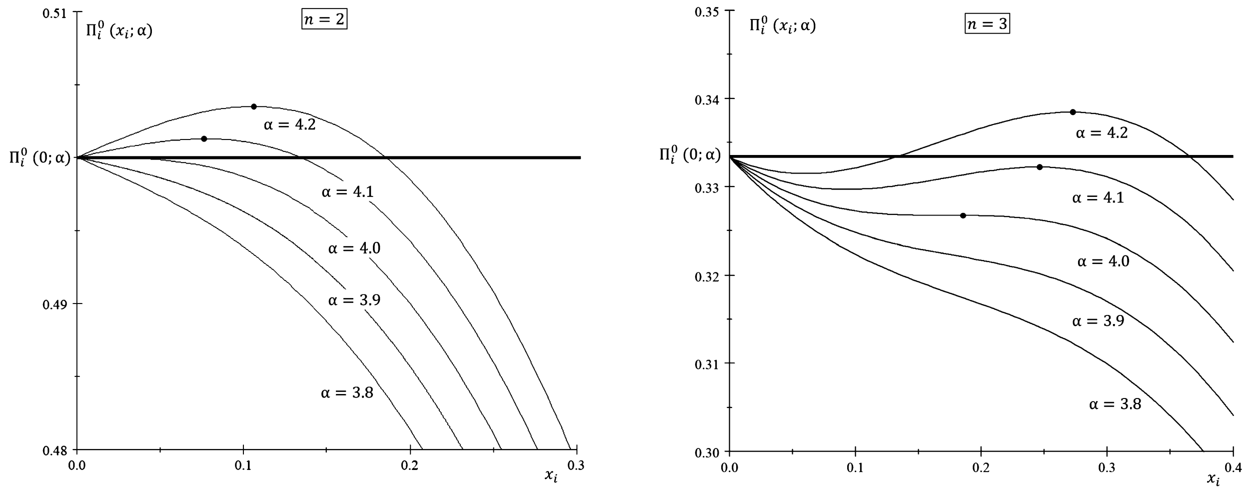

Fig. 1. Expected payoff against inactive opponents.

Lemma 2 *(Activity). At most one contestant is always active.*

Proof. See the Appendix. □

Thus, in any equilibrium, either all contestants choose the zero bid with positive probability, or there is precisely one contestant that is always active. In fact, both cases are possible, as will be seen below. The lemma thereby captures a dimension in which the equilibrium prediction for the Hirshleifer contest crucially differs from that for the Tullock contest.

Lemma 2 is obtained from an analysis of optimality conditions at small positive effort levels. As noted above, marginal returns are strictly increasing up to the inflection point where the probability of winning equals one half. This property of the contest technology is shown to imply that the second-order condition necessary for optimality cannot hold at small positive bids if two or more contestants are always active. Therefore, at most one contestant can be always active. Clearly, this argument also suggests that the conclusion of Lemma 2 is not restricted to the Hirshleifer contest but holds more generally for any difference-form contest with a sufficiently well-behaved distribution of noise.

3. High and intermediate levels of noise (leading to pure-strategy equilibria)

This section deals with the case where the decisiveness parameter α is relatively small, i.e., there is a substantial degree of noise in the contest technology, resulting in PSNE. Incentives may be so weak that no contestant bothers to exert positive effort (Subsection 3.1). Alternatively, with a bit less noise, one contestant may dominate all other contestants (Subsection 3.2). The findings are discussed and compared to the case of the Tullock contest (Subsection 3.3). Finally, we offer some illustrations (Subsection 3.4).

3.1. Multilateral peace

A PSNE x^* reflects *multilateral peace* if $x_1^* = \ldots = x_n^* = 0$. By Lemma 2, multilateral peace is the only symmetric PSNE feasible. To understand the conditions for this to be an equilibrium, suppose that all opponents $j \neq i$ of contestant i choose an effort of zero. Then, contestant i's expected payoff is given by

$$\Pi_i^0(x_i;\alpha) \equiv \Pi_i(x_i, \mathbf{0}_{n-1}) = \frac{V_i}{1+(n-1)\exp(-\alpha x_i)} - x_i, \tag{3}$$

where $\mathbf{0}_{n-1} = (0,\ldots,0) \in \mathbb{R}^{n-1}$. Fig. 1 outlines the shape of $\Pi_i^0(x_i;\alpha)$ as a function of the bid x_i for $n = 2$ (left panel) and $n = 3$ (right panel), where we normalized the valuation to $V_i = 1$.[6] The figure suggests that, for any given n and V_i, exerting zero effort is optimal for contestant i if and only if α is sufficiently small, i.e., if and only if there is enough noise.

This is indeed the case. A straightforward examination of marginal payoffs shows that $\Pi_i^0(\cdot;\alpha)$ is strictly declining for $\alpha \leq 4/V_i$.[7] For $\alpha > 4/V_i$, however, there is a unique interior local maximum $\widetilde{x}_i(\alpha) > 0$. In that case, an application of the envelope theorem shows that the payoff at the interior local maximum, $\Pi_i^0(\widetilde{x}_i(\alpha);\alpha)$, is strictly increasing in α. Therefore, there is a threshold value $\alpha_i^* \geq 4/V_i$, so that bidding zero is optimal for contestant i if and only if $\alpha \leq \alpha_i^*$ (with indifference to $\widetilde{x}_i(\alpha)$ at $\alpha = \alpha_i^*$ if $n \geq 3$). The

6 The graphs for higher n look very much like that for $n = 3$.

7 As suggested by the right panel, for $\alpha = 4/V_1$ and $n \geq 3$, there is a saddle point in the interior, i.e., marginal payoffs vanish at a positive effort level. Note, however, that $\Pi_i^0(\cdot;\alpha)$ is strictly declining also in that case.

equilibrium property then obviously hinges on the minimum of contestants' thresholds, $\alpha^* = \min_{i \in N} \alpha_i^* = \alpha_1^*$, where the pivotality of contestant 1's optimality condition should be intuitively plausible. Arguing along these lines, we arrive at the following result.

Proposition 1 *(Multilateral peace). Consider an n-player Hirshleifer contest with valuations $V_1 \geq \ldots \geq V_n > 0$. Then, there is a threshold value $\alpha^* \equiv \alpha^*(n, V_1) \geq 4/V_1$ such that the following holds true:*

(i) *Multilateral peace (i.e., $x_1^* = \ldots = x_n^* = 0$) is an equilibrium if and only if $\alpha \leq \alpha^*$;*
(ii) *in that case, contestant i's equilibrium payoff equals $\Pi_i^* = V_i/n$, for any $i \in N$;*
(iii) *multilateral peace is the unique equilibrium (even within the set of MSNE) if $\alpha < \alpha^*$;*[8]
(iv) *$\alpha^*(n, V_1)$ is strictly increasing in n, and strictly declining in V_1.*

Proof. See the Appendix. □

Thus, for any given number of contestants and any given highest valuation, there is a critical value of the decisiveness parameter α such that multilateral peace is an equilibrium if and only if α remains weakly below that value. For $n = 2$, we have $\alpha^* = 4/V_1$, so that we retrieve Hirshleifer's (1989) classic observation that bilateral peace is a PSNE if and only if $\alpha \leq 4/V_1$. For $n \geq 3$, the threshold value α^* may be characterized as the unique solution $\alpha > 4/V_1$ of the indifference relationship

$$\Pi_1^0(0; \alpha) = \Pi_1^0(\widetilde{x}_1(\alpha); \alpha). \tag{4}$$

An inspection of equation (4) reveals that $\alpha^*(n, V_1) = a(n)/V_1$, where $a(n)$ depends on the number of contestants alone. In particular, as stated in part (iv) of the proposition, α^* is strictly declining in V_1, i.e., multilateral peace becomes less likely as the highest valuation for the contested object rises. The proof of Proposition 1 shows that $a(n) \in (4, \frac{n^2}{n-1})$ for $n \geq 3$. Moreover, $a(n)$ is strictly increasing in n, as follows from part (iv). For instance, $a(3) \approx 4.12$, $a(4) \approx 4.29$, $a(5) \approx 4.46$, etc.[9] Thus, the entry of an additional contestant, unless stronger than contestant 1, makes it easier to sustain multilateral peace. We will elaborate on this point later in the paper.[10]

3.2. One-sided dominance

Multilateral peace breaks down as an equilibrium when $\alpha > \alpha^*$. Then, some relatively strong contestant $i \in N$, in particular the strongest contestant $i = 1$, has an incentive to deviate to $x_i = \widetilde{x}_i(\alpha) > 0$. Once contestant i switches to that positive bid level, however, the incentive for the other contestants to become active is often softened. Thus, a new type of equilibrium arises.

A PSNE x^* reflects *one-sided dominance* by contestant i if $x_i^* > 0$ while $x_{-i}^* = 0$. Our next result characterizes the conditions under which one-sided dominance is an equilibrium.

Proposition 2 *(One-sided dominance). Consider an n-player Hirshleifer contest with valuations $V_1 \geq \ldots \geq V_n > 0$. Then, there is a contestant $i^* \in N$, as well as threshold values $\alpha_i^{**} \equiv \alpha^{**}(n, V_i, \max_{j \neq i} V_j) \geq \alpha_i^*$, for $i \in \{1, \ldots, i^*\}$, such that the following holds true:*

(i) *For $n \geq 3$, one-sided dominance by contestant $i \in \{1, \ldots, i^*\}$ is an equilibrium if and only if $\alpha \in [\alpha_i^*, \alpha_i^{**}]$; for $n = 2$, one-sided dominance by contestant $i^* = 1$ is an equilibrium if and only if $V_1 > V_2$ and $\alpha \in (\alpha_1^*, \alpha_1^{**}]$;*
(ii) *if contestant $i \in \{1, \ldots, i^*\}$ dominates the other contestants, then equilibrium efforts are given by $x_i^* = \widetilde{x}_i(\alpha)$ with*

$$\widetilde{x}_i(\alpha) = \frac{1}{\alpha} \ln\left(\frac{n-1}{2}\left\{\alpha V_i - 2 + \sqrt{\alpha V_i(\alpha V_i - 4)}\right\}\right), \tag{5}$$

and by $x_j^* = 0$ for all $j \in N \backslash \{i\}$, while expected payoffs are given as

$$\Pi_i^* = \left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\alpha V_i}}\right) V_i - \widetilde{x}_i(\alpha), \tag{6}$$

$$\Pi_j^* = \frac{1}{n-1}\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\alpha V_i}}\right) V_j \qquad (j \in N \backslash \{i\}); \tag{7}$$

(iii) *$\alpha^{**}(n, V_i, \max_{j \neq i} V_j)$ is strictly increasing in n and V_i, as well as strictly declining in $\max_{j \neq i} V_j$; in particular, $\alpha_1^* \leq \ldots \leq \alpha_{i^*}^* \leq \alpha_{i^*}^{**} \leq \ldots \leq \alpha_1^{**}$, with $\alpha_i^* < \alpha_{i+1}^*$ and $\alpha_{i+1}^{**} < \alpha_i^{**}$ if $V_{i+1} < V_i$, for any $i < i^*$; moreover, $\alpha_1^* < \alpha_1^{**}$ (unless $n = 2$ and $V_1 = V_2$);*

[8] The equilibrium set for $\alpha = \alpha^*$ depends on n. For $n = 2$, bilateral peace can still be seen to be the unique MSNE via a straightforward generalization of the proof. For $n \geq 3$, however, there are additional equilibria, both asymmetric PSNE (cf. Proposition 2) and a continuum of MSNE (cf. Proposition 6).

[9] If n is considered a continuous parameter, then $a(n)$ may alternatively be characterized as the unique solution of a simple initial value problem. For details, see footnote (29) in the Appendix.

[10] Multilateral peace is, in fact, an equilibrium in strictly dominant strategies if either $n = 2$ or $\alpha \in (0, \alpha^*)$. Indeed, in those cases, exerting zero effort is the unique best response not only to $\mathbf{0}_{n-1}$, but also to any other strategy profile. This is obvious for $\alpha \leq 4/V_1$, and follows from relationship (19) in the Appendix for $\alpha > 4/V_1$. If $n \geq 3$ and $\alpha = \alpha^*$, however, then multilateral peace is merely an equilibrium in weakly dominant strategies.

(iv) *there are no PSNE other than those identified by Proposition 1 and part (i) above.*

Proof. See the Appendix. □

Part (i) says that there is a contestant i^* such that, for any $i \in \{1, \dots, i^*\}$, there is a nonempty interval $[\alpha_i^*, \alpha_i^{**}]$ of values for the decisiveness parameter α such that, if $n \geq 3$, one-sided dominance by contestant i is an equilibrium if and only if α lies in that interval. If $n = 2$, however, then there are several particularities. First, one-sided dominance is an equilibrium only if $V_1 > V_2$, i.e., it is inconsistent with homogeneous valuations.[11] Second, one-sided dominance is necessarily by contestant $i^* = 1$. We will see below that this is not true for general n. Finally, that type of equilibrium obtains if and only if $\alpha \in (\alpha_1^*, \alpha_1^{**}]$. Indeed, in the exceptional case where $n = 2$ and $\alpha = \alpha_1^*$, contestant 1's payoff function $\Pi_1^0(\cdot; \alpha)$ against $x_{-1} = \mathbf{0}_{n-1}$ has a unique global maximum at the zero bid (as illustrated in Fig. 1), so that one-sided dominance by contestant 1 is not an equilibrium in that case. In general, the interpretation of the boundaries of the interval $[\alpha_i^*, \alpha_i^{**}]$ is as follows. For $\alpha < \alpha_i^*$, contestant i would prefer to bid zero. On the other hand, for $\alpha > \alpha_i^{**}$, one of the strongest contestants in the set $N \backslash \{i\}$ would find it profitable to overbid contestant i. In the proof, we show that α_i^{**} is well-defined for any $i \leq i^*$, where we rely on an argument similar to the one used in the previous subsection to identify α_i^*. Note, however, that there is a countervailing effect here. Indeed, an increase in α not only renders overbidding of any standing bid less costly, but also unambiguously raises the impact $\widetilde{X}_i(\alpha) = \exp(\alpha \widetilde{x}_i(\alpha))$ of the dominating bid $\widetilde{x}_i(\alpha)$. We show that the first effect is always stronger than the second at any critical level of α where the strongest competitor of contestant i (or one of the strongest competitors if there are several with equal valuations) is just indifferent between staying inactive and overbidding contestant i. This single-crossing property ensures then the existence of the threshold α_i^{**}. Clearly, each $\alpha_i^{**} = \alpha^{**}(n, V_i, \max_{j \neq i} V_j)$ is a function of n, V_i, and $\max_{j \neq i} V_j$ alone, because the optimality condition of the strongest competitor of contestant i is pivotal for one-sided dominance by i to be an equilibrium. To illustrate, suppose that $V_i = \max_{j \neq i} V_j = 1$, i.e., that the respective valuations of the dominating contestant and the pivotal contestant are identical as well as normalized. Then, $\alpha^{**}(3) \approx 4.66$, $\alpha^{**}(4) \approx 5.21$, $\alpha^{**}(5) \approx 5.61$, etc. We also mention that $\alpha^{**}(n, V_i, \max_{j \neq i} V_j)$ is homogeneous of degree negative one in $(V_i, \max_{j \neq i} V_j) \in \mathbb{R}^2_{>0}$, in straightforward extension of the fact that $\alpha^*(n, V_i)$ has this property w.r.t. V_i.

Regarding part (ii), we remark that the necessary first-order condition for the unique interior optimum delivers the effort level of the dominant contestant as $x_i^* = \widetilde{x}_i(\alpha)$. The level of the dominating bid $\widetilde{x}_i(\alpha)$ is strictly increasing in V_i and n, but obviously independent of $\max_{j \neq i} V_j$. The comparative statics of one-sided dominance with respect to the decisiveness parameter α is less immediate, however. For instance, a numerical exercise for the case of homogeneous valuations reveals that, in the relevant range where the equilibrium exists, the dominating bid $\widetilde{x}_1(\alpha)$ is strictly increasing in α for $n \in \{3, 4\}$, hump-shaped for $n \in \{5, 6\}$, and strictly declining for $n \geq 7$. Thus, contrary to intuition, if the number of contestants is small, then a sharper technology may induce the dominating contestant to choose an even higher effort level. As for the equilibrium payoffs, the active contestant i receives a payoff weakly exceeding the "fair share", i.e., $\Pi_i^* \geq V_i/n$. Indeed, contestant i must find it weakly profitable to depart from multilateral peace. Moreover, as contestant i's positive effort bites into the cake available for distribution, her probability of winning rises, so that less than the fair share is left for each of the subdued contestants (all of which win with identical probability). Therefore, $\Pi_j^* < V_j/n$ for any $j \neq i$. It is straightforward to check that the equilibrium payoff of the dominating contestant is strictly increasing in α and V_i, yet independent of $\max_{j \neq i} V_j$. In contrast, the equilibrium payoff of any inactive contestant $j \neq i$ is strictly declining in α and V_i, as well as strictly increasing in V_j. Finally, all contestants strictly lose payoff when an inactive contestant enters, i.e., when n grows. Indeed, in response to entry, the dominating bid rises, which causes the probability of winning for each of the inactive contestants to decline.

The comparative statics of the threshold values $\alpha_i^{**} = \alpha^{**}(n, V_i, \max_{j \neq i} V_j)$ is summarized in part (iii). In general, dominance by contestant i is strictly more likely to obtain when the number of contestants is larger, the valuation of contestant i is bigger, or the valuation of the strongest competitor of contestant i is lower. In particular, the intervals $[\alpha_i^*, \alpha_i^{**}]$ over which one-sided dominance by contestant i is an equilibrium for $n \geq 3$ are nested for $i \in \{1, \dots, i^*\}$, i.e., the interval for any contestant (strictly) contains the interval for any other contestant with a (strictly) lower valuation. Moreover, unless $n = 2$ and $V_1 = V_2$, we generally have $\alpha_1^* < \alpha_1^{**}$, i.e., one-sided dominance by contestant 1 (and possibly other contestants, as discussed below) is always feasible. As a result, for $n \geq 3$, as α increases, the set of PSNE, or equivalently, the set of candidates for the role of the dominating contestant, may first expand and then shrink again.

Finally, part (iv) says that there are no PSNE other than those already discussed, viz. multilateral peace and one-sided dominance by some contestant $i \in \{1, \dots, i^*\}$. In particular, given nestedness, there is no PSNE for $\alpha > \alpha_1^{**}$.

A notable implication of the proof of Proposition 2 is that the identity of the dominating contestant is, in general, undetermined. In fact, this follows from a simple continuity argument with respect to the vector of valuations, starting from the case of homogeneous valuations $V_1 = \dots = V_n \equiv V$, with $\alpha \in (\alpha_1^*, \alpha_1^{**})$, where the contestants dominated by some contestant $i \in \{2, \dots, n\}$ find it strictly optimal to remain inactive, and then slightly raising V_1, for instance. In particular, the present discussion implies the possibility of a multiplicity of payoff-inequivalent PSNE even if there is a single strongest contestant.

[11] While we offer an independent proof, this observation follows alternatively from the uniqueness of the MSNE (Ewerhart and Sun, 2018).

Corollary 1 *(Dominance by a weaker contestant). Let $V_1 \geq V_2 \geq \ldots \geq V_n > 0$ with $n \geq 3$, and assume that $\alpha \in (\alpha_1^*, \alpha_1^{**})$. Take any $i \in \{2, \ldots, n\}$. Then, for V_i/V_1 sufficiently close to (but still strictly smaller than) one, there exists a PSNE in which contestant i is active, while all the other contestants remain inactive.*[12]

Proof. See the Appendix. □

3.3. Discussion

Taken together, Propositions 1 and 2 offer a fairly comprehensive characterization of the set of PSNE in the n-player Hirshleifer contest. Notably, the striking difference in the equilibrium prediction between the Tullock and Hirshleifer contests extends to the case of more than two players. In the Tullock contest, inactivity by all players ("multilateral peace") is never an equilibrium because, regardless of the tie-breaking rule in place, at least $(n-1)$ contestants would receive the prize with probability strictly smaller than one. For those contestants, however, a deviation to any small but positive bid would guarantee the prize, in conflict with the equilibrium property. Similarly, just one player being active ("one-sided dominance") is not an equilibrium in a Tullock contest because the dominating player would always have a strict incentive to lower her positive bid.[13]

3.4. Historical illustrations

Accounts of military history are full of examples that may serve as illustrations of pure-strategy equilibria involving inactive contestants. More than two thousand years ago, Caesar observed that his Roman legions and the Gallic opponents repeatedly avoided combat because that meant an uphill battle for the attacker (Lawrence, 2017, p. 15). In a similar vein, von Clausewitz (1832) 1976, Section I.1.17) concluded that "defense is the stronger form of fighting than attack" and that it "is this which explains most periods of inaction that occur in war." Consistent with our assumptions, he argued that effort is not certain to lead to success because of what he called "frictions in war," i.e., the combined effect of numerous unforeseen incidents. The primacy of defense was also reflected in feudal castles in late medieval Europe (Levy, 1984, p. 230), while its counterpart, the ineffectiveness of attack, became notorious in World War I (Hirshleifer, 2000, p. 786). Effectiveness of inactivity has its role even in modern warfare that relies on tanks and airplanes, as exemplified by the siege of Tobruk (Dupuy, 1980, p. 327).

An example for the break-down of multilateral peace is found in ancient China, during the transition of the Eastern Zhou dynasty from the Spring and Autumn Period (770 B.C. – 481 B.C.) to the Warring States Period (481 B.C. – 221 B.C.), both with dozens of states coexisting. Largely due to changes in the politico-military organization (emergence of counties and prefectures with responsibilities for military recruitment and administration) and significant improvements in warfare technologies (in the form of crossbows, cloud ladders, armor, helmets, as well as specially trained cavalry), the relatively peaceful former period transited into the much more violent latter period (Lewis, 1999). Indeed, military conflicts among the so-called "states organized for warfare" became more frequent and more intensive. This example might illustrate our finding, discussed in more detail below, that an increase in decisiveness may transform a PSNE outcome in which at most one contestant is active into a MSNE outcome in which several or even all contestants are active.[14]

4. Low levels of noise (leading to mixed-strategy equilibria)

For small noise, the pure-strategy equilibria considered in the previous section cease to exist. Intuitively, overbidding an active opponent becomes less costly, so that dominating others becomes more difficult. Instead, we find equilibria in which at least one contestant randomizes. We start with the case of homogeneous valuations, where we examine both symmetric (Subsection 4.1) and asymmetric equilibria (Subsection 4.2). Thereafter, we discuss the case of heterogeneous valuations (Subsection 4.3).

4.1. Homogeneous valuations: symmetric equilibria

Suppose that valuations are *homogeneous*, i.e., $V_1 = \ldots = V_n \equiv V > 0$. As usual, we call an equilibrium (pure or mixed) *symmetric* if all players use the same strategy. In this case, it follows from general arguments that the equilibrium strategy randomizes over a finite number of $L \geq 1$ bids

$$y^{(1)} > \ldots > y^{(L)} = 0, \tag{8}$$

[12] An exact formula for the lower cutoff value for the set of values V_i/V_1 for which one-sided dominance by contestant $i \in \{2, \ldots, n\}$ is an equilibrium for some $\alpha > 0$ is provided in the Appendix.

[13] For an insightful discussion of this argument, which extends to contest success functions that are homogeneous of degree zero, see Corchón (2000, Appendix).

[14] These and other historical examples also illustrate the importance of understanding the nature of coalitions in difference-form contests (even if unstable, cf. Footnote 5). While this topic is beyond the scope of the present analysis, we refer the reader to Cubel and Sanchez-Pages (2022), whose analysis captures both between-group contests and within-group distribution.

where each $y^{(l)}$ is selected with probability $q^{(l)} > 0$, for $l \in \{1, \ldots, L\}$.[15] If $L = 1$, then the symmetric equilibrium is in pure strategies, and necessarily multilateral peace, as seen above. If $L \geq 2$, however, then contestants randomize strictly.

Below, we offer a general result on symmetric equilibria in n-player Hirshleifer contests with homogeneous valuations.

Proposition 3 *(Symmetric MSNE). Consider an n-player Hirshleifer contest with homogeneous valuations $V_1 = \ldots = V_n \equiv V > 0$. Then, the following holds true:*

(i) *A symmetric equilibrium μ^* exists for any $\alpha > 0$;*
(ii) *$L \geq 2$ if and only if $\alpha > \alpha^*$;*
(iii) *the number L respects the lower bound given by $L \geq \left(\frac{(n-1)\alpha V}{n^2}\right)^{\frac{1}{n-1}}$;*
(iv) *each contestant i's payoff satisfies $\Pi_i^* \leq \frac{n}{(n-1)\alpha}$.*

Proof. See the Appendix. □

This proposition characterizes the structure of symmetric MSNE in the n-player Hirshleifer contest with homogeneous valuations. Part (i) establishes the existence of a symmetric MSNE for any $\alpha > 0$. Part (ii) says that $L \geq 2$ if and only if $\alpha > \alpha^*$. This means that multilateral peace (where $L = 1$) is the unique symmetric MSNE for $\alpha \leq \alpha^*$, while necessarily $L \geq 2$ if $\alpha > \alpha^*$. Part (iii) puts a more general lower bound on the number of mass points in the equilibrium bid distribution. As can be seen, the lower bound is strictly increasing and unbounded in α. Thus, as $\alpha \to \infty$, the number of mass points in the support of the equilibrium strategy will ultimately surpass any finite bound. Remarkably, the upper bound on the equilibrium payoff given in part (iv) does not depend on V.

The following example illustrates the symmetric MSNE in the simplest case where the support of the equilibrium strategy has precisely two elements, i.e., in the case $L = 2$.

Example 1 *(The case $L = 2$).* Let $n \geq 2$. Consider an equilibrium strategy that places probability $q^{(1)} > 0$ on a positive bid $y^{(1)} > 0$, and a complementary probability $q^{(2)} = 1 - q^{(1)} > 0$ on the zero bid $y^{(2)} = 0$. Then, we have two equations that jointly characterize $y^{(1)}$ and $q^{(1)}$, viz. the first-order condition at the interior bid $y^{(1)}$, and the indifference condition between $y^{(1)}$ and $y^{(2)}$. For instance, if $n = 3$ and $V = 1$, then this type of equilibrium exists for $\alpha \in (4.12, 6.98)$.[16] For smaller values of α, contestants have an incentive to deviate to the zero bid. On the other hand, for larger values of α, contestants have an incentive to deviate to a bid level strictly between zero and $y^{(1)}$.

An interesting conjecture concerns the uniqueness of the symmetric MSNE for $n \geq 3$.[17] Our efforts to tackle that question led to the following preliminary observation.

Proposition 4 *(Partial uniqueness). Suppose that μ_1^* and μ_1^{**} are two symmetric equilibrium strategies in an n-player Hirshleifer contest with homogeneous valuations $V_1 = \ldots = V_n \equiv V > 0$. Suppose also that μ_1^* and μ_1^{**} can be ranked in terms of first-order stochastic dominance. Then, $\mu_1^* = \mu_1^{**}$.*

Proof. See the Appendix. □

Thus, if multiple symmetric equilibria exist in the Hirshleifer contest with homogeneous valuations, then they are pairwise not comparable in terms of first-order stochastic dominance. The proof of Proposition 4 combines methods from the theory of two-person zero-sum games with results from the theory of monotone comparative statics. It might, therefore, be of independent interest.

4.2. Homogeneous valuations: asymmetric equilibria

We now turn to the analysis of asymmetric equilibria while keeping the assumption that valuations are homogeneous. As discussed in the Introduction, there are no asymmetric equilibria in this case for $n = 2$. For $n \geq 3$, however, the set of asymmetric equilibria can be quite large even for homogeneous valuations, as will be illustrated below by three examples. For convenience, we use the normalization $V = 1$.

Example 2 *(Two identically randomizing players and one inactive player).* Let $n = 3$. Suppose that contestants 1 and 2 use an identical mixed strategy that selects $y_1^{(1)} = y_2^{(1)} > 0$ with probability $q_1^{(1)} = q_2^{(1)} \in (0, 1)$ and the zero bid otherwise, while contestant 3 remains

15 Given the analyticity of the payoff functions on an open neighborhood of the strategy interval, any optimal mixed strategy in the n-player Hirshleifer contest has finite support. For a formal statement, see Ewerhart and Sun (2018). Similar techniques have been applied by Ewerhart (2015b, 2021), Sun (2017), and Levine and Mattozzi (2022), in particular.

16 For all the numerical examples reported in this paper, we used standard spreadsheet software and double-checked the results using Wolfram Mathematica 12.2.

17 The case $n = 2$ is settled (Ewerhart and Sun, 2018).

inactive. Then, as in Example 1, a first-order condition and an indifference relationship jointly characterize $y_1^{(1)}$ and $q_1^{(1)}$. This MSNE exists for $\alpha \in (4.12, 7.02)$.[18] For smaller values of α, contestants 1 and 2 individually have an incentive to reduce their efforts to zero. For larger values of α, however, both would prefer some bid strictly between zero and $y_1^{(1)}$.

Example 3 *(One always active player, one randomizing player, and $(n-2)$ inactive players).* Let $n \geq 3$. Suppose that contestant 1 chooses a positive bid $y_1^{(1)} > 0$ with probability one, while contestant 2 randomizes between the zero bid and some bid $y_2^{(1)} > y_1^{(1)}$. Suppose also that contestants $3, \ldots, n$ all remain inactive. In this case, we have three equilibrium conditions, viz. the respective first-order conditions for $y_1^{(1)}$ and $y_2^{(1)}$, as well as the indifference relationship for contestant 2. For instance, if $n = 3$, this MSNE exists for $\alpha \in (4.58, 4.66)$.[19] For smaller values of α, contestant 1 has an incentive to deviate to zero. For larger values of α, however, contestant 2 would prefer placing all probability weight on the zero bid.

Example 4 *(One always active and two identically randomizing players).* Let $n = 3$. Suppose that contestant 1 chooses a pure strategy $y_1^{(1)} > 0$, while contestants 2 and 3 identically randomize between the zero bid and $y_2^{(1)} = y_3^{(1)} > y_1^{(1)}$. This MSNE exists for $\alpha \in (4.66, 4.86)$. For smaller values of α, contestants 2 and 3 would individually prefer placing all probability weight on the zero bid, while for larger values of α, contestant 1 would wish to become inactive.[20]

4.3. Heterogeneous valuations

Let $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ be a MSNE in an n-player Hirshleifer contest with valuations $V_1 \geq \ldots \geq V_n > 0$. In this case, each contestant $i \in N$ randomizes over a finite set of bids

$$y_i^{(1)} > \ldots > y_i^{(L_i)} \geq 0, \tag{9}$$

for some $L_i \geq 1$, such that $y_i^{(l_i)}$ is selected with probability $q_i^{(l_i)} > 0$, for $l_i \in \{1, \ldots, L_i\}$. Let L_i^+ denote the number of positive elements in the support of μ_i^*. From Lemma 2, we know that $L_i^+ = L_i - 1$ holds for at least $(n-1)$ players. The following result establishes an upper bound on L_i^+ in terms of the cardinalities of the supports for the other contestants.

Proposition 5 *(Support inequality). $L_i^+ \leq \prod_{j \neq i} L_j$, for any $i \in N$.*

Proof. See the Appendix. □

This inequality says that the number of pure strategies over which a player can possibly randomize cannot be too heterogeneous across players. The proof exploits the fact that a contestant's first-order condition for the effort levels used with positive probability may be interpreted as a polynomial equation with known degree. The fundamental theorem of algebra then imposes an upper bound on the number of solutions.[21]

In the case of equilibria where only one contestant randomizes, while all other contestants use a pure strategy, Proposition 5 implies that the former randomizes between a zero bid and a single positive bid.

Corollary 2. *Suppose that μ^* is an equilibrium in an n-player Hirshleifer contest with valuations $V_1 \geq \ldots \geq V_n > 0$. Suppose that some contestant $i \in N$ randomizes strictly while all other contestants $j \neq i$ choose a pure strategy. Then, contestant i randomizes between the zero bid and a single positive bid level.*

Proof. Immediate from Proposition 5. □

Corollary 2 sheds light on the set of equilibria with the property that $L_j = 1$ for all $j \neq i$. In view of Lemma 2, at most one contestant $j \neq i$ can be always active. Therefore, there are two cases. Either there is an always active player j choosing a pure strategy, as in Example 3 above. Or else, all the players that choose a pure strategy remain inactive, as exemplified by the following result.

Proposition 6 *(One randomizing player and $(n-1)$ inactive players). Let $V_1 \geq \ldots \geq V_n > 0$ with $n \geq 3$, and $i \in N$ such that $V_i = V_1$. Then, at $\alpha = \alpha_1^*$, there is a continuum of payoff-inequivalent MSNE in which contestant i randomizes, choosing $y_i^{(1)} = \widetilde{x}_i(\alpha_i^*) > 0$ with probability $q_i^{(1)} \in [0, 1]$, and the zero bid otherwise, while the other $(n-1)$ contestants all remain inactive. Conversely, for any MSNE at $\alpha = \alpha_1^*$, there is some contestant $i \in N$ with $V_i = V_1$ randomizing in the way just described, while all others remain inactive.*

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[18] For instance, if $\alpha = 4.5$, then $y_1^{(1)} = 0.348$ and $q_1^{(1)} = 0.36$.

[19] At $\alpha = 4.6$, e.g., one finds $y_1^{(1)} = 0.300$, $y_2^{(1)} = 0.513$, and $q_2^{(1)} = 0.163$.

[20] It may be noted that all our examples of asymmetric MSNE are semi-mixed, i.e., at least one contestant uses a pure strategy. However, in the setup of Example 4, around $\alpha = 4.86$, there is an asymmetric equilibrium in which all three contestants randomize.

[21] This method of proof is restricted to the specific case of the Hirshleifer contest. In Ewerhart (2021), the theory of Pólya frequency functions is employed to derive similar inequalities for more general classes of noise distributions in the case $n = 2$.

Proof. See the Appendix. □

Thus, for $n \geq 3$, a continuum of MSNE exists when the parameter α lies precisely at the threshold value α_1^*. The polar cases where $q_i^{(1)} = 0$ and $q_i^{(1)} = 1$ correspond to special cases of Propositions 1 and 2, respectively. Moreover, equilibrium payoffs for intermediate values $q_i^{(1)} \in (0, 1)$ may be expressed as convex combinations of the expressions provided there. The continuum of equilibria is due to the fact, familiar from the theory of bimatrix games, that the set of beliefs to which a pure strategy is a best response is convex.

5. Vanishing levels of noise

In this section, we study the structure of MSNE in the case of vanishing noise. Thus, we examine equilibrium bid distributions for α arbitrarily large but still finite.

Proposition 7 *(Arbitrarily small noise). Let $V_1 \geq \ldots \geq V_n > 0$ and $\varepsilon > 0$. Then, for α sufficiently large, any MSNE μ^* of the n-player Hirshleifer contest has the following properties:*

(i) $\left|V_2 - y_i^{(1)}\right| < \varepsilon$ *for at least two contestants $i \in N$ (including, in particular, $i = 1$ if $V_1 > V_2$);*
(ii) $\left|\Pi_1^* - (V_1 - V_2)\right| < \varepsilon$, *and $\Pi_i^* < \varepsilon$ for any $i \in N \backslash \{1\}$;*
(iii) *either $y_i^{(1)} < \varepsilon$ or $y_i^{(1)} > V_2 - \varepsilon$, for any $i \in N$;*
(iv) *if $V_1 = V_2$, then $\mu_i^*(\{0\}) < \varepsilon$ for at least two contestants $i \in N$.*

Proof. See the Appendix. □

The proposition provides information about equilibrium strategies and expected payoffs in the case where the noise in the contest technology becomes arbitrarily small.

Part (i) says that at least two contestants bid arbitrarily close to the valuation of the second-ranked contestant. Moreover, the contestant with the single highest valuation, if any, definitely does so. Intuitively, effort levels close to V_2 are necessary because, without such bids, every contestant j with valuation $V_j \geq V_2$ could earn a substantial rent by overbidding all the other contestants. However, as we show, such rents are possible for at most one contestant. Moreover, at least two contestants must be involved because a single high bidder would obviously have an incentive to bid less aggressively. Note also that bidding weakly above V_2 is individually rational only for contestant 1 and only if $V_1 > V_2$.[22]

By part (ii), the equilibrium rent of contestant 1 is ε-close to $V_1 - V_2$ for α sufficiently large. Moreover, any rent of the other contestants is dissipated for $\alpha \to \infty$. Here, the main argument is that rents weakly exceeding $V_1 - V_2 + \varepsilon$ or ε, respectively, are feasible for some contestant j only if her bid distribution is bounded by $V_2 - \varepsilon$. But then, one of the other contestants would find it strictly profitable to overbid contestant j.

Part (iii) says that, in the limit, each contestant either bids up to V_2 or ultimately becomes inactive. Again, the proof relies on the idea that any contestant $i \neq 1$ bidding nearly up to V_2 would prefer to slightly overbid any contestant whose highest bid lies in the interval $[\varepsilon, V_2 - \varepsilon]$. However, an additional argument is needed because the rent of the moderate bidder cannot be assumed to be substantial.

Finally, part (iv) shows that, provided that $V_1 = V_2$, the probability of bidding zero vanishes as $\alpha \to \infty$ for at least two contestants. Intuitively, too much weight on the zero bid would allow others to profitably make use of small positive bids.

Thus, overall, the structure of equilibria in the n-player Hirshleifer contest for α arbitrarily large shares several important properties of those of the n-bidder all-pay auction.[23]

6. Large populations

In this brief section, we change the perspective by keeping the decisiveness parameter of the technology fixed while letting the number of contestants grow. Then, provided that valuations are uniformly bounded, it turns out that multilateral peace is the unique equilibrium in any Hirshleifer contest with sufficiently many contestants. Intuitively, as n becomes larger, it gets increasingly more difficult for any of the contestants to take a dominating position.

Proposition 8. *Let $\alpha > 0$ and $\overline{V} > 0$ be given. Then, there exists a threshold value $n^{\#}$ such that for any $n \geq n^{\#}$, multilateral peace is the unique MSNE in any n-player Hirshleifer contest with parameter α and valuations $\overline{V} \geq V_1 \geq \ldots \geq V_n > 0$.*

Proof. See the Appendix. □

[22] Clearly, if attention is restricted to symmetric MSNE for homogeneous valuations $V_1 = \ldots = V_n = V > 0$, then it easily follows from (i) that all bidders bid up to V.

[23] In extension of results of Levine and Mattozzi (2022), we therefore conjecture that sequences of MSNE in n-player Hirshleifer contests as $\alpha \to \infty$, if convergent in distribution, must converge to some MSNE of the corresponding all-pay auction.

7. Conclusion

In this paper, we have examined the equilibrium set of the n-player Hirshleifer contest with heterogeneous valuations, a canonical contest with additive noise. In line with the findings for contests with multiplicative noise such as the Tullock contest, the equilibrium prediction depends heavily on the level of noise in the contest technology. Notwithstanding, the analysis adds strong support to Hirshleifer's (1989) main conclusion, viz. that the nature of the equilibrium prediction is strikingly different across these two cases.

The results derived above are driven by the two characteristic features of the Hirshleifer contest technology that have been highlighted already in the introduction: (i) Effectiveness of inactivity, and (ii) increasing returns to marginal effort up to the inflection point of winning with probability one half. These features shape the equilibrium behavior over the entire range of the decisiveness parameter. In particular, two polar regimes emerge. In one regime, the noise in the technology is substantial, so that effort is not effective. Then, inaction may be the optimal choice for all but at most one contestant. In another regime, however, noise is small, i.e., effort is effective. Then, there is "an enormous gain when your side's forces increase from just a little smaller than the enemy's to just a little larger" (Hirshleifer, 1989, p. 103). As a result, at least two contestants enter into a bidding war. Then, however, the odds of winning for the other contestants depend little on their efforts, so that inactivity may again be the optimal choice. In the present paper, we have tried to disentangle these effects a bit, with the objective of shedding light on a plausible (even though less commonly employed) class of contests.

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

Appendix A. Proofs

This Appendix contains technical proofs omitted from the body of the paper.

Proof of Lemma 1. Each player i's set of pure strategies, $[0, V_i]$, is an interval, hence compact and nonempty. Moreover, payoff functions are continuous. Therefore, the claim follows directly from Glicksberg's (1952) theorem. □

Proof of Lemma 2. To provoke a contradiction, suppose there are two contestants $i, j \in N$ with $i \neq j$ that are always active in some MSNE μ^*. Consider an arbitrary pure-strategy profile (x_i, x_{-i}). Then, the second derivative of contestant i's payoff with respect to x_i is given by

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = \alpha^2 p_i(1-p_i)(1-2p_i)V_i, \tag{10}$$

where $\Pi_i = \Pi_i(x_i, x_{-i})$ and $p_i = p_i(x_i, x_{-i})$, with

$$p_i(x_i, x_{-i}) = \frac{\exp(\alpha x_i)}{\exp(\alpha x_i) + \sum_{j \neq i} \exp(\alpha x_j)}. \tag{11}$$

Hence, $\partial^2 \Pi_i / \partial x_i^2 \geq 0$ if $x_i \leq x_j$, and even $\partial^2 \Pi_i / \partial x_i^2 > 0$ if $x_i < x_j$. Similarly, $\partial^2 \Pi_i / \partial x_i^2 > 0$ if both $x_i \leq x_j$ and $n \geq 3$. Let $\underline{y}_i > 0$ and $\underline{y}_j > 0$ denote the *smallest* bids used by contestants i and j with positive probability in μ_i^* and μ_j^*, respectively.[24] Then, without loss of generality, we may assume that $\underline{y}_i \leq \underline{y}_j$. Taking the expectation over μ_{-i}^*, one obtains $E_{\mu_{-i}^*}[\partial^2 \Pi_i(\underline{y}_i, x_{-i})/\partial x_i^2] \geq 0$, where the inequality is strict if either $\underline{y}_i < \underline{y}_j$, or contestant j randomizes strictly, or $n \geq 3$. However, from the second-order condition, $E_{\mu_{-i}^*}[\partial^2 \Pi_i(\underline{y}_i, x_{-i})/\partial x_i^2] \leq 0$. Therefore, $\underline{y}_i = \underline{y}_j$, contestant j uses a pure strategy, and $n = 2$. Applying the same argument with the roles of contestants i and j exchanged, contestant i is likewise seen to use a pure strategy. Therefore, $(\underline{y}_i, \underline{y}_j)$ is a (symmetric) interior PSNE in a two-player Hirshleifer contest. But this is impossible, as noted by Hirshleifer (1989, p. 107). The claim follows. □

Proof of Proposition 1. (i) *(Contestant i's optimality condition)* Consider the marginal payoff of an arbitrary contestant $i \in N$ against $x_{-i} = \mathbf{0}_{n-1}$,

$$\frac{\partial \Pi_i^0(x_i; \alpha)}{\partial x_i} = \frac{(n-1)X_i \alpha V_i}{(X_i + n - 1)^2} - 1, \tag{12}$$

[24] A smallest positive bid always exists in equilibrium strategies of active contestants. See footnote 15.

where $X_i = \exp(\alpha x_i)$, as before. The right-hand side of (12) vanishes at the solutions of the quadratic equation

$$X_i^2 + (2 - \alpha V_i)(n-1)X_i + (n-1)^2 = 0. \tag{13}$$

Hence, there are three cases. First, if $\alpha < 4/V_i$, then there is no solution. In that case, therefore, $\Pi_i^0(\cdot;\alpha)$ is strictly declining, and bidding zero is optimal. Second, if $\alpha = 4/V_i$, then there is precisely one solution, viz. $X_i = n-1$. In that case, $\Pi_i^0(\cdot;\alpha)$ is still strictly declining but has a saddle point at $x_i = \frac{\ln(n-1)}{4}V_i$. Notably, that saddle point lies at the boundary for $n = 2$, and in the interior for $n > 2$. Either way, $\Pi_i^0(\cdot;\alpha)$ is strictly declining for $\alpha = 4/V_i$, and bidding zero is still optimal. Third and finally, if $\alpha > 4/V_i$, then equation (13) admits two solutions. In that case, therefore, $\Pi_i^0(\cdot;\alpha)$ has a unique interior local maximum at

$$\widetilde{x}_i(\alpha) = \frac{1}{\alpha}\ln\left(\frac{n-1}{2}\left\{\alpha V_i - 2 + \sqrt{\alpha V_i(\alpha V_i - 4)}\right\}\right) > 0. \tag{14}$$

Thus, bidding zero is optimal for contestant i if and only if either (a) $\alpha V_i \leq 4$ or (b) $\alpha V_i > 4$ and $\Delta_i(\alpha) \equiv \Pi_i^0(\widetilde{x}_i(\alpha);\alpha) - \Pi_i^0(0;\alpha) \leq 0$. *(Identification of α_i^*)* We apply the envelope theorem w.r.t. the transformed decision variable $X_i = \exp(\alpha x_i)$, and obtain

$$\frac{\partial \Delta_i(\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left\{\frac{X_i V_i}{X_i + n - 1} - \frac{\ln X_i}{\alpha} - \frac{V_i}{n}\right\}\Bigg|_{X_i = \widetilde{X}_i(\alpha)} = \frac{\ln X_i}{\alpha^2} = \frac{\widetilde{x}_i(\alpha)}{\alpha} > 0, \tag{15}$$

where $\widetilde{X}_i(\alpha) = \exp(\alpha \widetilde{x}_i(\alpha))$. Thus, $\Delta_i(\alpha)$ is strictly increasing in α. Moreover, as α grows large, contestant i's marginal payoff at the zero bid, $\partial \Pi_i^0(0;\alpha)/\partial x_i = \frac{(n-1)}{n^2}\alpha V_i - 1$, is positive for $\alpha > \frac{n^2}{(n-1)V_i}$, so that $\Delta_i(\alpha) > 0$ for any such α. Hence, for each contestant $i \in N$, there is a unique threshold $\alpha_i^* \in [\frac{4}{V_i}, \frac{n^2}{(n-1)V_i}]$ such that bidding $x_i = 0$ is a best response to $x_{-i} = \mathbf{0}_{n-1}$ if and only if $\alpha \in (0, \alpha_i^*]$.[25] *(Pivotality of contestant 1)* From the optimality conditions just derived, it follows that multilateral peace is a PSNE if and only if $\alpha \in (0, \alpha^*]$, where $\alpha^* = \min_{i \in N}\{\alpha_i^*\}$. To see that contestant 1 is pivotal, we show that $\alpha_1^* \leq \ldots \leq \alpha_n^*$. Indeed, from $V_{i+1} \leq V_i$, with $i \in \{1, \ldots, n-1\}$, we get

$$\frac{\Pi_i^0(\xi;\alpha) - \Pi_i^0(0;\alpha)}{V_i} - \frac{\Pi_{i+1}^0(\xi;\alpha) - \Pi_{i+1}^0(0;\alpha)}{V_{i+1}} = \left(\frac{1}{V_{i+1}} - \frac{1}{V_i}\right)\xi \geq 0, \tag{16}$$

for any $\xi \geq 0$, so that $\alpha_i^* \leq \alpha_{i+1}^*$, as claimed. In particular, $\alpha^* = \alpha_1^*$. This proves the claim. (ii) The claim is immediate. (iii) We claim that there does not exist any MSNE other than multilateral peace if $\alpha < \alpha^*$.[26] Suppose to the contrary that μ^* is a MSNE in which at least one player randomizes strictly. Let $i \in N$ be one of the players that uses the smallest positive bid in the contest, $y_i^{\min} > 0$, with positive probability. Then, $E_{\mu_{-i}^*}[\Pi_i(0, x_{-i})] \leq E_{\mu_{-i}^*}[\Pi_i(y_i^{\min}, x_{-i})]$. On the other hand, since $\alpha < \alpha^* = \min_{i \in N}\{\alpha_i^*\}$, we know from the proof of part (i) that $\Pi_i(0, \mathbf{0}_{n-1}) > \Pi_i(x_i, \mathbf{0}_{n-1})$ for any $x_i > 0$. Hence, in particular, $\Pi_i(0, \mathbf{0}_{n-1}) > \Pi_i(y_i^{\min}, \mathbf{0}_{n-1})$.[27] Now, for any bid vector x_{-i}, a straightforward calculation shows that

$$\begin{aligned}&\left(\Pi_i(0, x_{-i}) - \Pi_i(y_i^{\min}, x_{-i})\right) - \left(\Pi_i(0, \mathbf{0}_{n-1}) - \Pi_i(y_i^{\min}, \mathbf{0}_{n-1})\right)\\ &\quad = \frac{\left(X_{-i} - n + 1\right)\left(Y_i^{\min} - 1\right)\left((n-1)X_{-i} - Y_i^{\min}\right)V_i}{n\left(1 + X_{-i}\right)\left(Y_i^{\min} + n - 1\right)\left(Y_i^{\min} + X_{-i}\right)},\end{aligned} \tag{17}$$

where $Y_i^{\min} = \exp(\alpha y_i^{\min})$ and $X_{-i} = \sum_{j \neq i}\exp(\alpha x_j)$. Note that the right-hand side of equation (17) is weakly positive for any nonzero x_{-i} used with positive probability in μ_{-i}^* because, in that case, we have $(n-1)X_{-i} \geq \exp(\alpha x_j) \geq \exp(\alpha y_i^{\min}) = Y_i^{\min}$ for some $j \neq i$. Further, both sides of equation (17) vanish for $x_{-i} = \mathbf{0}_{n-1}$. Hence, taking expectations,

$$0 \geq E_{\mu_{-i}^*}[\Pi_i(0, x_{-i}) - \Pi_i(y_i^{\min}, x_{-i})] \geq E_{\mu_{-i}^*}[\Pi_i(0, \mathbf{0}_{n-1}) - \Pi_i(y_i^{\min}, \mathbf{0}_{n-1})] > 0, \tag{18}$$

a contradiction. This proves the claim.[28] (iv) (*α^* is strictly increasing in n*) We treat $n \geq 2$ as a continuous variable. Then, by an application of the envelope theorem w.r.t. $x_1 = \widetilde{x}_1(\alpha)$,

$$\frac{\partial \Delta_i(\alpha)}{\partial n} = -\frac{\widetilde{X}_1(\alpha)V_1}{(\widetilde{X}_1(\alpha) + n - 1)^2} + \frac{V_1}{n^2} \underset{\text{(FOC)}}{=} -\frac{1}{(n-1)\alpha} + \frac{V_1}{n^2}. \tag{19}$$

[25] Obviously, $\alpha_i^* = 4/V_i$ for $n = 2$. For $n > 2$, we can show that $\alpha_i^* \in (\frac{4}{V_i}, \frac{n^2}{(n-1)V_i})$. Indeed, the above-mentioned saddle point lies in the interior for $n > 2$, implying that $\Pi_i^0(\frac{\ln(n-1)}{4}V_i;\alpha) < V_i/n$ at $\alpha = 4/V_i$. Hence, $\alpha_i^* > 4/V_i$ in this case. Further, $\Pi_i^0(\cdot;\alpha)$ has a local minimum at $x_i = 0$ if $\alpha = \frac{n^2}{(n-1)V_i}$ and $n > 2$, because $\partial^2\Pi_i^0(0;\alpha)/\partial x_i^2 = \frac{(n-1)(n-2)}{n^3}\alpha^2 V_i > 0$ in that case.

[26] For a direct proof that there is no PSNE other than multilateral peace if $\alpha < \alpha^*$, one notes that, by Lemma 2, there exists a contestant $i \in N$ such that $x_{-i}^* = \mathbf{0}_{n-1}$ in any PSNE. But $\alpha^* = \alpha_1^* \leq \alpha_i^*$. Therefore, $\alpha < \alpha_i^*$, which implies that contestant i's unique best response to $x_{-i} = \mathbf{0}_{n-1}$ is $x_i = 0$. Thus, multilateral peace is indeed the unique PSNE if $\alpha < \alpha^*$.

[27] In fact, if $n = 2$, then this inequality holds even for $\alpha = \alpha^*$, which explains why the proof extends to that case, as we discuss in footnote (8).

[28] Alternatively, the uniqueness claim may be deduced from the fact that multilateral peace is an equilibrium in strictly dominant strategies if $\alpha < \alpha^*$.

Moreover, as noted above, the right-hand side of equation (19) is negative at $\alpha^*(n, V_1)$ if $n > 2$ (the analysis there is valid without change for continuous n). Hence, totally differentiating relationship (4) and recalling (15) shows that $d\alpha^*/dn = -(\partial\Delta_1/\partial n)/(\partial\Delta_1/\partial\alpha) > 0$. In particular, $\alpha^*(n+1, V_1) > \alpha^*(n, V_1)$ for any integer $n \geq 2$, as claimed. (α^* *is strictly declining in* V_1) For this, it suffices to note that, for any $\alpha > 0$ fixed, the graph of the transformed function $\overline{\Pi}_1^0(\cdot;\alpha)$, defined by

$$\overline{\Pi}_1^0(\xi;\alpha) \equiv \alpha\Pi_1^0(\xi/\alpha;\alpha) = \frac{\exp(\xi)\alpha V_i}{\exp(\xi)+n-1} - \xi, \tag{20}$$

results from a uniform scaling of the graph of $\Pi_1^0(\cdot;\alpha)$. Since the critical points of $\overline{\Pi}_1^0(\cdot;\alpha)$ depend on αV_1 and n alone, this reveals that $\alpha^*(n, V_1)V_1 \equiv a(n)$ depends on n alone, and thereby proves the final claim. The proposition follows. □

Proof of Proposition 2. The proof is divided into eight steps.

Step 1. (*The optimality condition of the pivotal contestant*) Consider the candidate equilibrium in which some contestant $i \in N$ chooses $\widetilde{x}_i(\alpha) > 0$, while all other contestants $j \neq i$ remain inactive. As shown in the proof of Proposition 1, the optimality condition for contestant i holds if and only if $\alpha \geq \alpha_i^*$, where the inequality is understood to be strict for $n = 2$. Assuming this to be the case, we check the optimality condition of the pivotal contestant $j \neq i$. Using the same argument as above, the pivotal contestant may be chosen as $j = 2$ if $i = 1$, and as $j = 1$ if $i \in \{2, \ldots, n\}$. Let

$$\widehat{\Pi}_j^{(i)}(x_j;\alpha) = \frac{X_j V_j}{\widetilde{X}_i(\alpha) + X_j + n - 2} - x_j \tag{21}$$

denote contestant j's expected payoff from bidding $x_j \geq 0$ against $x_{-j} = (\widetilde{x}_i(\alpha), \mathbf{0}_{n-2})$, where $\widetilde{X}_i(\alpha) = \exp(\alpha\widetilde{x}_i(\alpha))$ and $X_j = \exp(\alpha x_j)$. As in the proof of Proposition 1, an examination of the derivative $\partial\widehat{\Pi}_j^{(i)}(x_j;\alpha)/\partial x_j$ shows that $\widehat{\Pi}_j^{(i)}(\cdot;\alpha)$ is strictly declining for $\alpha \leq 4/V_j$, so that contestant j optimally remains inactive in that case. For $\alpha > 4/V_j$, however, $\widehat{\Pi}_j^{(i)}(x_j;\alpha)$ has a unique interior local maximum at

$$\widehat{x}_j^{(i)}(\alpha) = \frac{1}{\alpha}\ln\left(\frac{n-2+\widetilde{X}_i(\alpha)}{2}\left\{\alpha V_j - 2 + \sqrt{\alpha V_j(\alpha V_j - 4)}\right\}\right). \tag{22}$$

Thus, conditional on i's optimality being satisfied, bidding $x_j = 0$ is optimal for contestant j against $x_{-j} = (\widetilde{x}_i(\alpha), \mathbf{0}_{n-2})$ if and only if either (a) $\alpha \leq 4/V_j$, or (b) $\alpha > 4/V_j$ and $\Delta_j^{(i)}(\alpha) \equiv \widehat{\Pi}_j^{(i)}(\widehat{x}_j^{(i)}(\alpha);\alpha) - \widehat{\Pi}_j^{(i)}(0;\alpha) \leq 0$.

Step 2. (*The single-crossing property*) Recall that the payoff difference $\Delta_j^{(i)}(\alpha)$ has been defined for $\alpha \in A_j^{(i)}$, where $A_j^{(i)} = \{\alpha \,|\, \alpha \geq \alpha_i^*,\ \alpha > 4/V_j\}$ if $n \geq 3$, and $A_j^{(i)} = \{\alpha \,|\, \alpha > \max\{\alpha_i^*, 4/V_j\}\}$ if $n = 2$. We will show below that $\partial\Delta_j^{(i)}(\alpha)/\partial\alpha > 0$ holds at any $\alpha \in A_j^{(i)}$ at which contestant j is indifferent between $x_j = \widehat{x}_j^{(i)}(\alpha)$ and $x_j = 0$, i.e., at any $\alpha \in A_j^{(i)}$ such that $\Delta_j^{(i)}(\alpha) = 0$.

Step 3. (*Computation of* $\partial\Delta_j^{(i)}/\partial\alpha$) Applying the envelope theorem w.r.t. the transformed decision variable X_j yields

$$\frac{\partial\Delta_j^{(i)}(\alpha)}{\partial\alpha} = \frac{\widehat{x}_j^{(i)}(\alpha)}{\alpha} + \frac{\partial\widetilde{X}_i(\alpha)}{\partial\alpha}\cdot\frac{\partial}{\partial X_i}\left\{\frac{\widehat{X}_j^{(i)}(\alpha)V_j}{\widehat{X}_j^{(i)}(\alpha)+X_i+n-2} - \frac{V_j}{X_i+n-1}\right\}\Bigg|_{X_i=\widetilde{X}_i(\alpha)} \tag{23}$$

$$= \frac{\widehat{x}_j^{(i)}(\alpha)}{\alpha} + \frac{\partial\widetilde{X}_i(\alpha)}{\partial\alpha}\cdot\left(-\frac{\widehat{X}_j^{(i)}(\alpha)V_j}{(\widehat{X}_j^{(i)}(\alpha)+\widetilde{X}_i(\alpha)+n-2)^2} + \frac{V_j}{(\widetilde{X}_i(\alpha)+n-1)^2}\right) \tag{24}$$

$$= \frac{\widehat{x}_j^{(i)}(\alpha)}{\alpha} + \frac{\partial\widetilde{X}_i(\alpha)}{\partial\alpha}\cdot\frac{\left(\widehat{X}_j^{(i)}(\alpha) - (\widetilde{X}_i(\alpha)+n-2)^2\right)(\widehat{X}_j^{(i)}(\alpha)-1)V_j}{(\widehat{X}_j^{(i)}(\alpha)+\widetilde{X}_i(\alpha)+n-2)^2(\widetilde{X}_i(\alpha)+n-1)^2}, \tag{25}$$

where $\widehat{X}_j^{(i)}(\alpha) = \exp(\alpha\widehat{x}_j^{(i)}(\alpha))$. From $\Delta_j^{(i)}(\alpha) = 0$,

$$\widehat{x}_j^{(i)}(\alpha) = \frac{(\widehat{X}_j^{(i)}(\alpha)-1)(\widetilde{X}_i(\alpha)+n-2)V_j}{(\widetilde{X}_i(\alpha)+\widehat{X}_j^{(i)}(\alpha)+n-2)(\widetilde{X}_i(\alpha)+n-1)}. \tag{26}$$

Exploiting relationship (26) and noting that

$$\frac{\partial\widetilde{X}_i(\alpha)}{\partial\alpha} = \frac{\widetilde{X}_i(\alpha)V_i}{\sqrt{\alpha V_i\left(\alpha V_i - 4\right)}} \underset{\text{(FOC)}}{=} \frac{\widetilde{X}_i(\alpha)}{\alpha}\cdot\frac{\widetilde{X}_i(\alpha)+n-1}{\widetilde{X}_i(\alpha)-n+1}, \tag{27}$$

one finds

$$\frac{\partial \Delta_j^{(i)}}{\partial \alpha} = \frac{\widehat{x}_j^{(i)}}{\alpha} \cdot \left(1 + \frac{\widetilde{X}_i \left(\widehat{X}_j^{(i)} - (\widetilde{X}_i + n - 2)^2 \right)}{(\widetilde{X}_i + n - 2)(\widehat{X}_j^{(i)} + \widetilde{X}_i + n - 2)(\widetilde{X}_i - n + 1)} \right) \tag{28}$$

$$= \frac{\widehat{x}_j^{(i)}}{\alpha} \cdot \frac{\widetilde{X}_i \widehat{X}_j^{(i)} + (\widetilde{X}_i + n - 2)\left((\widetilde{X}_i - n + 1)\widehat{X}_j^{(i)} - (\widetilde{X}_i + n - 2)(n-1) \right)}{(\widetilde{X}_i + n - 2)(\widehat{X}_j^{(i)} + \widetilde{X}_i + n - 2)(\widetilde{X}_i - n + 1)}, \tag{29}$$

where we dropped the arguments to save space.

Step 4. (*Lower bounds $\widetilde{X}_i$ and $\widehat{X}_j^{(i)}$*) To be able to put a sign on $\partial \Delta_j^{(i)}/\partial \alpha$, we derive lower bounds on $\widetilde{X}_i$ and $\widehat{X}_j^{(i)}$, respectively. As for $\widetilde{X}_i$, recall that $\widetilde{X}_i = (n-1)\lambda(\alpha V_i)$, where $\lambda(z) = \frac{1}{2}(z - 2 + \sqrt{z(z-4)}) \geq 1$ for $z \geq 4$ is strictly increasing. But since $\alpha \geq \alpha_i^*$, we obtain $\alpha V_i \geq \alpha_i^* V_i = a(n)$. Hence, we have the lower bound

$$\widetilde{X}_i \geq (n-1)\lambda(a(n)). \tag{30}$$

As for $\widehat{X}_j^{(i)}$, note that contestant j's unique optimal bid against the strategy profile $x_{-j} = (\widetilde{x}_i(\alpha), \mathbf{0}_{n-2})$ is the zero bid if $\alpha < a(n - 1 + \widetilde{X}_i)/V_j$, provided that the function $a(n)$ defined in the body of the paper is extended to continuous arguments in a straightforward way.[29] As this would be in conflict with $\Delta_j^{(i)}(\alpha) = 0$, we conclude that $\alpha V_j \geq a(\widetilde{X}_i + n - 1)$. Hence, noting that $\widehat{X}_j^{(i)} = (\widetilde{X}_i + n - 2)\lambda(\alpha V_j)$, one obtains the lower bound (in fact, an equality)

$$\widehat{X}_j^{(i)} \geq (\widetilde{X}_i + n - 2)\lambda(a(\widetilde{X}_i + n - 1)). \tag{32}$$

Step 5. (*Proof of the single-crossing property*) For $n = 2$, the numerator of the second fraction in (29) reduces to $\widetilde{X}_i^2 \cdot (\widehat{X}_j^{(i)} - 1) > 0$, so that indeed $\partial \Delta_j^{(i)}/\partial \alpha > 0$ in that case. If $n = 3$, then inequality (30) implies $\widetilde{X}_i \geq 2\lambda(a(3)) \approx 2.8$, while (32) implies $\widehat{X}_j^{(i)} \geq (2.8 + 1)\lambda(a(2.8 + 2))) \approx 7.2$. Using $\widehat{X}_j^{(i)} > 7$, the numerator of the second fraction in (29) satisfies

$$\widetilde{X}_i \widehat{X}_j^{(i)} + (\widetilde{X}_i + 1)\left((\widetilde{X}_i - 2)\widehat{X}_j^{(i)} - 2(\widetilde{X}_i + 1) \right) \geq 5\widetilde{X}_i^2 - 4\widetilde{X}_i - 16 > 0, \tag{33}$$

because the quadratic term on the right-hand side is positive for $\widetilde{X}_i > 2.3$. Therefore, $\partial \Delta_j^{(i)}/\partial \alpha > 0$ holds also for $n = 3$. To deal with the remaining case where $n \geq 4$, we will show that

$$(\widetilde{X}_i - n + 1)\widehat{X}_j^{(i)} \geq (\widetilde{X}_i + n - 2)(n-1), \tag{34}$$

which is sufficient in view of (29). In fact, from (32), it suffices to check that $(\widetilde{X}_i - n + 1)\lambda(a(\widetilde{X}_i + n - 1)) \geq n - 1$. But from (30), $\widetilde{X}_i \geq (n-1)\lambda(a(n)) \geq (n-1)\lambda(a(4))$. Moreover, $a(4) \approx 4.3$ and hence, $\lambda(a(\widetilde{X}_i + n - 1)) \geq \lambda(a(4)) \approx 1.7$. Thus, $(\lambda(a(4)) - 1) \cdot \lambda(a(4)) \approx 1.2 > 1$, which proves (34) for any $n \geq 4$. We have shown, therefore, that $\partial \Delta_j^{(i)}/\partial \alpha > 0$ holds at any $\alpha \in A_j^{(i)}$ such that $\Delta_j^{(i)}(\alpha) = 0$. In particular, $\Delta_j^{(i)}(\cdot)$ changes sign at most once on the interval $A_j^{(i)}$ and, if so, from negative to positive.

Step 6. (*One-sided dominance by contestant 1*) We first deal with the case $n \geq 3$. Then, $\alpha_2^* \geq \alpha_1^*$ and $\alpha_2^* > 4/V_2$, so that $\alpha_2^* \in A_2^{(1)}$. Hence, letting $\alpha = \alpha_2^*$, for any fixed $x_1 \geq 0$, the payoff function

$$\Pi_2(x_2, x_1, \mathbf{0}_{n-2}) = \frac{\exp(\alpha x_2)V_2}{\exp(\alpha x_1) + \exp(\alpha x_2) + n - 2} - x_2 \tag{35}$$

of the pivotal contestant $j = 2$ admits a unique interior local optimum at some $x_2 = b_2(x_1)$. Moreover, $b_2(0) = \widetilde{x}_2(\alpha_2^*)$ and $b_2(\widetilde{x}_1(\alpha_2^*)) = \widehat{x}_2^{(1)}(\alpha_2^*)$. By the envelope theorem,

$$\frac{\partial}{\partial x_1}\left\{ \frac{B_2(X_1)V_2}{X_1 + B_2(X_1) + n - 2} - b_2(x_1) - \frac{V_2}{X_1 + n - 1} \right\}$$
$$= \frac{\partial X_1}{\partial x_1} \cdot \left(-\frac{B_2(X_1)V_2}{(X_1 + B_2(X_1) + n - 2)^2} + \frac{V_2}{(X_1 + n - 1)^2} \right) \tag{36}$$

[29] Thus, for continuous arguments $n > 2$, the function $a \equiv a(n) > 4$ uniquely solves the indifference relationship $\exp(a\xi)/(\exp(a\xi) + n - 1) - \xi = 1/n$, where $\xi = (1/a) \cdot \ln((n-1)\lambda(a))$. Alternatively, implicit differentiation of the indifference relationship shows that $a(n)$ uniquely solves the initial value problem

$$\frac{\partial a}{\partial n} = \frac{(\frac{1}{n-1} - \frac{a}{n^2})a}{\ln\left(\frac{n-1}{2}(a - 2 + \sqrt{a(a-4)}) \right)} \qquad (n \geq 3), \tag{31}$$

where $a(3) = 4.1185$.

$$\underset{(\mathrm{FOC})}{=} \underbrace{\frac{\partial X_1}{\partial x_1}}_{>0} \cdot \left(-\frac{1}{(X_1+n-2)\alpha_2^*} + \frac{V_2}{(X_1+n-1)^2} \right) < 0, \tag{37}$$

where $B_2(X_1) = \exp(\alpha_2^* b_2(x_1))$, and we used

$$\frac{(X_1+n-1)^2}{X_1+n-2} > \frac{n^2}{n-1} > a(n) = \alpha_2^* V_2. \tag{38}$$

Integrating over the interval $[0, \widetilde{x}_1(\alpha_2^*)]$, and subsequently exploiting

$$\frac{\widetilde{X}_2(\alpha_2^*)V_2}{\widetilde{X}_2(\alpha_2^*)+n-1} - \widetilde{x}_2(\alpha_2^*) - \frac{V_2}{n} = 0, \tag{39}$$

we arrive at $\Delta_2^{(1)}(\alpha_2^*) < 0$. On the other hand, for obvious economic reasons, $\lim_{\alpha\to\infty} \Delta_2^{(1)}(\alpha) = V_2 > 0$.[30] Hence, in view of the single-crossing property, there exists a unique threshold value $\alpha_1^{**} > \alpha_2^* \geq \alpha_1^*$ such that one-sided dominance by contestant 1 is an equilibrium if and only if $\alpha \in [\alpha_1^*, \alpha_1^{**}]$. The case $n=2$ is dealt with as follows. If $V_1 = V_2 \equiv V$, then dominance by contestant 1 implies $\alpha > 4/V$. Then, however, contestant 2's payoff function against $x_1 = \widetilde{x}_1(\alpha)$ has a strict local minimum at $x_2 = 0$, in conflict with her optimality condition. Thus, one-sided dominance is never an equilibrium for $n=2$ and $V_1 = V_2$. If $V_1 > V_2$, however, then $\alpha_2^* = 4/V_2 > 4/V_1 = \alpha_1^*$, and hence, $\alpha_2^* + \varepsilon \in A_2^{(1)}$ for any $\varepsilon > 0$. Then, the proof proceeds as above, with $\alpha_2^* + \varepsilon$ replacing α_2^* throughout, and lowering $\varepsilon > 0$, if necessary, to satisfy the inequality corresponding to (38). For the case $n=2$ and $V_1 > V_2$, this identifies $\alpha_1^{**} > \alpha_2^*$, and shows that one-sided dominance by contestant 1 is an equilibrium if and only if $\alpha \in (\alpha_1^*, \alpha_1^{**}]$.

Step 7. (*One-sided dominance by contestant* $i \in \{2, \dots, i^*\}$) In straightforward adaption of the argument given above for the case of homogeneous valuations, one shows that one-sided dominance by contestant 2 is generally not feasible if $n=2$. We may therefore assume, without loss of generality, that $n \geq 3$. Take some $i \in \{2, \dots, n\}$. Then, the pivotal contestant is $j=1$. Moreover, from $\alpha_i^* \geq \alpha_1^* > 4/V_1$, we see that $A_j^{(i)} = A_1^{(i)} = [\alpha_i^*, \infty)$. Suppose for the moment that $\Delta_1^{(i)}(\alpha_i^*) > 0$. Then, given the single-crossing property, $\Delta_1^{(i)}(\alpha) > 0$ for all $\alpha \in A_1^{(i)}$, so that one-sided dominance by contestant i cannot be an equilibrium. Moreover, for any $k \in \{i+1, \dots, n\}$, we know that $\alpha_k^* \geq \alpha_i^*$ and $\widetilde{X}_k(\alpha_k^*) = (n-1)\lambda(a(n)) = \widetilde{X}_i(\alpha_i^*)$. Therefore, for any $x_1 \geq 0$,

$$\widehat{\Pi}_1^{(k)}(x_1;\alpha_k^*) - \widehat{\Pi}_1^{(k)}(0;\alpha_k^*) = \frac{\exp(\alpha_k^* x_1)V_1}{\exp(\alpha_k^* x_1) + \widetilde{X}_k(\alpha_k^*)} - x_1 - \frac{V_1}{1+\widetilde{X}_k(\alpha_k^*)} \tag{40}$$

$$\geq \widehat{\Pi}_1^{(i)}(x_1;\alpha_i^*) - \widehat{\Pi}_1^{(i)}(0;\alpha_i^*). \tag{41}$$

Evaluating at $x_1 = \widehat{x}_1^{(i)}(\alpha_i^*)$ shows that $\Delta_1^{(k)}(\alpha_k^*) > 0$. Thus, one-sided dominance by contestant $k > i$ cannot be an equilibrium either. Let, therefore, $i^* \in N$ be the largest $i \in N$ such that $\Delta_j^{(i)}(\alpha_i^*) \leq 0$, where as before, j refers to the pivotal contestant if i is dominating. Then, exploiting the single-crossing property and the limit behavior of $\Delta_1^{(i)}(\alpha)$ for $\alpha \to \infty$ as above, we find that, for any $i \in \{2, \dots, i^*\}$, there is a unique threshold value $\alpha_i^{**} \equiv \alpha^{**}(n, V_i, V_1) \in A_1^{(i)}$ such that $\Delta_1^{(i)}(\alpha) \leq 0$ if $\alpha \leq \alpha_i^{**}$, whereas $\Delta_1^{(i)}(\alpha_i^{**}) > 0$ if $\alpha > \alpha_i^{**}$. Thus, one-sided dominance by contestant $i \in \{2, \dots, i^*\}$ is an equilibrium if and only if $\alpha \in [\alpha_i^*, \alpha_i^{**}]$. On the other hand, one-sided dominance by any contestant $i > i^*$ is never an equilibrium.

Step 8. (*Proof of parts (i) through (iv)*) (i) The claim has been shown in the previous two steps. (ii) The expression for the dominating bid is taken from the proof of Proposition 1. The remaining equations are then immediate. (iii) To prove the strict monotonicity of $\alpha^{**} \equiv \alpha^{**}(n, V_i, \max_{j\neq i} V_j)$ with respect to V_i, one notes that $\partial\Delta_j^{(i)}/\partial V_i = (\partial\Delta_j^{(i)}/\partial\widetilde{X}_i) \cdot (\partial\widetilde{X}_i/\partial V_i)$. But $\partial\widetilde{X}_i/\partial V_i > 0$ is immediate from $\widetilde{X}_i = (n-1)\lambda(\alpha V_i)$. Moreover, using the envelope theorem,

$$\frac{\partial\Delta_j^{(i)}}{\partial\widetilde{X}_i} = -\frac{\widehat{X}_j^{(i)} V_j}{(\widehat{X}_j^{(i)} + \widetilde{X}_i + n - 2)^2} + \frac{V_j}{(\widetilde{X}_i + n - 1)^2} \tag{42}$$

$$\underset{(\mathrm{FOC})}{=} -\frac{1}{(\widetilde{X}_i + n - 2)\alpha} + \frac{V_j}{(\widetilde{X}_i + n - 1)^2} < 0, \tag{43}$$

where the concluding inequality is derived from the fact that marginal payoffs must be negative at $x_j = 0$ to allow for indifference, given that $\widehat{\Pi}_j^{(i)}(\cdot;\alpha)$ has precisely one point of inflection. Next, to see that α^{**} is strictly increasing in n, it suffices to note that $\widetilde{X}_i$ and n are perfect substitutes in the contest technology, so that $\partial\Delta_j^{(i)}/\partial n = (\partial\Delta_j^{(i)}/\partial\widetilde{X}_i) \cdot (1 + \partial\widetilde{X}_i/\partial n) < 0$, where we used $\partial\Delta_j^{(i)}/\partial\widetilde{X}_i < 0$ and $\partial\widetilde{X}_i/\partial n > 0$. Finally, α^{**} is strictly decreasing in V_j, as can be shown in analogy to (16). Given the general comparative statics

[30] Indeed, as $\alpha \to \infty$, contestant 2 wins with probability $\lambda(\alpha V_2)/(\lambda(\alpha V_2)+1) \to 1$ when choosing $x_2 = \widehat{x}_2^{(1)}(\alpha)$, while she wins with probability $1/((n-1)(\lambda(\alpha V_1)+1)) \to 0$ when remaining inactive. Moreover, as $\lambda(\cdot)$ has a linear asymptotics, $\widehat{x}_2^{(1)}(\alpha) = \frac{1}{\alpha}\ln(((n-2)+(n-1)\lambda(\alpha V_1))\lambda(\alpha V_2)) \to 0$ as $\alpha \to \infty$.

properties of α^{**} established above, we conclude that, in particular, $\alpha^{**}_{i+1} = \alpha^{**}(n, V_{i+1}, V_1) \leq \alpha^{**}(n, V_i, V_1) = \alpha^{**}_i$ for $i \in \{2, \dots, n-1\}$. Similarly, $\alpha^{**}_2 = \alpha^{**}(n, V_2, V_1) \leq \alpha^{**}(n, V_1, V_1) \leq \alpha^{**}(n, V_1, V_2) = \alpha^{**}_1$. Hence, $\alpha^*_1 \leq \dots \leq \alpha^*_{i^*} \leq \alpha^{**}_{i^*} \leq \dots \leq \alpha^{**}_1$. Moreover, if $V_{i+1} < V_i$ for some $i < i^*$, then $\alpha^*_{i+1} = a(n)/V_{i+1} > a(n)/V_i = \alpha^*_i$ and $\alpha^{**}_{i+1} = \alpha^{**}(n, V_{i+1}, V_1) < \alpha^{**}(n, V_i, V_1) = \alpha^{**}_i$. It remains to be shown that $\alpha^*_1 < \alpha^{**}_1$ unless $n = 2$ and $V_1 = V_2$. If $n \geq 3$, we know already that $\alpha^*_1 \leq \alpha^*_2 < \alpha^{**}_1$, which proves the claim. But if $n = 2$ and $V_1 > V_2$, then similarly, $\alpha^*_1 = 4/V_1 < 4/V_2 = \alpha^*_2 < \alpha^{**}_1$. (iv) The claim is immediate from Lemma 2 in combination with what has been shown above. □

Proof of Corollary 1. Consider the strategy profile characterized by Proposition 2 where player i is active. It suffices to check that, for V_i/V_1 sufficiently close to unity, contestant i has no incentive to become inactive, while contestant 1 has no incentive to overbid contestant i. But this follows from $\alpha^*_1 < \alpha < \alpha^{**}_1$ and the continuity of α^*_i and α^{**}_i in V_1 and V_i. The claim follows.[31] □

The following lemma establishes an upper bound on the equilibrium payoff for any contestant that uses the zero bid with positive probability. This lemma will be used in the proofs of Proposition 3(iv) and Lemma A.4.

Lemma A.1. *Let μ^* be a MSNE in an n-player Hirshleifer contest with valuations $V_1 \geq \dots \geq V_n > 0$. If $i \in N$ satisfies $\mu^*_i(\{0\}) > 0$, then $\Pi^*_i \leq \frac{n}{(n-1)\alpha}$.*

Proof. Clearly, $p_i(0, x_{-i}) \leq \frac{1}{n}$ for any $x_{-i} \in \mathbb{R}^{n-1}_{\geq 0}$. Therefore, contestant i's marginal payoff at $x_i = 0$ is bounded from below by

$$\frac{\partial \Pi_i}{\partial x_i} = \alpha(1 - p_i)p_i V_i - 1 \geq \alpha \frac{n-1}{n} p_i V_i - 1 = \alpha \frac{n-1}{n} \Pi_i - 1. \tag{45}$$

Taking expectations with respect to μ^*_{-i}, and subsequently exchanging differentiation and integration, one obtains $\partial E_{\mu^*_{-i}}[\Pi_i]/\partial x_i \geq \frac{(n-1)\alpha}{n} \Pi^*_i - 1$. But $\partial E_{\mu^*_{-i}}[\Pi_i]/\partial x_i \leq 0$ from the KKT condition at $x_i = 0$, which proves the claim. □

Proof of Proposition 3. (i) The proof is analogous to that of Lemma 1, with Becker and Damianov (2006, Thm. 1) replacing Glicksberg's theorem. (ii) *(Sufficiency)* Suppose that $\alpha > \alpha^*$. By part (i), there exists a symmetric MSNE. This equilibrium cannot be a PSNE by Proposition 2(iv). Hence, $L \geq 2$, as claimed. *(Necessity)* It is claimed that multilateral peace is the unique symmetric MSNE if $\alpha \leq \alpha^*$. For $\alpha < \alpha^*$, this follows from Proposition 1(iii). The argument extents to $\alpha = \alpha^*$ and $n = 2$, as we have seen. Suppose therefore that $\alpha = \alpha^*$ and $n \geq 3$. If the equilibrium is not multilateral peace then, by the symmetry of the equilibrium, any contestant i uses $y^{\min}_i = y^{(L-1)} > 0$ with positive probability. Therefore, exploiting that $n \geq 3$, the second inequality in (18) becomes strict, while the third inequality holds weakly. The resulting contradiction shows that a symmetric MSNE other than multilateral peace is indeed not feasible for $\alpha = \alpha^*$ and $n \geq 3$. Clearly, this proves the claim. (iii) Take a symmetric MSNE μ^* in the n-player Hirshleifer contest with parameter $\alpha > 0$. Then, as discussed, there exist effort levels $y^{(1)} > \dots > y^{(L)} = 0$, for some $L \geq 1$, with corresponding probabilities $q^{(1)}, \dots, q^{(L)} \in [0, 1]$, such that each contestant $i \in N$ chooses $y^{(l)}$ with probability $q^{(l)}$, for any $l \in \{1, \dots, L\}$. Take some $l \in \{1, \dots, L\}$. By the KKT condition at the optimum $x_1 = y^{(l)}$, we get for contestant 1 that

$$1 \geq \frac{\partial E_{\mu^*_{-1}}[p_1(x_1, x_{-1})V]}{\partial x_1} \tag{46}$$

$$= \alpha V E_{\mu^*_{-1}}[p_1(x_1, x_{-1})(1 - p_1(x_1, x_{-1}))] \tag{47}$$

$$= \alpha V \sum_{l_2=1}^{L} \dots \sum_{l_n=1}^{L} \left(\prod_{i=2}^{n} q^{(l_i)} \right) p_1(x_1, y^{(l_2)}, \dots, y^{(l_n)})(1 - p_1(x_1, y^{(l_2)}, \dots, y^{(l_n)})) \tag{48}$$

$$\geq \alpha V \left(q^{(l)}\right)^{n-1} \frac{1}{n} \left(1 - \frac{1}{n}\right), \tag{49}$$

where the inequality in (49) is obtained by dropping all terms corresponding to scenarios in which some contestant $j \neq 1$ exerts an effort different from $y^{(l)}$. Rewriting yields

$$q^{(l)} \leq \left(\frac{n^2}{(n-1)\alpha V} \right)^{\frac{1}{n-1}}, \tag{50}$$

for any $l \in \{1, \dots, L\}$. Since $q^{(1)} + \dots + q^{(L)} = 1$, this proves the claim. (iv) By Lemma 2 and symmetry, $\mu^*_i(\{0\}) > 0$ for all $i \in N$. Therefore, the claim follows directly from Lemma A.1. □

[31] More specifically, it follows from the definition of the threshold value $\alpha^*(n, V) = a(n)/V$ that one-sided dominance by contestant i is an equilibrium *for some* $\alpha > 0$ if and only if

$$\frac{\max_{j \neq i} V_j}{V_i} \leq \rho(n) \equiv \frac{a((n-1)(\lambda(a(n)) + 1))}{a(n)}, \tag{44}$$

where the inequality is understood to be strict if $n = 2$. Computations show that $\rho(2) = 1$, $\rho(3) \approx 1.07$, $\rho(4) \approx 1.17$, $\rho(5) \approx 1.20$, etc.

The following two lemmas are used in the proof of Proposition 4.

Lemma A.2. *Let $i, j_1, j_2 \in N$ be such that $i \neq j_1 \neq j_2 \neq i$. Then, for any $x_{-(j_1,j_2)} \in \mathbb{R}^{n-2}_{\geq 0}$, the mapping $(x_{j_1}, x_{j_2}) \mapsto \Pi_i(x_1, \ldots, x_n)$ exhibits strictly increasing differences in (x_{j_1}, x_{j_2}).*

Proof. Let $X_k = \exp(\alpha x_k)$, for $k \in N$, and $X = \sum_{k=1}^n X_k$. Then,

$$\frac{\partial^2 \Pi_i}{\partial x_{j_1} \partial x_{j_2}} = \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} \left(\frac{X_i V_i}{X} - x_i \right) = \frac{2\alpha^2 X_{j_1} X_{j_2} X_i V_i}{X^3} > 0. \tag{51}$$

The claim follows. □

Lemma A.3. *Suppose that $f(x_1, \ldots, x_m)$, with $m \geq 2$, is symmetric and exhibits pairwise weakly increasing differences. Then, for any $y \geq x$,*

$$f(y, \ldots, y) \geq m f(y, x, \ldots, x) - (m-1) f(x, \ldots, x). \tag{52}$$

Proof. By induction. For $m = 2$, weakly increasing differences imply $f(y, y) - f(y, x) \geq f(x, y) - f(x, x)$, while $f(y, x) = f(x, y)$ by symmetry. Thus, $f(y, y) \geq 2f(y, x) - f(x, x)$, which is just (52) for $m = 2$. Suppose that the claim has been shown for some $m \geq 2$. Take some f with $m + 1$ arguments. Then, letting $g(x_1, \ldots, x_m) = f(y, x_1, \ldots, x_m)$ and $h(x_1, x_2) = f(x_1, x_2, x, \ldots, x)$, both g and h are symmetric and exhibit pairwise weakly increasing differences. Therefore, using the induction hypothesis (52) for g and h, respectively, one obtains

$$g(y, \ldots, y) \geq m g(y, x, \ldots, x) - (m-1) g(x, \ldots, x) \tag{53}$$

$$= m h(y, y) - (m-1) h(y, x) \tag{54}$$

$$\geq (m+1) h(y, x) - m h(x, x). \tag{55}$$

Hence, the claim holds for $m + 1$ and, therefore, for all $m \geq 2$. □

Proof of Proposition 4. Given a mixed strategy profile $\mu = (\mu_1, \ldots, \mu_n)$, we denote by

$$\widetilde{\Pi}_i(\mu) = E_\mu \left[\Pi_i(x_1, \ldots, x_n) - \frac{V}{n} + \frac{1}{n-1} \sum_{j \neq i} x_j \right] \tag{56}$$

a normalized version of contestant i's expected payoff.[32] Clearly,

$$\sum_{i=1}^n \widetilde{\Pi}_i(\mu) = 0. \tag{57}$$

Let $\boldsymbol{\mu}^*_{-1} = (\underbrace{\mu^*_1, \ldots, \mu^*_1}_{(n-1) \text{ times}})$, and $\widehat{\boldsymbol{\mu}}^*_{-1} = (\mu^{**}_1, \underbrace{\mu^*_1, \ldots, \mu^*_1}_{(n-2) \text{ times}})$. By assumption, $(\mu^*_1, \boldsymbol{\mu}^*_{-1})$ is an equilibrium. Hence,

$$\widetilde{\Pi}_1(\mu^*_1, \boldsymbol{\mu}^*_{-1}) \geq \widetilde{\Pi}_1(\mu^{**}_1, \boldsymbol{\mu}^*_{-1}) = -(n-1)\widetilde{\Pi}_1(\mu^*_1, \widehat{\boldsymbol{\mu}}^*_{-1}), \tag{58}$$

where the equality follows from (57) and the symmetry of the contest. By Lemma A.2 and Echenique (2003, Lemma 4), $f(\mu_{-i}) \equiv \widetilde{\Pi}_i(\mu^*_i, \mu_{-i})$ exhibits pairwise weakly increasing differences w.r.t. the $(n-1)$ variables $\{\mu_j\}_{j \neq i}$. Hence, using Lemma A.3,

$$\widetilde{\Pi}_1(\mu^*_1, \boldsymbol{\mu}^{**}_{-1}) \geq (n-1)\widetilde{\Pi}_1(\mu^*_1, \widehat{\boldsymbol{\mu}}^*_{-1}) - (n-2)\widetilde{\Pi}_1(\mu^*_1, \boldsymbol{\mu}^*_{-1}), \tag{59}$$

where $\boldsymbol{\mu}^{**}_{-1} = (\underbrace{\mu^{**}_1, \ldots, \mu^{**}_1}_{(n-1) \text{ times}})$. Adding up (59) and (58) yields

$$(n-1)\widetilde{\Pi}_1(\mu^*_1, \boldsymbol{\mu}^*_{-1}) \geq -\widetilde{\Pi}_1(\mu^*_1, \boldsymbol{\mu}^{**}_{-1}) \geq -\widetilde{\Pi}_1(\mu^{**}_1, \boldsymbol{\mu}^{**}_{-1}) = (n-1)\widetilde{\Pi}_1(\mu^{**}_1, \boldsymbol{\mu}^{**}_{-1}). \tag{60}$$

As this holds analogously for players 2 through n, relationship (57) implies that $\widetilde{\Pi}_i(\mu^*_1, \boldsymbol{\mu}^*_{-1}) = \widetilde{\Pi}_i(\mu^{**}_1, \boldsymbol{\mu}^{**}_{-1})$, for $i \in N$.[33] Therefore, all inequalities above are equalities. But then, necessarily $\mu^*_1 = \mu^{**}_1$ (this follows from a straightforward extension of Echenique's (2003) result to the case of strictly increasing differences), as has been claimed. □

[32] A similar strategic equivalence, connecting all-pay auctions and zero-sum games, has been noted by Pavlov (2023).

[33] Another way to derive this equation is to note that the normalized payoffs correspond to a symmetric n-person zero-sum game in which symmetric equilibrium payoffs are necessarily zero.

Proof of Proposition 5. The first-order condition for contestant i's problem reads

$$\sum_{x_{-i}\in\mathrm{supp}(\mu^*_{-i})}\left(\frac{\alpha X_i X_{-i} V_i}{\left(X_i+X_{-i}\right)^2}\prod_{j\neq i}\mu^*_j(\{x_j\})\right)=1, \tag{61}$$

where $X_i=\exp(\alpha x_i)$ and $X_{-i}=\sum_{j\neq i}\exp(\alpha x_j)$. Multiplying by $\prod_{x_{-i}\in\mathrm{supp}(\mu^*_{-i})}\left(X_i+X_{-i}\right)^2$ yields a polynomial equation of degree $D=2\cdot\prod_{j\neq i}L_j$ in the unknown X_i. By the fundamental theorem of algebra, this equation has at most D solutions. But any two neighboring interior maxima are separated by a local minimum. Hence, $E_{\mu^*_{-i}}[\Pi_i(x_i;x_{-i})]$ admits at most $\prod_{j\neq i}L_j$ interior maxima. □

Proof of Proposition 6. (*Equilibrium property*) Provided that all contestants $j\neq i$ are inactive, it follows from $n\geq 3$, $V_i=V_1$, and $\alpha=\alpha^*_1$ that contestant i is indifferent between her pure best responses $x_i=0$ and $x_i=\widetilde{x}_i(\alpha^*_i)$. Hence, contestant i chooses an optimal randomized strategy. Consider a specific deviation $x_j>0$ by some contestant $j\neq i$. From Proposition 1, $\Pi_j(x_j,\mathbf{0}_{n-1})\leq\Pi_j(0,\mathbf{0}_{n-1})$. Moreover, using again $n\geq 3$, $V_i=V_1$, and $\alpha=\alpha^*_1$, Proposition 2 ensures that $\Pi_j(x_j,\widetilde{x}_i(\alpha^*_i),\mathbf{0}_{n-2})\leq\Pi_j(0,\widetilde{x}_i(\alpha^*_i),\mathbf{0}_{n-2})$. Hence,

$$\begin{aligned}&(1-q_i^{(1)})\Pi_j(x_j,\mathbf{0}_{n-1})+q_i^{(1)}\Pi_j(x_j,\widetilde{x}_i(\alpha^*_i),\mathbf{0}_{n-2})\\&\quad\leq(1-q_i^{(1)})\Pi_j(0,\mathbf{0}_{n-1})+q_i^{(1)}\Pi_j(0,\widetilde{x}_i(\alpha^*_i),\mathbf{0}_{n-2}).\end{aligned} \tag{62}$$

As $x_j>0$ was arbitrary, it is optimal for contestant j to remain inactive, which proves the equilibrium property. *(Characterization of the equilibrium set)* Take some MSNE μ^*. Let $N_1=\{i\in N:V_i=V_1\}$ denote the set of contestants whose valuation equals V_1. For any contestant $j\in N\backslash N_1$, we have $\alpha^*_j>\alpha^*_1=\alpha$, so that exerting an effort of zero is a strictly dominant strategy for j. Therefore, if all contestants $i\in N_1$ are inactive, then μ^* corresponds to multilateral peace, and we are done. Assume next that some contestant $i\in N_1$ is active. Then, since $\alpha=\alpha^*$ and $n\geq 3$, it is not feasible that any other contestant $j\in N_1\backslash\{i\}$ is active, because in that case, the second inequality in (18), possibly after exchanging the roles of i and j, would hold strictly, while the third inequality would hold weakly. Thus, only contestant i is active in μ^*, which shows that the equilibrium takes the claimed form. □

The following three lemmas prepare the proof of Proposition 7. The first lemma is an existence result identifying, for α sufficiently large, a contestant bidding arbitrarily close to V_2.

Lemma A.4. *Suppose that $V_1\geq V_2=\ldots=V_m>V_{m+1}\geq\ldots\geq V_n>0$ for some $m\geq 2$. Let $b\in[V_{m+1},V_m)$, where $V_{n+1}=0$ if $m=n$. Then, for α sufficiently large, any MSNE μ^* has the property that there is a contestant $i\in\{1,\ldots,m\}$ such that*

$$\mu^*_i([0,b])\leq\left(\frac{2+b/V_2}{3}\right)^{\frac{1}{n-1}}. \tag{63}$$

Proof. By Lemma 2, at most one bidder is always active in any μ^*. Hence, given that $m\geq 2$, there is always some contestant $j\in\{1,\ldots,m\}$ that bids zero with positive probability. To provoke a contradiction, suppose that

$$\prod_{i\neq j}\mu^*_i([0,b])>\frac{2+b/V_2}{3}. \tag{64}$$

Then, by bidding $\frac{b+V_2}{2}$, contestant j obtains an expected payoff arbitrarily close to

$$\frac{2+b/V_2}{3}\times V_2-\frac{b+V_2}{2}=\frac{V_2-b}{6}>0, \tag{65}$$

in conflict with Lemma A.1. Thus, there exists at least one $i\neq j$ such that (63) holds true. But since $\mu^*_i([0,b])=1$ for any $i\geq m+1$, necessarily $i\leq m$. □

The next two lemmas capture the intuition that, as the contest becomes excessively decisive, competing lower bids have a negligible impact on the probability of winning.

Lemma A.5. *Let $\delta>0$ and $\varepsilon>0$. Then, for α large enough,*

$$p_i(x_i+\varepsilon,0,x_{-i,j})-p_i(x_i+\varepsilon,x_i,x_{-i,j})<\delta \tag{66}$$

for any $i\in N$, $x_i\geq 0$, and $x_{-i,j}\in\mathbb{R}^{n-2}_{\geq 0}$.[34]

[34] As usual, for a bid vector $x\in\mathbb{R}^n_{\geq 0}$, we write $x_{-i,j}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)\in\mathbb{R}^{n-2}_{\geq 0}$, so that $x=(x_i,x_j,x_{-i,j})$.

Proof. Let $X_{-i,j} = \sum_{k \neq i,j} \exp(\alpha x_k)$. Then,

$$\begin{aligned} & p_i(x_i+\varepsilon, 0, x_{-i,j}) - p_i(x_i+\varepsilon, x_i, x_{-i,j}) \\ &= \frac{\exp(\alpha(x_i+\varepsilon))}{\exp(\alpha(x_i+\varepsilon)) + 1 + X_{-i,j}} - \frac{\exp(\alpha(x_i+\varepsilon))}{\exp(\alpha(x_i+\varepsilon)) + \exp(\alpha x_i) + X_{-i,j}} \quad (67) \\ &= \frac{\exp(\alpha(x_i+\varepsilon))(\exp(\alpha x_i)-1)}{(\exp(\alpha(x_i+\varepsilon)) + \underbrace{1+X_{-i,j}}_{>0})(\exp(\alpha(x_i+\varepsilon)) + \underbrace{\exp(\alpha x_i)+X_{-i,j}}_{>0})} \quad (68) \\ &< \frac{\exp(\alpha x_i)-1}{\exp(\alpha(x_i+\varepsilon))} < \exp(-\alpha\varepsilon). \quad (69) \end{aligned}$$

This proves the claim. □

Lemma A.6. *Let $\delta > 0$. Then, for α large enough, in any MSNE μ^* of the n-player Hirshleifer contest, we have*

$$p_i(y_j^{(1)} + \delta, x_j, x_{-i,j}) > p_j(y_j^{(1)}, x_i, x_{-i,j}) - \delta, \quad (70)$$

for all $x_j \in supp\{\mu_j^*\}$, $x_i \in supp\{\mu_i^*\}$, and $x_{-i,j} \in \mathbb{R}^{n-2}_{\geq 0}$.

Proof. There are two cases. If $x_i > y_j^{(1)}$, then $x_i > x_j$, so that

$$p_i(y_j^{(1)} + \delta, x_j, x_{-i,j}) > p_j(y_j^{(1)}, x_i, x_{-i,j}) > p_j(y_j^{(1)}, x_i, x_{-i,j}) - \delta. \quad (71)$$

If, however, $x_i \leq y_j^{(1)}$, then by Lemma A.5, for α large enough,

$$p_i(y_j^{(1)} + \delta, x_j, x_{-i,j}) \geq p_j(y_j^{(1)} + \delta, 0, x_{-i,j}) - \delta > p_j(y_j^{(1)}, x_i, x_{-i,j}) - \delta. \quad (72)$$

This proves (70). □

Proof of Proposition 7. (i) (*At least two contestants bid close to V_2*) By Lemma A.4, for any α sufficiently large and any MSNE μ^*, there exists a contestant $i \in N$ such that $V_i \geq V_2$ and $y_i^{(1)} \geq V_2 - \frac{\varepsilon}{2}$. We show that, possibly after raising α further, there exists another contestant $j \neq i$ such that $y_j^{(1)} > V_2 - \varepsilon$. Suppose not. Then, $y_j^{(1)} \leq V_2 - \varepsilon$ for any $j \in N \backslash \{i\}$. But then, by the optimality condition at i's highest bid $x_i = y_i^{(1)} > 0$,

$$0 = \alpha V_i E_{\mu^*_{-i}}[p_i(1-p_i)] - 1 \leq \alpha V_i E_{\mu^*_{-i}}[1-p_i] - 1 \leq \frac{(n-1)\alpha V_i}{\exp(\alpha\varepsilon/2) + (n-1)} - 1, \quad (73)$$

where $p_i = p_i(x_i, x_{-i})$. As the ratio in (73) goes to zero for large α, we arrive at a contradiction. Hence, there exists $j \neq i$ such that $y_j^{(1)} > V_2 - \varepsilon$. Next, it is obvious that $y_k^{(1)} < V_2$ for any $k \neq 1$. Hence, arguing as above, $y_1^{(1)} < V_2 + \varepsilon$ for α large enough. Thus, $\left|V_2 - y_i^{(1)}\right| < \varepsilon$ and $\left|V_2 - y_j^{(1)}\right| < \varepsilon$, with $j \neq i$. (*If $V_1 > V_2$, then contestant 1, in particular, is bidding close to V_2*) Assume that $V_1 > V_2$. We have seen already that $y_1^{(1)} < V_2 + \varepsilon$. We prove that $y_1^{(1)} > V_2 - \varepsilon$.[35] To provoke a contradiction, suppose that $y_1^{(1)} \leq V_2 - \varepsilon$. From the derivation above, $y_i^{(1)} > V_2 - \varepsilon$. Hence, $i \neq 1$. Moreover, since $V_i \geq V_2$, we see that $V_i = V_2$. Assume that contestant i uses the bid $x_i = y_1^{(1)} + \delta$, where $\delta > 0$. Then, for α large enough, $\Pi_1^* > V_1 - V_2 - \delta$, because no contestant $i \neq 1$ bids weakly above V_2. From Lemma A.6, taking expectations over μ^*,

$$\frac{\Pi_i^* + y_1^{(1)} + \delta}{V_2} \geq E_{\mu^*_{-i}}[p_i(y_1^{(1)} + \delta, x_{-i})] > E_{\mu^*_{-1}}[p_1(y_1^{(1)}, x_{-1})] - \delta > \frac{V_1 - V_2 - \delta + y_1^{(1)}}{V_1} - \delta, \quad (74)$$

for α sufficiently large. Rewriting yields $\Pi_i^* > \frac{(V_1-V_2)(V_2-y_1^{(1)})}{V_1} + o(1) \geq \frac{(V_1-V_2)\varepsilon}{V_1} + o(1)$, where $o(1)$ refers to a term vanishing as $\alpha \to \infty$. Hence, Π_i^* remains bounded away from zero, in conflict to what as been shown above. Thus, if $V_1 > V_2$, then contestant 1 bids ε-close to V_2, for α sufficiently large. (ii) *(Contestant 1's equilibrium payoff is close to $V_1 - V_2$)* It has been seen above that $\Pi_1^* > V_1 - V_2 - \varepsilon$. On the other hand, by part (i), for α large enough, there exists a subset $N^* \subseteq N$ consisting of at least two contestants, such that $y_i^{(1)} \geq V_2 - \varepsilon$ for all $i \in N^*$. Hence, $\Pi_i^* < V_i - V_2 + \varepsilon$, for all $i \in N^*$. In particular, if $V_1 > V_2$, then $1 \in N^*$ and $\Pi_1^* < V_1 - V_2 + \varepsilon$. Thus, $\left|\Pi_1^* - (V_1 - V_2)\right| < \varepsilon$. (*The equilibrium payoff of the other contestants is close to zero*) Clearly, we may assume without loss of generality that $\varepsilon > 0$ is small enough such that $V_{m+1} - V_2 + \frac{\varepsilon}{2} \leq 0$, where m is defined as in Lemma A.4 above. But since $\Pi_i^* > 0$, this implies $V_i = V_2$ and $\Pi_i^* < \frac{\varepsilon}{2}$ for all $i \in N^* \backslash \{1\}$. We show that, possibly after raising α further, $\Pi_j^* < \varepsilon$ for any $j \in N \backslash N^*$. Suppose not.

[35] The following argument is adapted from Baye et al. (1990).

Then, $\Pi_j^* \geq \varepsilon$ for some $j \in N \backslash N^*$. Suppose that some contestant $i \in N^* \backslash \{1\}$ overbids j using the bid $y_j^{(1)} + \delta$, where $\delta = \frac{\varepsilon}{2(1+V_2)} > 0$. Then, from Lemma A.6, we see that $E_{\mu_{-i}^*}[p_i(y_j^{(1)} + \delta, x_{-i})] \geq E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_{-j})] - \delta$, resulting in $\frac{\varepsilon}{2} > \Pi_i^* \geq \Pi_j^* - \delta V_2 - \delta \geq \frac{\varepsilon}{2}$, which is impossible. (iii) (*Contestants not bidding close to V_2 ultimately drop out*) To provoke a contradiction, suppose that $y_j^{(1)} \in [\varepsilon, V_2 - \varepsilon]$ for some $j \in N$. By Lemma A.4, there exist $\widetilde{\varepsilon} > 0$, not dependent on α, such that $\mathrm{pr}\{x_i \leq V_2 - \frac{\varepsilon}{2}\} < 1 - \widetilde{\varepsilon}$ for some $i \in N$ such that $V_i = V_2$. Since $y_j^{(1)} < V_2 - \varepsilon$, this implies $\mathrm{pr}\{x_i \leq y_j^{(1)} + \frac{\varepsilon}{2}\} < 1 - \widetilde{\varepsilon}$. We know that $\Pi_j^* = E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_i, x_{-i,j})]V_j - y_j^{(1)} > 0$. But,

$$E_{\mu_{-j}^*}\left[p_j(y_j^{(1)}, x_i, x_{-i,j})\right] = \underbrace{\mathrm{pr}\{x_i \leq y_j^{(1)} + \tfrac{\varepsilon}{2}\}}_{<1-\widetilde{\varepsilon}} \cdot E_{\mu_{-j}^*}\left[p_j(y_j^{(1)}, x_i, x_{-i,j})\middle| x_i \leq y_j^{(1)} + \tfrac{\varepsilon}{2}\right] \tag{75}$$

$$+ \underbrace{\mathrm{pr}\{x_i > y_j^{(1)} + \tfrac{\varepsilon}{2}\}}_{\leq 1} \cdot \underbrace{E_{\mu_{-j}^*}\left[p_j(y_j^{(1)}, x_i, x_{-i,j})\middle| x_i > y_j^{(1)} + \tfrac{\varepsilon}{2}\right]}_{=o(1)}.$$

Moreover, by Lemma A.6,

$$E_{\mu_{-j}^*}\left[p_j(y_j^{(1)}, x_i, x_{-i,j})\middle| x_i \leq y_j^{(1)} + \tfrac{\varepsilon}{2}\right] < E_{\mu_{-i}^*}\left[p_i(y_j^{(1)} + \delta, x_j, x_{-i,j})\right] + \delta, \tag{76}$$

where $\delta = o(1)$. Hence, by bidding $y_j^{(1)} + \delta$, contestant i receives a payoff of

$$E_{\mu_{-i}^*}[\Pi_i(y_j^{(1)} + \delta, x_{-i})] \geq \left(\frac{(\Pi_j^* + y_j^{(1)})/V_j - o(1)}{1 - \widetilde{\varepsilon}} - \delta\right)V_2 - y_j^{(1)} - \delta > \underbrace{y_j^{(1)}}_{\geq \varepsilon} \frac{\widetilde{\varepsilon}}{1 - \widetilde{\varepsilon}} + o(1), \tag{77}$$

in conflict with $\Pi_i^* = o(1)$, which follows from part (ii). This proves the claim. (iv) (*If $V_1 = V_2$ then at least two contestants are always active in the limit*) Suppose that $V_1 = V_2$ but, regardless of α, there exists a contestant i such that $\mu_j^*(\{0\}) \geq \varepsilon$ for all $j \neq i$. Then, contestant i could bid slightly above zero and claim a positive rent bounded away from zero as $\alpha \to \infty$. That, however, conflicts with part (ii) above. □

Proof of Proposition 8. For any $n \geq n^{\#}(\alpha) \equiv \exp(\alpha \overline{V}) + 1$, let μ^* be a MSNE in the n-player Hirshleifer contest with parameter α. Take any $i \in N$ and $x_i \in \mathrm{supp}\{\mu_i^*\}$. Then, exploiting that $\exp(\alpha x_i) < \exp(\alpha \overline{V}) \leq n - 1$, contestant i's bid x_i wins against any $x_{-i} \in \mathrm{supp}\{\mu_{-i}^*\}$ with probability $p_i(x_i, x_{-i}) \leq \exp(\alpha x_i)/(\exp(\alpha x_i) + n - 1) < \frac{1}{2}$. But then, from equation (10), $\partial^2 E_{\mu_{-i}^*}[\Pi_i]/\partial x_i^2 > 0$. Therefore, $x_i = 0$, as claimed. □

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